Electronics

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TEXT BOOK 145 TEXT BOOK
ELECTRICITY & ELECTRONICS
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1 Vectors

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1.1. SCALARS AND VECTORS

The physical quantities which are completely defined by magnitude alone, are called *scalars*. They are represented by *Latin* or *Greek* letters. They obey the ordinary laws of algebra. Examples are mass, charge, time, temperature, etc. In addition, there are other physical quantities which are completely determined by a magnitude and a direction in space and follow the parallelogram law of addition. These are called *vectors* and are represented by *boldface letters* or *Greek letters with arrow over them*. They are represented geometrically by an arrow pointing in the direction of the vector and of length equal to its magnitude. Examples are velocity, displacement, force, momentum, electric and magnetic fields, etc. The vectors may be classified as :(1) polar vector and (2) axial vector. For the quantity, in which more linear action in a particular direction is involved, the vector is called a *polar vector* (*e.g.*, displacement, velocity, force, etc). For the quantity, in which rotary action of some kind takes place about an axis, the vector is called *axial vector* (e.g., angular velocity, angular acceleration, etc.).

A vector which has no magnitude is called a *null* or *zero* vector. If it is added with any vector **A**, the vector **A** will always remain unchanged, *i.e.*,

$$\mathbf{0}+\mathbf{A}=\mathbf{A}. \qquad \ldots (1)$$

A vector whose module is unity is called *unit vector*. It is represented by a letter with a cap over it. On multiplying a scalar with the unit vector we get a vector in the direction of the unit vector, *i.e.*,

$$\mathbf{A}=A\hat{A}. \qquad \ldots (2)$$

Two vectors **A** and **B** are said to be equal if and only if they have the same magnitude and the same direction, *i.e.*,

$$\mathbf{A}=\mathbf{B} \text{ or } A\hat{A}=B\hat{B}, \text{ if } A=B \text{ and } \hat{A}=\hat{B}. \qquad \ldots (3)$$

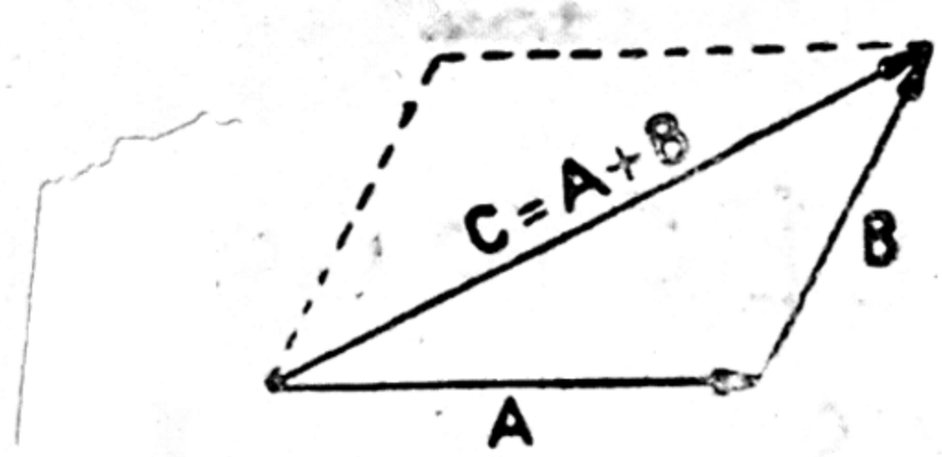


Fig. 1.1

The *vector sum* of two vectors can be obtained as follows : Draw the vector **A** and **B**, as in Fig. 1.1, at any point, with the initial point of **B** (tail of **B**) at the end point of **A** (head of **A**). The vector sum **C** is the vector from the initial point of **A** upto the end point of **B**. We thus have

$$\mathbf{C}=\mathbf{A}+\mathbf{B}. \qquad \ldots (4)$$

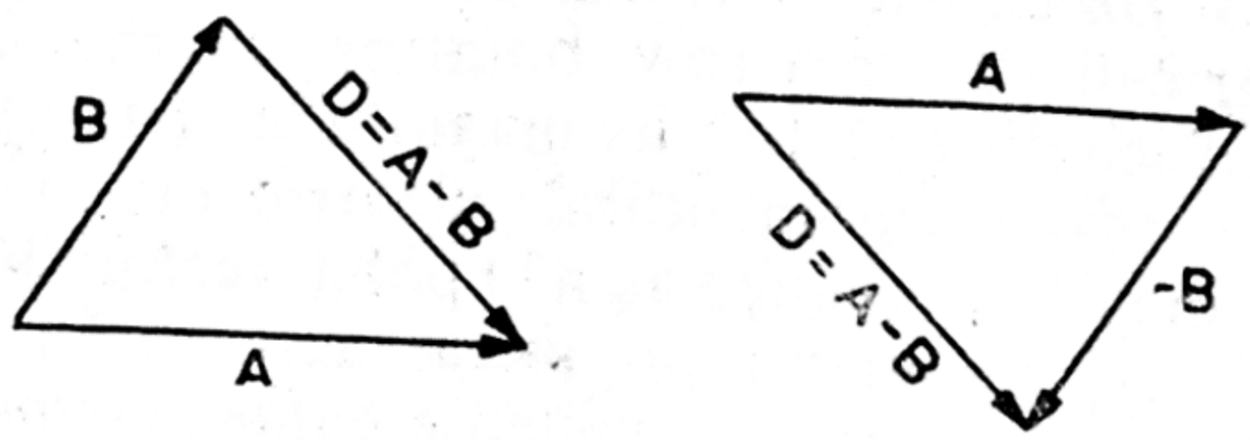
C is the diagonal of the parallelogram with sides **A** and **B** and its magnitude and direction can be found by trigonometry (Fig 1.1)

The resultant **C** may also be obtained by adding vector **B** with vector **A** (*i.e.*, $\mathbf{B}+\mathbf{A}=\mathbf{C}$). Thus the *vector addition is commutative.* This method may be extended for addition of three or more vectors. The summation is independent of the order and grouping of terms, *e.g.*,

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}. \qquad \ldots (5)$$

Hence the *vector sum is associative.*

For *vector subtraction,* vectors **A** and **B** are drawn from the



(*a*) (*b*)

Fig 1.2

common origin and the difference $\mathbf{A}-\mathbf{B}$ is the vector from the head of **B** to head of **A** [Fig 1.2 (*a*)]. Thus

$$\mathbf{D}=\mathbf{A}-\mathbf{B}. \qquad \ldots (6)$$

As the vector $-\mathbf{B}$ is the vector of magnitude *B* but possesses an opposite direction hence the difference $\mathbf{A}-\mathbf{B}$ may be obtained by adding to the vector **A** a vector $-\mathbf{B}$ [Fig 1.2 (b)].

If **A** and **B** are parallel vectors, then the magnitude of **C** is equal to the scalar sum of the magnitudes of **A** and **B** and the magnitude of **D** is equal to the scalar difference.

The *converse process* is often useful, a vector **C** may be resolved into two vectors **A** and **B**, such that **C** is the diagonal of a parallelogram of the adjacent side **A** and **B**. Generally **A** and **B** are chosen to be at right angles. Any vector in space may be resolved into three components parallel to the axes of cartesian ordinates.

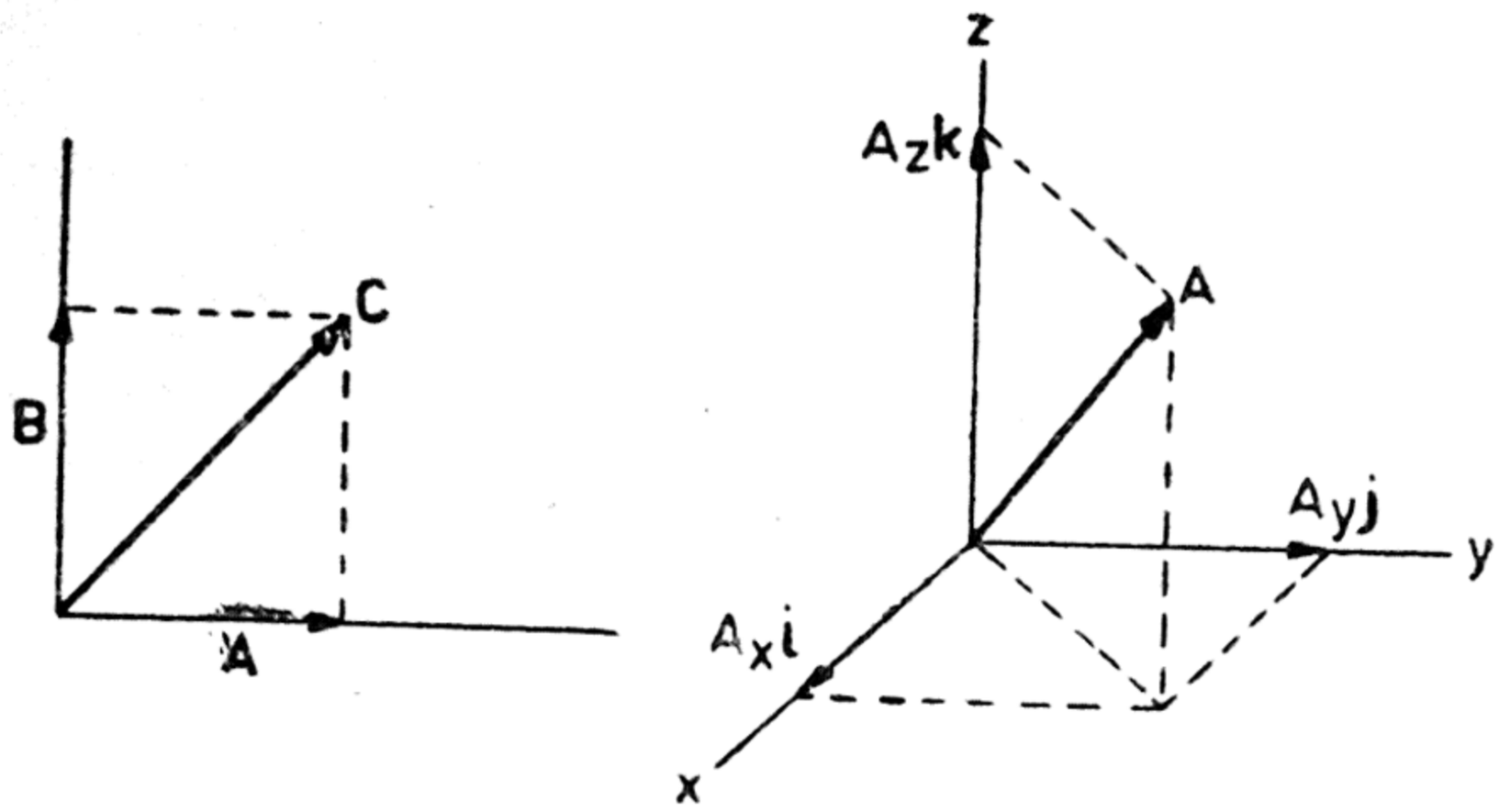


Fig. 1.3

If A_x, A_y and A_z are the magnitudes of components of the vector **A** parallel to the axes x, y and z respectively, then

$$\mathbf{A}=A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}. \qquad \ldots(7)$$

where **i**, **j** and **k** are vectors of unit length along the axes x, y and z respectively. The sum of two vectors may thus be written as

$$\mathbf{A}+\mathbf{B}=(A_x+B_x)\mathbf{i}+(A_y+B_y)\mathbf{j}+(A_z+B_z)\mathbf{k}. \qquad \ldots(8)$$

1.2 MULTIPLICATION OF VECTORS

When a vector **A** is multiplied by a scalar b, a new vector **C** is formed whose direction is the same as that of the original vector **A** and whose magnitude is the product of the scalar b and the magnitude A of the original vector **A**. Thus

$$\mathbf{C}=b\mathbf{A}=b[A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}]. \qquad \ldots(9)$$

Similarly

$$(a+b)\ \mathbf{A}=a\mathbf{A}+b\mathbf{A}. \qquad \ldots(10)$$

Following two types of vector multiplication have been defined.

(*i*) **Scalar Product.** The scalar product of two vectors **A** and **B** is a scalar quantity. It is indicated by a dot, placed between the two vectors and thus is also known as *dot-product*. Its magnitude is equal to the product of the magnitudes of two vectors and the cosine of the angle between them.

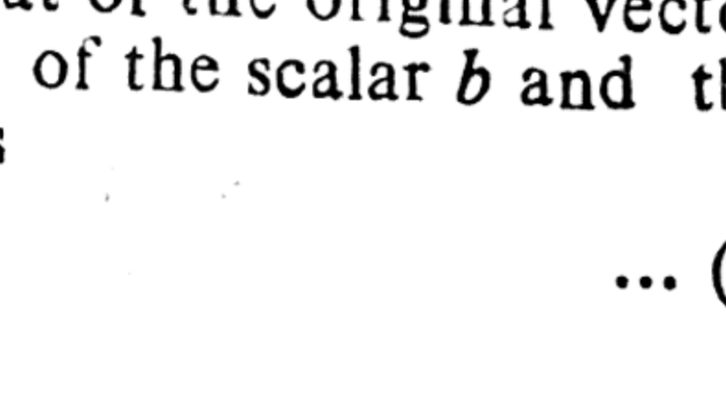


Fig. 1.4

$$\mathbf{A}.\mathbf{B}=AB\cos\theta=\mathbf{B}.\mathbf{A} \qquad \ldots(11)$$

This relation shows that the scalar product is commutative. As $B\cos\theta$ = projection of **B** along **A**, hence the scalar product may be defined as *the scalar quantity numerically equal to the magnitude of one vector multiplied by the component of the other along the first.*

The scalar product of two perpendicular vectors is zero. Thus for the unit vectors **i**, **j** and **k**, we have

$$\mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{k}=\mathbf{k}.\mathbf{i}=0 \quad \text{and} \quad \mathbf{i}.\mathbf{i}=\mathbf{j}.\mathbf{j}=\mathbf{k}.\mathbf{k}=1. \qquad ...(12)$$

$$\therefore \quad \mathbf{A}.\mathbf{B}=(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}).(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k})$$

$$=A_xB_x+A_yB_y+A_zB_z. \qquad ...(13)$$

Relation (11) shows that

$\mathbf{A}.\mathbf{B}=AB\cos 0=AB$, if $\mathbf{A} \parallel \mathbf{B}$

$\mathbf{A}.\mathbf{B}=AB\cos\pi=-AB$ if $\theta=\pi$

$\mathbf{A}.\mathbf{B}=0$ if either $\theta=\pi/2$ or both or at least one vector is zero.

(*ii*) **Vector Product.** The *vector product of two vectors is a vector quantity.* It is indicated by placing a cross between the vectors, hence is also called *cross product.* Its magnitude is equal to the product of the magnitudes of two vectors and the sine of the

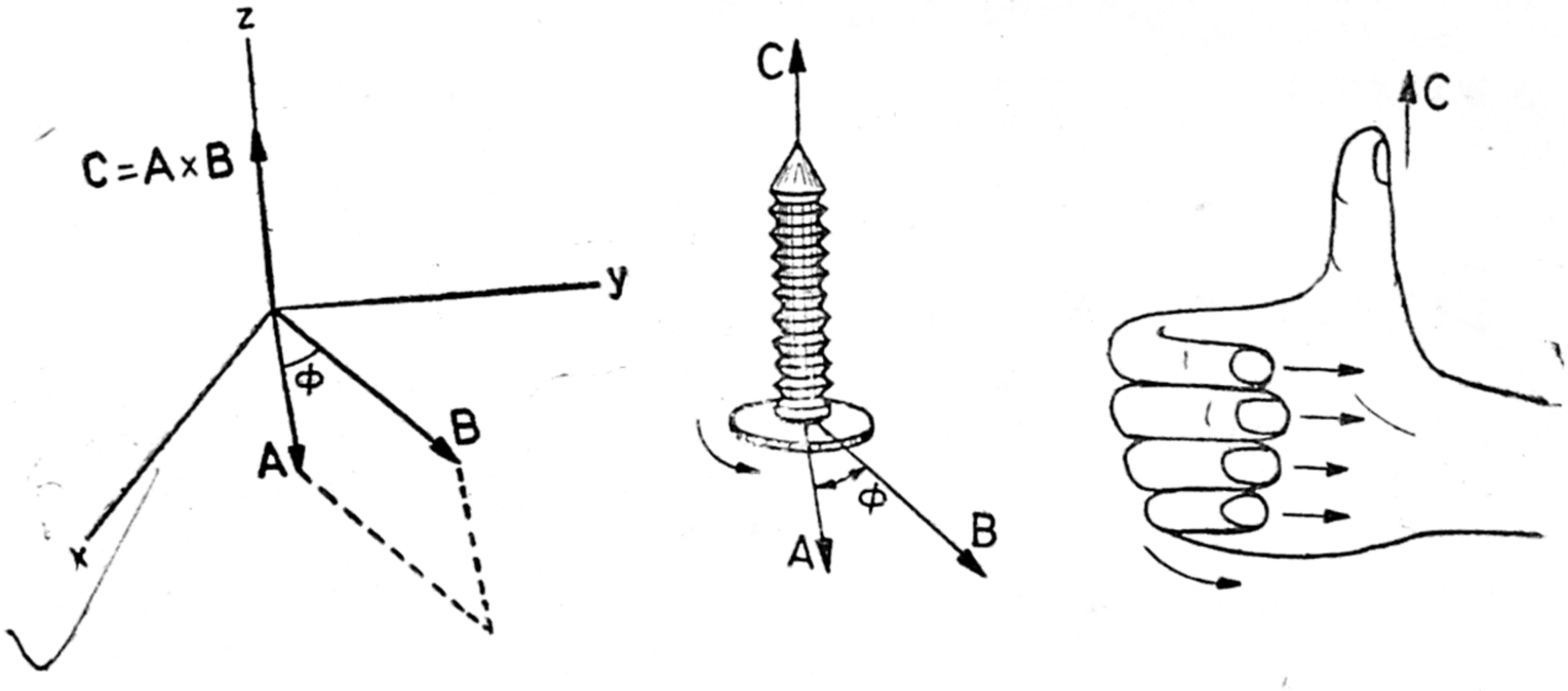


Fig. 1.5.

angle between them. *Its direction is ⊥ to the plane containing these vectors and is governed by the right handed screw if turned from the first vector towards the second vector.* If vectors **A** and **B** are in $x-y$ plane, the resultant vector $\mathbf{C}=\mathbf{A}\times\mathbf{B}$ is ⊥ to both **A** and **B** and is therefore in the z-direction.

$$\mathbf{C}=\mathbf{A}\times\mathbf{B}=AB\sin\phi\,\hat{n}=-\mathbf{B}\times\mathbf{A}. \qquad ...(14)$$

where $\hat{n}$ indicates the direction of normal.

Since $AB\sin\theta$ represents the area of the parallelogram having **A** and **B** as adjacent sides. Hence vector product of two vectors may be defined as a vector of magnitude equal to the area of the parallelogram with these vectors as its adjacent sides and in the direction ⊥ to this area (Fig 1.5).

Another way to obtain the direction of a vector product is the *right hand rule.* If we wrap fingers of the right hand around the axis, which is ⊥ to the plane of vectors **A** and **B**, so that the curled fingers follow the rotation of **A** into **B**, the extended right thumb will give the direction of the vector product $\mathbf{A}\times\mathbf{B}$.

Since $\mathbf{A}\times\mathbf{B}\neq\mathbf{B}\times\mathbf{A}$, hence the vector product is not *commutative*. Relation (14) shows that

$$\mathbf{A}\times\mathbf{B}=0 \qquad \text{if } \mathbf{A}\parallel\mathbf{B} \text{ or } \mathbf{A}=\mathbf{B}$$

$$\mathbf{A}\times\mathbf{B}=AB\hat{n} \qquad \text{if } \mathbf{A}\perp\mathbf{B}.$$

For unit vectors, we have

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{k}=\mathbf{i},\ \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

and
$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0. \qquad \ldots(15)$$

In cartesian co-ordinates, the vector product may be expressed as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\mathbf{i}(A_yB_z-A_zB_y)+\mathbf{j}(A_zB_x-A_xB_z)+\mathbf{k}(A_xB_y-A_yB_x).$$

1.3. MULTIPLE PRODUCT

$$(i)\ \mathbf{A}.(\mathbf{B}+\mathbf{C})=\mathbf{A}.\mathbf{B}+\mathbf{A}.\mathbf{C} \qquad \ldots(16)$$

$$(ii)\ \mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}. \qquad \ldots(17)$$

$$(iii)\ (\mathbf{A}.\mathbf{B})\mathbf{C}=(AB\cos\theta)\mathbf{C}=ABC\cos\theta \text{ along vector } \mathbf{C}. \qquad \ldots(18)$$

$$(iv)\ \mathbf{A}.(\mathbf{B}\times\mathbf{C})=\mathbf{B}.(\mathbf{C}\times\mathbf{A})=\mathbf{C}.(\mathbf{A}\times\mathbf{B})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix} \qquad \ldots(19)$$

It is a scalar quantity, equal in magnitude to the volume of the parallelopiped whose sides are formed from the three vectors **A**, **B**, and **C**. On account of this reason, such a vector product is called *scalar triple product*. It has following properties.

(*a*) If **A**, **B**, and **C** are coplaner or parallel to the same plane.

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C})=0.$$

(*b*) If **A**,**B**, and **C** are orthogonal to each other,

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C})=ABC.$$

(*c*) If any two of the three are identical or parallel.

$$\mathbf{A}.(\mathbf{A}\times\mathbf{C})=\mathbf{A}.(\mathbf{B}\times\mathbf{A})=\mathbf{A}.(\mathbf{B}\times\mathbf{B})=0.$$

$$(v)\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}.\mathbf{C})\mathbf{B}-(\mathbf{A}.\mathbf{B})\ \mathbf{C}. \qquad \ldots(20)$$

It is a vector quantity and is thus called *vector triple product*
One can prove very easily that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0, \qquad \ldots(21)$$

1.4. DIFFERENTIATION AND INTEGRATION OF VECTORS

Vector quantities are often expressed as functions of scalar variables. For example, the electric field **E** can be expressed as a function of the position co-ordinates x, y and z. The vector may be differentiated and integrated with respect to these variables. Thus if $\mathbf{F}(u)$ is a vector function of the scalar variable u, the differential of **F** with respect to a scalar variable u is defined as

$$\frac{d\mathbf{F}}{du} = \underset{\Delta u \to 0}{\text{limit}} \frac{\Delta \mathbf{F}}{\Delta u} = \underset{\Delta u \to 0}{\text{limit}} \frac{\mathbf{F}(u+\Delta u)-\mathbf{F}(u)}{\Delta u} . \qquad \ldots(22)$$

The derivative is a vector in the direction of $\Delta \mathbf{F}$ as Δu approaches zero. The derivative of the sum of two vectors is the sum of the derivatives. The derivative of a product is the same as that for the derivative of a product of scalars, with due regard for the change in sign if the order of the vectors is changed in the case of the vector product, such as :

If $\mathbf{C}=\mathbf{A}+\mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{\mathbf{A}} + \dot{\mathbf{B}}. \qquad \ldots(23)$$

If $\mathbf{C}=b\mathbf{A}$, then

$$\frac{d\mathbf{C}}{dt} = \frac{db}{dt}\mathbf{A} + b\frac{d\mathbf{A}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{b}\mathbf{A} + b\dot{\mathbf{A}}. \qquad \ldots(24)$$

If $\mathbf{C}=\mathbf{A} \cdot \mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt} = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \dot{\mathbf{B}}. \qquad \ldots(25)$$

If $\mathbf{C}=\mathbf{A} \times \mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt} = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{\mathbf{A}} \times \mathbf{B} + \mathbf{A} \times \dot{\mathbf{B}}. \qquad \ldots(26)$$

We can differentiate the scalar triple product or vector triple product with the ordinary processes of differentiation, keeping the order of vectors as unchanged.

The process of integration is the limit approached in a summation of simple products. When a force $\mathbf{F}$ acts for a small distance $d\mathbf{l}$, the work done $dW=\mathbf{F}.d\mathbf{l}$. Total work done over a large distance is given by

$$W = \int \mathbf{F} \cdot d\mathbf{l} = \int F \cos\theta \, dl. \qquad \ldots(27)$$

This integral is known as line integral of $\mathbf{F}$ along the curve. Similarly we have surface and volume integrals.

1.5. SCALAR AND VECTOR FIELDS

A physical quantity which can be expressed as a continuous function of the position of the point in a region of space is known as a *point or position function*. The region in which it specifies the physical quantity is known as a *field*. A scalar function ϕ which has values throughout a region constitutes a field known as *scalar field*. Examples are the density fields or temperature fields. Vector func-

tion **F** which has values throughout a region constitutes a field known as *vector field.* Examples are the *velocity field* or the *force field.*

1.6. THE GRADIENT OF A SCALAR FIELD

Let ϕ be a scalar point function, which is continuous in the given region of space. In this region, consider two level surfaces (the level surface of a scalar point function is a surface for all points of which the function has the same value, *e.g.*, isothermal or equipotential surfaces) S_1 and S_2 very close together, Fig 1.6. If these surfaces are characterised by scalar functions ϕ and $\phi+\delta\phi$ respectively. Reference to the origin O, we then have

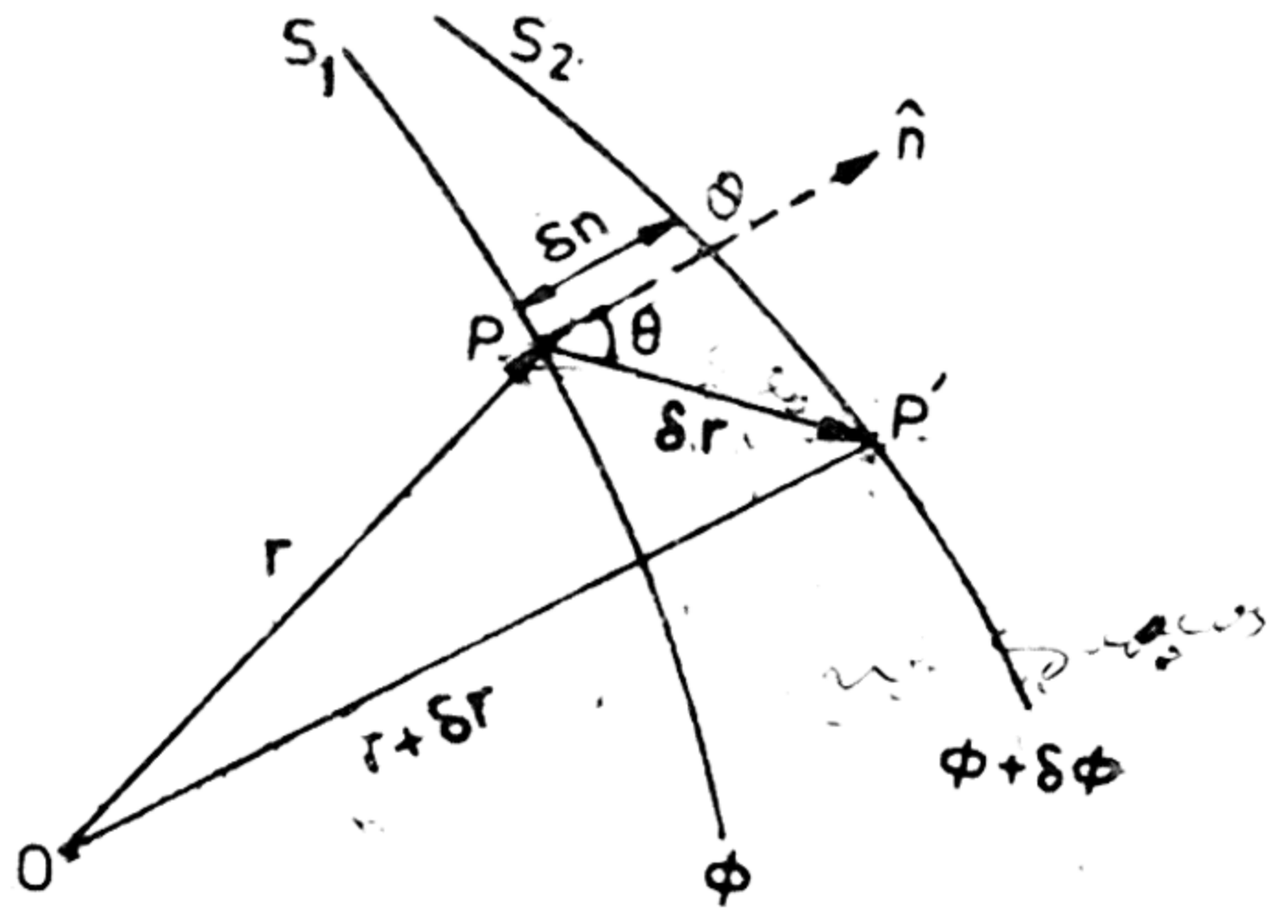


Fig. 1.6

$$\mathbf{OP} = \mathbf{r}, \quad \mathbf{OP'} = \mathbf{r}+\delta\mathbf{r},$$

hence $\mathbf{PP'} = \delta\mathbf{r}$.

The rate of increase of ϕ at point P in the direction $\mathbf{PP'}$ is $\delta\phi/\delta r$ with the limit $\delta r \to 0$. The least distance between the surfaces is PQ in the direction of unit normal vector $\hat{n}$.

Here n represents a unit normal at P to the level surface of ϕ through P in the direction of ϕ increasing. Thus the maximum rate of increase of ϕ is $\partial\phi/\partial n$ and is along $\hat{n}$. If the normal at P makes an angle θ with $\mathbf{PP'}$, then we have

$$\delta n = \delta r \cos\theta \text{ or } \delta n = \hat{n} \cdot \delta\mathbf{r}.$$

$$\therefore \quad \frac{\delta\phi}{\delta r} = \frac{\delta\phi}{\delta n}\frac{\delta n}{\delta r} = \frac{\delta\phi}{\delta n}\cos\theta,$$

As $\cos\theta \leqslant 1$, hence $\delta n \leqslant \delta r$ and $\delta\phi/\delta r < \delta\phi/\delta n$,

$$\text{or} \quad \underset{\delta r\to 0}{\text{Limit}} \frac{\delta\phi}{\delta r} \leqslant \frac{\partial\phi}{\partial n} \quad \ldots(28)$$

Thus the vector $(\partial\phi/\partial n)\,\hat{n}$ gives the greatest rate of increase of ϕ at the point P and is known as *gradient* or *grad of a scalar point function* ϕ. Hence

$$\text{grad}\,\phi = (\partial\phi/\partial n)\,\hat{n}. \quad \ldots(29)$$

It shows that the gradient of a scalar field is a *vector field*. This vector has magnitude equal to the most rapid rate of change of the scalar field and is in the direction in which this rate is maximum.

$$\text{As } \frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial n}.$$

$$\therefore \quad \text{grad } \phi = \frac{\partial \phi}{\partial n}\hat{n} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n}\hat{n} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n}\hat{n} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial n}\hat{n}$$

$$= \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}, \qquad \ldots(30)$$

where $\frac{\partial x}{\partial n}\hat{n}$, $\frac{\partial y}{\partial n}\hat{n}$ and $\frac{\partial z}{\partial n}\hat{n}$ are the unit vectors **i**, **j** and **k** along the axes x, y and z respectively.

$$\therefore \text{ grad } \phi = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]\phi = \nabla\phi, \qquad \ldots(31)$$

where the differential operator ∇ is pronounced as **del** or **nabla**. It is defined as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \qquad \ldots(32)$$

As a differential operator the ∇ vector by itself can be characterized neither by magnitude nor by direction. It only assumes a definite value when it is applied to some field quantity. If it is applied to a scalar ϕ, then we have $\nabla\phi \equiv \text{grad } \phi$. If the scalar function ϕ represents temperature, then $\nabla\phi = \text{grad } \phi$ is a temperature gradient. It is a vector quantity. Its direction being that in which the temperature changes most rapidly.

Important Deductions

(1) Gradient of sum of two scalar functions is the vector sum of their gradients, *i.e.*,

$$\nabla(u+v) = \nabla u + \nabla v.$$

(2) $\nabla(uv) = \nabla uv + v\nabla u.$

(3) $\nabla f(u) = f'(u)\nabla u.$ $\qquad \ldots(33)$

1.7. THE DIVERGENCE OF A VECTOR FIELD

For a physical concept of the divergence of a vector point function, **v**, let us consider an infinitesimal element of volume with sides Δx, Δy and Δz parallel to the axes of x, y and z, respectively

(Fig 1.7). Let the vector point function **v** at the middle of this element has components of magnitude v_x, v_y and v_z along these axes.

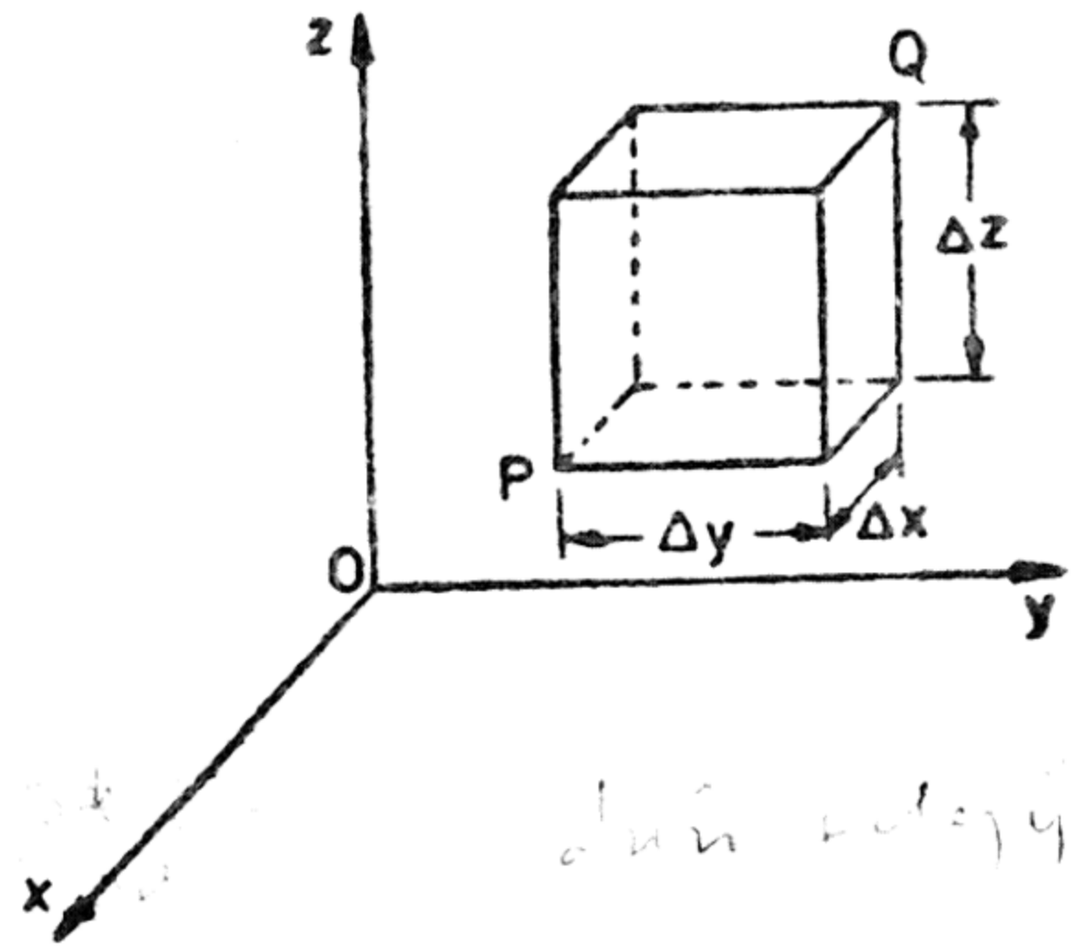


Fig. 1.7.

Consider the two faces of the volume element each with area $\Delta x\,\Delta z$ perpendicular to the y-axis. On the left hand face of the volume, the value of v_y at the middle of the face becomes

$$v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y.$$

Since the face area is infinitesimally small, hence the above value may be taken as the value all over the face. Similarly, on the right hand face, the value of v_y is

$$v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y.$$

We define the flux of a vector field through any face as the scalar product of the vector area of the face and the vector **v**, *i.e.*, the product of the area of the face and the normal component of the vector upon it. Thus the flux entering the element in the y-direction

$$= \left(v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\,\Delta z.$$

and flux leaving the element in the y-direction

$$= \left(v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\,\Delta z.$$

∴ Excess of flux leaving the element over that entering it in the y-direction is

$$\left(v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\,\Delta z - \left(v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\,\Delta z$$

$$= \frac{\partial v_y}{\partial y}\Delta x\,\Delta y\,\Delta z,$$

Similarly the net outward flux in the x-direction

$$= (\partial v_x/\partial x)\Delta x\,\Delta y\,\Delta z.$$

and in the z-direction, it is

$$= (\partial v_z/\partial z)\Delta x\,\Delta y\,\Delta z.$$

$\therefore$ Total net outward flux diverging from or leaving the element

$$= -\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

The amount of this flux per unit volume is defined as the divergence of the vector point function **v** and is thus written as

$$\text{div } \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \qquad \ldots(34)$$

This expression does not contain the dimensions of the box. Since the divergence is the amount of net flux, it is essentially a scalar.

Since

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\ (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$$

$$= \nabla . \mathbf{A}.$$

$$\text{Hence div } \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla . \mathbf{v}. \qquad \ldots(35)$$

Since the operator ∇ is invariant, *i.e.* independent of any system of axes, hence the divergence of any vector is also *invariant*. If the divergence exists at a point in a fluid of density ρ_m, the quantity $[\nabla . (\rho_m \mathbf{v})]$ of fluid is known as the strength of the **v** flow or simply the flux of **v**. This represents the excess of the outward over the inward flow or the divergence of the fluid. The quantity $-\nabla . (\rho_m \mathbf{v})$ represents the excess of the inward over the outward flow, or the convergence of the fluid. Due to this reason $\nabla . (\rho_m \mathbf{v})$ or $\nabla . \mathbf{F}$ is named divergence of $\rho_m \mathbf{v}$ or **F**, denoted by *div*.

In general we can write it for a vector function **F** as

$$\text{div}\mathbf{F} = \nabla .\ \mathbf{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z. \qquad \ldots(36)$$

It is a scalar quantity and can be represented as

$$\nabla . \mathbf{F} = \Sigma \mathbf{i} .\ (\partial \mathbf{F}_x/\partial x). \qquad \ldots(37)$$

Equation (36) can be written as

$$\text{div } \mathbf{F} = \underset{V \to 0}{\text{limit}} \frac{1}{V} \int \mathbf{F} .\ d\mathbf{S}. \qquad \ldots(38)$$

Thus the divergence of a vector is the limit of its surface integral per unit volume as the volume V enclosed by the surface goes to zero. It is a scalar quantity. It represents the net amount of flux coming out of a volume element.

The divergence of a field is positive at any point if either the fluid is expanding and its density is decreasing at that point or there is a *source of fluid*. The divergence is negative if the fluid is contracting and its density is increasing or the point is a *sink*. If the net flow of fluid is zero, divergence is zero.

If there is no source or sink and the density of the fluid is not changing, the fluid is called *incompressible*. The vector field satisfying this condition is said to be *solenoidal field*. Hence for incompressible fluid or for a solenoidal field, divergence of a vector field **F** is always zero, *i.e.*, div $\mathbf{F}=\nabla . \mathbf{F}=0$.

The flux may be flow of liquid, flow of heat or electric flux. Thus in the electric flux, the existence of a finite positive value of the divergence at a point shows that there must be a positive charge at the concerned point.

1.8. LINE INTEGRAL

Let $\mathbf{F}(x,y,z)$ be a vector function defined throughout some region of space with two positions A and B anywhere in this region (Fig 1.8). The line integral of **F** along the curve of some path that runs between A and B is defined as the integral of tangential component of **F** along the curve. Thus

$$\text{line integral}=\int_A^B \mathbf{F}.\, d\mathbf{l}. \qquad \ldots(39)$$

This means : Divide the path into short segments, each segment is represented by a vector $d\mathbf{l}$ and take the scalar product of the path segment vector with the vector **F**. Now add these products for the whole path. If the segments are made shorter, this sum can be represented by the integral from one end to other.

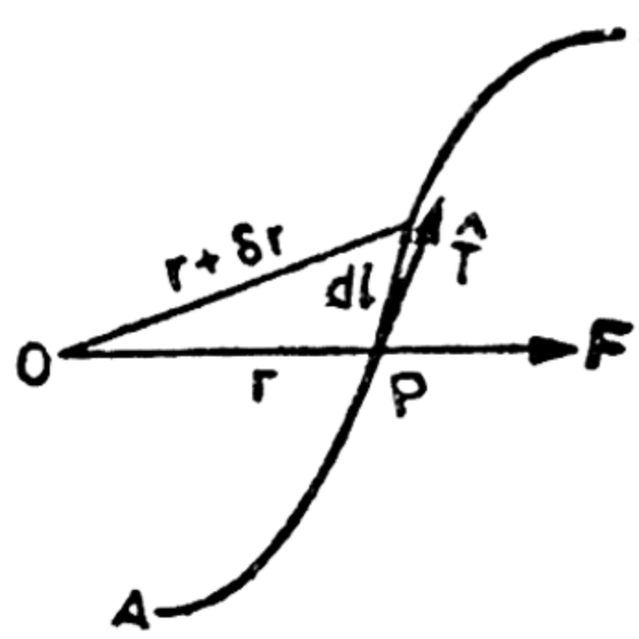


Fig. 1.8

If $\hat{T}$ is a unit vector tangent to the curve at any point P with a position vector **r**. Its value is given as

$$\hat{T}=d\mathbf{r}/dl.$$

$$\therefore \quad \text{Line integral}=\int \mathbf{F}.\, d\mathbf{l}=\int \mathbf{F}.\, \hat{T}\, dl=\int \mathbf{F}.\, \frac{d\mathbf{r}}{dl}\cdot dl$$

$$=\int \mathbf{F}.\, d\mathbf{r} \qquad (\ldots 40)$$

Thus we see that $d\mathbf{l}$ is equivalent to $d\mathbf{r}$. For the three dimensional case, we have

$$\mathbf{F}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k} \text{ and } d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}.$$

$$\therefore \qquad \int \mathbf{F}.\, d\mathbf{r}=\int (F_x dx+F_y dy+F_z dz). \qquad \ldots(41)$$

The line integral may or may not depend upon the path of integration. When it is independent of integration, the field defined by **F** is known as **conservative field.** In this case

$$\int_{C_1} \mathbf{F}.\,d\mathbf{r} = \int_{C_2} \mathbf{F}.d\mathbf{r}, \qquad \ldots(42)$$

where C_1 and C_2 are the two curves with the same end points A and B. If we travel from A to B through curve C_1 and come back to A through C_2, then for conservative field **F**, we get

$$\oint \mathbf{F}.d\mathbf{r} = \int_{C_1} \mathbf{F}.d\mathbf{r} + \int_{C_2} \mathbf{F}.d\mathbf{r} = 0. \qquad \ldots(43)$$

Let $\mathbf{F} = \nabla\,\phi$, therefore

$$\int_B^A \mathbf{F}.d\mathbf{r} = \int_A^B \nabla\phi.d\mathbf{r} = \int_A^B d\phi = \phi_B - \phi_A, \ldots(44)$$

which is independent of path but depends only on the end points, A and B.

For any closed path the above relation reduces to

$$\oint \mathbf{F}.d\mathbf{r} = \int_A^B \mathbf{F}.d\mathbf{r} + \int_B^A \mathbf{F}.d\mathbf{r}$$

$$= \phi_B - \phi_A + \phi_A - \phi_B = 0. \qquad \ldots(45)$$

Thus a conservative field is that field which may be defined as a gradient of a scalar field and for which the closed line integral is always zero. Such fields are called *irrotational fields*. As these are obtained from the scalar by taking gradients, hence are called *scalar potential fields*. Since space is divided up into layers or laminae by the level or equipotential surfaces of the function ϕ, hence the fields are also called the *lamellar vector fields*.

1.9. SURFACE INTEGRAL

If **F** represents a velocity vector **v** of a fluid, the dot product **v**.δ**S** represents the rate of flow of fluid through an area δ**S** (whose direction is perpendicular to the surface) and is also known as *normal velocity flux*. Consider some vector field **F** in space and in this space some arbitrary closed surface. Now divide the whole surface into little patches which are so small that over any one patch the surface is practically flat and the vector field is same at all points on it (Fig 1.9). The dot product **F**. δ**S** is the flux through the patch of surface area δ**S**. If we add all these products for all patches we get the flux Φ through the entire surface.

Fig. 1.9

$$\mathbf{\Phi} = \Sigma \mathbf{F}.\delta\mathbf{S}. \qquad \ldots(46)$$

If the patches are of very small area, the flux Φ can be written as the surface integral, *i.e.*,

$$\Phi = \int_{\text{Entire surface}} \mathbf{F}.d\mathbf{S}. \qquad ...(47)$$

Also the surface is a two-dimensional quantity, we generally use double integral, such as

$$\Phi = \iint_{\mathbf{s}} \mathbf{F}.d\mathbf{S}. \qquad ...(48)$$

Since $d\mathbf{S} = \mathbf{i}dS_x + \mathbf{j}dS_y + \mathbf{k}dS_z$, hence we have

$$\Phi = \iint (F_x dS_x + F_y dS_y + F_z dS_z). \qquad ...(49)$$

1.10. GAUSS'S DIVERGENCE THEOREM

This theorem states that *the surface integral of the normal component of a vector point function* $\mathbf{F}$ *taken over a closed surface* S *enclosing a volume* V *is equal to the volume integral of the divergence of the vector* $\mathbf{F}$ *taken through the volume* V, *i.e.*,

$$\iint_s \mathbf{F}.d\mathbf{S} = \iiint_{\mathbf{v}} \text{div } \mathbf{F} dV. \qquad ...(50)$$

In rectangular co-ordinates

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}, \; dV = dx\, dy\, dz,$$

$dS_x = dy\, dz$, $dS_y = dx\, dz$ and $dS_z = dx\, dy$.

The equation (50) may thus be written as

$$\iint_s (F_x dS_x + F_y dS_y + F_z dS_z) = \iiint_{\mathbf{v}} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx\, dy\, dz \qquad ...(51)$$

To prove this theorem, let us first integrate the second term of right hand side with respect to y. For this we consider a rectangu-

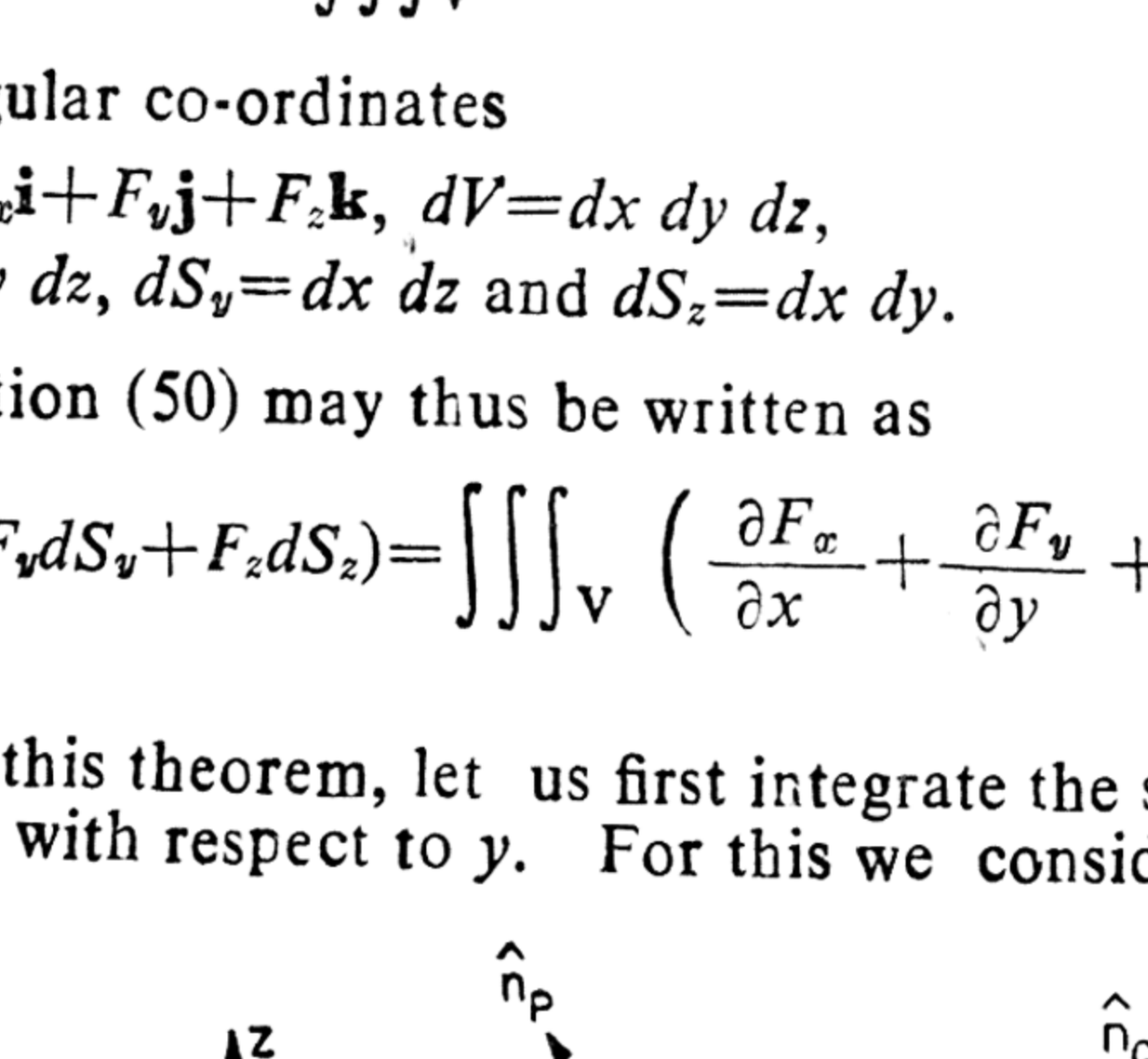


Fig. 1.10.

lar prism PQ with cross sectional area $dx.dz$ and edges parallel to y-axis. We thus get

$$\iiint_V \frac{\partial F_y}{\partial y}\, dx.dy.dz = \iint \Big[F_y\,(x, y_2, z) - F_y(x, y_1, z) \Big] dxdz \qquad \ldots(52)$$

Here x, y_1, z are the co-ordinates of end P and x, y_2, z of the end. Q. Let n_p and n_q be the unit vectors along the normals to the elements of areas dS_P and dS_Q, the areas where the elementary prism PQ meets the close surface. The area dS has a projection on the x-z plane of value $dS_y = dx.\,dz$. As n_p makes an obtuse angle with the y-axis, hence the projection of dS_P on x-z plane

$$= (-\mathbf{i}.n_p)\, dS_P = dx.\,dz = -(dS_y)_P.$$

Similarly for the area dS_Q the projection on x-z plane

$$= (\mathbf{i}.n_Q) dS_Q = dx.dz = (dS_y)_Q.$$

Hence the equation (52) becomes

$$\iiint_V \frac{\partial F_y}{\partial y}\, dx.dy.dz = \iint \Big[(F_y dS_y)_Q + (F_y dS_y)_P \Big].$$

Thus dividing the whole volume into the large number of rectangular prisms with edges parallel to y-axis and of small cross sectional areas. Hence on adding all contributions we get equation (51) as

$$\iiint_V \frac{\partial F_y}{\partial y} dy.dx.dz = \iint F_y dS_y. \qquad \ldots(53)$$

Similarly we can show that

$$\iiint_V \frac{\partial F_x}{\partial x}\, dx.\,dy.\,dz = \iint F_x dS_y \qquad \ldots(54)$$

and

$$\iiint_V \frac{\partial F_z}{\partial z} dz.dx.dy = \iint F_z dS_z. \qquad \ldots(5)$$

On adding these equations, we get

$$\iiint_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx.\,dy.dz.$$

$$= \iint_S (F_x dS_x + F_y dS_y + F_z dS_z).$$

This is *Gauss's divergence theorem.* It is very useful when we transform the surface integral of vector function over a closed surface into a volume integral. This theorem also shows that if the surface integral of a vector **F** is equal to the volume integral of a scalar function P over the volume enclosed by the surface, then we get

$$P = \text{div } \mathbf{F}. \qquad \ldots(56)$$

1.11. ROTATIONAL VECTOR FIELDS, CURL OF A VECTOR

We have read that for irrotational fields the line integral around a closed path is zero. There are also other vector fields for which the line integral around a closed path is not zero, such fields are known as *rotational vector fields*.

The curl is an important property for these vector fields. Let us first understand the *physical significance* of a curl of a vector field.

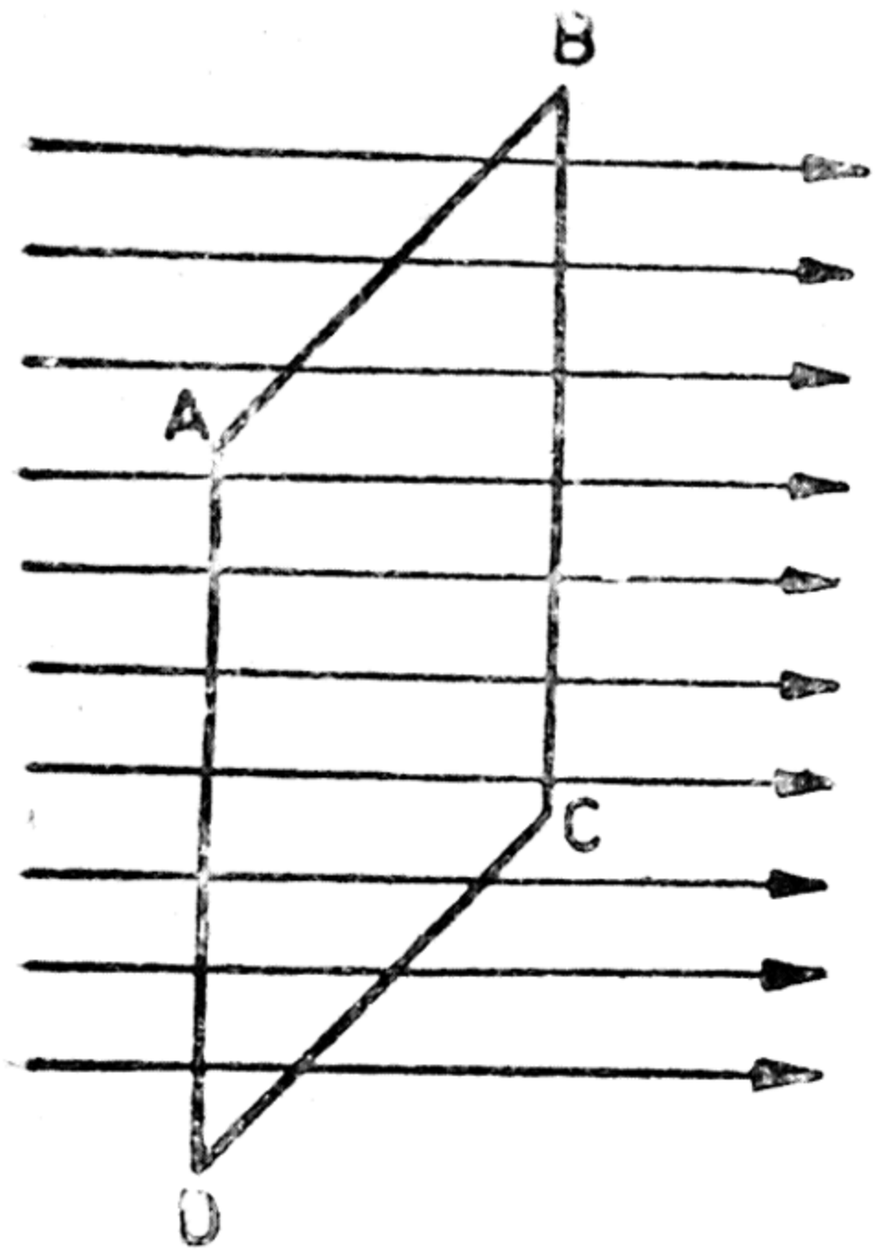


Fig. 1.11

Consider a small rectangular plane *ABCD* placed in a very small region of rotational vector field. When the plane is in such a way that the vector field is normal to it, *i.e.*, normal to each side of this plane, the line integral along its all sides is

$$\oint \mathbf{F}.d\mathbf{l} = \int_A^B \mathbf{F}.d\mathbf{l} + \int_B^C \mathbf{F}.d\mathbf{l} + \int_C^D \mathbf{F}.d\mathbf{l} + \int_D^A \mathbf{F}.d\mathbf{l}$$

$$= 0+0+0+0 = 0. \qquad \ldots(57)$$

If this plane is rotated to 90° such that the vector field is parallel to it, *i.e.*, to the sides *AB* and *CD* then the line integral

$$\oint \mathbf{F}.d\mathbf{l} = \int_A^B \mathbf{F}.d\mathbf{l} + \int_B^C \mathbf{F}.d\mathbf{l} + \int_C^D \mathbf{F}.d\mathbf{l} + \int_D^A \mathbf{F}.d\mathbf{l}$$

$$= \int_A^B \mathbf{F}.d\mathbf{l} - \int_C^D \mathbf{F}.d\mathbf{l}. \qquad \ldots(58)$$

As **F** is continuously varying with the position, hence **F** for *AB* is different than that for *CD* and the net closed line integral is of a finite value. Thus the closed line integral along *ABCD* depends upon its orientation with respect to the vector field. There is a certain orientation of the area for which the line integral is maximum. This maximum line integral at any point in such a vector field around a closed curve, expressed for unit area is called the **curl of a vector field** at that point. It is a vector quantity directed along the normal to the test area, towards which a right handed screw moves when turned in the sense in which the line integral along the boundary of the test area is maximum. Hence curl **F** is defined as

$$\text{curl } \mathbf{F} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \int \mathbf{F}.d\mathbf{l}. \qquad \ldots(59)$$

To understand clearly the curl of a vector and its direction, let us consider a small paddle wheel in the path of the flow of water

in a canal. On account of the viscous property of the water, the velocity of the water layer decreases downwards, it is nearly zero at the bottom and is maximum at the top. The wheel will thus turn

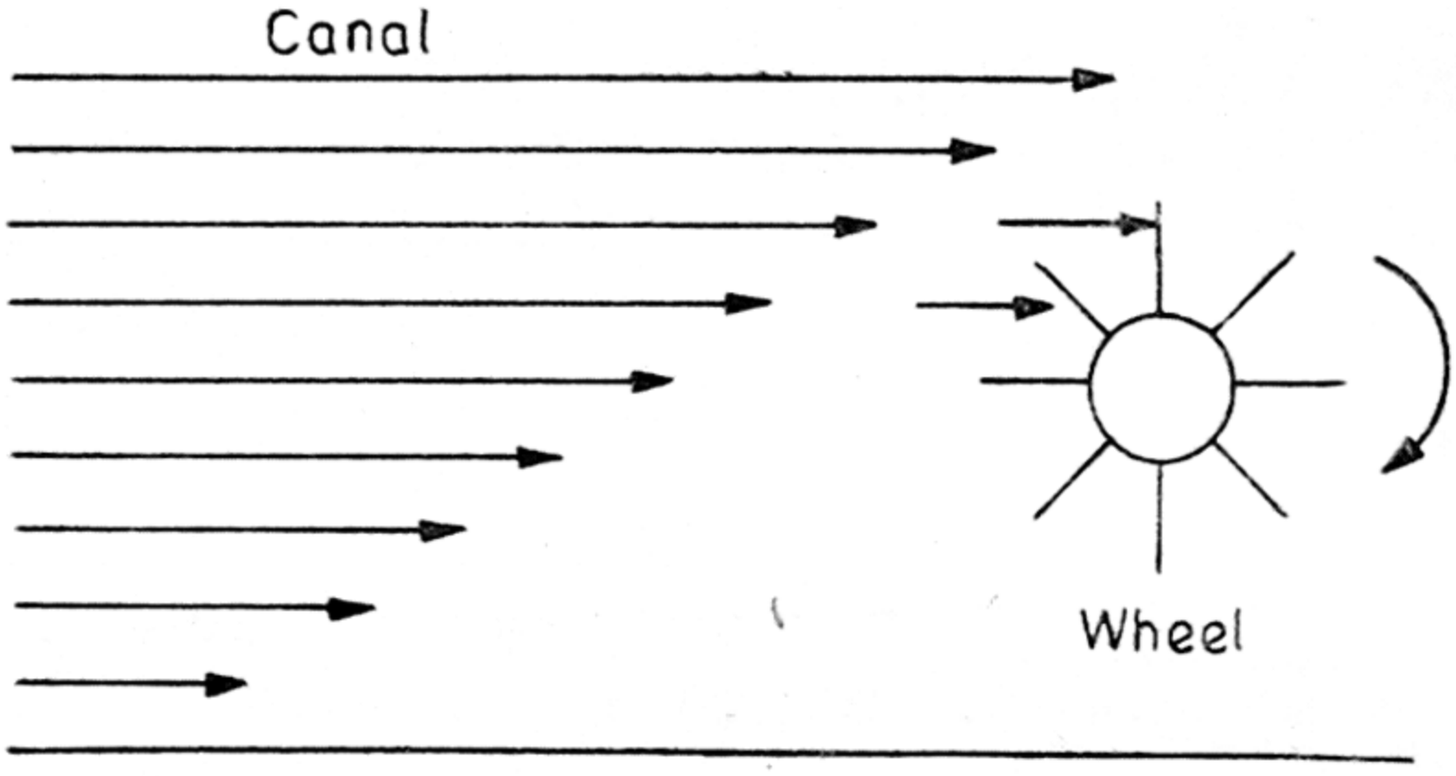


Fig. 1.12.

in the clockwise direction, as shown in Fig. 1.12, showing that there is a small circulation of water round about the wheel. The axis about which the wheel rotates gives the direction of the **curl.**

Let us find out the curl of a vector field **F** in terms of the cartesian components. Consider a rectangular area *ABCD* perpendicular to the axis of *y*, having sides $\triangle x$ and $\triangle z$. (Fig 1.13). Because of the small sides we can assume that the numerical value of the component of **F** at the middle point of any side is the average value along that side. Let F_x, F_y, and F_z be the components of vector **F** along *x*-, *y*-, and *z*- axes respectively at the origion O. Thus average values along sides *AB*, *BC*, *CD* and *DA* are

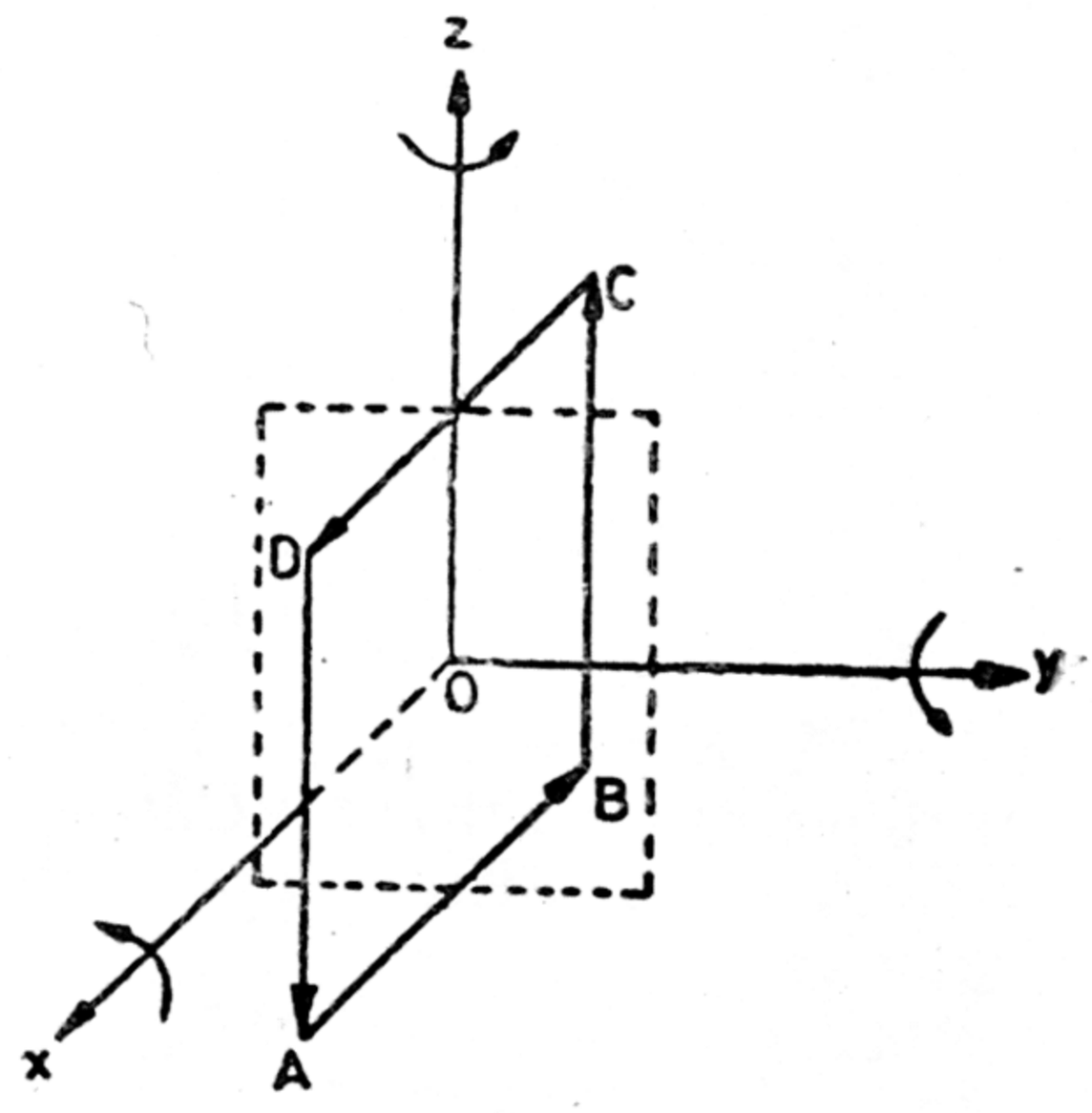


Fig. 1.13.

$$\left[F_x+\frac{\partial F_x}{\partial z}(-\tfrac{1}{2}\triangle z)\right],\left[F_z+\frac{\partial F_z}{\partial x}(-\tfrac{1}{2}\triangle x)\right],$$

$$\left[F_x+\frac{\partial F_x}{\partial z}(\tfrac{1}{2}\triangle z)\right] \text{ and } \left[F_z+\frac{\partial F_z}{\partial x}(\tfrac{1}{2}\triangle x)\right] \text{ respectively.}$$

As $$\int_A^B \mathbf{F}.d\mathbf{l}=\left[F_x+\frac{\partial F_x}{\partial z}(-\tfrac{1}{2}\triangle z)\right](-\triangle x).$$

$$\int_B^C \mathbf{F}.d\mathbf{l}=\left[F_z+\frac{\partial F_z}{\partial x}(-\tfrac{1}{2}\triangle x)\right](\triangle z),$$

$$\int_C^D \mathbf{F}.d\mathbf{l}=\left[F_x+\frac{\partial F_x}{\partial z}(\tfrac{1}{2}\triangle z)\right](\triangle x),$$

and

$$\int_D^A \mathbf{F}.d\mathbf{l}=\left[F_z+\frac{\partial F_z}{\partial x}(\tfrac{1}{2}\triangle x)\right](-\triangle z).$$

Hence the total line integral of **F** along the boundary *ABCD*

$$\oint \mathbf{F}.d\mathbf{l}=\int_A^B \mathbf{F}.d\mathbf{l}+\int_B^C \mathbf{F}.d\mathbf{l}+\int_C^D \mathbf{F}.d\mathbf{l}+\int_D^A \mathbf{F}.d\mathbf{l}$$

$$=(\partial F_x/\partial z-\partial F_z/\partial x)\triangle x\triangle z.$$

The value of this integral per unit area is $(\partial F_x/\partial z-\partial F_z/\partial x)$. It is in the direction normal to the area *ABCD*, *i.e.* in the *y*—direction. It is thus the *y*-component of the curl of a vector function **F**. Hence

$$(\text{curl } \mathbf{F})_y=(\partial F_x/\partial z-\partial F_z/\partial x)\,\mathbf{j},$$

where **j** is the unit vector along *y*-axis.

Similarly we have

$(\text{curl } \mathbf{F})_x=(\partial F_z/\partial y-\partial F_y/\partial z)\,\mathbf{i}$ and $(\text{curl } \mathbf{F})_z=(\partial F_y/\partial x-\partial F_x/\partial y)\mathbf{k}$.

$\therefore$ curl $\mathbf{F}=(\text{curl } \mathbf{F})_x+(\text{curl } \mathbf{F})_y+(\text{curl } \mathbf{F})_z$

$$=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}. \quad \ldots(60)$$

This may be conveniently written in the form of a determinant as

$$\text{curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \quad \ldots(61)$$

Since the vector product $\nabla\times\mathbf{F}$ of the operator ∇ and the vector **F** may be written as

$$\nabla\times\mathbf{F}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\mathbf{F}_x\mathbf{i}+\mathbf{F}_y\mathbf{j}+\mathbf{F}_z\mathbf{k})$$

$\therefore$ curl $\mathbf{F}=\nabla\times\mathbf{F}=$ Del cross **F**.

Since the operation is independent of the system of the curl of a vector is *invariant*.

The name curl suggests that a vector field **F** at a point of space has circulation there. As the curl is associated in hydrodynamics with the rotation of a fluid hence is sometimes called *rotation* or simply *rot*. *For any conservative vector field the curl is zero at all points of space, as the line integral for a closed path is zero for such a field.*

1.12. STOKE'S THEOREM

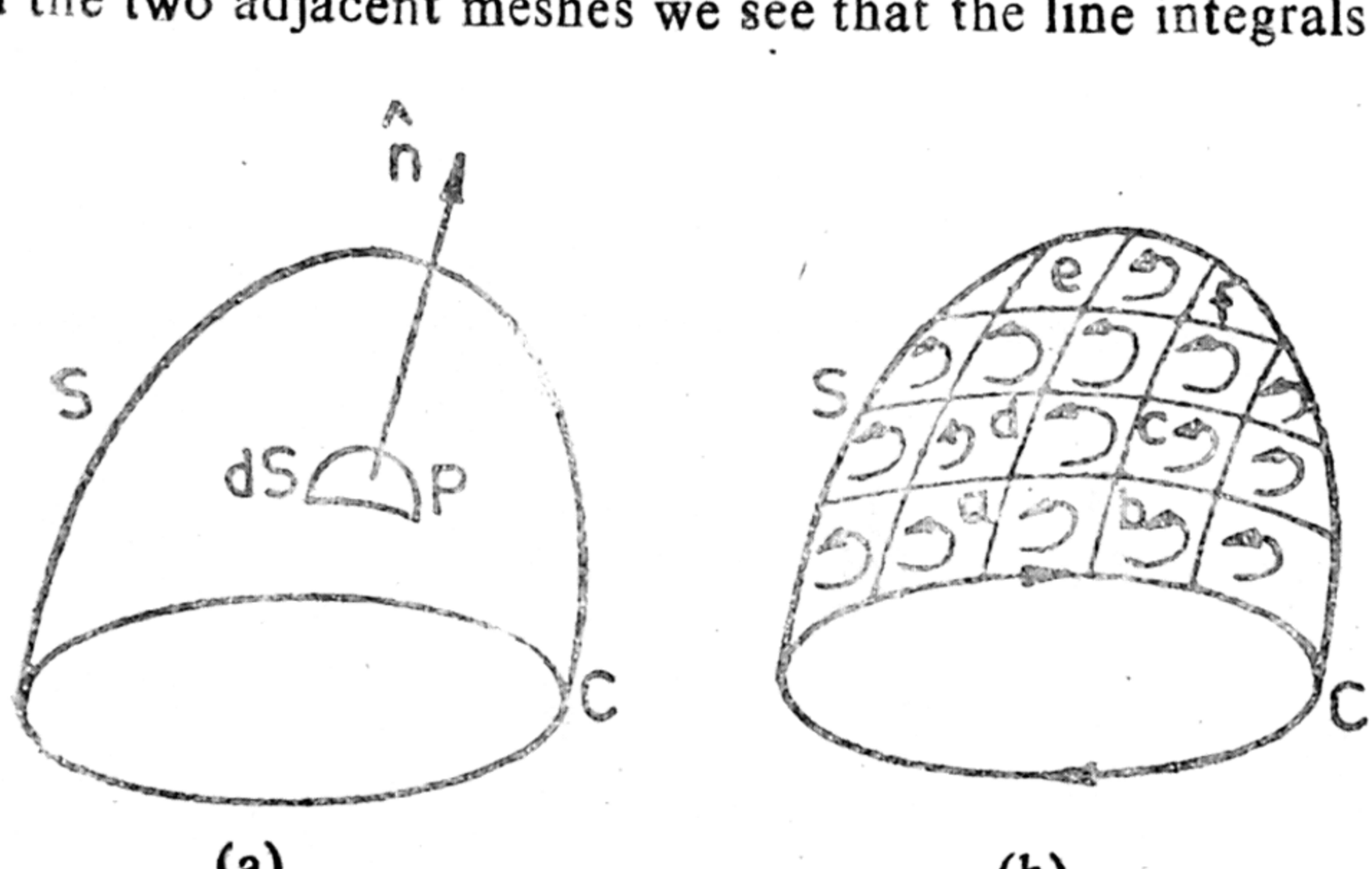
This theorem states that *the line integral of tangential component of a vector field* **F** *around a closed curve is equal to the surface integral of the normal component of its curl over the surface S bounded by that curve C, i.e.*

$$\oint_C \mathbf{F}.d\mathbf{l} = \oint_S \text{curl } \mathbf{F}.d\mathbf{S}. \qquad \ldots(63)$$

Consider a surface area $d\mathbf{S}$ at any point P on the surface S having closed boundary C (Fig 1.14 a). If $\hat{n}$ is the unit normal vector on this area, then $d\mathbf{S}=\hat{n}dS$ and the Stoke's theorem may also be expressed as

$$\int_C \mathbf{F}.d\mathbf{l} = \iint_S \hat{n}.\,(\text{curl}\mathbf{F})\,dS.$$

Let the surface S be subdivided by sets of curves lying on the surface and joining the points on the curve C so as to form a net work as shown in (Fig 1.14 b). If we consider the line integrals of **F** around the two adjacent meshes we see that the line integrals along



(a) (b)

Fig. 1.14

the common side of the two meshes cancel each other as one has to travel in opposite directions. In the case of meshes *abcd* and *dcfe*, the common side *dc* does not contribute to the line integral and the line integral along *abcd*+line integral along *dcfe*=line

integral along *abfe*. In general every mesh has a side common with its neighbouring meshes, hence the sum of the line integrals of **F** around all the meshes forming a network on the whole surface S is the line integral of **F** along the contour of the curve C, *i.e.*

$$\int_o \mathbf{F}.d\mathbf{l} = \sum_{\text{all meshes}} \oint \mathbf{F}.d\mathbf{l}. \qquad \ldots(64)$$

Let us consider a single mesh $ABCD$ which is a parallelogram with the sides AB and AD represented by infinitesimal vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$. Let the mean values of vector field **F** along the sides AB, BC, CD and DA be $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ respectively, then

$$\oint \mathbf{F}.d\mathbf{l} = \int_A^B \mathbf{F}.d\mathbf{l} + \int_B^C \mathbf{F}.d\mathbf{l} + \int_C^D \mathbf{F}.d\mathbf{l} + \int_D^A \mathbf{F}.d\mathbf{l}$$

$$= \mathbf{F}_1.\delta\mathbf{a} + \mathbf{F}_2.\delta\mathbf{b} - \mathbf{F}_3.\delta\mathbf{a} - \mathbf{F}_4.\delta\mathbf{b}$$

$$= \delta\mathbf{b}.(\mathbf{F}_2 - \mathbf{F}_4) - \delta\mathbf{a}.(\mathbf{F}_3 - \mathbf{F}_1). \qquad \ldots(65)$$

We know that the change in **F** corresponding to a displacement $\delta\mathbf{r}$ is given by

$$\delta\mathbf{F} = \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z} \right) \mathbf{F} = (\delta\mathbf{r} \,.\, \nabla)\, \mathbf{F}. \qquad \ldots(66)$$

Hence we can write

$$\mathbf{F}_2 - \mathbf{F}_4 = (\delta\mathbf{a}.\nabla)\, \mathbf{F}.$$

and

$$\mathbf{F}_3 - \mathbf{F}_1 = (\delta\mathbf{b}.\nabla)\, \mathbf{F}. \qquad \ldots(67)$$

Substituting these values in equation (65), we get

$$\oint \mathbf{F}.d\mathbf{l} = \delta\mathbf{b}.(\delta\mathbf{a}.\nabla)\mathbf{F} - \delta\mathbf{a}.(\delta\mathbf{b}.\nabla)\, \mathbf{F}$$

$$= [\delta\mathbf{b}(\delta\mathbf{a}.\nabla) - \delta\mathbf{a}(\delta\mathbf{b}.\nabla)]\, .\mathbf{F}.$$

Using vector identities

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}.\mathbf{a})\, \mathbf{b} - (\mathbf{c}.\mathbf{b})\mathbf{a}$$

and $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \mathbf{b}.(\mathbf{c}\times\mathbf{a}) = \mathbf{c}.(\mathbf{a}\times\mathbf{b})$,

we get

$$\oint \mathbf{F}.d\mathbf{l} = [(\delta\mathbf{a}\times\delta\mathbf{b})\times\nabla].\mathbf{F}$$

$$= (\delta\mathbf{a}\times\delta\mathbf{b})\,.\,(\nabla\times F). \qquad \ldots(68)$$

Putting $\delta\mathbf{a}\times\delta\mathbf{b} = \delta\mathbf{S}$ = Area of $ABCD$, we get

$$\oint \mathbf{F}.d\mathbf{l} = (\nabla\times\mathbf{F})\,.\,\delta\mathbf{S}. \qquad \ldots(69)$$

Summing these results for all the meshes, we get

$$\oint \mathbf{F}.d\mathbf{l} = \sum_{\text{all the meshes}} \mathbf{F}.d\mathbf{l} = \iint_s (\nabla\times\mathbf{F}).d\mathbf{S}.$$

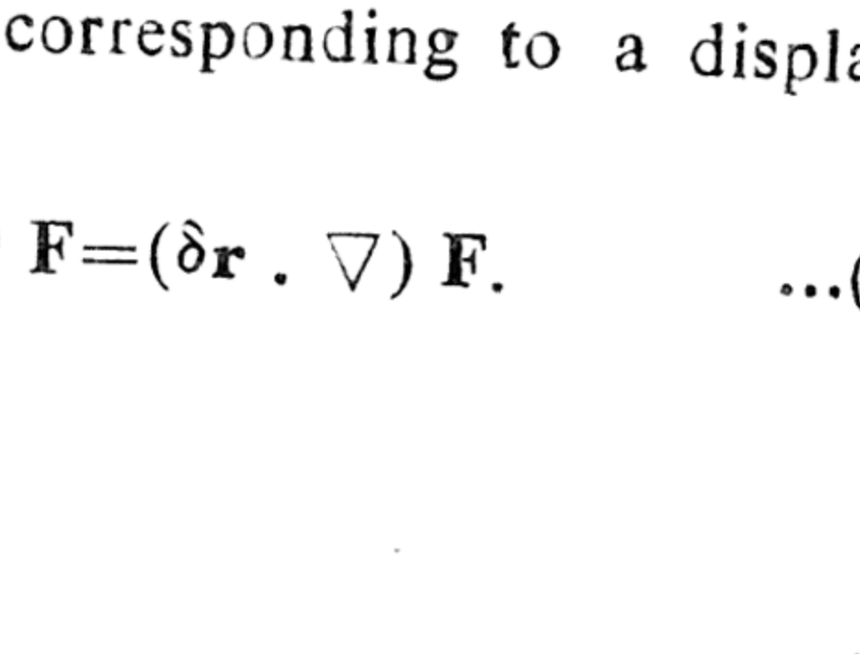


Fig. 1.15.

or $$\oint \mathbf{F}.d\mathbf{l} = \iint_S \hat{n}.\ \text{curl}\ \mathbf{F}\ dS. \qquad \ldots(70)$$

Hence the Stoke's theorem.

Like the divergence theorem this theorem is very useful in the evaluation of integrals.

1.13. GREEN'S THEOREM

Green's theorem can be derived from Gauss's divergence theorem as follows. Substituting vector field **F** as the product of one scalar ϕ and grad of another scalar ψ, *i.e.*, $\mathbf{F}=\phi\nabla\psi$, we get

$$\nabla.\mathbf{F} = \frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\phi\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\phi\frac{\partial\psi}{\partial z}\right)$$

$$=\phi\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}$$

$$=\phi\nabla^2\psi+\nabla\phi.\nabla\psi. \qquad \ldots(71)$$

Hence Gauss's theorem reduces to

$$\iiint_V \phi\nabla^2\psi+\nabla\phi.\nabla\psi)\,dV = \iint_S (\phi\nabla\psi).d\mathbf{S}. \qquad \ldots(72)$$

This is known as the first form of Green's theorem. After inter-changing ϕ and ψ, we have

$$\iiint_V (\psi\nabla^2\phi+\nabla\psi.\nabla\phi)\,dV = \iint_S (\psi\nabla\phi).d\mathbf{S} \qquad \ldots(73)$$

On subtracting Eq. (73) from Eq. (72), we get

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\ dV = \iint_S (\phi\nabla\psi-\psi\nabla\phi).d\mathbf{S} \ldots(74)$$

This is referred to as the *second form of Green's theorem.* These are of extreme importance in the fields of electrodynamics and hydrodynamics.

Some Useful Vector Relations involving the vector ∇

$(\mathbf{u}.\nabla)\ \phi=\mathbf{u}.(\nabla\ \phi)$

$(\mathbf{u}.\nabla)\ \mathbf{r}=\mathbf{u}.$

$\text{div}\ (\mathbf{u}+\mathbf{v})=\nabla.(\mathbf{u}+\mathbf{v})=\nabla.\mathbf{u}+\nabla.\mathbf{v}.$

$\text{curl}\ (\mathbf{u}+\mathbf{v})=\nabla\times(\mathbf{u}+\mathbf{v})=\nabla\times\mathbf{u}+\nabla\times\mathbf{v}$

$\text{div}\ (\phi\mathbf{u})=\nabla.(\phi\mathbf{u})=\phi\ \text{div}\ \mathbf{u}+\mathbf{u}.\ (\text{grad}\ \phi).$

$\text{curl}\ (\phi\mathbf{u})=\nabla\times(\phi\mathbf{u})=\phi\ \text{curl}\ \mathbf{u}+(\text{grad}\ \phi)\times\mathbf{u}.$

$$\text{div } (\mathbf{u}\times\mathbf{v})=\mathbf{v}.\ \text{curl } \mathbf{u}-\mathbf{u}.\ \text{curl } \mathbf{v}.$$

$$\text{curl } (\mathbf{u}\times\mathbf{v})=(\mathbf{v}.\ \text{grad})\ \mathbf{u}-(\mathbf{u}.\ \text{grad})\ \mathbf{v}+\mathbf{u}\,\text{div } \mathbf{v}-\mathbf{v}\ \text{div } \mathbf{u}.$$

$$\text{grad } (\mathbf{u}.\mathbf{v})=\mathbf{u}\times\text{curl } \mathbf{v}+\mathbf{v}\times\text{curl } \mathbf{u}+(\mathbf{u}.\ \text{grad})\mathbf{v}+(\mathbf{v}.\text{grad})\mathbf{u}.$$

Second order differential operators

$$\text{div curl } \mathbf{u}=0$$

$$\text{div (grad } \phi)=\nabla.(\nabla\ \phi)=\nabla^2\phi=\partial^2\phi/\partial x^2+\partial^2\phi/\partial^2 y+\partial^2\phi/\partial z^2.$$

$$\text{curl grad } \phi=0$$

$$\text{curl curl } \mathbf{u}=\text{grad div } \mathbf{u}-\nabla^2\mathbf{u}.$$

Exercises

Example 1. *Prove that the four points* $4\mathbf{i}+5\mathbf{j}+\mathbf{k}$, $-(\mathbf{j}+\mathbf{k})$, $3\mathbf{i}+9\mathbf{j}+4\mathbf{k}$ *and* $4(-\mathbf{i}+\mathbf{j}+\mathbf{k})$ *are coplaner.*

Let the given vectors represent the points A, B, C and D respectively, then

$$\mathbf{a}=\mathbf{AB}=\text{Position vector of B}-\text{Position vector of A}$$

$$=-(\mathbf{j}+\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-(4\mathbf{i}+6\mathbf{j}+2\mathbf{k}).$$

Similarly $\mathbf{b}=\mathbf{AC}=3\mathbf{i}+9\mathbf{j}+4\mathbf{k}-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-\mathbf{i}+4\mathbf{j}+3\mathbf{k}.$

and $\mathbf{c}=\mathbf{AD}=4(-\mathbf{i}+\mathbf{j}+\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-8\mathbf{i}-\mathbf{j}+3\mathbf{k}.$

If the points A, B, C and D are coplaner, then vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ will also be coplaner. It is possible only when

$$\mathbf{a}\ .\ (\mathbf{b}\times\mathbf{c})=0$$

$$\text{Now } \mathbf{a}.\ (\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=\begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}=0$$

Hence the points A, B, C, D are coplaner.

Example 2. *For a position vector* $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, *show that*

(*a*) *div* $\mathbf{r}=3$,

(*b*) *div* $(r^n\ \mathbf{r})=(3+n)\ r^n$,

(*c*) *curl* $\mathbf{r}=0$,

(*d*) *curl* $(\mathbf{r}/r^3)=0$.

(*a*) We know that

$$\text{div } \mathbf{F}=\nabla\ .\mathbf{F}=\left(\mathbf{i}\ \frac{\partial}{\partial x}+\mathbf{j}\ \frac{\partial}{\partial y}+\mathbf{k}\ \frac{\partial}{\partial z}\right)\cdot\ (F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k})$$

$$\text{div } \mathbf{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}.$$

In the present case $\mathbf{F}=\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, therefore

$$\text{div } \mathbf{r}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=1+1+1=3.$$

(*b*) In this case $\mathbf{F}=r^n\,\mathbf{r}=r^n(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$,

$$\therefore \text{div } \mathbf{F}=\text{div } (r^n\,\mathbf{r})=\frac{\partial}{\partial x}(r^n x)+\frac{\partial}{\partial y}(r^n y)+\frac{\partial}{\partial z}(r^n z)$$

$$=r^n+xnr^{n-1}\frac{\partial r}{\partial x}+r^n+ynr^{n-1}\frac{\partial r}{\partial y}+r^n+znr^{n-1}\frac{\partial r}{\partial z}$$

$$=3r^n+nr^{n-1}\left[x\frac{\partial r}{\partial x}+y\frac{\partial r}{\partial y}+z\frac{\partial r}{\partial z}\right]$$

$$=3r^n+nr^{n-1}\left[\frac{x^2}{r}+\frac{y^2}{r}+\frac{z^2}{r}\right]=3r^n+nr^{n-1}.\frac{r^2}{r}$$

$$=(3+n)\,r^n.$$

(*c*) Since curl of a vector field $\mathbf{F}$ is given as

$$\text{curl } \mathbf{F}=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}.$$

In this case $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, hence

$$\text{curl } \mathbf{F}=\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)\mathbf{k}.$$

Since x, y, z are independent co-ordinates, hence

$$\frac{\partial x}{\partial y}=\frac{\partial y}{\partial x}=\frac{\partial z}{\partial y}=\frac{\partial y}{\partial z}=\frac{\partial x}{\partial z}=\frac{\partial z}{\partial x}=0.$$

and $\text{curl } \mathbf{F}=0\mathbf{i}=0\mathbf{j}+0\mathbf{k}=0.$

(*d*) In this case

$$\mathbf{F}=\frac{\mathbf{r}}{r^3}=\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r^3}$$

$$\therefore \text{curl } \mathbf{F}=\mathbf{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right]+\mathbf{j}\left[\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)-\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)\right]$$

$$+\mathbf{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right)-\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]$$

$$\text{curl } \mathbf{F}=\mathbf{i}\left[-\frac{3}{r^4}\left(z\frac{\partial r}{\partial y}-y\frac{\partial r}{\partial z}\right)\right]+\mathbf{j}\left[-\frac{3}{r^4}\left(x\frac{\partial r}{\partial z}-z\frac{\partial r}{\partial x}\right)\right]$$
$$+\mathbf{k}\left[-\frac{3}{r^4}\left(y\frac{\partial r}{\partial x}-x\frac{\partial r}{\partial y}\right)\right]$$

Since $r=(x^2+y^2+z^2)^{1/2}$, hence $\partial r/\partial x=x/r$, $\partial r/\partial y=y/r$ and $\partial r/\partial z=z/r$. Substituting these values in the above relation, we have

$$\text{curl } \mathbf{F}=\mathbf{i}\left[-\frac{3}{r^4}\left(\frac{zy}{r}-\frac{yz}{r}\right)\right]+\mathbf{j}\left[-\frac{3}{r^4}\left(\frac{xz}{r}-\frac{zx}{r}\right)\right]$$
$$+\mathbf{k}\left[-\frac{3}{r^4}\left(\frac{yx}{r}-\frac{xy}{r}\right)\right]=0.$$

Example 3. *If* $\mathbf{r}$ *is a position vector and* $\mathbf{a}$ *is a constant vector then prove that*

$$\nabla(\mathbf{a}.\mathbf{r})=\mathbf{a}.$$

Using the vector identity

$$\text{grad } (\mathbf{A}.\mathbf{B})=\mathbf{A}\times\text{curl } \mathbf{B}+\mathbf{B}\times\text{curl } \mathbf{A}+(\mathbf{A}.\text{ grad})\mathbf{B}+(\mathbf{B}.\text{ grad})\mathbf{A},$$

we have

$$\nabla(\mathbf{a}.\mathbf{r})=\mathbf{a}\times(\nabla\times\mathbf{r})+\mathbf{r}\times(\nabla\times\mathbf{a})+(\mathbf{a}.\nabla)\mathbf{r}+(\mathbf{r}.\nabla)\mathbf{a}. \quad \ldots(i)$$

As $\mathbf{a}$ is a constant vector, hence curl $\mathbf{a}=0$. We know the vector identities curl $\mathbf{r}=0$ and

$$(\mathbf{A}.\nabla)\mathrm{B}=A_x\frac{\partial\mathbf{B}}{\partial x}+A_y\frac{\partial\mathbf{B}}{\partial y}+A_z\frac{\partial\mathbf{B}}{\partial z}. \quad \ldots(ii)$$

In the present case if we assume $\mathbf{A}=\mathbf{a}$ and $\mathrm{B}=\mathbf{r}$, then we have

$$(\mathbf{a}.\nabla)\mathbf{r}=a_x\frac{\partial\mathbf{r}}{\partial x}+a_y\frac{\partial\mathbf{r}}{\partial y}+a_z\frac{\partial\mathbf{r}}{\partial z}$$
$$=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}=\mathbf{a}.$$

If $\mathbf{A}=\mathbf{r}$ and $\mathbf{B}=\mathbf{a}$ in equation (*ii*), then we have

$$(\mathbf{r}.\nabla)\mathbf{a}=x\frac{\partial\mathbf{a}}{\partial x}+y\frac{\partial\mathbf{a}}{\partial y}+z\frac{\partial\mathbf{a}}{\partial z}$$
$$=0. \text{ (as } \mathbf{a} \text{ is a constant vector)}$$

Substituting these values in Eq. (*i*), we get

$$\nabla(\mathbf{a}.\mathbf{r})=0+0+\mathbf{a}+0=\mathbf{a}.$$

Hence the result.

Example 4. (*a*) *Calculate grad* $\phi(\mathbf{r})$. (*b*) *Prove that grad is* $\perp$ *to the surface for which* ϕ *is constant.*

(*a*) grad $\phi=\mathbf{i}\partial\phi/\partial x+\mathbf{j}\,\partial\phi/\partial y+\mathbf{k}\,\partial\phi/\partial z$,

where $\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial r}.\frac{\partial r}{\partial x}=\phi'\frac{\partial r}{\partial x}$, $\frac{\partial\phi}{\partial y}=\phi'\frac{\partial r}{\partial y}$ and $\frac{\partial\phi}{\partial z}=\phi'\frac{\partial r}{\partial z}$.

As $r=(x^2+y^2+z^2)^{1/2}$, hence $\partial r/\partial x=2x/2(x^2+y^2+z^2)^{1/2}=x/r$, $\partial r/\partial y=y/r$ and $\partial r/\partial z=z/r$.

$$\therefore \quad \text{grad } \phi(r)=\mathbf{i}\,\phi'\,\frac{x}{r}+\mathbf{j}\phi'\frac{y}{r}+\mathbf{k}\phi'\,\frac{z}{r}$$

$$=\phi'\frac{\mathbf{r}}{r}=\frac{\partial\phi\mathbf{r}}{\partial r\, r}=\frac{\partial\phi}{\partial r}\hat{\mathbf{r}}$$

$$(b)\quad d\phi=\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz=(\text{grad }\phi).\,d\mathbf{r}.$$

Let $d\mathbf{r}$ lies along the surface for which $\phi=$const, then obviously $d\phi=0$ and (grad ϕ). $d\mathbf{r}=0$. It shows that grad ϕ and $d\mathbf{r}$ are mutually $\perp$ to each other for a surface $\phi=$const.

Example 5. *Prove the vector identities :*

(*i*) *div curl* $\mathbf{F}=0$. (*ii*) *curl grad* $\phi=0$.

$$(i)\ \text{div curl } \mathbf{F}=\frac{\partial}{\partial x}\,(\text{curl }\mathbf{F})_x+\frac{\partial}{\partial y}\,(\text{curl }\mathbf{F})_y+\frac{\partial}{\partial z}\,(\text{curl }\mathbf{F})_z$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right]+\frac{\partial}{\partial y}\left[\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right]+\frac{\partial}{\partial z}\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]$$

$$=\frac{\partial^2 F_z}{\partial x\partial y}-\frac{\partial^2 F_y}{\partial x\partial z}+\frac{\partial^2 F_x}{\partial y\partial z}-\frac{\partial^3 F_z}{\partial y\partial x}+\frac{\partial^2 F_y}{\partial z\partial x}-\frac{\partial^2 F_x}{\partial z\partial y}$$

$$=0.$$

It means that *vector field whose divergence is everywhere zero can be expressed as the curl of some other suitable vector field. The fields whose divergence is zero have their field lines always forming closed curves or the vector field is solenoidal.*

$$(ii)\ (\text{curl grad }\phi)_x=\left[\frac{\partial}{\partial y}\,(\text{grad }\phi)_z-\frac{\partial}{\partial z}\,(\text{grad }\phi)_y\right]\mathbf{i}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\mathbf{i}=0.$$

Similarly we can prove that y-and z-components are also zero.

$\therefore$ curl grad$\phi=0$.

It means that *a vector field whose curl is everywhere zero can be expressed as the gradient af another suitable scalar field. The field is known as non-curl, irrotational, or scalar potential field. All the conservative fields are of this type.*

Example 6. *A rigid body is rotating with constant angular velocity ω about a fixed axis, if* $\mathbf{v}$ *is the velocity of a point of the body, prove that curl* $\mathbf{v}=\omega$. *Give its physical significance.*

Consider a rigid body which is rotating with an angular velocity ω about an axis OA, where O is a fixed point in the body. Then any point P on the body moves in a circular path about OA with a tangential linear velocity $\mathbf{v}$ $(=\omega\times\mathbf{r})$, where $\mathbf{r}$ is the distance of point P from the fixed point O. Since the angular velocity ω is a vector constant for all points on the body and can be written as

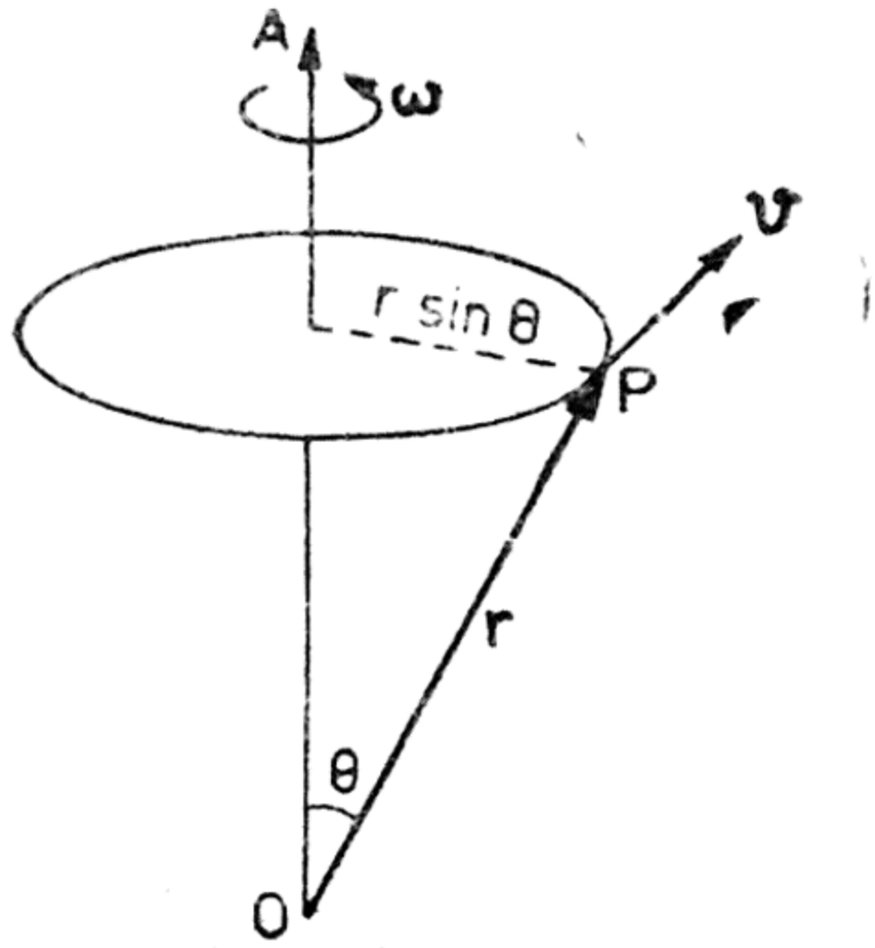


Fig. 1.16.

$\omega=\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k}$, hence the components ω_x, ω_y, ω_z are independent of the co-ordinates $\mathbf{r}$ of the point.

$$\therefore\quad \mathbf{v}=\omega\times\mathbf{r}=(\omega_y z-\omega_z y)\mathbf{i}+(\omega_z x-\omega_x z)\,\mathbf{j}+(\omega_x y-\omega_y x)\,\mathbf{k}$$
and $\nabla\times\mathbf{v}=2\omega_x\mathbf{i}+2\omega_y\mathbf{j}+2\omega_z\mathbf{k}=2\omega$.

Hence when the rigid body is in motion the curl of its linear velocity at any point is equal to twice its angular velocity.

Thus we see that if a motion of a rigid body is such that the velocity has a curl, the ultimate particles of the body are in rotation with an angular velecity, which is equal to half of that curl. The motion is described as *rotational* or *vortical*.

Example 7. *For a position vector* $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, *find the values of* (*i*) *grad* $(1/\mathbf{r})$ (*ii*) *and grad* r^m.

$$\text{grad}\ \frac{1}{r}=\nabla\left(\frac{1}{r}\right)=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$=\mathbf{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}+\mathbf{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y}+\mathbf{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$=-\frac{1}{r^2}\left[\mathbf{i}\,\frac{\partial r}{\partial x}+\mathbf{j}\,\frac{\partial r}{\partial y}+\mathbf{k}\,\frac{\partial r}{\partial z}\right]$$

Since $\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$, hence $r^2=\mathbf{r}.\mathbf{r}=x^2+y^2+z^2$

and $\partial r/\partial x=x/r$, $\partial r/\partial y=y/r$, $\partial r/\partial z=z/r$.

Substituting these values in the above relation, we get

$$\text{grad}\left(\frac{1}{r}\right)=-\frac{1}{r^2}\left[\mathbf{i}\,\frac{x}{r}+\mathbf{j}\,\frac{y}{r}+\mathbf{k}\,\frac{z}{r}\right]$$

$$\text{grad}\left(\frac{1}{r}\right)=-\frac{1}{r^3}\left[\,\mathbf{i}x+\mathbf{j}y+\mathbf{k}z\,\right]=-\frac{\mathbf{r}}{r^3}.$$

Similarly

$$\text{grad } r^m=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)\left(r^m\right)$$

$$=mr^{m-1}\left[\,\mathbf{i}\,\frac{\partial r}{\partial x}+\mathbf{j}\,\frac{\partial r}{\partial y}+\mathbf{k}\,\frac{\partial r}{\partial z}\right]$$

$$=mr^{m-1}\left[\,\mathbf{i}\,\frac{x}{r}+\mathbf{j}\,\frac{y}{r}+\mathbf{k}\,\frac{z}{r}\right]$$

$$=mr^{m-2}\left[\,\mathbf{i}x+\mathbf{j}y+\mathbf{k}z\,\right]$$

$$=mr^{m-2}\,\mathbf{r}.$$

Example 8. Prove that curl curl $\mathbf{A}=$grad div $\mathbf{A}-\nabla^2\mathbf{A}$.

We know that

$$\text{curl }\mathbf{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{k}.$$

and $\nabla=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)$. Therefore

$$\text{curl curl }\mathbf{A}=\nabla\times\nabla\times\mathbf{A}$$

$$=\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i}\right.$$

$$\left.+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_y}{\partial x}\,\frac{\partial A_x}{\partial y}\right)\mathbf{k}\right].$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{k}-\frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{j}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{k}+\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{i}$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{j}-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{i}$$

Let us consider only the x component and call it curl $_x$ curl **A**.

$$\text{curl}_x \text{ curl } \mathbf{A}=\left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\right]i$$

$$=\left[\frac{\partial^2 A_y}{\partial y\partial x}-\frac{\partial^2 A_x}{\partial y\partial y}-\frac{\partial^2 A_x}{\partial z\partial z}+\frac{\partial^2 A_z}{\partial z\partial x}\right]\mathbf{i}.$$

Adding and subtracting $\partial^2 A_x/\partial x^2$ in this expression, we get

$$\text{curl}_x \text{ curl } \mathbf{A}=\left[\frac{\partial^2 A_x}{\partial x\partial x}+\frac{\partial^2 A_y}{\partial y\partial x}+\frac{\partial^2 A_z}{\partial z\partial y}-\left(\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}\right)\right]\mathbf{i}$$

$$=\left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_x\right]\mathbf{i}$$

$$=\mathbf{i}\ \frac{\partial}{\partial x}(\text{div } \mathbf{A})-\nabla^2(A_x\mathbf{i})$$

$$=\text{grad}_x\ (\text{div } \mathbf{A})-\nabla^2(A_x\mathbf{i})$$

Similarly, we have

$$\text{curl}_y \text{ curl } \mathbf{A}=\text{grad}_y\ (\text{div } \mathbf{A})-\nabla^2\ (A_y\mathbf{j})$$

and $$\text{curl}_z \text{ curl } \mathbf{A}=\text{grad}_z\ (\text{div } \mathbf{A})-\nabla^2\ (A_z\mathbf{k})$$

Therefore the summation gives

$$\text{curl curl } \mathbf{A}=\text{grad div } \mathbf{A}-\nabla^2\mathbf{A}.$$

This relation can also be derived by using the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}\ (\mathbf{a}.\mathbf{c})-(\mathbf{a}.\mathbf{b})\ \mathbf{c}.$$

or $$\nabla\times\nabla\times\mathrm{A}=\nabla\ (\nabla.\mathbf{A})-(\nabla.\nabla)\ \mathbf{A}$$

$$=\text{grad div } \mathbf{A}-\nabla^2\mathbf{A}.$$

Oral Questions—

1. Show how three equal magnitude vectors would have to be oriented if they were to add to give zero. Can this be done with three unequal vectors ? Two unequal vectors ?

2. Find the dot and cross products of two similar vectors ?

3. Why is the work a scalar quantity and the moment of the force a vector quantity ?

4. If ∇ operates v when u is assumed as constant, then which of the following is the correct form

$$(\nabla\ .\ \mathbf{u})\mathbf{v} \text{ or } (\mathbf{u}\ .\ \nabla)\mathbf{v}\ ?$$

5. Find the surface integral of *a* curl of *a* vector field when the surface is closed.

6. If two vectors **A** and **B** are not parallel in question $a\mathbf{A}+b\mathbf{B}=0$, then find the values of a and b.

Problems—

1. Prove that the line joining the vertex of a parallelogram and the mid-point of the opposite side trisects the diagonal.

2. Prove that $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$, $\mathbf{a}\times\mathbf{a}=0$ and $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2(\mathbf{a}\times\mathbf{b})$.

3. If **A** and **B** are the sides of a parallelogram and **C** and **D** are the diagonals, show that $C^2+D^2=2(A^2+B^2)$.

4. Prove that the components of **F** in the plane of **a** and **F**, along and perpendicular to **a** are $\frac{\mathbf{a}\cdot\mathbf{F}}{a^2}\,\mathbf{a}$ and $\frac{(\mathbf{a}\times\mathbf{F}\times\mathbf{a})}{a^2}$ respectively

5. Find grad v, where $v=r^m$ and $r=(x^2+y^2+z^2)^{1/2}$

[mr^{m-2}]

6. Find div **F** and curl **F**, where $\mathbf{F}=\text{grad}\,(x^3+y^3+z^3-3xyx)$.

[$6(x+y+z)$ and 0]

7. Evaluate $\nabla^2 r$ where $r=(x^2+y^2+z^2)^{1/2}$.

8. Show that the potential function $\phi=q(x^2+y^2+z^2)^{-1/2}$ satisfies the Laplace's equation.

2

Electric Charge and Electrostatic Field

2.1. ELECTRIC CHARGE

It is a well established fact that bodies which are not in contact can interact with each other without the use of an intervening medium. The best known interaction of this kind is *gravitation*. In certain circumstances the interaction between separated bodies can be such that the forces produced on them completely overshadow the gravitational attraction. In these cases we say that these bodies are magnetic or charged. We can show that a glass rod rubbed with silk will repel the second glass rod rubbed with silk and will attract a hard rubber rod rubbed with fur. Two hard rubber rods rubbed with fur will repel each other. We explain these facts by saying that rubbing a rod gives it an electric charge and the charges on the glass rod and on the hard rubber must be different in nature. It is also clear that *like charges repel and unlike charges attract*. Benjamin Franklin, American Physicist, named the kind of charge that appears on the glass positive and the kind that appears on the hard rubber negative. He also observed that equal negative and positive charges are obtained at the same time by the rubber on the one hand and by the body rubbed on the other. Electric effects are not limited to glass or rubber, any substance rubbed with any other under suitable conditions will become charged. Now-a-days it is assumed that the matter contains equal amount of negative (electrons) and positive (protons) electricity. In the process acquiring the electric charge, a small amount of one kind of electricity is transferred from one to the other body. In this way one body would become positive and other negative.

Charge is quantized. The atomic theory of matter has shown that fluids (*e.g.*, water and air) are not continuous, but are made up of atoms. Milikan's oil drop experiment shows that there is the exact equality in the charges carried by all charged drops. The charge is written as ne, where n is an integer number. The symbol 'e' is known as electronic charge. It is also observed that

electron and proton have charges equal in magnitude but opposite in sign. No one has been able to detect a charge smaller in magnitude than the charge of the electron *e*. In addition the magnitudes of all other charges are found to be integer multiples of the magnitude of the charge on the electron. Thus we say that charge exists in discrete packets rather than in continuous amounts and hence is said to be *quantized*. The basic packet, or quantum, of charge has magnitude *e*. All charged elementary particles, known upto this time carry charges of precisely the same magnitude and have their anti-particles. It shows that the *quantization of charge is a deep and universal law of nature*. There are two kinds of charge quanta. Both have the same magnitude. One is that on the electron and other is that on the proton. Higher theories of quantum mechanics are required to explain the quantization of charge.

The quantum of charge *e* is so small that the *graininess* of electricity does not show up in most of the experiments. For example in an ordinary 220 volts 60 watt light bulb, 1.7×10^{18} elementary charges (electrons) enter and leave filament every second.

Conservation of charge. We know that the total charge in an *isolated system* remains constant. By isolated we mean that no matter is allowed to enter or leave the system. The charges can be neither created nor destroyed. An equal quantity of each sign positive and negative is simultaneously produced or disappeared. This is known an *conservation of charge* and is valid both for large scale events and at the atomic and nuclear level, no exceptions have been found. An interesting example of charge conservation is the creation of pair of electron and positron when high energy photons enter a thin walled box in vacuum. Experimentally it was observed that the electric charges of electron and positron were equal in magnitude and opposite in sign. These particles are related to one another as *particle* to *antiparticle*. The reverse process, in which energy appears in the form of two gamma rays when an electron and a positron are brought close to each other, is also observed. This process is known as *annihilation* process. In these processes the net charge is zero both before and after the event so that *charge is conserved*.

Another example of the conservation of charge is found in *radioactive decay*. Few decay processes are :

$$_{92}U^{238} \rightarrow {}_{90}Th^{234} + {}_{2}He^{4}$$
$$_{88}Ra^{226} \rightarrow {}_{86}Rn^{222} + {}_{2}He^{4}.$$

In these processes the totol amount of charge present before disintegration (92 and 88 for the first and second processes respectively) is the same as that present after the disintegration.

Charge conservation is also valid in *nuclear reactions*. Few examples are :

$$_{20}Ca^{44} + _{1}H^{1} \rightarrow _{21}Sc^{44} + _{0}n^{1}$$
$$_{2}He^{4} + _{7}N^{14} \rightarrow _{8}O^{17} + _{1}H^{1}$$
$$_{3}Li^{7} + _{1}H^{1} \rightarrow 2_{2}He^{4}.$$

Here we see that the sum of the atomic numbers (*i.e.*, no. of protons in the atom) before the reaction is exactly equal to the sum of the atomic numbers after the reaction. It means that total charge remains constant or *charge is conserved*.

The law of conservation of charge is also true for relativistic motion. In other words we can say that the total electric charge of an isolated system is *relativistically invariant*.

Unit of charge. One cannot explain the charge in terms of any other known property. In macroscopic charging process the number of electrons involved is very large and we use a unit of charge, the *coulomb*. The *coulomb* is defined as the amount of charge that flows through a wire per second if there is a steady current of one ampere in the wire. In nuclear or atomic problems or in microscopic processes, the unit of charge is taken as the charge on the electron, *i.e.*, *electronic charge*, *e*, which is equal to 1.6021×10^{-19} coulomb.

Conductors and Insulators. For the purpose of electrostatic theory all substances can be divided into two main classes : *conductors and insulators*. In conductors electric charges are free to move from one place to another, whereas in insulators they are tightly bound to their respective atoms. In an uncharged body there are an equal number of positive and negative charges.

The examples of conductors of electricity are the metals, human body and the earth and that of insulators are glass, hard rubber and plastics. In metals, the free charges are free electrons, known as conduction, metallic or free electrons. But in electrolytes, each molecule of electrolyte separates into +vely and —vely charged parts which can move independently of each other. Although there is no perfect insulator or perfect conductor. The insulating ability of fused quartz is about 10^{25} times as great as that of copper and hence used as perfect insulator. The concepts of perfect insulator and of perfect conductor are useful in electrostatic problems. There are a number of substances that are neither good conductors of electricity nor good insulators. These substances are called *semiconductors*.

Charge and Matter. Every matter is consisted of neutral atoms. The atoms are made up of a dense +vely charged nucleus surrounded by the electrons in the orbital motion. The radius of the nucleus is of the order of fermi ($1\ f = 10^{-15} m$) and is about 10^{-5} times

smaller than the radius of the outer electron or that of an atom. In the nucleus protons and neutrons are packed by the strong *attractive forces*, known as *nuclear forces*. These forces are much stronger than the electrostatic repulsive forces between the protons.

The forces that bind the electrons of an atom to the nucleus, the forces that bind atoms together to form molecules and the forces that bind molecules together to form matter may be described with the help of Coulomb's law and the structure of the matter.

2.2 COULOMB'S LAW

Charles Augustin de Coulomb performed following experiment in I785 to find how the force varies with the distance between two point charges. In his experiment, two light balls were fixed at the ends of a light insulating rod and the whole system was suspended by means of a torsion fibre at its centre of mass (Fig 2.1). One ball *a* was given a charge and a third charged ball c supported on the insulating handle was brought near the ball *a*. The charge on *a* was repelled and the rod was rotated and the fibre was twisted. The suspension head was rotated to restore the original position. This experiment was repeated for different distances between balls *a* and *c* and corresponding angles through which the suspension head was rotated to restore the original position were observed. From these results, *Coulomb* gave a law, can be stated as :

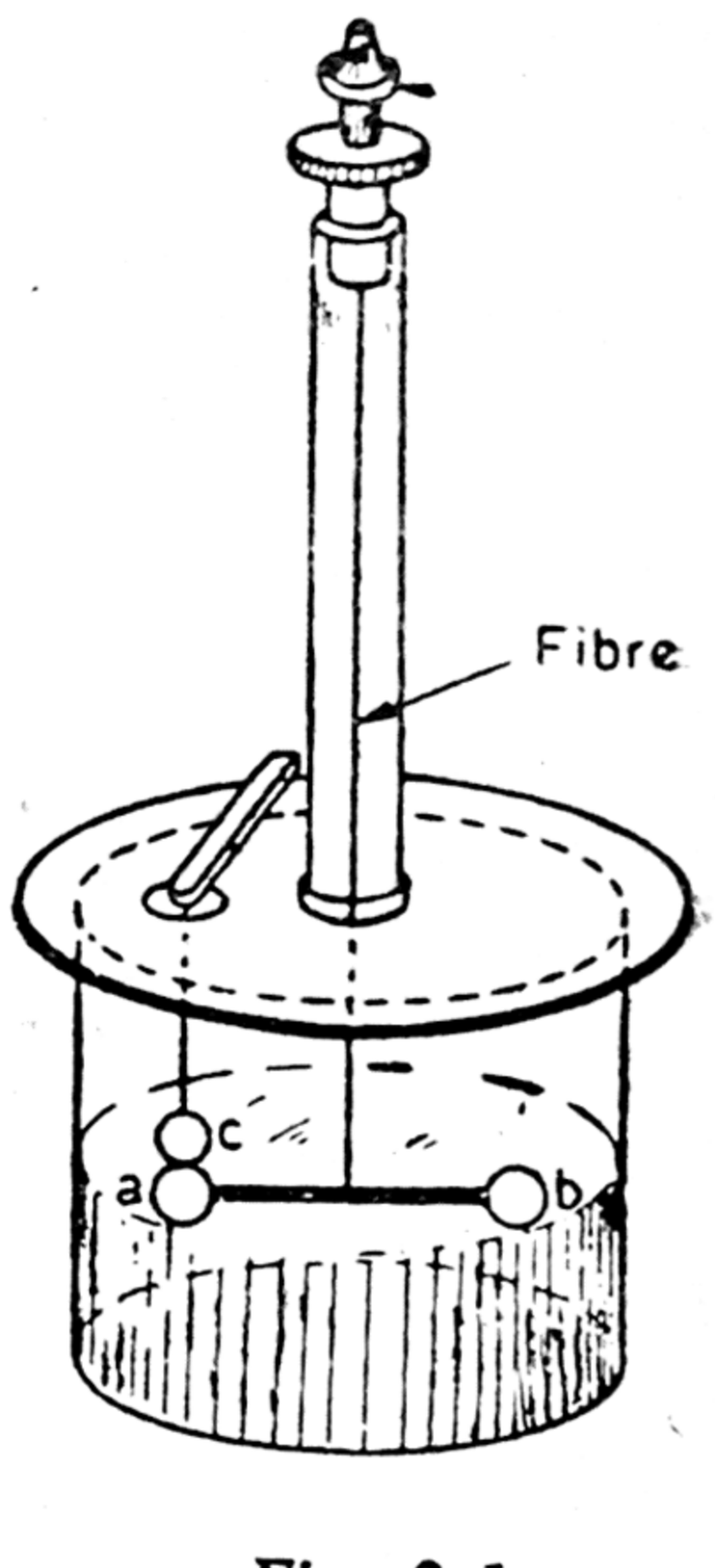


Fig. 2.*1.*

(*a*) The force which two charged bodies 1 and 2 (say), whose dimensions are small compared to their separation (*i.e.*, point charges). exert on one another has a direction of the line joining the charges and is inversely proportional to the square of their separation *r*. *i.e.*,

$$\mathbf{F} \propto \frac{1}{r^2} \hat{r}, \qquad (i)$$

where $\hat{r} = \mathbf{r}/r$ is the unit vector in the direction from charge 1 to charge 2 if the force **F** is acting on charge 2 due to charge 1 (Fig 2.2).

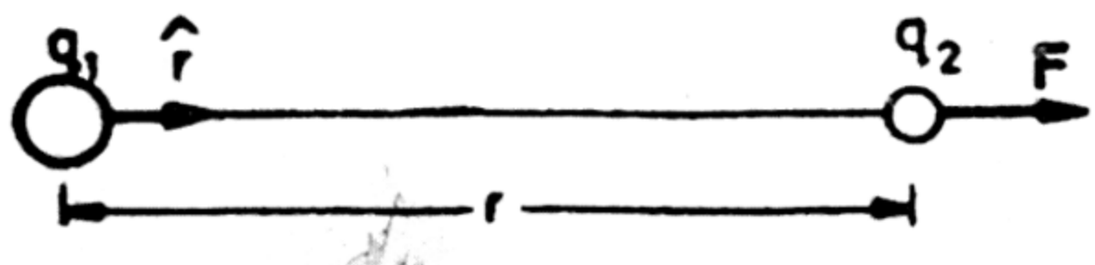


Fig. 2.2.

(*b*) This force is proportional to the product of the charges on these charged bodies.

$$F \propto q_1 q_2. \qquad (ii)$$

(*c*) This force acts as a repulsive force when q_1 and q_2 have same sign and as an attractive force when two signs are opposite.

(*d*) The force between any two charges is independent of the presence of other charges.

Combinning these relations we have

$$F \propto \frac{q_1 q_2}{r^2} \hat{r}. \qquad \ldots(1)$$

It is better to write $\mathbf{F}_2$ instead of $\mathbf{F}$ as we are concerned with the force on charge 2 due to charge 1. Hence the force $\mathbf{F}_1$ on charge 1 due to charge 2 will be $-\mathbf{F}_2$.

In Gaussian system of units the unit of charge is thus defined so that upon a quantity of electricity equal to itself, at a distance 1 cm., it exerts a force of 1 dyne. In this case constant of proportionality for the charges in vacuum or air becomes unity. In SI units the constant of proportionality is usually written in a more complex way as $1/4\pi\varepsilon_0$. Thus we have

$$\mathbf{F}_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}. \qquad (2)$$

There is no choice about the constant ε_0, *known as the permittivity of free space.* It must have that value which makes the right hand side equal to the left hand side. Thus we have

$$\varepsilon_0 = 8.85418 \times 10^{-12} \text{ coul}^2/\text{nt.m}^2 \text{ or } 1/4\pi\varepsilon_0 = 9 \times 10^9 \text{ nt.m}^2/\text{coul}^2.$$

If more than two charges are present, the force exerted on any one say q_1, by all the others q_2, q_3, q_4, etc., can be given by adding vectorially all the forces obtained by using Eq. (2).

$$\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \ldots, \qquad (3)$$

where $\mathbf{F}_{12}$ stands for the force exerted on q_1 by q_2. Thus we have

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} + \frac{q_1 q_3}{r_{13}^2} \hat{r}_{13} + \ldots \right] \qquad \ldots(4)$$

In the Gaussian system of units the constant of proportionality in Coulomb's law is arbitrarily chosen equal to unity and the units of charge, current, etc. thereby established. In SI units we start with the definition of the unit or current strength and from this the units of the other quantities such as voltage, charge, etc. were deduced.

Here the unit vectors $\hat{r}_{12}$ and $\hat{r}_{13}$ have the directions of the lines from q_2 to q_1 and q_3 to q_1 respectively.

For the force on q_1 due to a number of charges, we may write

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_1 q_i}{r_{1i}^2} \hat{r}_{1i}. \qquad \ldots(5)$$

This is just the *superposition principle* for forces.

Let us consider situations where the charge is spread over a region instead of being concentrated at particular points. To calculate the total force on q_1, let us consider the charge as made up of small charge elements and then the forces due to each element dq is added vectorially. As the charge distribution is continuous, hence the summation is replaced by integration. Thus we have

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \int q_1 \frac{dq}{r^2} \hat{r} = \frac{q_1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{r}, \qquad \ldots(6)$$

where $\hat{r}$ is a variable unit vector that points from each charge element dq toward the location of charge q. If the charge is distributed over a volume dv, this equation may be written as

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0} \int q_1 \frac{q\,dv}{r^2} \hat{r} \qquad \ldots(7)$$

where q is the charge density.

Coulomb's law applies to point charges. In the macroscopic sense a point charge is one whose spatial dimensions are very small compared with any other length used in the problem. It also applies to the interactions of elementary particles, such as electrons and protons. It is found to hold even for electrostatic repulsion between protons inside the nucleus, however nuclear forces dominate over this repulsion. The spontaneous emission of α-particles, breaking up of the nucleus into two large fragments, presence of more neutrons than the protons in the heavy nuclei are the Coulomb's repulsion effects. We do not know whether this law holds for very large astronomical distances or not.

Coulomb's law has the same form as Newton's law of gravitation. The analogy between these laws is given below :

Coulomb's Law	**Newton's gravitational law**
(1) It is based on experimental observations.	It is based on speculations concerning the fall of an apple towards earth.

(2) Force is directly proportional to the product of charges.	Force is directly proportional to the product of masses.
(3) Force varies inversely as the square of the distance between two charges.	Force varies inversely as the square of the distance between the masses.
(4) Force between the charges $F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$.	Force between two masses $F = G\frac{m_1 m_2}{r^2}$.
(5) Force is repulsive when charges are of similar nature and is attractive when they are unlike.	Force between two masses is always attractive.
(6) Force between any two charges is independent of the presence of other charges.	Force between any two masses is independent of the presence of other masses.
(7) ϵ_0 is constant for a free space.	G is a universal constant.
(8) ϵ_0 depends upon units of force, charge and distance.	G also depends upon units of force, mass and length.

2.3 ELECTRIC FIELD

The gravitational force can be explained by assuming that every point in space near the earth is associated with a field, known as *gravitational field.* Similarly the space surrounding a charged body is associated with a field known as *electric field.*

If a charge q_1 is placed at any point, it sets up an electric field in the space around itself. This field is indicated by the shaded region in Fig. 2.3. If a charge q_2 is placed in the field region of charge q_1, the former will experience a force **F**.

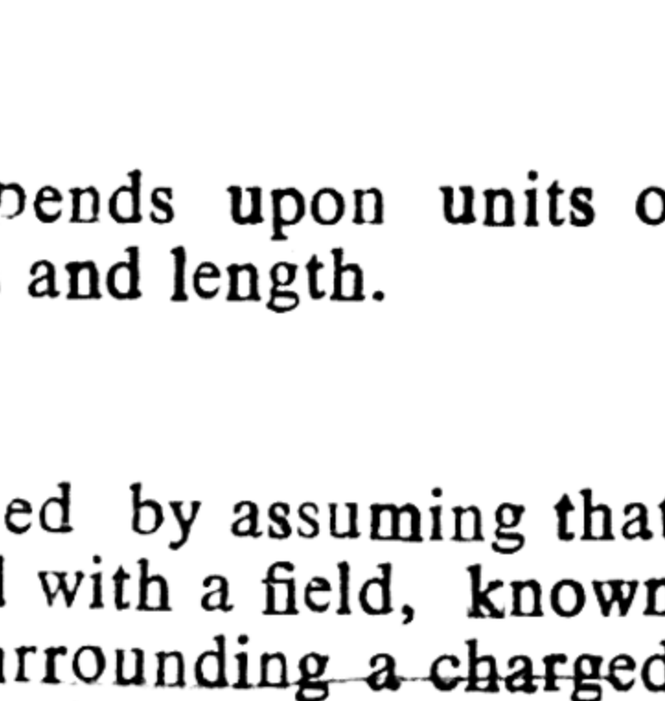


Fig. 2.3.

In this way the field plays an intermediary role in the forces between the charges.

Electric Field Strength. A test charge q_0 (assumed positive for convenience) is placed at any point in the region of any charge where we want to calculate the electric field strength. If this test

charge experiences a force **F**, then the electric field intensity or strength **E** at that point is defined as

$$\mathbf{E}=\mathbf{F}/q_0.$$

The direction of **E** is the direction of **F** and its unit is newton/coulomb. We must assume a test charge as small as possible so that it will not influence the behaviour of the primary charges that are responsible for the field to be determined. Thus Eq. (8) should be replaced by

$$\mathbf{E}=\underset{q_0\to 0}{Limit}\ \frac{\mathbf{F}}{q_0}. \qquad \ldots(9)$$

Actually the *test charge is fictitious*. We merely ask what would be the force on it, if placed at the observation point. The requirement that the test charge be vanishingly small compared with all sources of the field limits the practical validity of the definition, *i.e.* the definition is suitable for macroscopic phenomena only.

The force experienced by test charge q_0 placed at a distance r from a point charge q can be written at once from Coulomb's law as

$$\mathbf{F}=\frac{1}{4\pi\epsilon_0}\frac{qq_0}{r^2}\hat{r}.$$

$$\therefore \mathbf{E}=\underset{q_0\to 0}{Limit}\ \frac{\mathbf{F}}{q_0}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}. \quad =-\frac{q}{4\pi\epsilon_0}\nabla\left(\frac{1}{r}\right). \qquad \ldots(10)$$

Its direction is radial from q pointing outward if q was +ve. It is inward if q was −ve.

If we consider co-ordinates of the points instead of absolute distance between the charges, then Eq. (10) will become

$$\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\epsilon_0}q_1\frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}, \qquad \ldots(11)$$

where electric field is calculated at point $\mathbf{x}$ due to a point charge q_1 placed at point $\mathbf{x}_1$ and $|\mathbf{x}-\mathbf{x}_1|$ is the absolute distance between these points.

2.4. CALCULATION OF ELECTRIC FIELD STRENGTH

To find **E** for a group of point charges, we first calculate $\mathbf{E}_i$ due to each charge at the given point as if it was the only charge present and then add them vectorially.

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_3+\ldots=\sum_{i=1}^{n}\mathbf{E}_i=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_{0i}^2}\hat{r}_{0i},$$

where r_{0i} is the vector from the i^{th} charge in the system to the point in space for which **E** is being calculated.

If the charge distribution is continuous, the field strength can be computed by dividing the charge into infinitesimal elements dq and integrating the field contributions due to all the charge elements.

$$\mathbf{E}=\int d\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\hat{r}, \qquad ...(12)$$

where $\hat{r}$ is the unit vector pointing from dq toward the point in space for which **E** is being calculated.

(*a*) **Electric Dipole.** If two electric charges, equal in magnitude but opposite in signs coincide at any point the electric field strength due to these charges in the space around them vanishes. If the charges suffer a small relative displacement, there is an electric field of sensible magnitude in the neighbourhood of these charges. Such a combination of two charges is termed as *electric dipole.*

The electric field strength **E** at a point Q at a distance r_2 along the perpendicular bisector of the line joining the charges of magnitude q placed at a distance $2a$ apart is given by the relation

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2. \qquad ...(13)$$

From Eq. (10)

$$E_1=E_2=\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{a^2+r_2^2}.$$

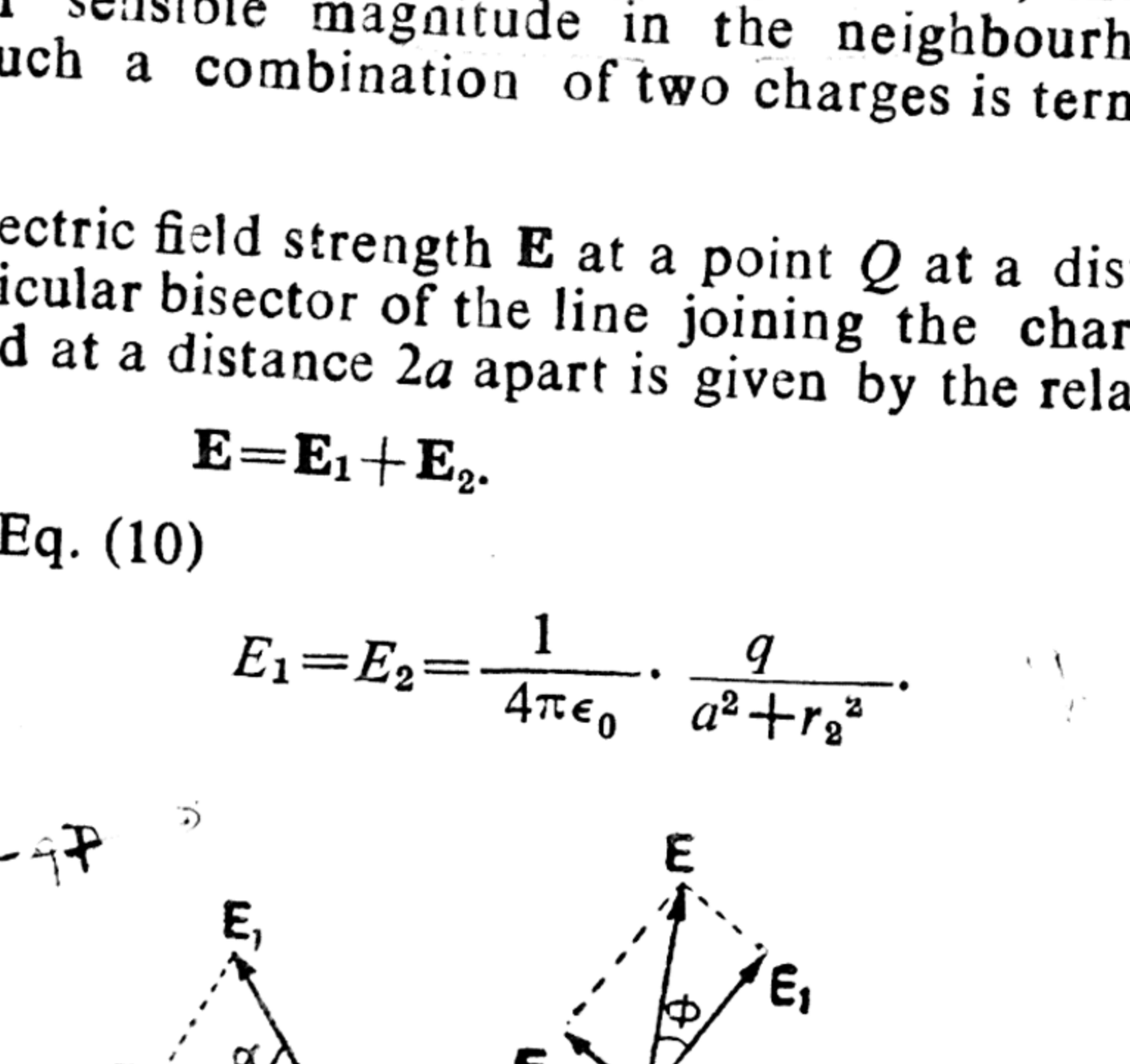


Fig. 2.4.

The vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$ points parallel to the line joining $+q$ and $-q$ and has the magnitude

$$E=2E_1 \cos \alpha$$

$$=2\times\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2+r_2^2}\times\frac{a}{\sqrt{a^2+r_2^2}}=\frac{2aq}{4\pi\varepsilon_0(a^2+r_2^2)^{3/2}}. \quad ...(14)$$

If $r_2 \gg a$, we can neglect a in the denominator and the above equation thus reduces to

$$E=\frac{1}{4\pi\varepsilon_0}\frac{(2a)(q)}{r_2^3}=\frac{1}{4\pi\varepsilon_0}\frac{p}{r_2^3}, \quad ...(15)$$

where $p=2aq$, the product of the magnitude of the charge q and the separation $2a$ between the charges and is known as the *electric dipole moment*. Thus we see that the electric field strength due to dipole at distant points along the perpendicular bisector varies as $1/r^3$. whereas for a point charge it varies as $1/r^2$ only. Its direction is opposite to that of dipole moment which is +ve in the direction from −ve charge to +ve charge.

The electric field strength $\mathbf{E}$ at a point P at a distance r_1 from the centre of the dipole along its axis is given by

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2.$$

The resultant vector $\mathbf{E}$ points parallel to $\mathbf{E}_1$ and antiparallel to $\mathbf{E}_2$, as $\mathbf{E}_2$ is smaller due to the greater distance of $-q$, *i.e.* along the axis of the dipole. The magnitude

$$E=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(r_1-a)^2}-\frac{q}{(r_1+a)^2}\right]=\frac{q}{4\pi\varepsilon_0}\cdot\frac{4ar_1}{(r_1^2-a^2)^2}. \quad ...(16)$$

As in a dipole the distance between two point charges is negligible in comparison to the distance r_1, *i.e.*, $a \ll r_1$, hence

$$E=\frac{2q(2a)}{4\pi\varepsilon_0 r_1^3}=\frac{1}{4\pi\varepsilon_0}\frac{2p}{r_1^3}. \quad ...(17)$$

Its direction is along $\mathbf{E}_1$, *i.e.*, along the direction of dipole moment. It is also inversely proportional to r_1^3 and proportional to the dipole moment p.

For the calculation of electric field strength E at point R having polar co-ordinates r and θ, let us resolve dipole moment $\mathbf{p}$ into two components, one parallel and other perpendicular to OR, the line joining point under consideration to the centre of dipole. The field E_1 due to the component $p\cos\theta$ is equal to $2p\cos\theta/4\pi\varepsilon_0 r^3$ and the field E_2 due to the component $p\sin\theta$ is equal to $p\sin\theta/4\pi\varepsilon_0 r^3$ which is perpendicular to the former. Hence magnitude of the resultant field $\mathbf{E}$ is given by

$$E=\frac{p}{4\pi\varepsilon_0 r^3}\sqrt{4\cos^2\theta+\sin^2\theta}=\frac{p}{4\pi\varepsilon_0 r^z}\sqrt{1+3\cos^2\theta}... \quad (18)$$

Its inclination ϕ to OR is given by

$$\tan \phi = \frac{p \sin\theta / 4\pi\varepsilon_0 r^3}{2p \cos\theta / 4\pi\varepsilon_0 r^3} = \frac{1}{2} \tan \theta. \qquad \ldots(19)$$

(*b*) **Charged Rod.** Let us consider a uniformly charged rod whose linear charge density λ (charge per unit length) is constant. We wish to find the electric field at point P due to this rod. ***Coulomb's law cannot be applied directly in this case as it applies only to point charges.*** However we can imagine the rod to be divided into a large number of small segments, so that each segment will act as a point charge. Coulomb's law can be applied to this segment. If Δx be the length of the segment at a distance x from point P,

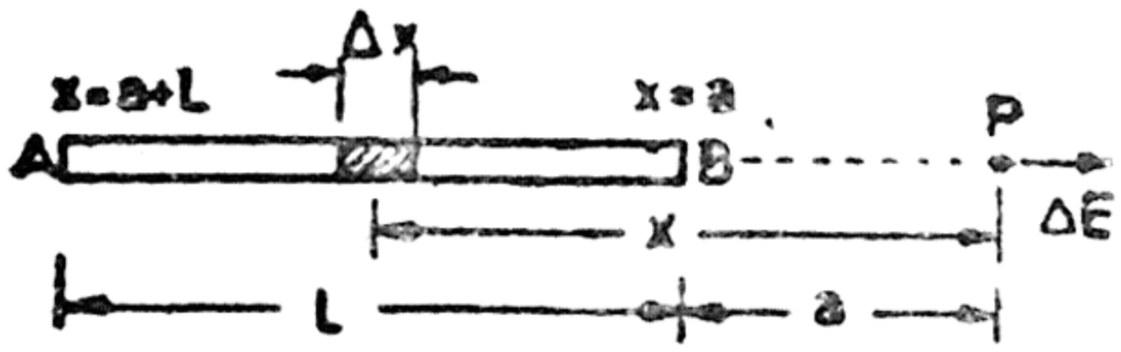


Fig. 2.5.

which is along the length of this rod (Fig 2.5). The electric field due to this segment at point P is given by

$$\Delta \mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda \Delta x}{x^2} x.$$

It is along **BP** direction. Since the fields due to all segments of the rod are in the same direction as Δ **E**, hence

$$E = \sum \Delta E = \sum \frac{\lambda}{4\pi\epsilon_0} \frac{\Delta x}{x^2}. \qquad \ldots(20)$$

If Δx is sufficiently small, we can replace the sum by an integral, thus we get

$$E = \int_{-(a+L)}^{-a} \frac{\lambda}{4\pi\varepsilon_0} \frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0} \int_{-(a+L)}^{-a} \frac{dx}{x^2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \left[\frac{1}{a} - \frac{1}{L+a} \right] \qquad \ldots(21)$$

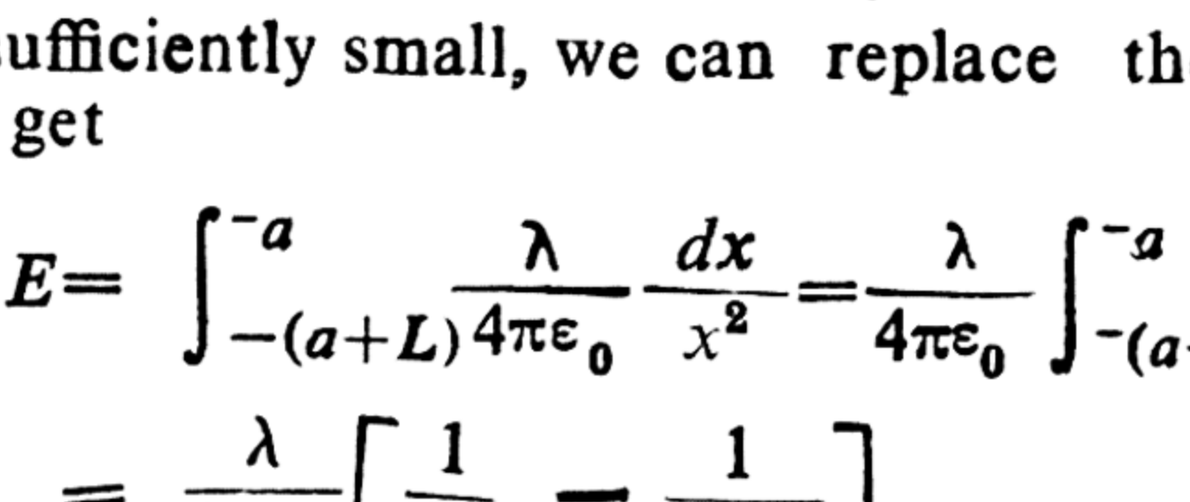


Fig. 2.6.

To find the electric field at a point to one side of the rod we can not write relation (20), as $\triangle E$ due to each segment of the rod will point in a different direction (Fig 2.6). The y—component

$$\triangle E_y = \triangle E \cos\theta.$$

As all the y—components are in the same direction, thus

$$E_y = \sum \triangle E \cos\theta = \sum \frac{\lambda \triangle x \cos\theta}{4\pi\varepsilon_0 r^2} = \int_a^{L+a} \frac{\lambda\, dx}{4\pi\varepsilon_0} \frac{\cos\theta}{r^2}.$$

Unfortunately, there are three variables (x, r and θ). To convert in one let us use the relation $r^2 = x^2 + b^2$ and $\cos\theta = b/r$.

$$\therefore E_y = \int_a^{L+a} \frac{\lambda b}{4\pi\varepsilon_0} \frac{dx}{(x^2+b^2)^{3/2}} = \frac{\lambda b}{4\pi\varepsilon_0} \left[\frac{1}{b^2} \frac{x}{(x^2+b^2)^{1/2}}\right]_a^{L+a}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{L+a}{[(L+a)^2+b^2]^{1/2}} - \frac{a}{(a^2+b^2)^{1/2}}\right]. \quad \ldots(22)$$

The x—component of **E** can be obtained in the similar way.

$$E_x = \sum \triangle E \sin\theta = \int_a^{L+a} \frac{\lambda}{4\pi\varepsilon_0} \frac{x\,dx}{(x^2+b^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{[(L+a)^2+b^2]^{1/2}} + \frac{1}{(a^2+b^2)^{1/2}}\right] \quad \ldots(23)$$

This relation shows that $E_x = 0$ for $a = -L/2$ or on the points lying on the perpendicular bisector of the rod. For these points, the value of $\triangle E_x$ due to the left hand segment cancels the $\triangle E_x$ due to the right hand segment at the same distance.

Another interesting property of the result will be obtained when point P is very far from rod, *i.e.*, $b \gg L$. In this case

$$E_y = \frac{\lambda}{4\pi\varepsilon_0} \frac{L}{b^2} = \frac{q}{4\pi\varepsilon_0 b^2} \text{ and } E_x = 0. \quad \ldots(24)$$

This relation is same as due to a point charge $q(=\lambda L)$. It is expected, as the rod will appear to be a point charge when viewed from far away.

In the special case when the rod is of infinite length, the field can be obtained by integrating with in the limits $x = -\infty$ to $x = +\infty$ and the result will be

$$E_y = \lambda/2\pi\varepsilon_0 b \text{ and } E_x = 0. \quad \ldots(25)$$

Charged Loop. Consider a circular loop of radius a carrying a uniform charge λ per unit length. It is very difficult to

compute **E** at any point in the space, however easy at points on its axis. Let us divide the loop in very small segments so that each segment can be assumed as a point charge. The electric field at a point P on the axis of the loop at a distance x from its centre due to the charge on the segment of length δl (Fig. 2.7) is given by

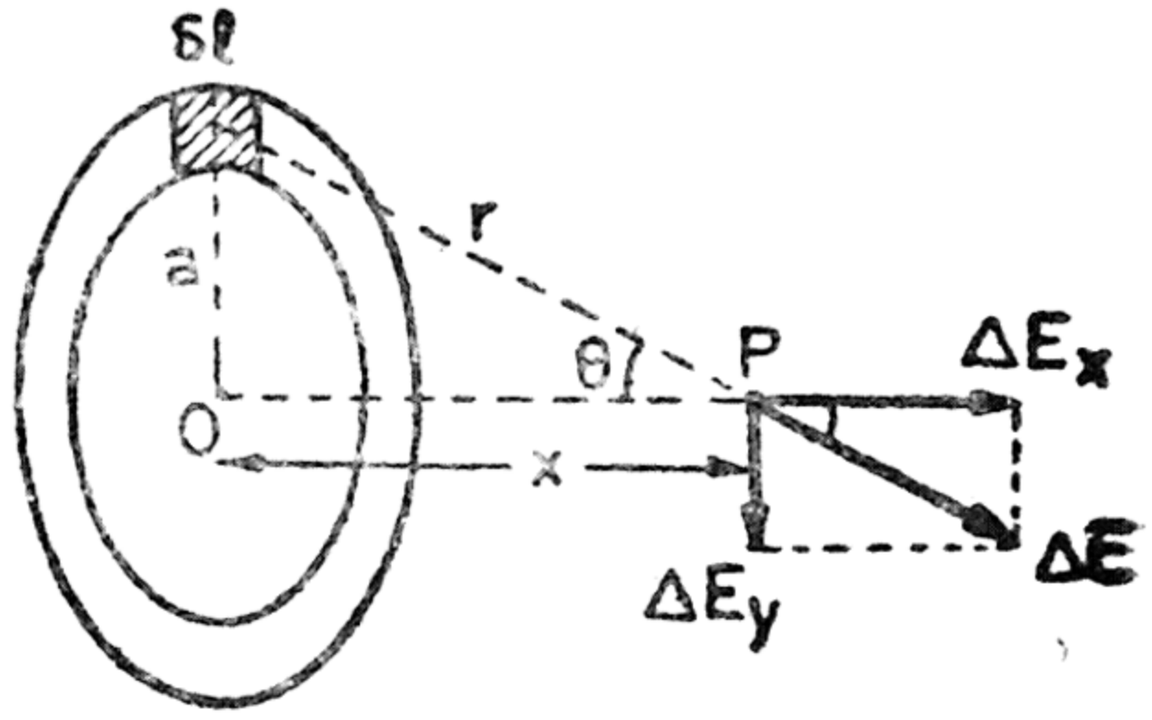


Fig. 2.7.

$$\Delta E = \frac{1}{4\pi\varepsilon_0}\frac{\lambda \delta l}{r^2}.$$

Its direction is along the line from δl to point P. The contribuions to **E** of the various parts of the ring will be in different directions. The y-component ΔE_y cancels with the contribution due to the oppositely placed element of the loop and the net y-component is thus zero. The resultant field **E** will therefore be

$$E = \sum \Delta E_x = \int \frac{\lambda \, dl}{4\pi\varepsilon_0 \, r^2} \cos\theta.$$

Using relations $r^2 = a^2 + x^2$ and $\cos\theta = x/r$, we get

$$E = \frac{\lambda}{4\pi\varepsilon_0}\frac{x}{(a^2+x^2)^{3/2}}\int dl = \frac{\lambda}{4\pi\varepsilon_0}\frac{x.2\pi a}{(a^2+x^2)^{3/2}}$$

$$= \frac{qx}{4\pi\varepsilon_0 (a^2+x^2)^{3/2}}. \qquad (\text{as } q = 2\pi a\lambda) \qquad ...(26)$$

At $x = \infty$, the field will thus be same as due to a point charge.

(d) Infinite Plane Distribution of Charge. Let us find the field at a point P, a height z above an infinite plane distribution of charge of uniform charge density σ. For this, let us consider a charge element ΔS at any point on the plane at a distance r from point P with the co-ordinates $x, y, 0$. (Fig 2.8).

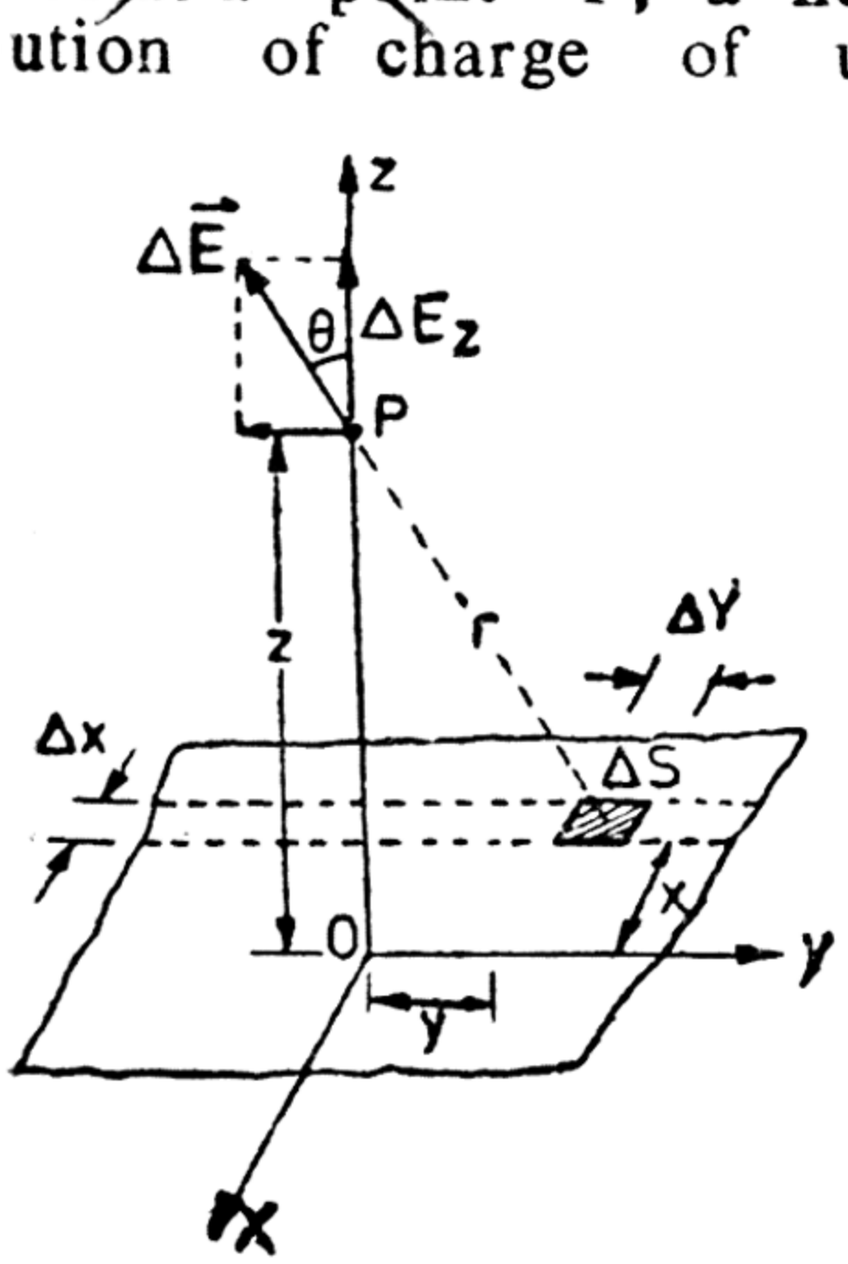


Fig. 2.8.

Hence field due to this element ΔS is given by

$$\Delta \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma \Delta S}{r^2}\hat{r}$$

$$\therefore \Delta E_z = \frac{\sigma}{4\pi\varepsilon_0}\frac{\Delta x \Delta y}{r^2} \times \frac{z}{r}$$

$$= \frac{\sigma z \Delta x \Delta y}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}}.$$

All the components along the z-direction are added

while perpendicular components are cancelled due to the symmetry of the plane of infinite size. Hence the total electric field due to the infinite plane of charge

$$E=E_z=\frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}\frac{z\,dx}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}dy\,\frac{2z}{y^2+z^2}=\frac{\sigma}{2\varepsilon_0}. \qquad \ldots(27)$$

Thus we see that the field strength at a point outside an infinite plane distribution of charge is independent of the distance z of P from the plane and is in the opposite direction on the other side of the plane and hence suffers a discontinuity at the plane.

The above relation can also be obtained by dividing the plane in circular rings. We leave it as an exercise for the students.

(e) Charged Disc. Let us calculate the field strength on the axis of a disc of radius R, at a distance x from its centre O. For this let us divide the disc into annular rings. The field at a point P (Fig 2.9) due to charge element δs on the annular ring of radius a and of thickness δa is given by

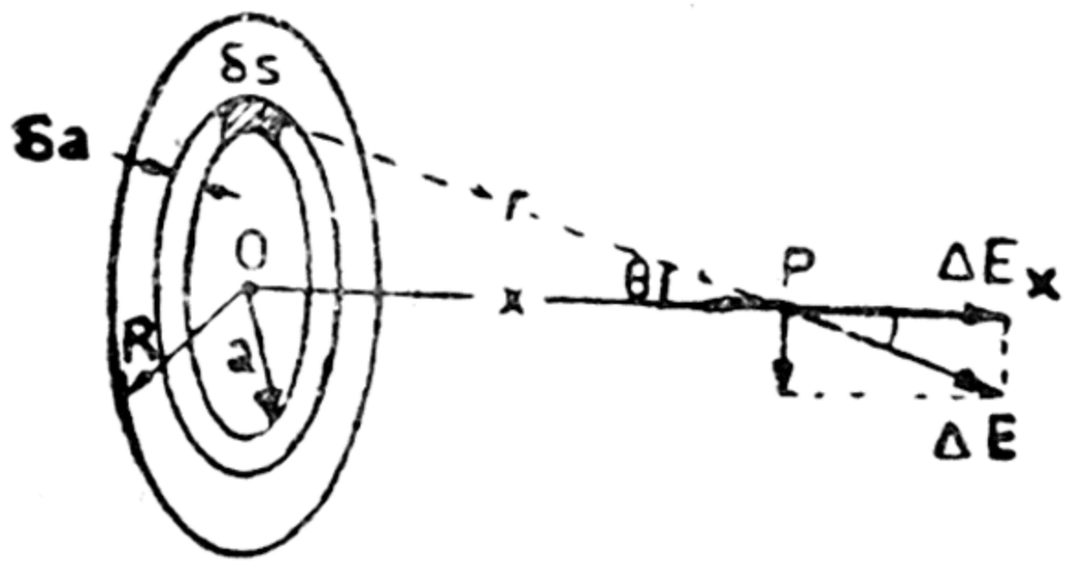


Fig. 2.9.

$$\Delta\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\delta s}{r^2}\hat{r}.$$

Its component along the axis of the disc or along the x direction

$$\Delta E_x=\frac{1}{4\pi\varepsilon_0}\frac{\sigma\delta s}{r^2}\frac{x}{r}.$$

For reasons of symmetry, the total field is along x-direction and is obtained by adding vectorially or by integrating. Assuming that the surface density of charge σ is a function of the distance from the centre of the disc, *i.e.*, $\sigma=ba$, where b is a constant of proportionality.

$$\therefore\ E=E_x=\frac{1}{4\pi\varepsilon_0}\int_0^R\frac{xba}{(a^2+x^2)^{3/2}}\int ds=\frac{bx}{2\varepsilon_0}\int_0^R\frac{a^2\,da}{(a^2+x^2)^{3/2}}$$

$$=\frac{bx}{2\varepsilon_0}\left[\log_e\frac{R+(R^2+x^2)^{1/2}}{x}-\frac{R}{(R^2+x^2)^{1/2}}\right] \qquad \ldots(28)$$

If the surface density of charge σ is constant, then

$$E=E_x=\frac{\sigma}{4\pi\varepsilon_0}\int_0^R\frac{x\,ds}{r^3}=\frac{\sigma}{4\pi\varepsilon_0}\int_0^R\frac{x\,2\pi a\,da}{(x^2+a^2)^{3/2}}$$

$$=\frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{(x^2+R^2)^{1/2}}\right]. \qquad \ldots(29)$$

2.5. EFFECT OF ELECTRIC FIELD ON A POINT CHARGE

The force **F** on a charged particle of mass m and charge q due to an electric field of strength **E** is given by

$$\mathbf{F}=q\mathbf{E}. \qquad \ldots(30)$$

This force will produce an acceleration in the charged particle, given by

$$\mathbf{a}=\mathbf{F}/m=(q/m)\mathbf{E}. \qquad \ldots(31)$$

This acceleration is of charged particle placed, not of that due to which the electric field **E** is. For example, the earth's gravitational field can not have any effect on the earth but only on a second body, say a stone, placed in that field.

2.6. LINES OF FORCE

When the vector **E** is known for all points in an electric field, the field is completely specified. The direction and magnitude of field at any point indicate how a small charge q_0 would begin to move if put there. By following this direction from point to point curves are obtained. These are termed as *lines of force*. Thus *the line of force in an electric field is a curve such that the tangent at any point on it gives the direction of the resultant electric field strength at that point. This is also the path on which a test charge will tend to move, if free to move.*

These lines are imaginary, their existence can be shown by sprinkling saw dust or gypsum salt in the electrostatic field. These particles acquire charges and place themselves along the lines of force. *The properties of electric lines of force are :*

1. No line of force originates or terminates in the space surrounding a charge. Every line of force is a continuous and smooth curve originating from a positive charge and ending on a negative charge.

2. The tangent to a line of force at any point gives the direction of **E** at that point.

3. They do not pass but leave or end on a charged conductor normally when the charges on the conductor are in equilibrium. Suppose the lines of force are not $\perp$ to the conductor surface. In this situation the component of electric field $\parallel$ to the surface would cause the electrons to move and would therefore give rise to a current. Since there is no current on the surface, hence lines of force are always $\perp$ to the conductor surface.

4. Lines of force never intersect. If it happens then we say that at the point of intersection the electric intensity is zero, otherwise it would have to be tangential to two different curves at the same instance which is impossible.

5. The number per unit area crossing a surface at right angle to the field direction at every point is proportional to the electric intensity. Hence the lines of force are closely spaced where the intensity is large and are widely separated where the intensity is small.

6. They repel each other in the case of two like charges.

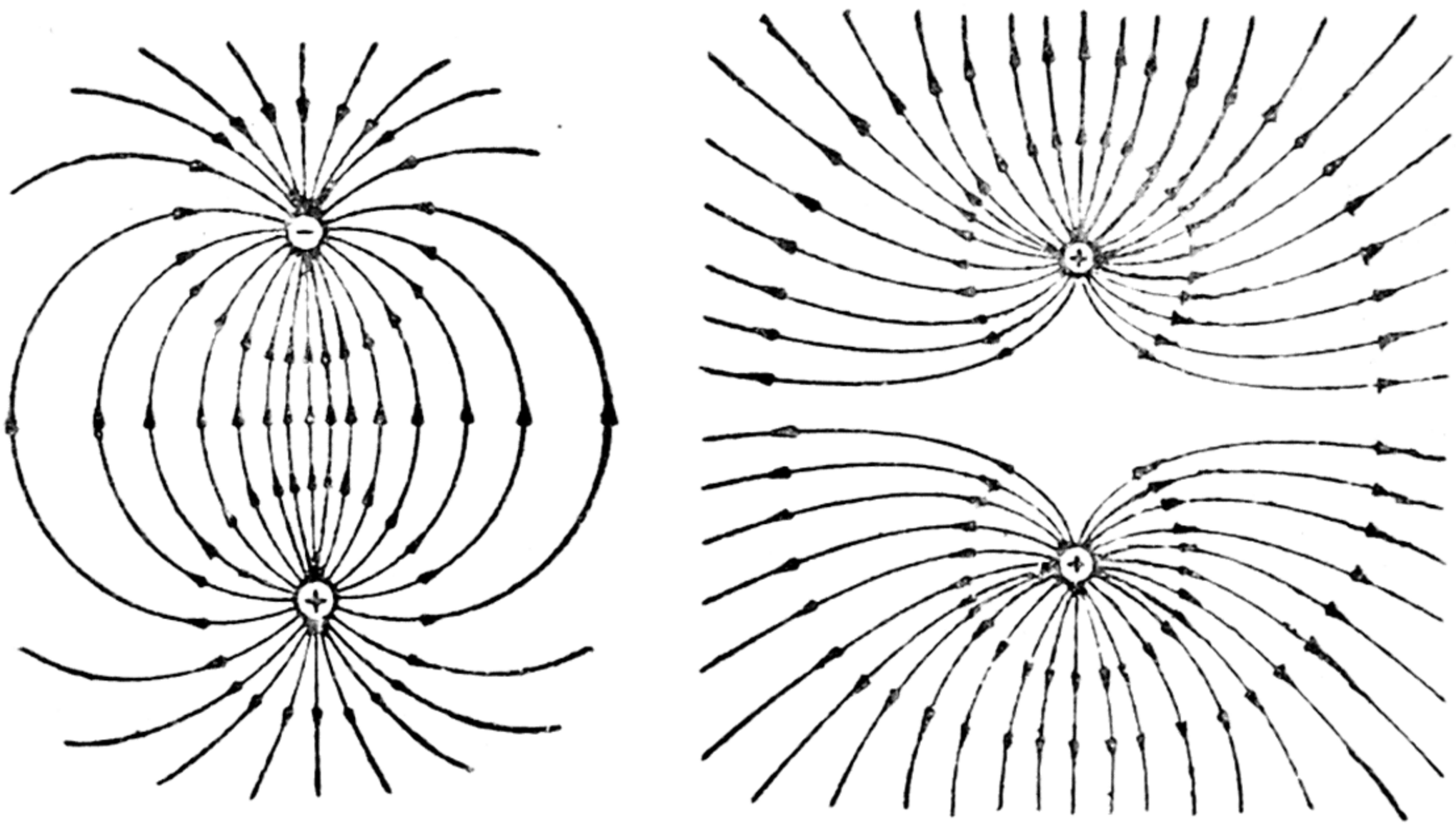


Fig. 2.10.

The direction of lines of force in two cases are shown in Fig. 2.10.

We define unit field (1 N/C) arbitrarily as corresponding to unit density of lines of force (one line/m^2). To determine the number of lines per unit charge, let us draw a spherical surface of radius r around the charge q as its centre. The electric field E at the surface is given by $E=q/4\pi\varepsilon_0 r^2$. As the density of lines of force is E, the total number of lines of force originating at q and crossing the spherical surface will be $4\pi r^2 E=q/\varepsilon_0$. Hence the *number of lines originating from a unit charge is* $1/\varepsilon_0$.

Electric lines of force are very useful as they give qualitative picture of the electric field distribution. There is a simple connection between the lines of force and inverse square law. We know that the density of lines at r meter from an isolated charge q is $q/4\pi\varepsilon_0 r^2$. As the area of the spherical surface subtended by a bundle of lines originating from the point charge is four times greater for a surface $2r$ meter away than for a surface at r meter distance. Thus the density of lines of force at $2r$ meter is just one quarter that at r meter in agreement with the value from inverse square law.

2.7. SOLID ANGLE

Solid angle is the analogue in three dimensions of the usual angle in two dimensions. In two dimensions, the unit angle is the

angle subtended by an arc of length equal to its radius and is known as *radian*. In three dimensions, a given area on a sphere subtends a certain solid angle at the centre. The unit solid angle is that subtended at the centre of a sphere by an area r^2 and is known as steradian. Since total surface area of the sphere is $4\pi r^2$, hence the total solid angle subtended at its centre is 4π steradians.

In Fig. 2.11, dS and dS' subtend the same solid angle. Thus the solid angle subtended by any surface dS at a point O a distance r away is given by

$$d\omega = dS \cos\theta / r^2, \qquad \ldots(32)$$

where $dS \cos\theta$ $(=dS')$ is the projection of the surface dS perpendicular to the radius vector r from the point O.

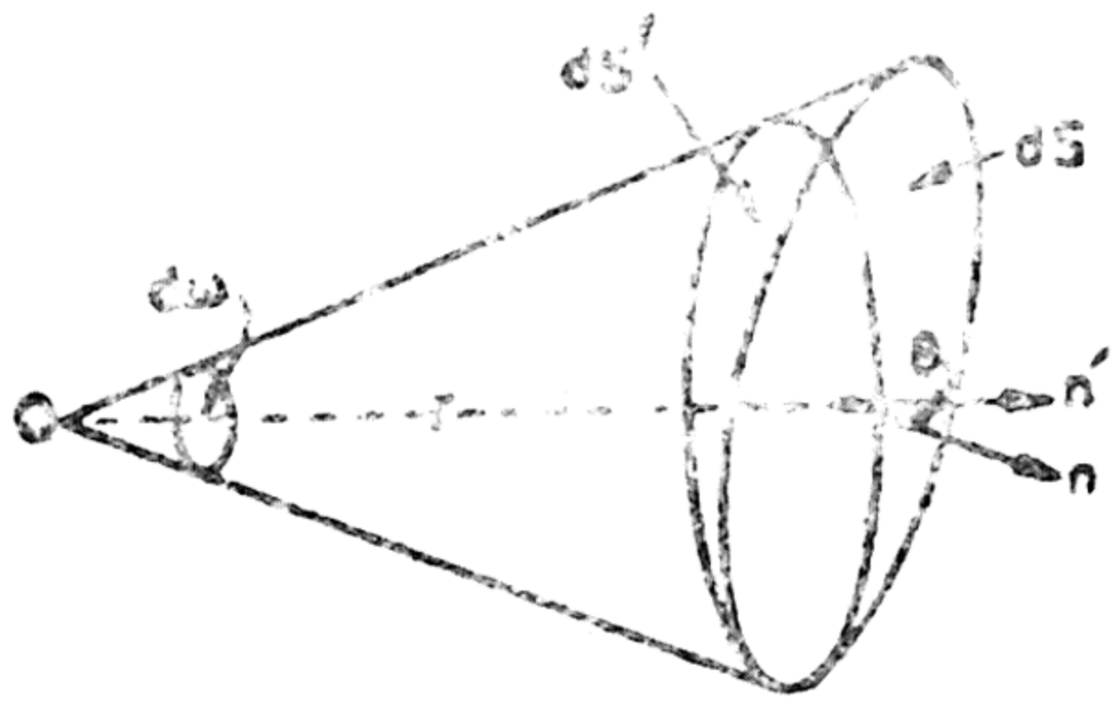


Fig. 2.11.

2.8. GAUSS'S LAW

Consider a single positive point charge q surrounded by a closed surface of arbitrary shape, as shown in Fig. 2.12. The electric

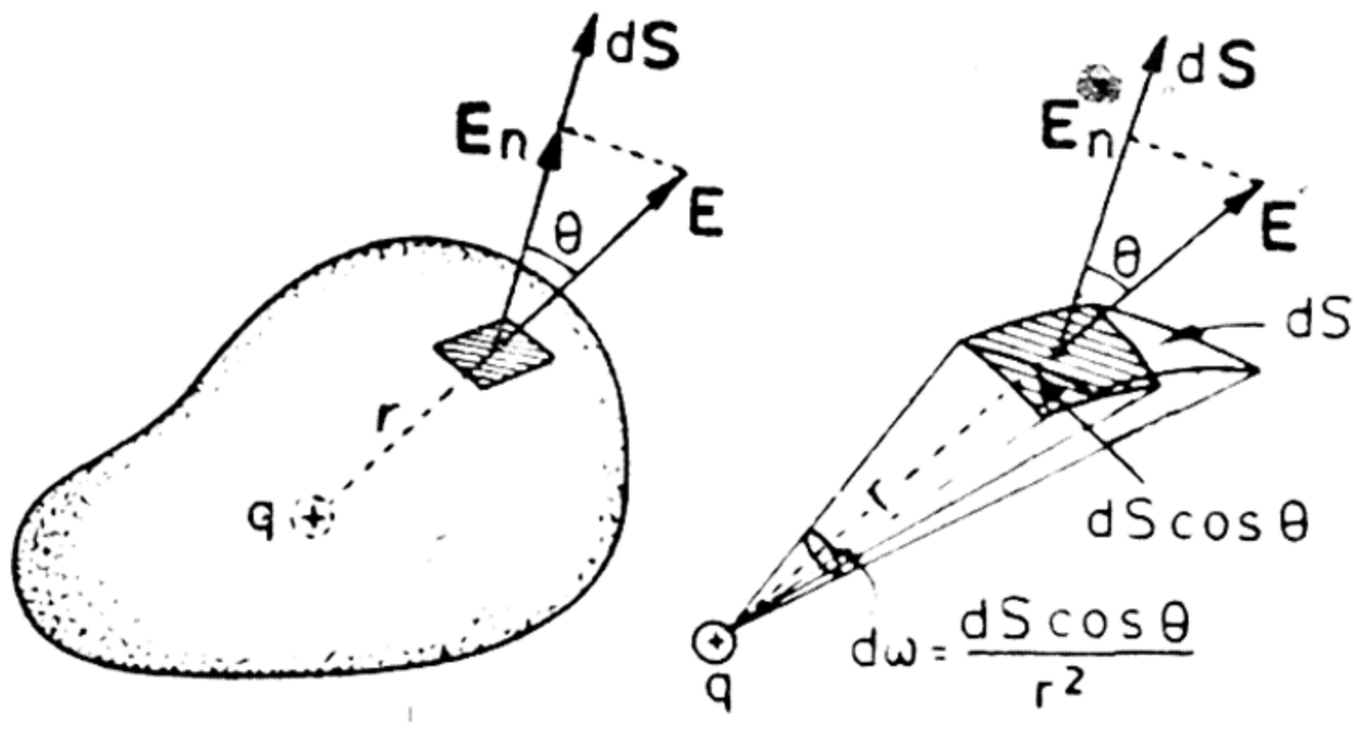


Fig. 2.12.

intensity **E** at every point of the surface is directed radially outward from the charge. Let us consider any sufficiently small area dS of the surface for which **E** can be considered to have same magnitude and direction. If θ be the angle between **E** and the outward normal to the surface at any point of this area. The product E_n, the component of **E** normal to the surface, and the area dS is

$$E_n\, dS = E \cos\theta\, dS = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta\, dS.$$

As $dS \cos\theta/r^2$ can be replaced by the term solid angle $d\omega$ subtended at the charge q by the area dS, hence

$$E_n\, dS = (q/4\pi\epsilon_0)\, d\omega.$$

If the surface is continuous we can integrate both sides of this equation over the entire closed surface and so we have

$$\oint_S E_n \, dS = \frac{q}{4\pi\varepsilon_0} \oint d\omega.$$

Regardless of the shape or size of the closed surface, $\oint d\omega$=Total solid angle surrounding the charge $q=4\pi$,

$$\therefore \quad \oint_S E_n \, dS = \frac{q}{4\pi\varepsilon_0} \; 4\pi = \frac{q}{\varepsilon_0} \; .$$

As the product $E_n \, dS = E \cos\theta \, dS = \mathbf{E}.d\mathbf{S}$, hence we have

$$\oint_s \mathbf{E}.d\mathbf{S} = q/\varepsilon_0. \qquad \ldots(33)$$

If the point charge is negative the field **E** is directed radially inward, the angle θ is greater than 90° or E_n is negative. The surface integral is thus negative. Hence the form of Eq.(33) is correct whatever be the sign of the charge.

When the point lies outside the surface, every elementary cone from O cuts the surface an even number of times. Let us draw a cone of solid angle $d\omega$ at point O, intercepting areas δS_1, δS_2, δS_3, δS_4, δS_5 and δS_6 at points A, B, C, D, E and F respectively. The corresponding outward normal over these areas are shown in Fig. 2.13. The angles θ_1, θ_3 and θ_5 are obtuse and θ_2, θ_4 and θ_6 are acute. Thus

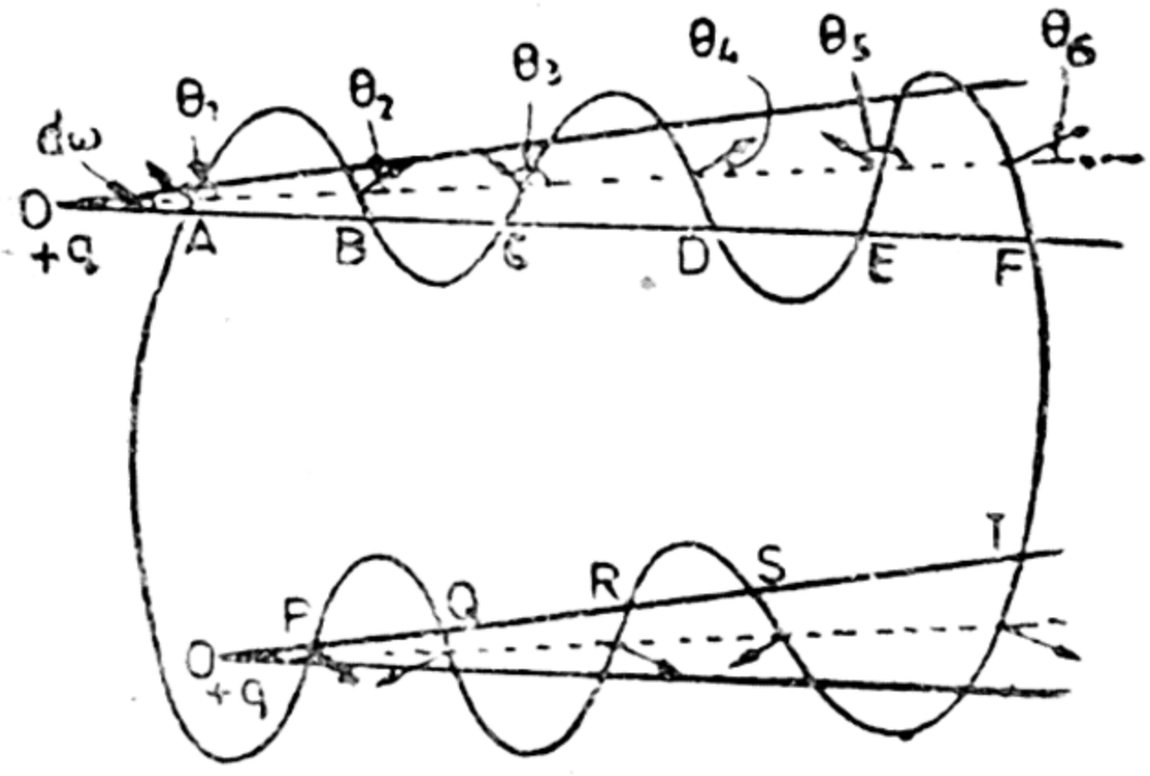


Fig. 2.13.

$$d\omega = \frac{\delta S_1 \cos(\pi-\theta_1)}{r_1^2} = -\frac{\delta S_1}{r_1^2} \cos\theta_1 = \frac{\delta S_2}{r_2^2} \cos\theta_2 = \text{etc.}$$

$$\therefore \quad \delta S_1 \cos\theta_1 = -r_1^2 \, d\omega \; ; \; \delta S_2 \cos\theta_2 = r_2^2 \, d\omega \; ; \ldots\text{etc.}$$

Hence the contribution from these six (say) areas to the surface integral is

$$E_1 \cos\theta_1 \, \delta S_1 + E_2 \cos\theta_2 \, \delta S_2 + E_3 \cos\theta_3 \, \delta S_3 + E_4 \cos\theta_4 \, \delta S_4 + \ldots$$

$$= \frac{q}{4\pi\varepsilon_0 r_1^2} (-r_1^2 d\omega) + \frac{q}{4\pi\varepsilon_0 r_2^2} (r_2^2 \, \delta\omega) + \frac{q}{4\pi\varepsilon_0 r_3^2} (-r_3^2 \, d\omega) + \ldots$$

$$= \frac{q}{4\pi\varepsilon_0} [-d\omega + d\omega - d\omega + d\omega - d\omega + d\omega] = 0. \qquad \ldots(34)$$

But the charge inside the closed surface is zero, hence the Eq. (33) holds good whether the charge inside the surface is +ve, −ve or zero.

Any arbitrary distribution of charges can be assumed as a system of point charges. We can use Eq. (33) for each point charge and sum over all charges. In this case charge q is equal to Σq or more accurately $\int \rho\, dV$. Thus

$$\oint_s \mathbf{E}.d\mathbf{S} = \frac{1}{\varepsilon_0}\int_v \rho dV. \qquad \ldots(35)$$

This is the equation which expresses Gauss's law : *The surface integral of the normal component of electric field* **E** *over any closed surface in an electrostatic field equals* $1/\varepsilon_0$ *times the total charge enclosed by the surface.*

The left hand side of Eq. (35) is called the flux of **E** across the surface and is represented by Φ. Thus

$$\Phi = \oint_s \mathbf{E}.d\mathbf{S} = \frac{1}{\varepsilon_0}\int_v \rho dV. \qquad \ldots(36)$$

The term flux is borrowed from hydrodynamics, where a similar integral in which **E** is replaced by the velocity vector **v** gives the net flow of fluid across a surface. The Gauss's law can now be stated as : *the outward flux of electric field across a closed surface equals* $1/\varepsilon_0$ *times the net charge contained in the volume enclosed by the surface.*

In terms of electric displacement density **D** (to be defined later), which is equal to $\varepsilon_0\mathbf{E}$, the Gauss's law can be stated as : *the net outward electric displacement through a closed surface is equal to the net charge contained in the volume enclosed by the surface, i.e.,*

$$\oint_s \mathbf{D} \,.\, d\mathbf{S} = \int_v \rho\, dV. \qquad \ldots(37)$$

Gauss's law can also be written in the other form by using divergence theorem $\oint_s \mathbf{D} \,.\, d\mathbf{S} = \int_v \nabla \,.\, \mathbf{D}\, dV$,

as
$$\int_v \nabla \,.\, \mathbf{D}\, dV = \int_v \rho dV. \qquad \ldots(38)$$

This holds for any volume whatsoever. As the volume considered is reduced to an elementary volume, this becomes the point relation

$$\nabla \,.\, \mathbf{D} = \rho, \qquad \ldots(39)$$

i.e., at every point in a medium the divergence of electric displacement is equal to the charge density at that point.

Graphical Interpretation of Gauss's Law. For a flow field the flux is measured by the number of stream lines that cut through such a surface. As the number of electric lines of force per unit area at right angles to their direction is proportional to E, the surface integral of **E** over a closed surface is proportional

to the total number of lines of force crossing the surface in an outward direction. The total number of lines of force is also proportional to the net charge within the surface.

For closed surfaces Φ is positive if the lines of force point outward everywhere (**E** is outward every where, $\theta < 90°$ and **E**.d**S** will be positive) and negative if they point inward (**E** is inward every where, $\theta > 90°$ and **E** . d**S** will be negative). Let us consider two equal and opposite point charges and their lines of forces as shown in Fig. 2.14. From the statement just given, Φ is positive for surface S_1 and is negative for surface S_2. Surface S_4 encloses both charges. The net number of lines of force crossing in an outward direction is zero and the net charge inside the closed surface is also zero. Similarly in surface S_3, number of lines of force in inward direction is same as those in outward direction or the net number of outward lines of force is zero. The charge inside this surface is also zero. In this way it is clear that Gauss's law is correct whatsoever be the charge inside the closed surface.

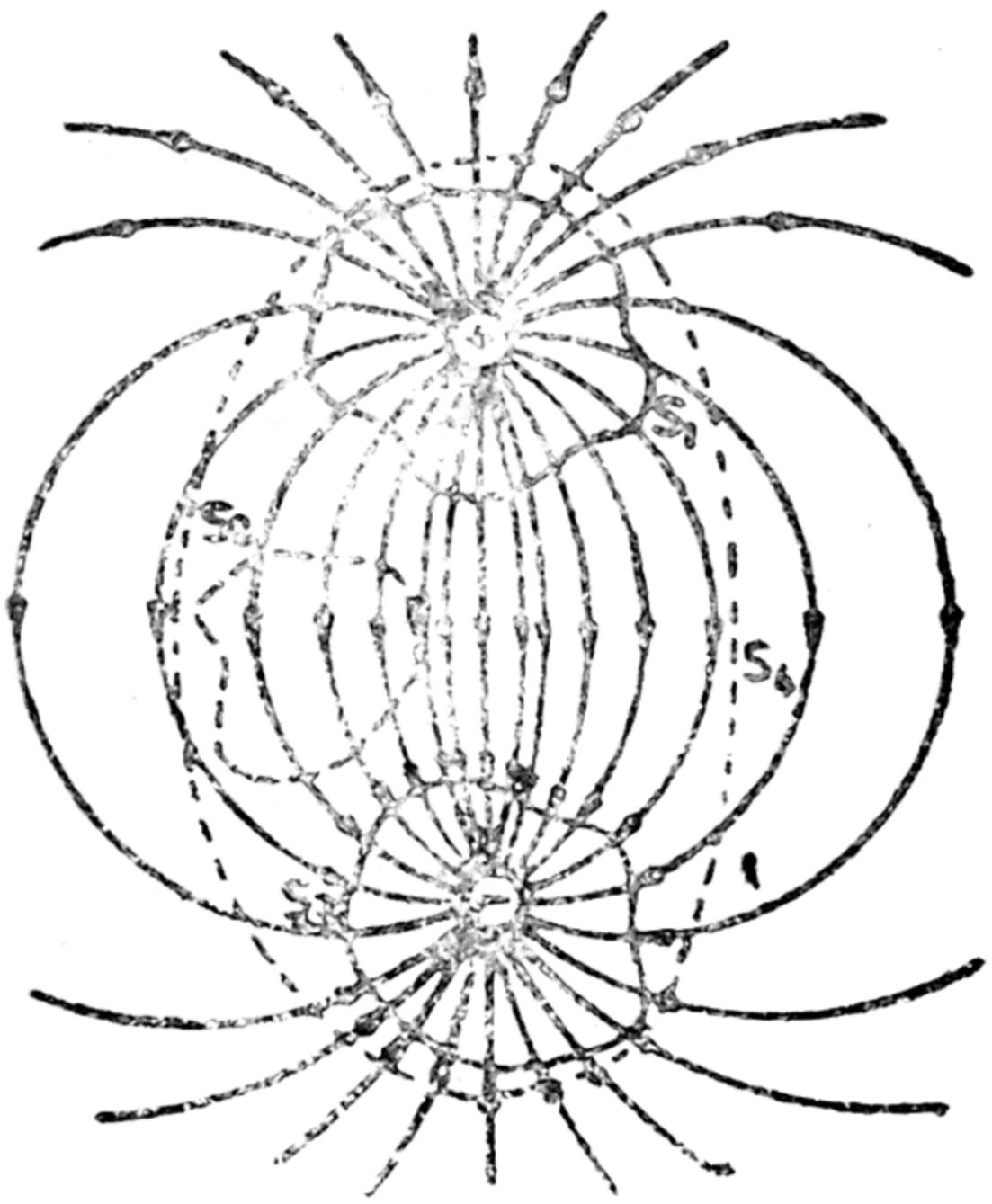


Fig. 2.14.

2.9. APPLICATIONS OF GAUSS'S LAW

As Gauss's law does not provide expression for **E** but provides only for its surface integral. To calculate **E** we must choose surface in such a way that the surface integral may be replaced by a product of which **E** is a factor. Let us discuss few simple cases :

(A) Point charge—The electric field due to a point charge is everywhere radial. Its magnitude is the same at all points at the same distance r from the charge. Hence we select gaussian surface (*a closed surface drawn in an electric field*) a spherical surface. It is also clear from Fig. 2.15 that both **E** and d**S** (direction of which is always perpendicular to the surface itself) at any point on the gaussian surface are directed radially outward. Hence

$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_S E_n dS = ES = 4\pi r^2 E.$

But from Gauss's law it should be $1/\varepsilon_0$ times the total charge inside the gaussian surface. Hence

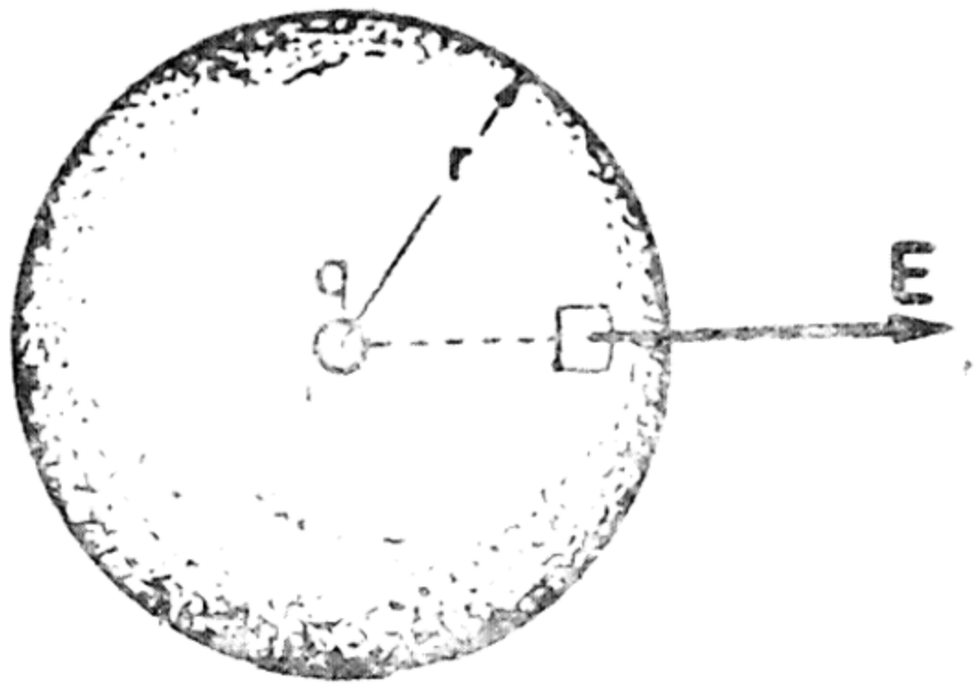


Fig. 2.15.

$$4\pi r^2 E = q/\varepsilon_0$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \ldots(40)$$

The force on the point charge q_0 if placed on this surface, *i.e.*, at a distance r from point charge q is then

$$F = q_0 E = qq_0/4\pi\varepsilon_0 r^2.$$

The force is acting in radially outward direction, hence can be represented vectorially as $\mathbf{F} = (qq_0/4\pi\varepsilon_0 r^2)\,\hat{r}$ ...(41)

It is nothing but *Coulomb's law.*

(B) Uniformly charged sphere—In the uniformly charged sphere of radius R and charged to q units, the charge density ρ at every point is constant. It is evident that the field due to a spherical charged surface at an external point P has the same symmetry as that of a point charge. Thus we can construct gaussian surface by drawing a concentric sphere of radius OP with centre O. At all points of this sphere the magnitude of electric field is the same and its direction is perpendicular to the surface. Thus

$\oint_S \mathbf{E}.d\mathbf{S} = \oint_S E_n dS = E4\pi r^2$ and from Gauss's law

$$4\pi r^2 E = \frac{q}{\varepsilon_0} \text{ or } E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \quad \ldots(42)$$

Hence the *electric field at any external point due to a uniformly charged sphere is the same as if the total charge is concentrated at its centre.*

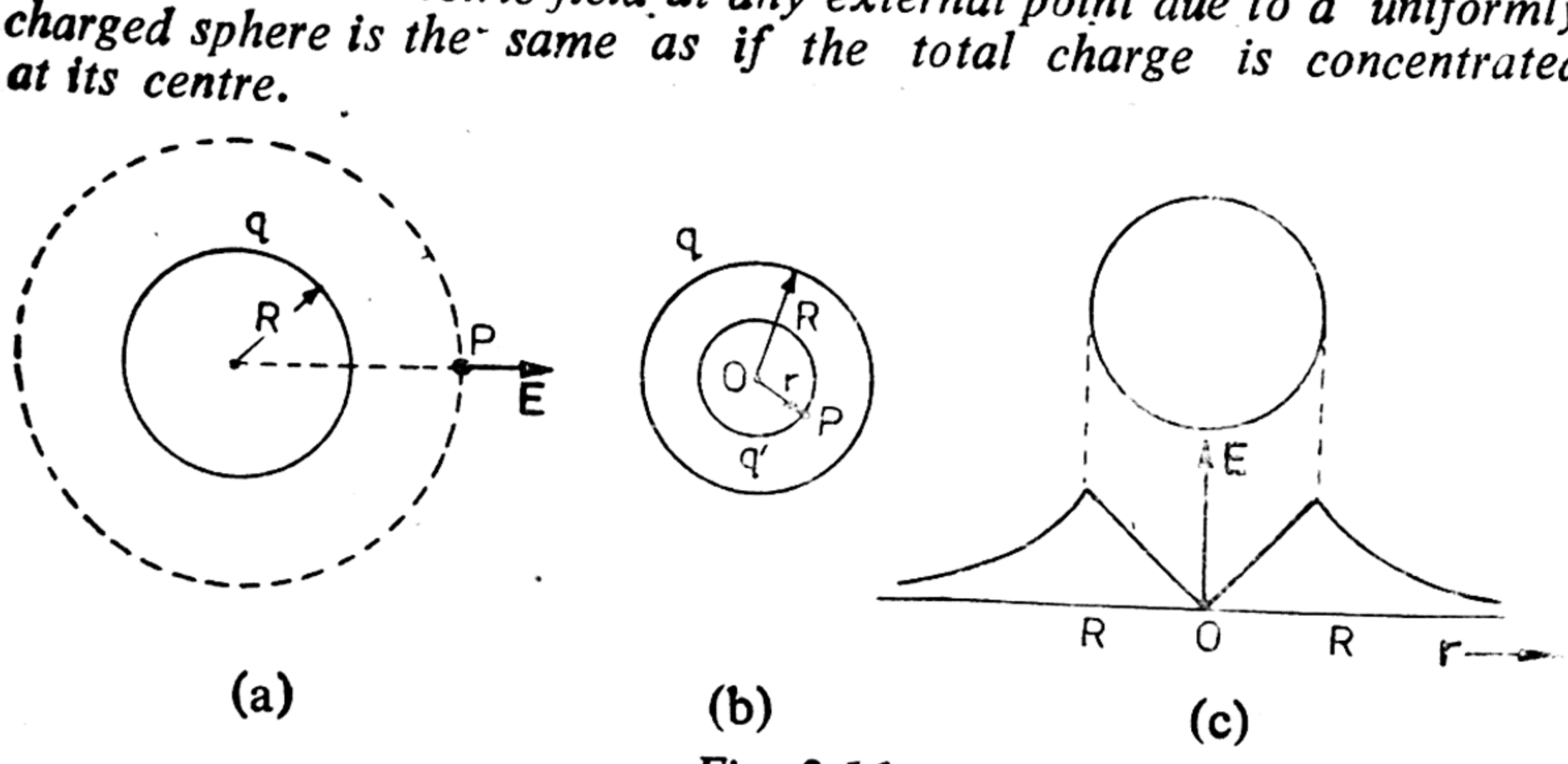


Fig. 2.16.

At the surface of sphere $r=R$, the electric field

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}. \qquad \ldots(43)$$

For the field inside the charged sphere let us draw gaussian surface of radius $r(<R)$, as shown in Fig. 2.16 (b). Thus

$$\oint_S \mathbf{E}.d\mathbf{S}=\oint_S E n dS=4\pi r^2 E$$

and from Gauss's law

$$4\pi r^2 E=q'/\varepsilon_0,$$

where q' is that part of charge which is with in the sphere of radius r. The part of q that lies outside this sphere makes no contribution to E at the surface of sphere of radius r. If ρ is the volume density of the charge which is constant, then

$$q'=\frac{4}{3}\pi r^3\rho=\frac{4}{3}\pi r^3\left(\frac{q}{\frac{4}{3}\pi R^3}\right)=\left(\frac{r^3}{R}\right)^3 q.$$

$$\therefore 4\pi r^2 E=\frac{q}{\varepsilon_0}\left(\frac{r}{R}\right)^3 \qquad \text{or } E=\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}.$$

In vector from

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\mathbf{r}. \qquad \ldots(44)$$

From Eqs (42), (43), and (44), it is clear that the electric field due to a uniformly charged sphere at the surface is maximum and is zero at the centre [Fig. 2.16 (c)].

(C) **An Isolated uniformly charged spherical conductor.** In an isolated charged spherical conductor any excess charge on it is distributed uniformly over its outer surface and there is no charge inside it. Hence this problem is same as that of charged spherical shell or hollow sphere. As in the previous case, the field at external points has the same symmetry as that of a point charge so we can construct a gaussian surface of radius $r> R$. The electric intensity calculated, as in the previous case, is given by the relation

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}, \qquad \ldots(45)$$

It is same as due to the uniformly charged sphere or the point charge of same magnitude if placed at the centre.

At the surface $r=R$,

$$E=\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}. \qquad \ldots(46)$$

At all points inside the charged spherical conductor or hollow spherical shell, electric field $\mathbf{E}=0$ as there is no charge inside such a sphere.

(D) Charged cylindrical conductor of infinite length. If the cylinder (say a wire) is long and of radus R and we are not too near to either end, then by symmetry the lines of force outside the cylinder are radial and perpendicular to its surface. The electric field has the same magnitude at all points at the same radial distance. Let us draw an imaginary co-axial cylinder (a gaussian surface) of length l through any point at a distance r from the axis of the cylinder as shown in Fig 2.17. Lines of force are parallel to the circular caps, hence there is no component of electric field normal to the end faces (the circular caps). Thus flux through these faces is zero and the total flux is due to curved surface only. If λ is the charge per unit length, *i.e.*, linear charge density, the charge with in the gaussian surface will be λl. Thus for the curved surface

$$\oint s\, \mathbf{E}.\, d\mathbf{S}=\oint s\;\; EdS \;=\; E2\pi rl$$

From Gauss's law

$$E\,2\pi rl=q/\varepsilon_0=\lambda l/\varepsilon_0$$

or $$E=\lambda/2\pi\varepsilon_0 r. \qquad \ldots(47)$$

If the charge is distributed throughout the volume and charge density is ρ, then

$$2\pi rl\; E=\rho(\pi R^2 l)/\varepsilon_0$$

or $$E=\rho R^2/2\varepsilon_0 r. \qquad \ldots(48)$$

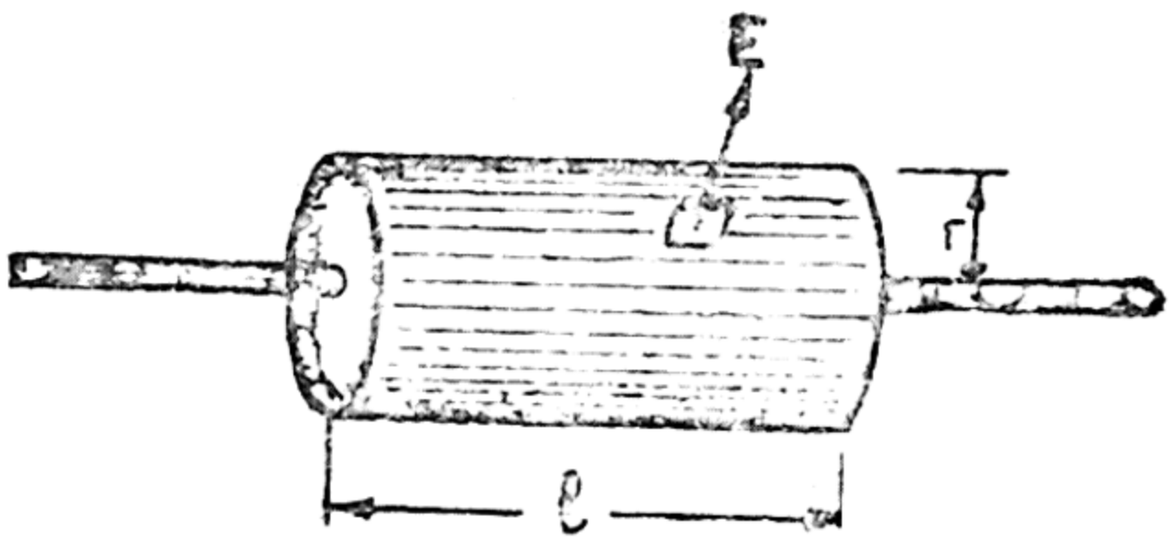


Fig. 2.17.

The electric field inside the charged cylinder will be zero if the charge is on its surface only, as net charge in the gaussian surface through this point is zero. The case is different when the electric charge is distributed uniformly with in the cylinder of radius R. To find $\mathbf{E}$ at an inner point P, a distance r apart from the axis of the cylinder, let us draw a gaussian surface passing through P, which is a cylinder of length l and of radius r (Fig 2.18). As the flat surfaces do not contribute to the flux and the flux is due to the curved surface for which $\mathbf{E}$ is $\perp$ to the surface or $\parallel$ to the surface vector $d\mathbf{S}$. Hence from Gauss's law we have

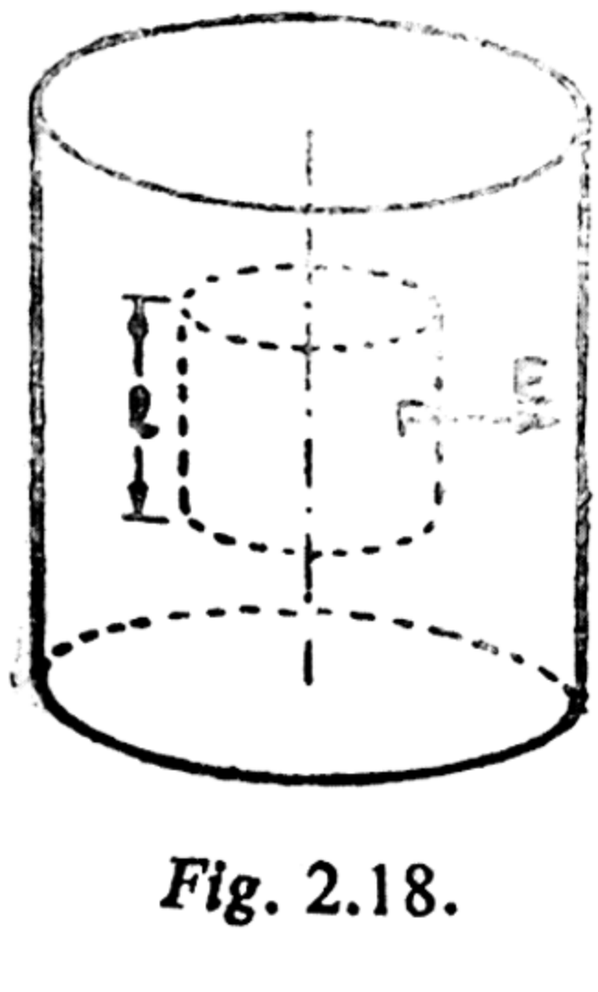


Fig. 2.18.

$$\oint s\; \mathbf{E}.d\mathbf{S}=\oint s\; EdS=E\;2\pi rl=q'/\varepsilon_0.$$

The charge q' inside this gaussian surface $=\pi r^2 l\rho$

$$\therefore\; E2\pi rl=\pi r^2 l\rho/\varepsilon_0 \text{ or } E=r\rho/2\varepsilon_0. \qquad \ldots(49)$$

Thus the electric intensity at a point inside an infinite uniformly charged cylinder is radially directed and varies as the distance from its axis. At $r=R$, relations (48) and (49) give the same result. The field decreases at points outside.

Here we see that the electric field due to a charged cylinder does not depend upon its radius, hence we can say that it is same as though the charge on the cylinder were concentrated in a line along its axis. It is right to state that the electric field so calculated is due to the entire charge on the cylinder although only a portion of total charge is used when we apply Gauss's law. We can very well understand that the existence of the entire charge on the cylinder is taken into account indirectly by considering the symmetry of the problem. If the cylinder is of small length we can not conclude that the field at one end of cylinder is same as at the centre or the field is symmetric. It means that the lines of force are everywhere not perpendicular to the cylinder. In this way the entire charge is used indirectly by taking field symmetry.

(E) Infinite plane sheet of charge. Fig. 2.19 shows a portion of a flat thin sheet, infinite in size with the constant surface charge density σ. By symmetry, since the sheet is infinite, the field must have the same magnitude and the opposite directions at two points equidistant from the sheet on opposite sides. To solve the given problem let us draw a cylinder A (gaussian surface) with one end on one side and other end on the other side and of cross sectional area S. No lines of force cross the cylindrical wall, *i.e.*, the component of **E** normal to curved surface is zero. Thus the surface integral of $\mathbf{E}.d\mathbf{S}$ over the entire surface of cylinder is reduced to the surface integral due to side surfaces. Hence Eq. (33) becomes

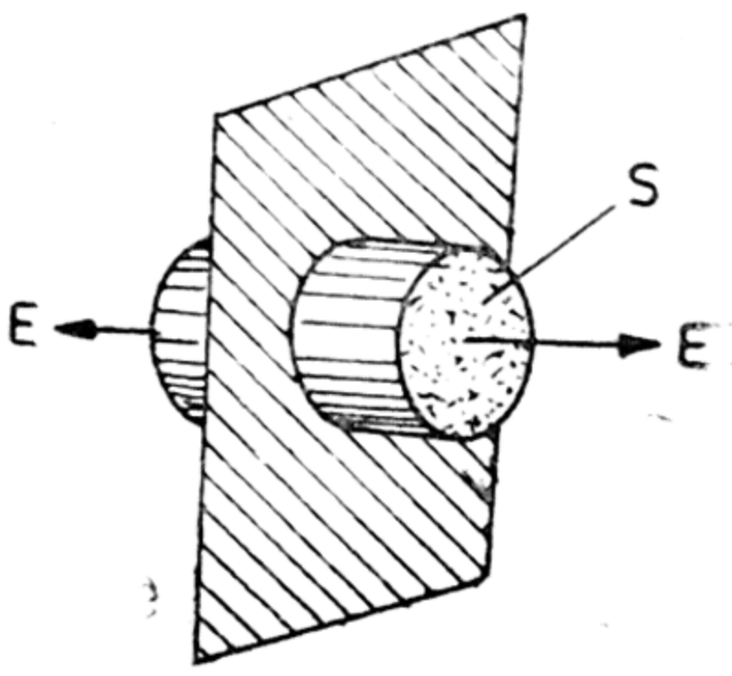


Fig. 2.19

$$\oint_s \mathbf{E}.d\mathbf{S}=q/\varepsilon_0 \text{ or } \oint_{s_1} EdS+\oint_{s_2} EdS=\sigma S/\varepsilon_0.$$

$$\therefore \quad 2ES=\sigma S/\varepsilon_0 \text{ or } E=\sigma/2\varepsilon_0. \qquad \ldots(50)$$

Thus we see that the magnitude of the field is independent of the distance from the sheet. Practically an infinite sheet of charge does not exist. *These results are correct for real charge sheets if points under consideration are not near the edges and the distances from the sheet are small compared to the dimensions of sheet.*

(F) Infinite charged conducting plate. When a charge is given to a conducting plate, it distributes itself over the entire outer surface of the plate. The surface density of charge σ is uniform and is the same on both surfaces if plate is of uniform thicknses and of

infinite size. The electric field at any point thus arises from the superposition of the fields of two sheets of charge, one on each surface of the plate. By symmetry it is very easy to say that the field is perpendicular to the plate and is in outward direction if the charge on the plate is positive and uniform. Draw a cylinder through any point P, as shown in Fig. 2.20, normal to the plate surface and cross sectional area S. The other end face of the cylinder lies inside the conductor. This face does not contribute to the flux as the field inside the conductor is zero. Curved surface of the cylinder also does not contribute to the flux as **E** is parallel to the curved surface. Thus the only face is the outside end face which contributes to the flux as **E** is $\perp$ to this surface. Hence from Gauss's law

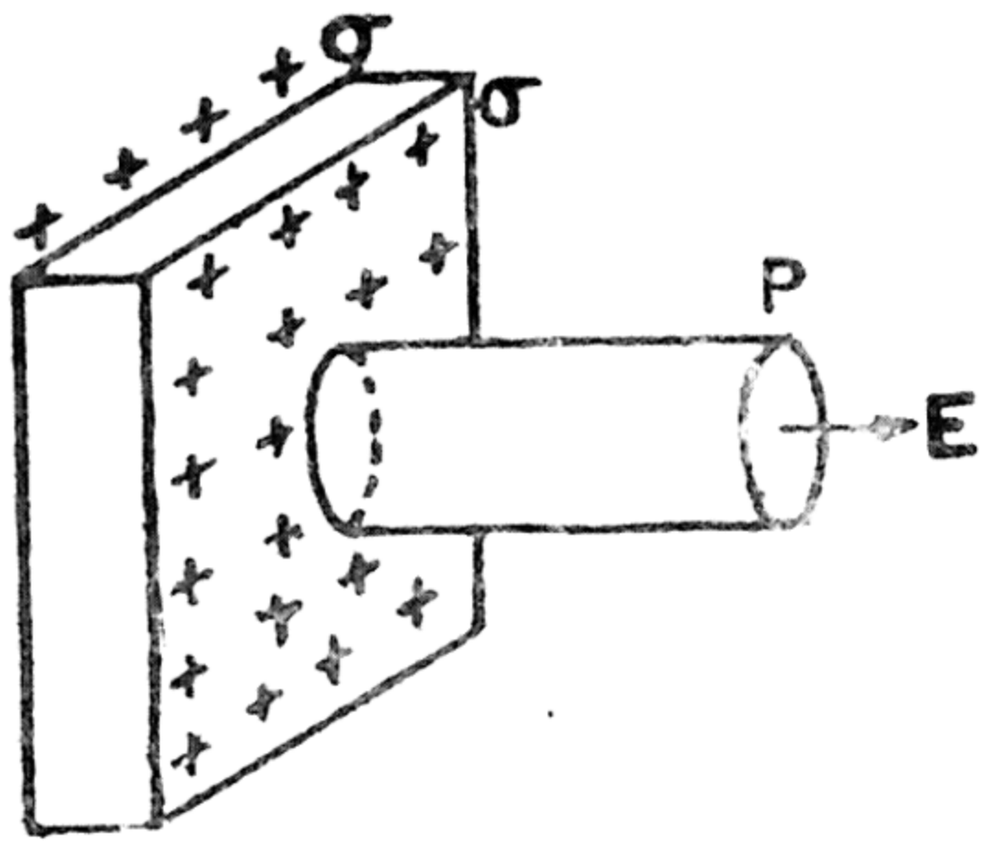


Fig. 2.20.

$$\oint_S \mathbf{E}.\, d\mathbf{S} = q/\varepsilon_0 \quad \text{or} \oint_S E dS = ES = \rho S/\varepsilon_0$$

$$\therefore \qquad E = \sigma/\varepsilon_0. \qquad \ldots(51)$$

This result shows that the field due to charged conducting plate is twice the field due to plane sheet of charge. It also has same limitations.

(G) Electric intensity between oppositely charged parallel plates. When two plane parallel infinite conducting plates, separated by a distance, are given equal and opposite charges, the field is uniform in the space between them. There is a small quantity of charge on the outer surfaces of the plates and a certain spreading or fringing (edge effect) of the field at the edges of the plates. We can assume fringing negligible as the plates are of the infinite size. Thus the field between plates can be taken as uniform.

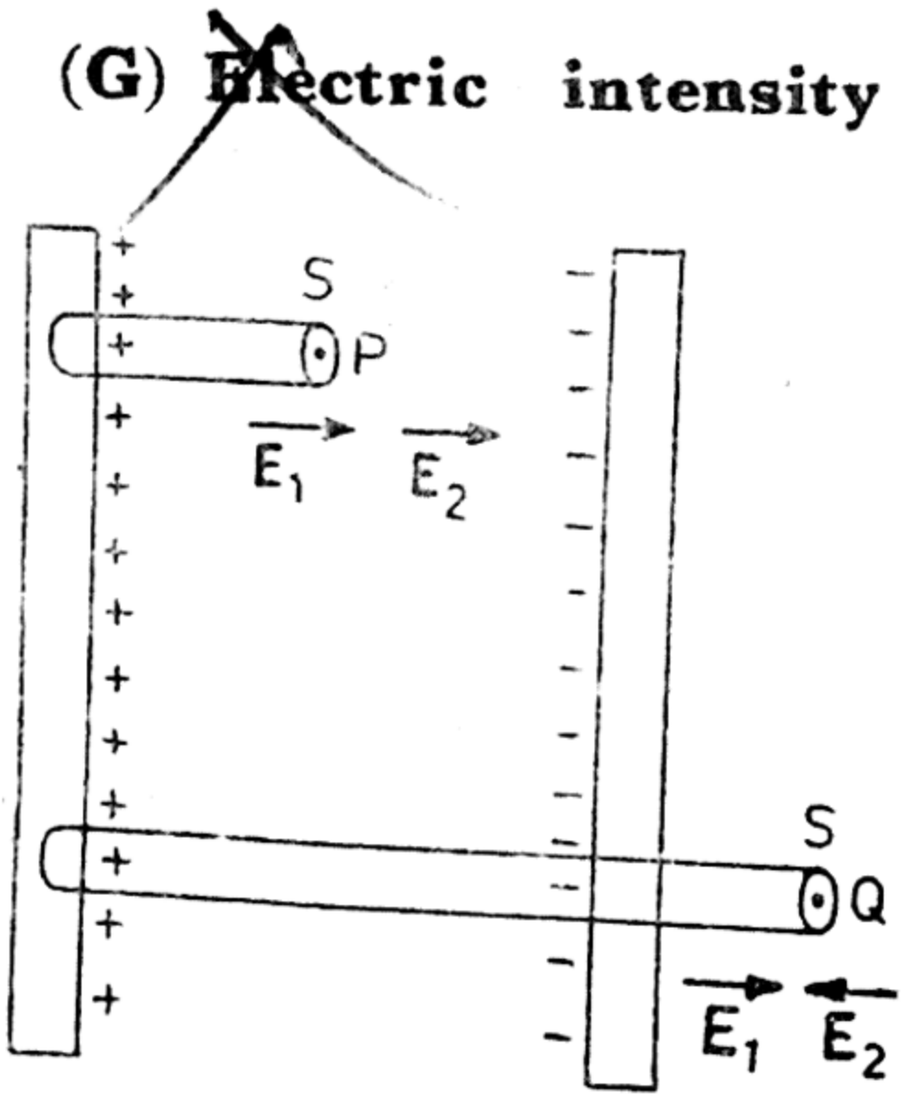


Fig. 2.21.

Draw a normal cylinder through a point P, between two plates and let the other end of this cylinder lying in the left conducting plate and crossing an area S. The problem is same as with an

infinite charged conducting plate and the electric field at point P is thus given by

$$E=\sigma/\epsilon_0. \qquad \ldots(52)$$

To find the electric intensity at an outside point Q, draw the cylinder through the point Q whose second end face lying on the first conductor. It is also clear that the surface of the cylinder contributing to the flux is the end face passing through the point Q. Hence from Gauss's law

$$\oint_S \mathbf{E}.\,d\mathbf{S}=\oint_S E dS=ES=(\sigma S-\sigma S)/\varepsilon_0$$

or $$E=0. \qquad \ldots(53)$$

The electric field at any point due to this system of plates may be written as the superposition of the fields due to infinite plane of sheets of charge. Therefore at point P, the electric field

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2 \text{ or } E=E_1+E_2$$

$$\therefore \quad E=\sigma/2\,\epsilon_0+\sigma/2\varepsilon_0=\sigma/\varepsilon_0.$$

For point Q, $\mathbf{E}_1$ and $\mathbf{E}_2$ are antiparallel, hence the net field

$$\mathbf{E}=0.$$

Thus *the electric intensity due to two oppositely charged infinite planes is* σ/ε_0 *at any point in between the planes and is zero for all external points.*

(H) Coulomb's theorem. Consider a charged conductor of irregular shape. In general, surface charge density will vary from point to point. At a small surface δS, let us assume it to be a constant, say σ. Let us construct a gaussian surface in the form of a cylinder AB of cross section δS, one end face of which is inside the conductor and other passes through a point P outside the surface close to it, as shown in Fig. 2.22. The end face B does not contribute to flux as $\mathbf{E}$ is zero everywhere inside the conductor. The curved surface also does not contribute to the flux as $\mathbf{E}$ is always normal to the charged conductor and hence parallel to the curved surface. Thus the only contribution to the flux is through the end face A which is outside the conductor, Thus from Gauss's law

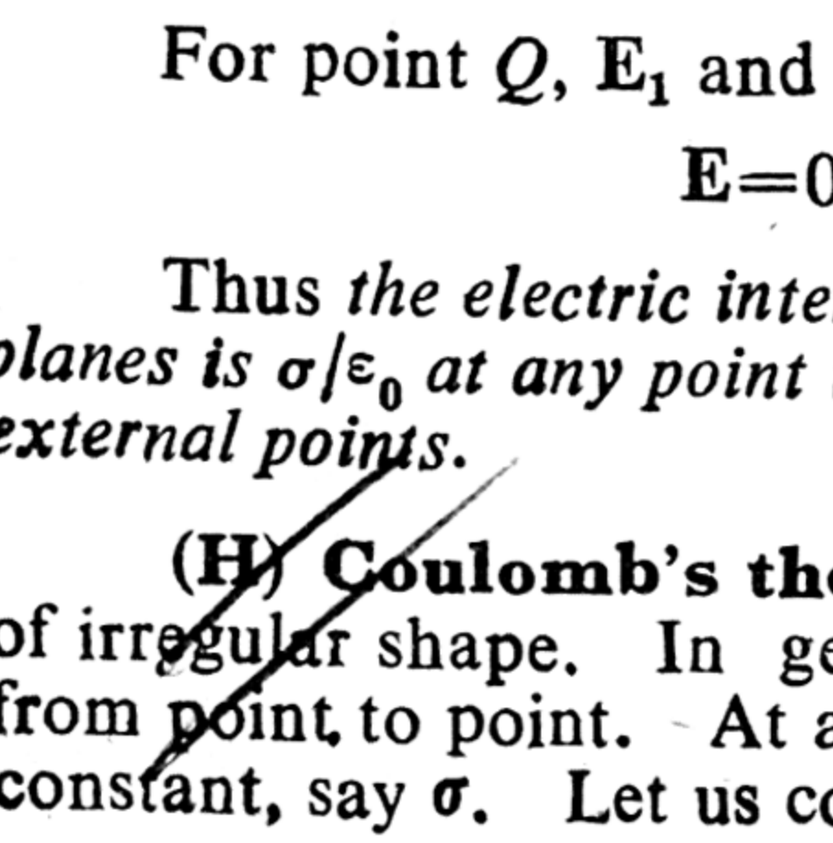


Fig. 2.22.

$$\oint_S \mathbf{E}.\,d\mathbf{S}=\oint_S E dS=E\delta S=\frac{\sigma \delta S}{\varepsilon_0}$$

or $$E=\sigma/\epsilon_0. \qquad \ldots(54)$$

This agrees with the results already obtained for spherical, cylindrical and plane surfaces and is known as *Coulomb's theorem.* This theorem states that the electric intensity at any point close to the charged conductor is $1/\varepsilon_0$ times the surface density of charge on the surface.

2.10. ELECTRIC FIELD AND CHARGE IN CONDUCTORS

Remembering that the conductors possess free electrons. If a resultant electric field exists in the conductor, these free charges will experience a force which will set a current flow. When no current flows, the resultant force and the electric field must be zero. Thus, under electrostatic conditions, the value of **E** at all points within a conductor is zero. This idea, together with the Gauss's law can be used to prove several interesting facts.

(*i*) Consider a charged conductor carrying a charge Q and no currents are flowing in it. Now consider a gaussian surface inside the conductor, everywhere on which $\mathbf{E}=0$. Thus from Gauss's law

$$\oint s\mathbf{E}.d\mathbf{S}=(1/\varepsilon_0)\ \Sigma q,$$

we get

$$\Sigma q=0 \text{ as } E=0. \qquad ...(55)$$

Thus the sum of all charges inside the gaussian surface is zero. This surface can be taken just inside the surface of the conductor, hence any charge on the conductor must be on the surface of the conductor. In other words, *under electrostatic conditions, the excess charge on a conductor resides on its outer surface.*

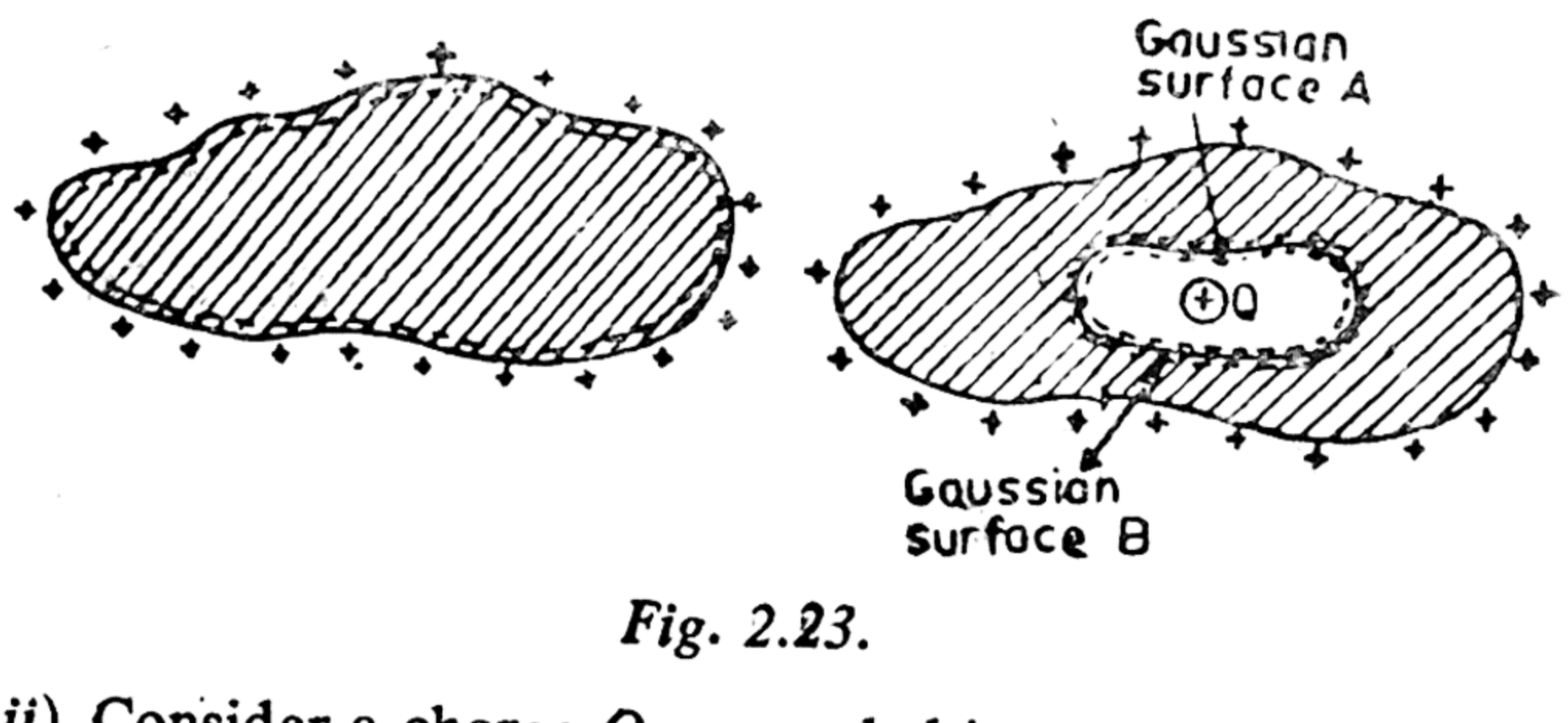


Fig. 2.23.

(*ii*) Consider a charge Q suspended in a cavity in a conductor. Let us apply Gauss's law to the gaussian surface A (within the cavity).

$$\oint s\mathbf{E}\ .\ d\mathbf{S}=(1/\varepsilon_0)\ \Sigma q=Q/\epsilon_0.$$

Consider a second gaussian surface B, just inside the conductor. $\mathbf{E}=0$ on this gaussian surface, as it is inside the conductor. Hence Gauss's law

$$\oint s\mathbf{E}\ .\ d\mathbf{S}=(1/\varepsilon_0)\ \Sigma q \text{ gives}$$

$$\Sigma q=0. \qquad ...(56)$$

This concludes that a charge of $-Q$ must reside on the metal surface of the cavity and thus the sum of this induced charge $-Q$ and the original charge Q is zero. In other words, *a charge Q suspended inside a cavity in a conductor induces an equal and opposite charge $-Q$ on the surface of the cavity.*

(*iii*) The same line of approach can be used to show that the field inside the cavity of a conductor is zero when no charge is suspended in it.

(*iv*) As field inside the conductor is zero, the lines of flux coming from Q cannot penetrate into the metal and must terminate on the surface of the cavity. Since lines of flux terminate on negative charge only, hence an equal negative charge must be induced on the surface of the cavity.

2.11 FIELD INSIDE A HOLLOW CONDUCTOR

Let us calculate the field at any point C inside a uniformly charged hollow sphere (spherical shell). The field at point C is obtained by taking the vector sum of all the contributions at this point. Draw through C a double cone of infinitesimal solid angle $\delta\omega$ cutting out areas dS_1 and dS_2 of the spherical shell at distances r_1 and r_2 from point C. From the properties of solid angles, we know that

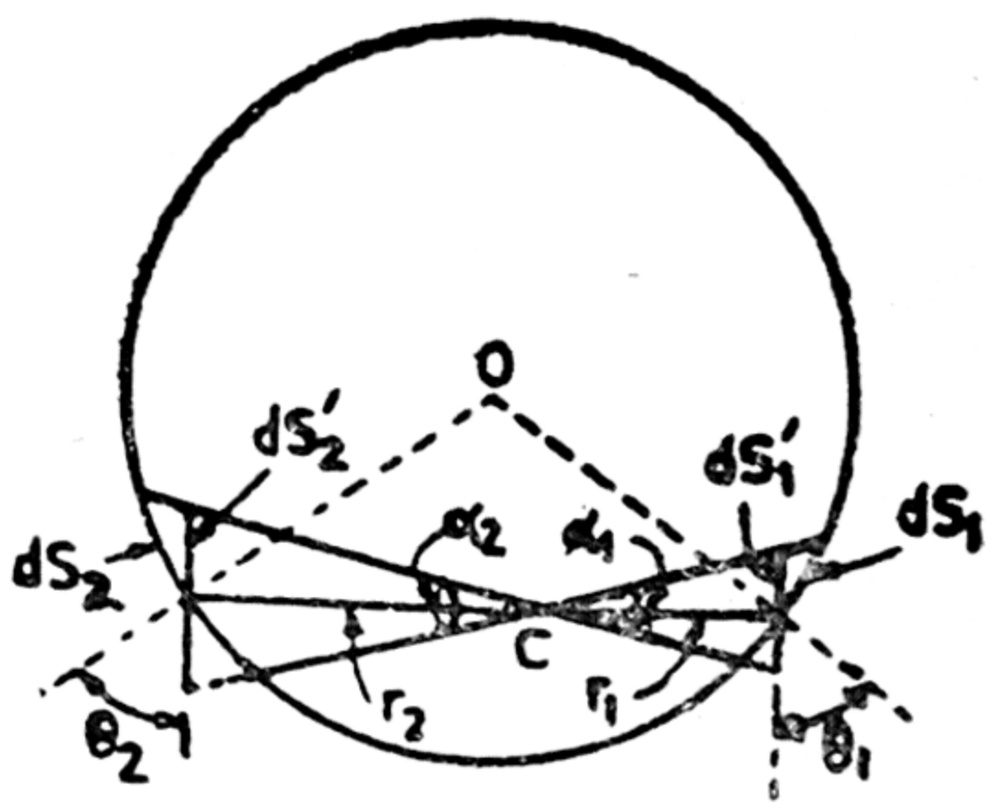


Fig. 2.24.

$$\delta\omega = \alpha_1 = \alpha_2 = \left(\frac{dS_1 \cos\theta_1}{r_1^2}\right)$$

$$= \left(\frac{dS_2 \cos\theta_2}{r_2^2}\right)$$

For small solid angles, the angle between the surface and the right section of the cone, θ_1 and θ_2 are equal, hence

$$\frac{dS_1}{r_1^2} = \frac{dS_2}{r_2^2} \quad \text{or} \quad \frac{\sigma dS_1}{r_1^2} = \frac{\sigma dS_2}{r_2^2}. \qquad \ldots(57)$$

If the inverse square law holds, the components of the field at C are

$$\Delta E_1 = \frac{1}{4\pi\varepsilon_0}\frac{\sigma dS_1}{r_1^2} \text{ and } \Delta E_2 = \frac{1}{4\pi\varepsilon_0}\frac{\sigma dS_2}{r_2^2} \qquad \ldots(58)$$

and are in opposite directions. Thus we see that ΔE_1 and ΔE_2 are equal in magnitude but opposite in directions, hence the field at point C due to the particular part of the total charge subtended by the pair of equal and opposite elements of solid angles, is

zero. Similarly the remaining surface of the sphere can be divided into pairs of elements, hence the electric field at point C due to the whole charge on the shell comes out to be zero.

Thus we see that the resultant electric intensity at this point C is zero, if the inverse square law holds good. Therefore *the inverse square law is proved if electric field inside the charged shell is zero.* This was experimentally shown by *Cavendish.* In his experiment a conducting sphere A was supported inside a bigger concentric sphere B. These were insulated from each other. The outer sphere B was first charged and connected with sphere A with the help of a wire H. The connection between these spheres was broken and the residual charge on the inner sphere was tested by electroscope. No charge was found on the sphere A. Thus he proved that the exponent n in the force (*i.e.* power of r in the relation for electric field) law was two. He repeated his experiment many times and found that n was between 2.02 and 1.98. Cavendish did not publish his results so that almost no body knew about them at that time. Maxwell repeated Cavendish's experiment with more accuracy and obtained values as $2 \pm 5 \times 10^{-5}$. In 1936, Plimpton and Lawton repeated the experiment again and set the limits as $2 \pm 2 \times 10^{-9}$. Thus we see that the inverse square law is correct, but not exactly.

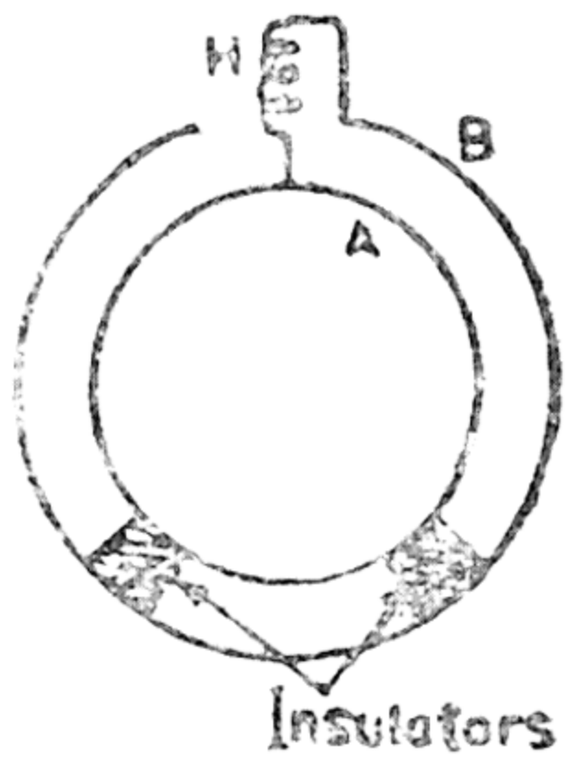


Fig. 2.25.

2.12. MECHANICAL FORCE ON THE CHARGED CONDUCTOR

We know that similar charges repel each other, hence the charge on any part of surface of the conductor is repelled by the charge on its remaining part. The surface of the conductor thus experiences a mechanical force as the charge is bound to the surface. The electric intensity at any point P near the conductor surface can beassumed as due to the small part of the surface of area say δS immediately in the neighbourhood of the point under consideration and due to the rest of the surface. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric intensities due to these parts respectively. We know from Coulomb's theorem that the total electric field $\mathbf{E}$ $(=\mathbf{E}_1+\mathbf{E}_2)$ has magnitude σ/ε_0 at any point P just outside the conductor and is zero at point Q just inside the conductor, *i.e.*,

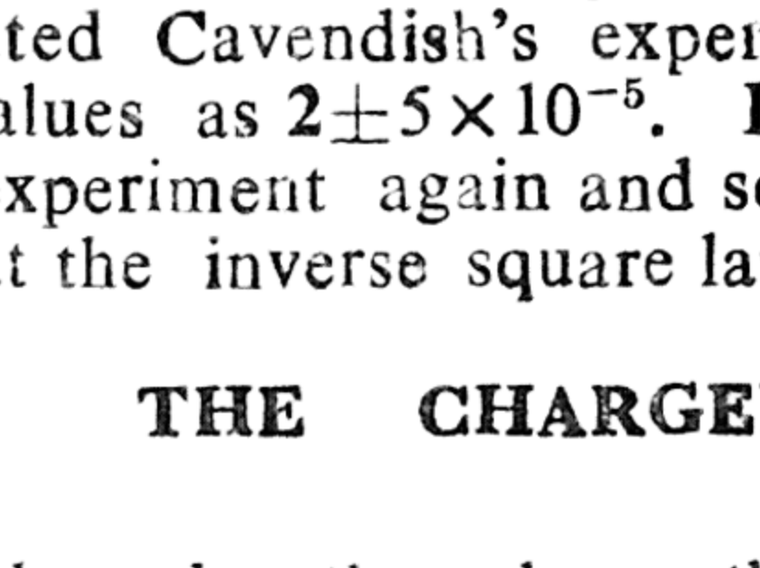


Fig. 2.26.

$$E = E_1 + E_2 = \sigma/\varepsilon_0 \qquad \text{at point } P$$

and

$$-E_1 + E_2 = 0 \qquad \text{at point } Q$$

$$\therefore \quad E_1 = E_2 = \sigma/2\varepsilon_0.$$

Hence the force experienced by the small surface of area δS due to the charge on the rest of the surface is

$$\sigma\delta S E_2 = \sigma\delta S\ \sigma/2\varepsilon_0 = \sigma^2\delta S/2\varepsilon_0.$$

Thus the mechanical force per unit area on the charged conductor

$$= \sigma^2/2\varepsilon_0 = (\varepsilon_0 E)^2/2\varepsilon_0 = \varepsilon_0 E^2/2. \qquad \ldots(59)$$

Its unit is newton/metre^2. It is directed along the outdrawn normal. If the medium is of dielectric constant k, the above relation becomes

$$\text{Outward mechanical force/area} = \tfrac{1}{2}\ \varepsilon_0 k E^2.$$

Electrostatic energy in the medium—When a conductor situated in a dielectric is charged, a force acts outward on the surface of the conductor. Imagine the unit area of the charged surface to be moved in the opposite direction through a small distance dx. The work done $dW = \frac{1}{2}\varepsilon_0 k E^2 \times dx$. Since the volume swept out by this imaginary area is dx, hence the work done per unit volume will be $\frac{1}{2}\varepsilon_0 k E^2$. This work is stored as an energy of the medium. Therefore the energy stored up in unit volume of the medium is

$$W = \tfrac{1}{2}\ \varepsilon_0 k E^2.\ \text{joule}/m^3. \qquad \ldots(60)$$

Let us consider few applications of the mechanical force.

(*i*) **Electrified soap bubble.** We have proved in mechanics that the excess of pressure inwards on a soap bubble of radius r due to surface tension T is given by

$$P = 4T/r.$$

If the bubble is now given a charge q, then the outward mechanical pressure

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(\frac{q}{4\pi r^2}\right)^2 = \frac{q^2}{32\pi^2\varepsilon_0 r^4}.$$

$\therefore$ Net excess of pressure

$$P = \frac{4T}{r} - \frac{q^2}{32\pi^2\varepsilon_0 r^4} = \frac{4}{r}\left[T - \frac{q^2}{128\pi^2\varepsilon_0 r^3}\right]. \qquad \ldots(61)$$

Thus we see that the effective value of surface tension of the bubble decreases and therefore the bubble will expand. The increase in radius of bubble can be obtained as follows :

We know that $P \propto 1/V$ or $P \propto 1/r^3\ (=k/r^3)$.

$$\therefore\ \frac{dP}{dr} = -\frac{3k}{r^4} = -\frac{3P}{r} \quad \text{or} \quad dr = -\left(\frac{r}{3P}\right) dP.$$

As change in pressure due to charge q, $dP=(q^2/32\pi^2\varepsilon_0 r^4)$, hence

$$dr=\frac{r}{3P}\cdot\frac{q^2}{32\pi^2\varepsilon_0 r^4}=\frac{q^2}{96\varepsilon_0\pi^2 Pr^3}\cdot \qquad ...(62)$$

(*ii*) **Formation of a cloud on charged particles.** We have proved in mechanics that the surface tension helps in the contraction of the liquid drop, the inward pressure is $2T/r$. If charge q is given to the drop, the outward mechanical pressure due to this charge is $q^2/32\pi^2\varepsilon_0 r^4$. Thus if the latter is greater than the former, the drop will tend to grow instead of contracting. On the other hand if r is small and the drop is uncharged, *i.e.*, the former is very large, the tendency to form drops on uncharged particles is extremely small. But when there is a charged particle present in saturated vapour which is suddenly cooled, drops of finite size will be formed. Thus a dust free super-saturated vapour may be used to detect the presence of charged particles. This principle was used by C.T.R. Wilson for the detection of radioactive particles. The discussion of the apparatus used by Wilson, known as Cloud Chamber is beyond the limit of this book,

2.13. DETERMINATION OF ELECTRONIC CHARGE

The first attempt to determine the electronic charge was made by J.J. Thomson in 1898. He used a method developed by C.T.R. Wilson and later on modified by H.A. Wilson. The method was based on the principle that the cloud of water droplets could be formed on charged ions under special conditions. This cloud falls down with a certain velocity. Its velocity was observed and the value of 'e' was determined. The value so obtained was 40% higher than the standard value determined later.

In 1909, R.A. Millikan not only demonstrated the discrete nature of electric charge but gave the correct value of the charge on an individual electron. In this experiment two actually parallel horizontal metal plates A and B are insulated from each

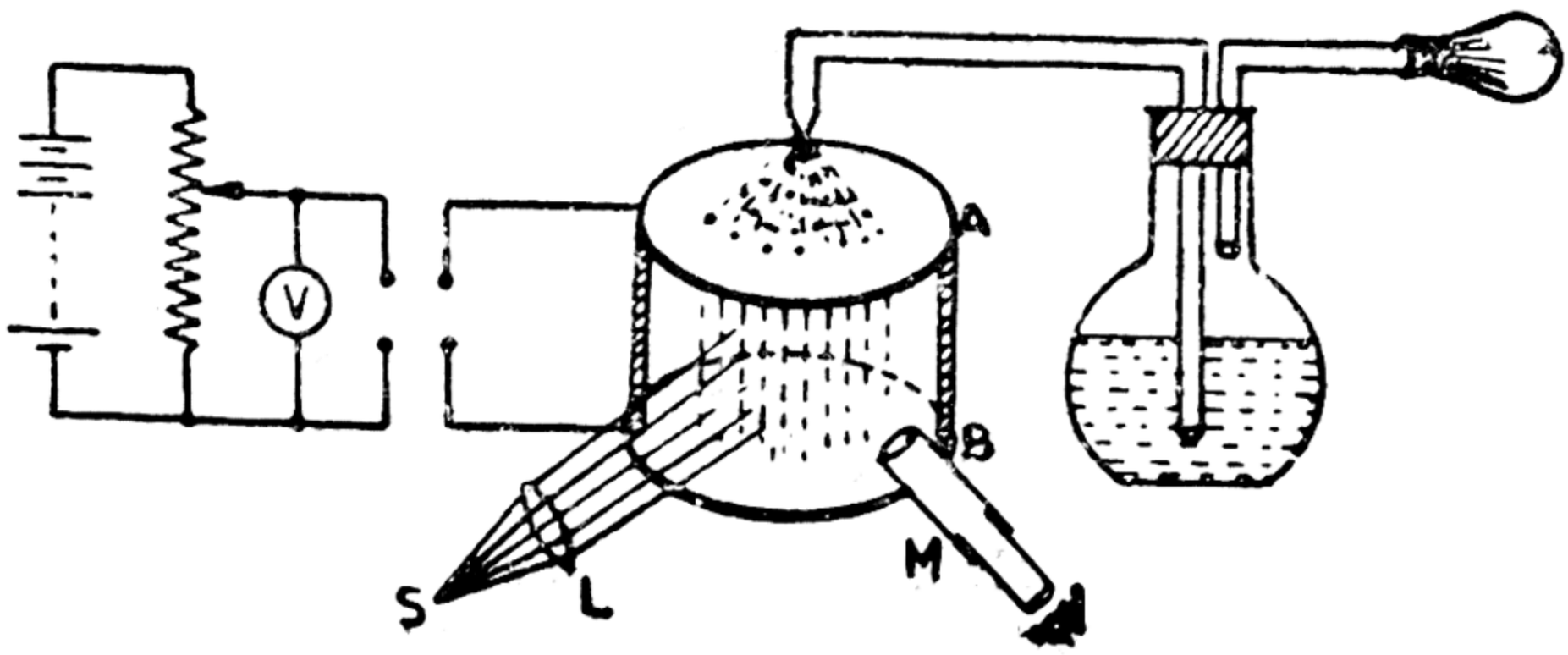


Fig. 2.27.

other and separated by ebonite or glass pieces of thickness about a cm. Potential difference is applied between these plates by connecting a power supply through a reversing switch with the help of which the direction of electric current can be changed or can be set zero. The upper plate *A* has some fine holes, through which few of the droplets of a non-volatile oil from an atomiser are allowed to fall. A majority of these drops becomes charged by friction at the atomiser nozzle during the time of spray. The droplets, illuminated by the light beam, appear as tiny bright stars falling slowly with a terminal velocity governed by their weights, viscous force and by buoyant force. These droplets are observed through a microscope *M* whose eyepiece carries a micrometer scale at the position of cross wires to measure the distance travelled by the droplets.

When electric field is not applied, a drop falls down under gravity, opposed by the viscous force and bouyancy force and acquires a steady terminal velocity *v*. If *a* be the radius of the oil drop, ρ the density of oil and σ the density of air then by Stokes law, we get

$$\text{Viscous force } 6\pi\eta a v = \text{Apparent weight of the drop}$$

$$= \tfrac{4}{3}\pi a^3 \rho g - \tfrac{4}{3}\pi a^3 \sigma g$$

or
$$a = [9\eta v / 2(\rho - \sigma)g]^{1/2}. \qquad \ldots(63)$$

If the charge on the droplet is *n* times the electronic charge, denoted by *e*, and the plates are maintained at a potential difference *V* such that $E\ (=V/d)$ is in downward direction. The force on the—vely charged drop will be in upward direction and having value *neE* or *neV/d*. On the other hand the downward velocity of+vely charged droplets will increase further and the velocity of uncharged particles remains same. If one of these droplets is selected for observations. Firstly this droplet is allowed to move under the electric field. This field is so chosen that the droplet moves with a constant velocity, known as terminal velocity. The velocity v' is measured with the help of a stop-watch. In equilibrium, downward forces=upward forces, *i.e.*,

$$\tfrac{4}{3}\pi a^3 \rho g = \tfrac{4}{3}\pi a^3 \sigma g + 6\pi\eta a v' + neE$$

$$neE = \tfrac{4}{3}\pi a^3(\rho - \sigma)\, g - 6\pi\eta a v' = 6\pi\eta a[v - v']$$

or
$$ne = 9\sqrt{2}\pi\eta^{3/2}\, (\rho - \sigma)^{-1/2}\, g^{-1/2}\, v^{1/2}(v - v')/E$$

$$= 9\pi \left[\frac{2\eta^3}{g(\rho - \sigma)} \right]^{1/2} \frac{v^{1/2}}{E} (v - v'). \qquad \ldots(64)$$

The value of *ne* is calculated for each set separately by observing velocities of the droplets in the presence and in the absence of an electric field. These values of charges $n_1 e$, $n_2 e$,...are divided by the highest common factor of all such values. Milikan measured

the charges of some thousands of drops and found that the charge on any one droplet was integral multiple of a basic charge e. He gave the value of e as 1.60207×10^{-19} coulomb.

Exercises

Example 1. *Prove that the gravitational force is negligible in comparison to electrostatic force in the hydrogen atom in which the electron and the proton are about 5.3×10^{-11} meter apart.*

From Coulomb's law, the electrostatic force between electron and proton

$$F_e = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

$$= (9.0 \times 10^9 \text{ newton. m}^2/\text{coul}^2) \quad \frac{(1.6 \times 10^{-19} \text{ coul})^2}{(5.3 \times 10^{-11} \text{ meter})^2}$$

$$= 8.2 \times 10^{-8} \text{ newton.}$$

From gravitational law, the gravitational force

$$F_g = G \frac{m_1 m_2}{r^2}$$

$$= (6\text{-}7 \times 10^{-11} \ N.\ m^2/kg^2) \quad \frac{(9.1 \times 10^{-31} kg)(1.7 \times 10^{-27} kg)}{(5.3 \times 10^{-11} \text{ meter})^2}$$

$$= 3.69 \times 10^{-47} \text{ newton.}$$

which is about 10^{-39} times smaller than the electrostatic force, thus can be neglected in comparison to the latter in nuclear or atomic problems.

Example 2. *Two pitch balls, each weight 100 mg and suspened from the same point by silk threads, 30 cms. long, are equally charged and repel each other to a distance of 10 cms. What is the charge on each ball ?*

Let two pitch balls each having a charge q are suspended from the same point P. Due to repulsive force they are at a distance r apart. Each ball is acted upon by following forces :

(a) Its weight acting vertically downward

$$mg = 100 \times 10^{-6} \text{ kg.} \times 9.8 \text{ meter/sec}^2$$
$$= 9.8 \times 10^{-4} \text{ newton.}$$

(b) Force of repulsion acting along **F**

$$F = \frac{q^2}{4\pi\varepsilon_0 r^2}.$$

$$= \frac{9 \times 10^9 \text{ nt m}^2}{\text{coul}^2} \times \frac{q^2}{(0.1 \text{ metre})^2}$$

$$= 9 \times 10^{11} \ q^2 \text{ newton.}$$

(c) The tension T of the string acting along the string, along **T**.

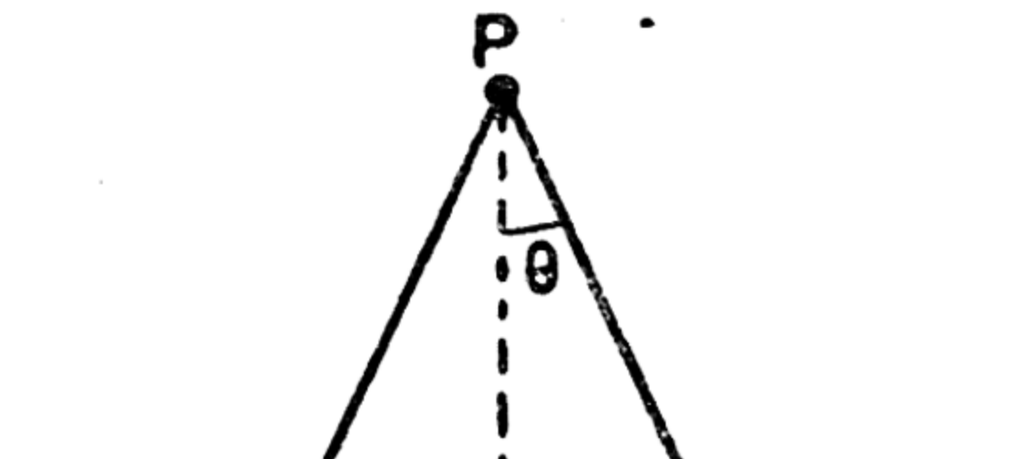


Fig. 2.28

In equilibrium, moment of these forces about any point say point of suspension P will be zero, hence

$$mg\,(r/2)\sin\theta = F(r/2)\cos\theta \quad \text{or} \quad F = mg\tan\theta$$

$$\therefore \quad 9\times10^{11}\,q^2 = 9.8\times10^{-4}\,\frac{5\times10^{-2}}{(30^2-5^2)^{1/2}\times10^{-2}}$$

or $$q^2 = 1.841\times10^{-16} \text{ and } q = 1.375\times10^{-8} \text{ coulomb.}$$

Example 3. *$ABCD$ is a square of 4 cm side, charges of 16×10^{-9}, -16×10^{-9} and 32×10^{-9} coulomb are placed at the points A, C and D respectively. Find the intensity of the field at point B.*

The electric field intensity $\mathbf{E}$ due to a point charge q at a distance r is given by the relation.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}\,\hat{r},$$

where r is along r increasing.

As $\mathbf{E}$ is a vector quantity, the total intensity at B is the vector sum of three intensities due to charges at A, C and D, where

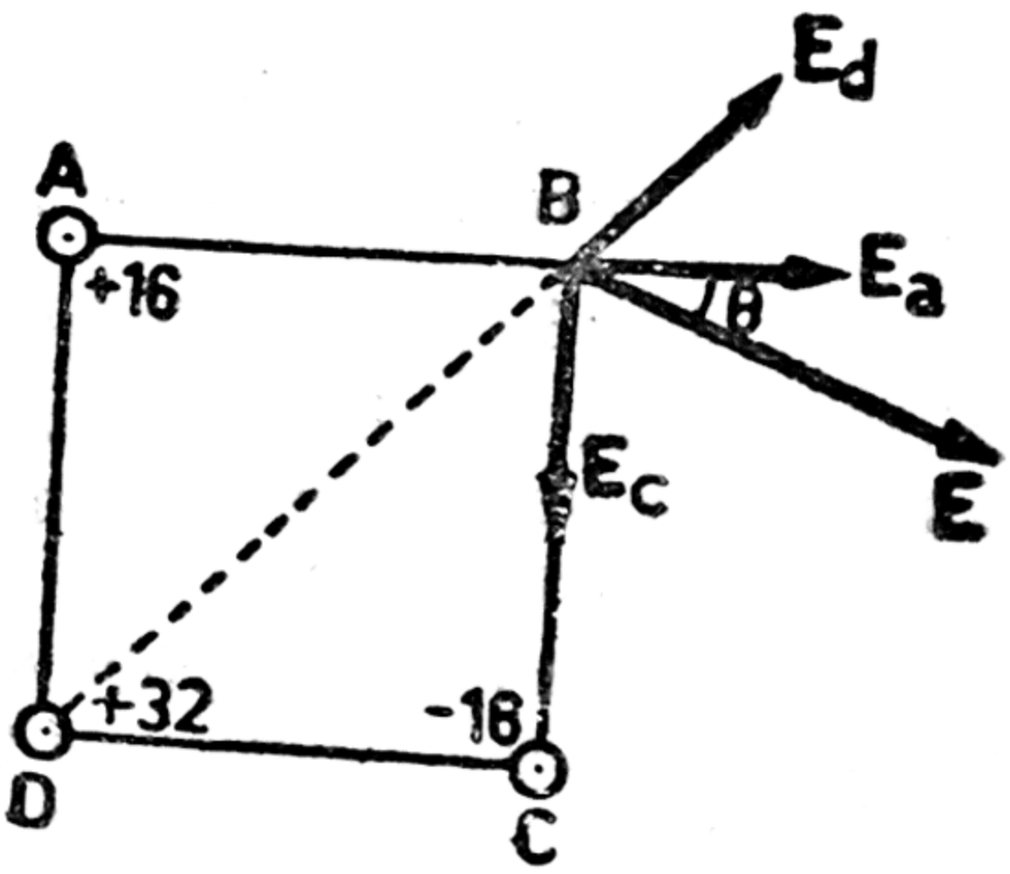


Fig. 2.29.

$$E_a = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}$$

$$= 9\times10^9 \times \frac{16\times10^{-9}}{(0.04)^2} = 9\times10^4 \text{ along } \mathbf{AB}$$

$$E_c = 9\times10^9 \times \frac{(-16\times10^{-9})}{(0.04)^2} = 9\times10^4 \text{ along } \mathbf{BC},$$

and $$E_d = 9\times10^9 \times \frac{32\times10^{-9}}{32\times10^{-4}} = 9\times10^4 \text{ along } \mathbf{DB}.$$

Resultant intensity $\mathbf{E} = \mathbf{E}_a + \mathbf{E}_c + \mathbf{E}_d$.

$$\therefore \quad E = \sqrt{(E_x^2 + E_y^2)} = \sqrt{[(E_a + E_d\cos 45°)^2 + (E_c - E_d\sin 45°)^2]}$$

$$= \sqrt{[E_a^2 + E_d^2 + E_c^2 + 2E_d(E_a\cos 45° - E_c\sin 45°)]}$$

$$= 9\sqrt{3}\times10^4 \text{ newtons/coul.}$$

$$\text{and} \quad \tan\theta=\frac{E_y}{E_a}=\frac{E_c-E_d\sin 45^\circ}{E_a+E_d\cos 45^\circ}=\frac{(1-1/\sqrt{2})9\times10^4}{(1+1/\sqrt{2})9\times10^4}$$

$$=0.171$$

$$\text{or} \quad \theta=9^\circ\ 44'.$$

Example 4. *An electron is constrained to move along the axis of the ring of charge q and of radius a. Show that the electron can perform oscillations whose frequency is given by*

$$\omega=\sqrt{eq/4\pi\varepsilon_0\, ma^3}.$$

Electric intensity **E** at a point *P* on the axis of the ring of radius *a* at a distance *x* from its centre, as given in article 2.4 (*c*), is given by

$$E=\frac{1}{4\pi\varepsilon_0}\ \frac{qx}{(a^2+x^2)^{3/2}}\ . \qquad \ldots(26)$$

It is along the axis of the ring.

The force on the electron if placed at point *P*

$$F=qex/4\pi\varepsilon_0\,(a^2+x^2)^{3/2}.$$

If $x \ll a$, then the force is proportional to the distance from the mean position *i e*, the centre of the ring. It means that the electron executes simple harmonic motion about the centre of the ring. The frequency is given by the relation

$$\omega=\sqrt{\frac{d^2x/dt^2}{x}}=\sqrt{\frac{F/m}{x}}=\sqrt{\frac{q\,e}{4\pi\varepsilon_0 ma^3}}.$$

Example 5. *Show the variation of electric field due to the positive charge in the gold atom (Z = 79).*

Gold atom is spherical and is of radius 1×10^{-10} meter. The positive charge of the atom is $79\times1.6\times10^{-19}$ coul. and is distributed uniformly within the nucleus of radius 6.9×10^{-15} meter. The electric intensity outside the nucleus is given by the relation (40), *i.e.*,

$$E=\frac{1}{4\pi\varepsilon_0}\ \frac{q}{r^2}\ .$$

At the surface of atom $r=10^{-10}$ meter

$$E=\frac{(9\times10^9\ \text{nt. m}^2/\text{coul}^2)\ (79\times1.6\times10^{-19}\ \text{coul})}{(10^{-10}\ \text{meter})^2}$$

$$=1.14\times10^{18}\ \text{nt/coul.}$$

At the nuclear surface $r=R=6.9\times10^{-15}$ meter and

$$E=\frac{(9\times10^{9}\text{ nt. m}^2/\text{coul}^2)\,(79\times1.6\times10^{-19}\text{ coul}}{(6.9\times10^{-15}\text{ meter})^2}$$

$$=2.39\times10^{21}\text{ nt./coul.}$$

Inside the nucleus the electric intensity is given by the relation

$$E=\frac{1}{4\pi\varepsilon_0}\,\frac{q}{R^3}\,r.$$

It is zero at the centre ($r=0$) and increases with r upto the maximum value 2.39×10^{21} nt./coul. at nuclear surface. Beyond it, decreases inversely as the square of the distance from the centre.

Example 6. *If the electric field is given by* $\vec{E}=8\hat{x}+4\hat{y}+3\hat{z}$, *calculate the electric flux through a surface of area 100 units lying in the x-y plane.*

In this problem $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors along x, y and z axes respectively, generally taken as **i**, **j** and **k**.

Since the electric flux is defined as $\Phi=\mathbf{E}.\mathbf{S}$. The vector surface area in the x-y plane will be in the direction of the out drawn normal, *i.e.*, the z-direction. Therefore

$$\mathbf{S}=100\;\hat{z}(=100\,\mathbf{k})$$

$$\Phi=\mathbf{E}\text{-}\mathbf{S}.=(8\hat{x}+4\hat{y}+3\hat{z}).\;100\;\hat{z}$$

$$=300\text{ units.}$$

Example 7. *Find the electric field at the centre of a uniformly charged semicircular arc.*

Let the charge per unit length of arc be λ and the radius of the arc be a. If the arc is divided into small segments, so that each will work as a point charge. The electric field at point P due to a segment of length $\triangle L$,

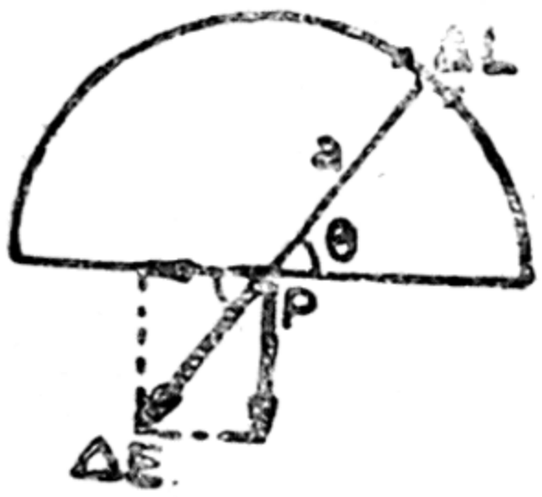


Fig. 2.30.

$$\Delta E=\frac{1}{4\pi\varepsilon_0}\,\frac{\lambda\Delta L}{a^2}.$$

Its x—component $\triangle E_x=\triangle E\cos\theta$ will be cancelled by the contribution from a symmetrically placed $\triangle L$ on the left half of the arc, hence net component $E_x=\Sigma\triangle E_x=0$. Thus the y-component contributes to E only.

$$\therefore \qquad E=E_y=\sum \Delta E_y=\int \frac{\lambda}{4\pi\varepsilon_0 a^2}.\, dl \sin\theta.$$

To convert two variables l and θ in one, let us use the relation $dl=a\, d\theta$ (arc=radius × angle subtended by the arc at the centre).

$$\therefore \qquad E=\frac{\lambda}{4\pi\varepsilon_0 a}\int_{-\pi}^{\pi} \sin\theta\, d\theta.$$

$$=\lambda/2\pi\varepsilon_0 a.$$

Example 8. *A delectric cylinder of radius a is infinitely long. It is non-uniformly charged such that volume charge density* ρ *varies directly as the distance from the cylinder. Calculate the electric intensity due to it, if* ρ *is zero at the axis and is* ρ_s *on the surface.*

To calculate net charge within the charged cylinder, let us divide it in large number of co-axial thin cylindrical shells. Consider a co-axial shell of radii x and $x+\delta x$ and of length l. If the thickness δx is very small so that the charge density can be assumed as constant at each point on it. Thus the charge on this shell

$$\delta q=2\pi x l\, \delta x\, \rho.$$

As ρ varies directly as the distance x, hence ρ can be written as cx, where c is a constant given by the relation

$$ca=\rho_s \text{ or } c=\rho_s/a.$$

The intensity at any point P can be calculated as given in article 2.9 (*d*). The field E at the point at a distance r outside the cylinder is given by

$$E\, 2\pi r l=\frac{q}{\varepsilon_0}=\frac{1}{\varepsilon_0}\int dq.$$

Charge inside the gaussian surface is the charge on the cylinder of length l, *i.e.*, the integral of dq must be over the whole cylinder.

$$\therefore \qquad E=\frac{1}{2\pi r l\varepsilon_0}\int_0^a 2\pi l c x^2\, dx=\frac{ca^3}{3r\varepsilon_0}=\frac{\rho_s a^2}{3r\varepsilon_0}.$$

At points inside the cylinder, E is given by

$$E\, 2\pi r l=\frac{q'}{\varepsilon_0}=\frac{1}{\varepsilon_0}\int dq=\frac{1}{\varepsilon_0}\int_0^r 2\pi l c x^2\, dx,$$

as the charge inside the gaussian surface is the charge on the cylinder of radius r and of length l. Therefore

$$E=cr^2/3\varepsilon_0=\rho_s r^2/3a\varepsilon_0.$$

At the points on the surface of the cylinder, $E=\rho_0 a/3\varepsilon_0$

In all the cases the field is in radially outward direction.

Example 9. *Calculate the electric field intensity due to a spherical charge distribution, given by*

$$\rho=\rho_0(1-r/a), \text{ when } r\leqslant a$$

and $\rho=0$ *when* $r>a$.

Find the value of r for which **E** *is maximum.*

Let us consider a thin spherical shell of radii x and $x+dx$. The volume of this shell $\delta V=4\pi x^2 dx$. To calculate electric field intensity let us assume a gaussian spherical surface passing through the point which is at a distance r from the centre. Using Gauss's law, we get

$$\int_s \mathbf{E}.\,d\mathbf{S}=\frac{1}{\varepsilon_0}\int dq=\frac{1}{\varepsilon_0}\int_0^a \rho_0\left(1-\frac{x}{a}\right)4\pi x^2 dx.$$

Here integration is between $x=0$ to $x=a$, as the whole charge is within the gaussian surface.

$$\therefore \qquad \oint E dS=4\pi r^2\, E=\pi\rho_0 a^3/3\varepsilon_0$$

or
$$E=\rho_0 a^3/12\varepsilon_0 r^2.$$

Field inside the charge distribution is given by

$$\int_s \mathbf{E}.\,d\mathbf{S}=\frac{1}{\varepsilon_0}\int dq=\frac{1}{\varepsilon_0}\int_0^r \rho_0\left(1-\frac{x}{a}\right)4\pi x^2 dx.$$

Here integration is between $x=0$ to $x=r$, as the charge inside the gaussian surface is the charge within the sphere of radius r

$$\therefore \qquad \oint E dS=4\pi r^2\, E=\frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{r^3}{3}-\frac{r^4}{4a}\right],$$

or
$$E=\frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3}-\frac{r^2}{4a}\right].$$

Electric field E will be maximum, when

$$\frac{dE}{dr}=0, \quad \text{or} \quad \frac{d}{dr}\left[\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^2}{4a}\right)\right]=0.$$

$$\therefore \quad r=2a/3.$$

Example 10. *If a heavy atom can be pictured as a spherical nucleus with charge* $+q$ *and radius a embeded in a much larger sphere of negative charge, the electrons. This negative charge is distributed uniformly throughout this sphere of radius b. Find the electric field intensity outside the nucleus.*

The total charge inside the gaussian surface passing through any point *outside the atom* is zero, hence by Gauss's law

$$\int \mathbf{E}.d\mathbf{S}=(1/\varepsilon_0)\Sigma q=0.$$

As surface area $d\mathbf{S}$ is always $\|$ to the radius vector, hence $\mathbf{E}$ cannot be $\perp$ to $d\mathbf{S}$ and thus zero.

At a point inside the atom but outside the nucleus and a distance r apart from the centre of the nucleus, the total charge within the gaussian sphere is $\Sigma q=q-(\frac{4}{3}\pi r^3)\,\rho$.

$$\therefore \qquad \int \mathbf{E}.d\mathbf{S}=\frac{1}{\varepsilon_0}\left[\, q-\frac{4}{3}\,\pi r^3.\left(\frac{q}{\frac{4}{3}\pi b^3}\right)\right]$$

$$E4\pi r^2=\frac{q}{\epsilon_0}\left[\, 1-\left(\frac{r}{b}\right)^3\right] \quad \text{or} \quad E=\frac{q}{4\pi\epsilon_0 r^2}\left[\, 1-\left(\frac{r}{b}\right)^3\right].$$

The first term is due to the nucleus, while the second term represents the cancelling field due to electrons. At point very close to nucleus the second term is negligible, the nucleus and innermost electrons of an atom are therefore not greatly affected by the outer electrons of the atom.

Example 11. *Two plane metal plates are placed parallel to each other, one carries a surface charge density $+\sigma$ and the other -2σ. Find the charge densities on the two surfaces of the third plane metal plate placed at the centre. The central plate is parallel to the other plates and is assumed connected to the earth.*

The lines of force coming from the plate on the left must end on the left side of the central plate. Thus the charge density must be $-\sigma$. Similarly the charge density on the right surface of the central plate must be $+2\sigma$, as it is due to the lines of force coming from the right plate having charge density -2σ. Thus the central plate has a net charge $+\sigma$.

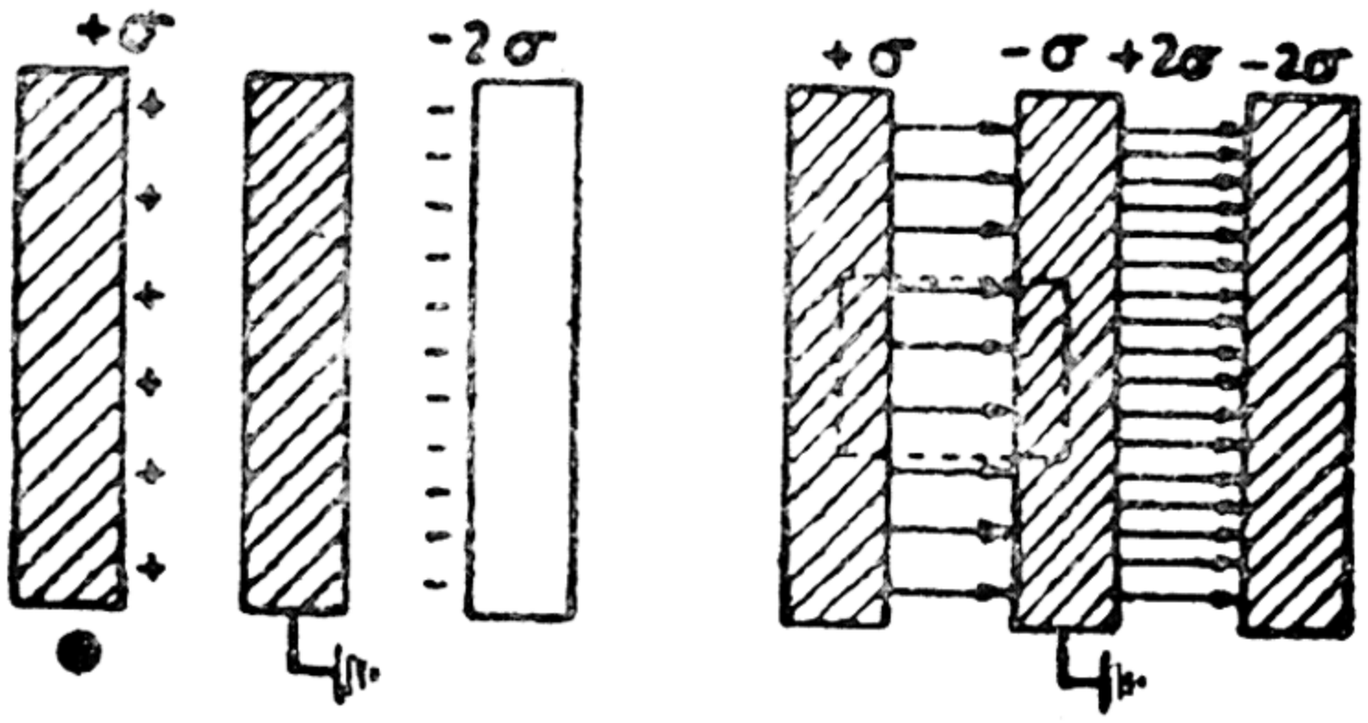


Fig. 2.31

The same result can be obtained by using Gauss's law. Consider a gaussian surface shown by the dotted lines in Fig. 2.31. Thus by Gauss's law.

$$\int \mathbf{E}.d\mathbf{S}=\frac{1}{\varepsilon_0}\Sigma q.$$

As the net flux coming out of the gaussian surface is zero, hence Σq must be zero. This can be true only if the charge on the left side of the central plate is equal and opposite to the charge on the left plate. Similarly we can show that the charge on the right side will be equal and opposite to the charge on the right plate.

Example 12. *Calculate the flux of the electric field strength through each of the faces of a closed cube of length l, if a charge q is placed at its centre and at one of its vertices.*

Symmetry of six faces of a cube about its centre ensures that the flux Φ of **E** through each of the faces will be same when charge q is placed at the centre.

$$\therefore \quad \Phi = 6\Phi s = q/\varepsilon_0 \text{ or } \Phi s = q/6\varepsilon_0.$$

In the second case, when q is at one of the vertices, the flux through each of the three faces meeting at this vertex will be zero, as **E** is parallel to these surfaces. The flux through other three faces will be same, say Φs. We know that one-eighth of the flux originating from charge q passes through this cube, as the faces of the cube meeting at the charge include one-eighth of the spherical surface drawn with charge q as centre.

$$\therefore \quad 3\Phi s = \frac{1}{8}\frac{q}{\varepsilon_0}, \text{ or } \Phi s = \frac{1}{24}\cdot\frac{q}{\varepsilon_0}.$$

Example 13. *Calculate the flux of* **E** *through (i) each of the bases, (ii) the curved surface of a right cylindrical closed surface of radius a and length l, due to the charge q situated at its geometrical centre.*

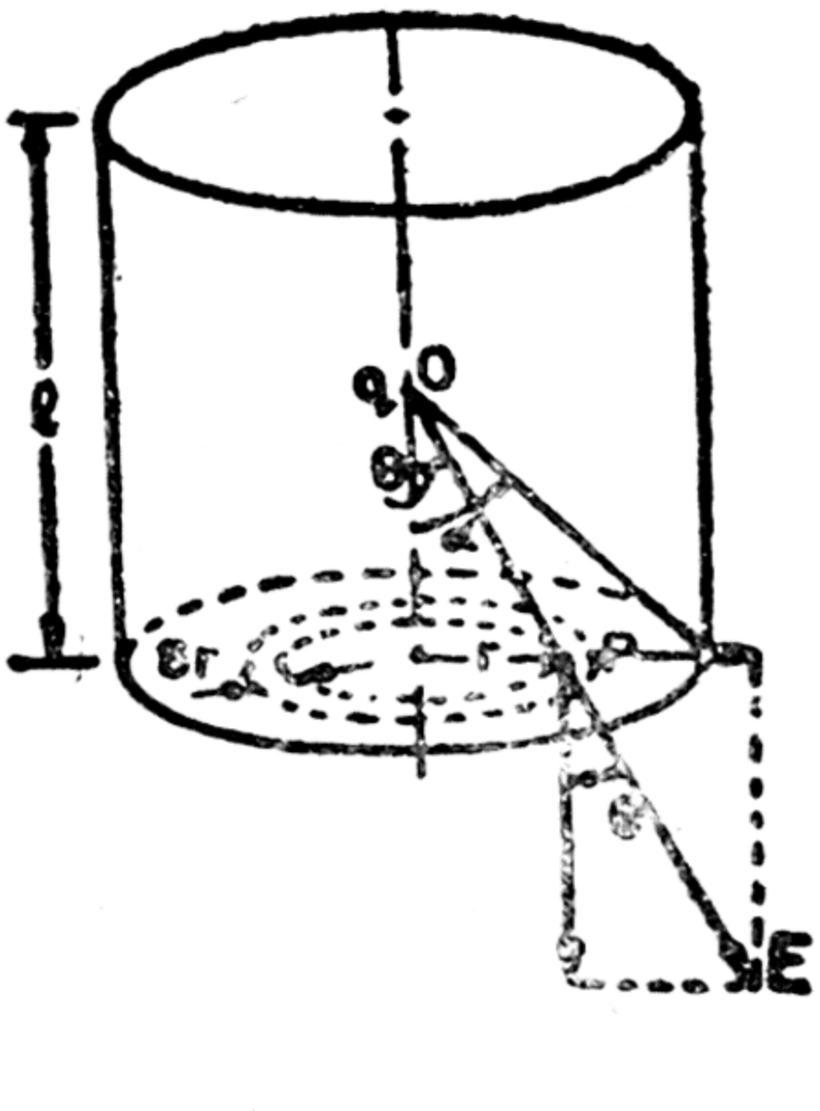


Fig. 2.32

To find the electric flux through the base of the cylinner, let us divide the base (say lower) into large number of concentricr ings. The electric field at any point P of an annular ring between radii r and $r+\delta r$ is

$$E = q/4\pi\varepsilon_0(r^2+l^2/4) \text{ along } \mathbf{OP}.$$

The flux through a very small element of this annular ring is $E\cos\theta \times$ area of the element. As $E\cos\theta$ is same for all elements on this annular ring, hence the flux through this annular ring

$$\delta\Phi = E\cos\theta.2\pi r\delta r$$

$$= \frac{q.2\pi r\delta r\cos\theta}{4\pi\varepsilon_0(r^2+l^2/4)} = \frac{qlr\delta r}{4\varepsilon_0(r^2+l^2/4)^{3/2}}$$

$\therefore$ Total flux through the base of the cylinder

$$\Phi=\int_0^a \frac{qlrdr}{4\varepsilon_0(r^2+l^2/4)^{3/2}}=\frac{q}{2\varepsilon_0}\left[1-\frac{l/2}{(a^2+l^2/4)^{1/2}}\right].$$

By symmetry the flux through the above base will also be same. As the total flux originating from a point charge is always q/ε_0, hence by Gauss's law

$$\Phi_{curved}+2\Phi_{bases}=q/\varepsilon_0$$

or $$\Phi_{curved}=\frac{q}{\varepsilon_0}-\frac{q}{\varepsilon_0}\left[1-\frac{l/2}{(a^2+l^2/4)^{1/2}}\right]=\frac{ql}{2\varepsilon_0(a^2+l^2/4)^{1/2}}.$$

Example 14. *Calculate electric flux for a cube of side a as shown in Fig. 2.33, where* $E_x=bx^{1/2}$, $E_y=E_z=0$, $a=10$ *cm and* $b=800$ *nt/coul-m*$^{1/2}$.

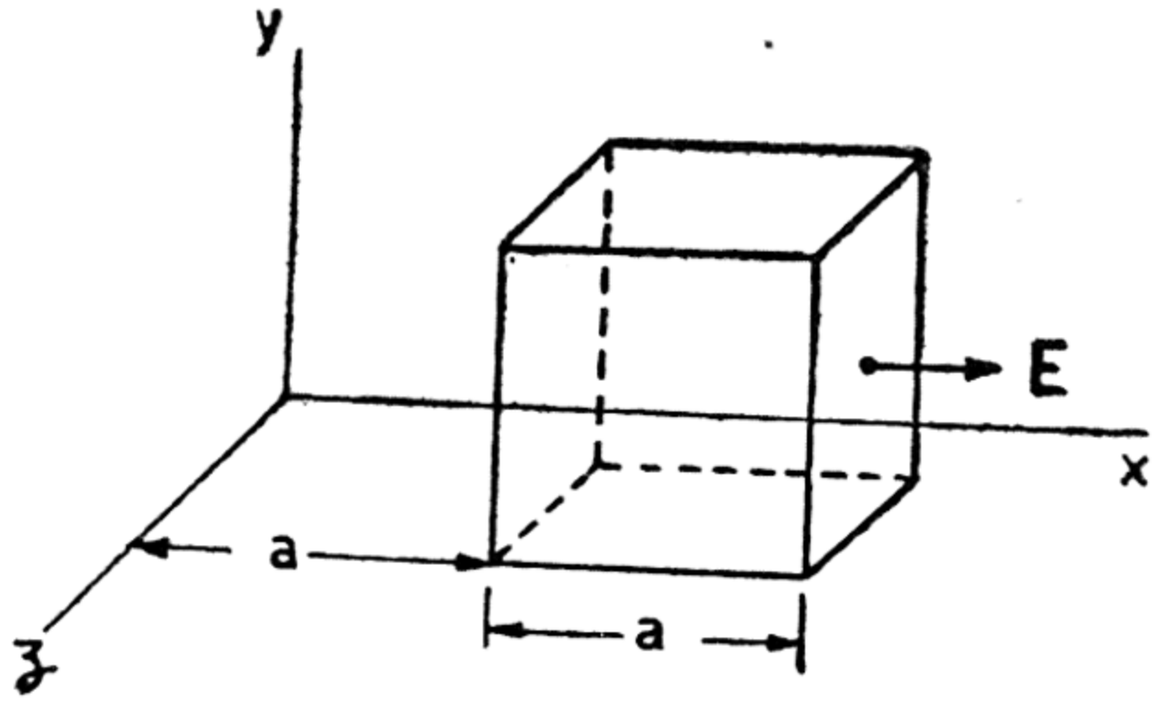


Fig. 2.33.

Since the electric field is acting only in x-direction and its y- and z-components are zero. The net electric flux is thus given as

$$\Phi=\Phi_{right\ surface}-\Phi_{left\ surface}$$
$$=\int\mathbf{E}.d\mathbf{S}-\int\mathbf{E}.d\mathbf{S}$$
$$=(E_x.a^2)_{right}-(E_x a^2)_{left}=a^2[E_{2a}-E_a]$$
$$=a^2[b\,(2a)^{1/2}-ba^{1/2}]$$
$$=ba^{5/2}\,(\sqrt{2}-1)=800\times(0.1)^{5/2}\,(\sqrt{2}-1)$$
$$=1.05 \text{ mks units.}$$

Example 15. *If the electric field near the earth's surface be 300 volts/meter directed downwards, what is the surface density of charge on the earth's surface ?*

The electric field just outside a charged conductor of surface charge density σ is given by

$$E=\sigma/\varepsilon_0.$$

In this problem the electric field near the earth's surface, which may be treated as a charged conductor, is given as 300 volts/meter. Since the permittivity of the free space $\varepsilon_0 = 8.85 \times 10^{-12}$ coul^2/nt-m^2. Thus the surface density of charge on the earth's surface

$$\sigma = \varepsilon_0 E = 8.85 \times 10^{-12} \times 300 = 2.66 \times 10^{-9} \text{ coul/m}^2.$$

Example 16. *A gold foil weighing 50 mg per sq cm is placed on a horizontal charged plate. What would be the density of charge so that the foil may just rise ?*

Mechanical force per unit area of a charged plate $= \sigma^2/2\varepsilon_0$.

In this problem outward mechanical force is balanced by the force due to the weight of gold foil. Hence at equilibrium

$$\sigma^2/2\varepsilon_0 = mg, \quad \text{or} \quad \sigma^2 = 2\varepsilon_0 \, mg,$$

where σ is the surface charge density and m is the mass per unit area of the foil.

$$\therefore \quad \sigma^2 = 2 \times 8.85 \times 10^{-12} \times 50 \times 10^{-2} \times 9.8$$

or
$$\sigma = 9.313 \times 10^{-6} \text{ coul/m}^2.$$

Example 17. *Derive an expression for the electric charge required to expand the bubble to twice its dimensions.*

Let P_1 be the internal pressure inside the bubble of radius r and surface tension T. If P be the atmospheric pressure, then

$$P_1 = P + 4T/r.$$

When the bubble after charging expands, the radius becomes $2r$, the volume becomes 8 times the initial volume. Therefore by Boyle's law the pressure inside the bubble reduces to $P_1/8$. This decrease in pressure is due to the mechanical outward pressure due to the charge q given to the bubble.

$\therefore$ Total outward pressure = total inward pressure

$$\frac{1}{8}\left(P + \frac{4T}{r}\right) + \frac{q^2}{32\varepsilon_0\pi^2(2r)^4} = P + \frac{4T}{2r}$$

$$q^2 = 64\pi^2\varepsilon_0 \, r^3[7Pr + 12T]$$

or
$$q = 8\pi\sqrt{\varepsilon_0 r^3(7Pr + 12T)}.$$

Example 18. *An insulated soap bubble 10 cm in radius is charged with 20 stat-coulomb. Find the increase in radius due to the charge. Given the atmospheric pressure $= 10^5$ nt/m^2.*

From Boyle's law $PV=\text{const}=k$

Pressure P of a sphere of radius $R=k/(\frac{4}{3}\pi R^3)=K/R^3$.

Differentiation gives $dP/dR=-3K/R^4=-3P/R$

or $$dR=-(R/3P)dP.$$

The change in pressure due to the charge q is given by

$$-dP=-\sigma^2/2\varepsilon_0=-(q/4\pi R^2)^2/2\varepsilon_0.$$

$$\therefore \quad dR=-\frac{R}{3P}dP=\frac{R}{3P}\cdot\frac{q^2}{32\pi^2\varepsilon_0 R^4}=\frac{q^2}{96\pi^2\varepsilon_0 PR^3}$$

Given $q=20$ stat coul $=20$ e.s.u. of charge

$$=20\times 3.36\times 10^{-10} \text{ coul.}$$

Thus the substitution of numerical values gives

$$dR=5.3\times 10^{-9} \text{ cm.}$$

Example 19. *In Milikan's experiment an oil drop of radius 10^{-4} cm. remains suspended between the plates which are 1 cm apart. If the drop has charge of 5e over it, calculate the potential difference between the plates. The density of oil may be taken as 1.5 gm./cc.*

Since the drop remains suspended between the plates, the viscous force on the drop is zero and the weight mg of the drop is balanced by the force due to electric field between the plates. If V be the potential difference between the plates, then

$$qE=mg \quad \text{or} \quad qV/d=\tfrac{4}{3}\pi r^3(\rho-\sigma)g.$$

In this problem $q=5e=5\times 1.6\times 10^{-19}$ coul, $d=1$ cm $=0.01$ m, $r=10^{-4}$ cm $=10^{-6}$ m, $\rho=1.5\times 10^3$ kg/m^3. Assuming the density of air as negligible in comparison the oil density, we get

$$V=\frac{0.01}{5\times 1.6\times 10^{-19}}\times\frac{4}{3}\times 3.14\times(10^{-6})^3\times 1.5\times 10^3\times 9.81$$

$$=770 \text{ volts.}$$

Oral Questions—

1. Given an insulated rod, how could you determine if it was charged ? How could you find the sign of the charge on it ?
2. The quantum of charge is 1.6×10^{-19} coul, is there any single quantum of mass ?
3. Give some examples of physical quantity to be (*a*) quantized or (*b*) conserved.
4. Whether the constant of proportionality in Coulomb's law is a measured value or a calculated value ?
5. A stone is placed in the gravitational field of earth. Can we also say that the earth lies in the gravitational field of the stone ?

6. Electric lines of force never cross, why ?

7. Two point charges are placed at a certain distance apart. The electric field strength is zero at one point between them. What can you conclude about the charges ?

8. What is the electric flux through a surface encloses an electric dipole ?

9. A charge q is placed inside an enclosure. Compare the total flux coming out of the walls of the enclosure, if it is spherical. cubic, rectangular, parallellopiped or hemisphere.

10. Show that no work is done in moving the test charge from point to point on the surface of the metal.

11. A man is placed with an electrical measuring instrument inside a large closed metal sphere. What will the man observe as (*a*) charge is placed on the sphere, or (*b*) a large charged object is brought close to the sphere ?

12. As you penetrate a uniform sphere of charge, find the effect on E while moving towards the centre.

13. A spherical rubber balloon carries a charge that is uniformly distributed over its surface. How does E vary for points (*a*) inside, (*b*) on the surface and (*c*) outside the balloon, as the balloon is blown up ?

14. In Milikan's apparatus, how can you find the sign of charge on the droplets from the atomiser.

Problems—

1. In the Bohr model of atomic hydrogen an electron of mass 9.11×10^{-31} kg revolves, about a nucleus consisting one proton, in a circular orbit of radius 5.29×10^{-11} m. If mass of proton is 1.67×10^{-27} kg, calculate the radial acceleration and angular velocity of the electron.

2. If two equally charged balls of identical masses of 0.20 gm are suspended from 50 cm long strings. Calculate the value of each charge, if the strings make an angle of 37° to the vertical. (3.2×10^{-7} *coul*)

3. Two charges Q and Q are placed at the diagonally opposite corners of a square, while charge q and q are placed at the remaining corners. If the resultant force on one charge Q is zero, find the relation between Q and q.

($Q = 2\sqrt{2}q$)

4. A pith ball covered with tin foil having a mass of m kg hangs by a fine silk thread l metre long in an electric field E. When the ball is given electric charge of q coulomb, it stands out d metre from the vertical line. Show that the strength of the electric field is given by

$$E = mgd/q\sqrt{l^2 - d^2} \text{ newton/coulomb.}$$

5. If a uniform sphere of charge q, radius R and charge density ρ has a narrow straight tunnel along its diameter, and a point charge $-q'$ is placed at the entrance to the tunnel and released. Show that its motion is simple harmonic. Find the period of this motion and the maximum velocity acquired by the charge $-q$. Given the mass of the charge q' as m.

$$2\pi[3\varepsilon_0 m/q'\rho]^{1/2},\ R[q'\rho/3\varepsilon_0 m]^{1/2}.$$

6. A thin circular ring of radius 20 cm is charged with a uniform charge density of λ coul/m. A small section of 1 cm length is removed from the ring. Find the electric field intensity at the centre of the ring. ($2.25\lambda \times 10^9$ *N/C*)

7. Prove that the electric field intensity due to a uniformly charged ring is maximum at a distance $1/\sqrt{2}$ times its radius form the centre on its axis.

8. The inner surface of a non-conducting hemi-spherical bowl of radius a has uniformly spread charge of surface density σ over it. Find the electric field at the centre of the flat surface of the bowl. $(\sigma/4\varepsilon_0\ N/C)$

9. A small sphere whose mass is 1×10^{-4} kg carries a charge of 3×10^{-10} coul and is attached to one end of a silk fibre 5 cm long. The other end of the fibre is attached to a large vertical conducting plate which has a surface charge of 25×10^{-6} C/m^2. Find the angle which the fiber makes with the vertical. (41^0)

10. Five thousand lines of force enter a certain volume of space and three thousand lines emerge from it. What is the total charge in coulombs within the volume? $(-1.77 \times 10^{-8}$ *coulomb*)

11. A spherical shell of outer and inner radii a and b respectively is concentric to a metal sphere (of radius c) inside the shell. Find the net charge on the outer surface of the spherical shell, if the electric field at a point P outside the shell at a distance 40 cm from the centre is $200 N/C$. $(3.55 \times 10^{-9}$ *coulomb*)

12. A cylinder of radius b is uniformly charged with a volume charge density ρ coul/m^2. Find expressions for the electric field as a function of r for inside and outside the cylinder, if the charge density varies as $\rho = \rho_0 r^3$ within the cylinder. $(\rho_0 r^4/5\varepsilon_0,\ \rho_0 b^5/5\varepsilon_0 r)$

13. Calculate the electric field intensity at a point (*i*) inside and (*ii*) outside the spherically symmetric charge distribution of radius a. The charge volume density varies as $\rho(r) = \rho_0\ (1 - r^3/a^3)^{1/2}$ inside and is zero outside the distribution.

$$\frac{2a^3\rho_0}{9\epsilon_0 r^2}\left[\ 1-\left(\ 1-\frac{r^3}{a^3}\right)^{3/2}\right],\ \frac{2a^3\rho_0}{9\varepsilon_0 r^2}.$$

14. A circular ring of radius a carries a charge which varies as $\lambda = \lambda_0 \sin\theta$. Find the electric field at the centre of the ring. $(\lambda_0/4a\varepsilon_0)$

15. A long conducting cylinder of length l carrying a total charge $+q$ is surrounded by a conducting cylindrical shell of total charge $-2q$. Calculate **E** (*i*) at point outside the shell and (*ii*) in the region between the cylinders.

$(q/2\pi\varepsilon_0 lr$ *radially inward.* $q/2\pi\varepsilon_0 lr$ *radially outward*)

16. Three infinite metal plates A, B, C are arranged parallel to each other. Plate B carries a uniform +ve charge having a charge density σ on both the sides. Plate A and C carry unknown charges, but the arrangement produces zero electric field in the region left to plate A and right to plate C, *i.e.*, outside this arrangement. Find (*i*) the electric intensity at points between A and B, (*ii*) charges on the two surfaces of plate A. $(\sigma_{right} = \sigma,\ \sigma_{left} = 0)$

17. Two parallel infinite wires carry uniform charges of λ_1 and λ_2 coul/m. If the separation of the wires is b, find the force on unit length of one as a result of the other. $(\lambda_1\ \lambda_2/2\pi\varepsilon_0 b)$

18. Two concentric thin metallic spherical shells of radii R_1 and R_2 where $R_1 < R_2$ bear charges q_1 and q_2 coulombs respectively. Using Gauss Theorem, show that

(*a*) The electric field intensity at radius $r < R_1$ is zero.

(*b*) The electric field intensity at radius r between R_1 and R_2 is $q_1/4\pi\varepsilon_0 r^2$

and (*c*) the electric field intensity at radius $r \leqslant R_2$ is $(q_1+q_2)/4\pi\epsilon_0 r^2$.

19. An infinitely long wire is stretched horizontally 4 meters above the surface of the earth. It has a charge of microcoulomb per cm of its length. Calculate the electric field at a point on earth vertically below the wire.

(4.5 × 10⁵ nt/coul)

20. The surface density of charge on a conductor, in dry air, is 2.67×10^{-13} coul cm^{-2}. What is the force per unit area due to this charge.

(4.03 × 10⁻⁷ newton/m²)

21. Find the greatest charge which can be carried by a metallic sphere of 10 cm diameter, if the dielectric strength of air is 2×10^4 volts/cm.

(0.55 micro coulomb)

22. An α-particle, approaching the surface of a gold nucleus, is at a distance equal to the nuclear radius (6.9×10^{-15} m) from the surface. If the α-particle is assumed as point charge and of mass 6.7×10^{-27} kg, calculate the force and acceleration of the α-particle. *(1.9×10^2 N, 2.8×10^{28} m/sec²)*

23. A 10 MeV α-particle moves head on towards a stationary nucleus having 80 protons. Caleulate the distance of its closest approach.

(1.44×10^{-15} m)

24. Electric charge of uniform density ρ per unit volume is distributed within a spherical shell of inner and outer radii '*a*' and '*b*' respectively. Calculate the electric field at a point distant *r* from the centre such that
(*i*) $a<r<b$ and (*ii*) $r<b$

$$\frac{\rho}{3\epsilon_9}\left[\frac{r^3-a^3}{r^2}\right], \frac{\rho}{3\epsilon_0}\left[\frac{b^3-a^3}{r^2}\right]$$

25. An oil droplet of mass 3×10^{-11} gm and of radius 2×10^{-4} cm carries 10 excess electrons. What is its terminal velocity (*a*) when falling in a region in which there is no electric field, (*b*) when falling in an electric field 3×10^5 *N/C*, directed downward ? Given the viscousity of air $= 1.80 \times 10^{-7}$ *N* sec./m².

(4.34×10^{-4} m/sec., 2.74×10^{-4} m/sec. upward)

26. A bubble of radius *a* is formed from a soap solution having a surface tension *T*. If the bubble is raised to a potential *V* by being touched with a wire from a static machine, show that the radius of the bubble increases to *r*, given by

$$P(r^3-a^3)+4T(r^2-a^2)-\tfrac{1}{2}\epsilon_0 V^2 r=0,$$

where *P* is the atmospheric pressure.

3

Electrical Potential

3.1. LINE INTEGRAL OF ELECTRIC INTENSITY

In mechanics, the line integral plays a very fundamental role in problems involving work. Its importance is equally great in electrostatic problems. We know that if a force **F** acts on a body while the body suffers a displacement $d\mathbf{l}$, the work done

$$dW = \mathbf{F} \cdot d\mathbf{l} = F \cos\theta \, dl, \qquad \ldots(1)$$

where θ is the angle between **F** and $d\mathbf{l}$ and $F\cos\theta$ is the component of F along the direction of displacement. For a path of finite length, work

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{l}, \qquad \ldots(2)$$

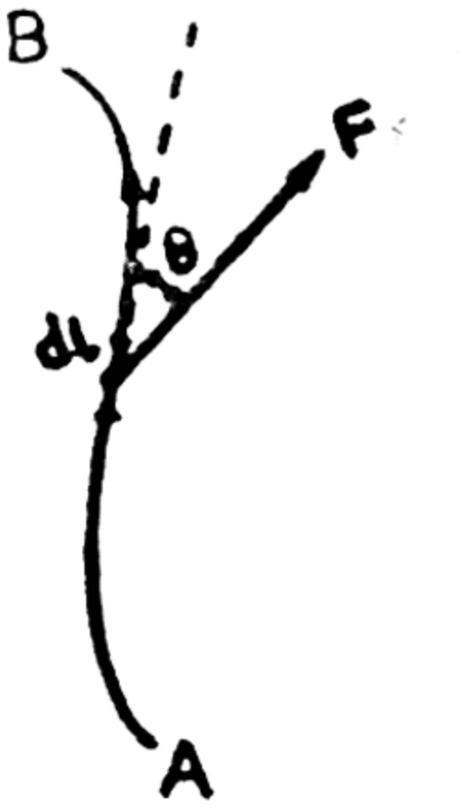


Fig. 3.1.

It is the another use of scalar product of two vectors, one was discussed in chapter 2 where it was applied to the surface integral. One should not confuse as the *integral for flux is taken over a surface and the integral for work is taken along a line.*

We know that the earth's gravitational field, a vector field, is a *conservative field. For a conservative field the potential energy of a system is independent of the path taken and is defined by its position only, or* in other words the *work necessary to move a body from one point to another is independent of the path taken between the two points.* If we can prove that the electrostatic field, *an electric field due to stationary charges,* is a conservative field, then we can say that the use of work and energy is as natural in *electrostatics* as in *mechanics.*

To show that the electrostatic field is a conservative field, let us consider the field due to a point charge q (which is very small so that its presence does not affect the electric field in which it is being placed). If **E** is the electric field at any point P due to the point charge q, the force on the test charge q_0 will be $q_0\mathbf{E}$. Hence the external force $\mathbf{F} = -q_0\mathbf{E}$ is required to displace the test charge q_0. External work necessary to move the test charge through a

differential distance $d\mathbf{l}$ making an angle θ with the force $\mathbf{F}$ is $dW=\mathbf{F}.d\mathbf{l}$. Hence the total external work to move the test charge

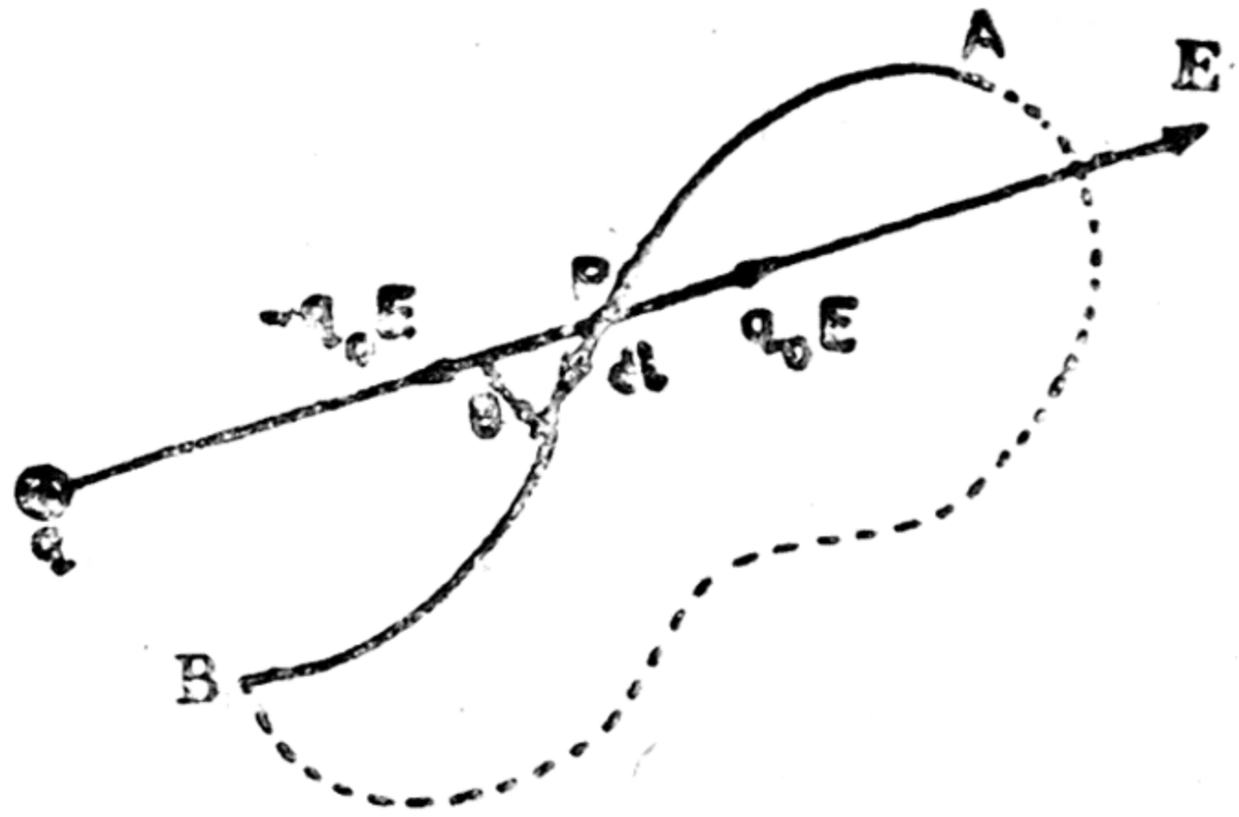


Fig. 3.2.

q_0 from A to B against the electric field due to charge q is given by

$$W=\int_A^B \mathbf{F}.d\mathbf{l}=-q_0\int_A^B \mathbf{E}.d\mathbf{l}$$

In Fig. 3.2, $\mathbf{E}$ point to the right and $d\mathbf{l}$, which is always in the direction of motion, points to the left. Therefore

$$\mathbf{E}.d\mathbf{l}=E\,dl\cos(180-\theta)=-Edl\cos\theta.$$

As we move a distance dl along the motion, we are moving in the direction of decreasing r, because r is measured from q as origin, therefore $dl\cos\theta=-dr$

and
$$W=-q_0\int_A^B \mathbf{E}\,.\,d\mathbf{l}=-q_0\int_A^B E\,dr. \qquad ...(3)$$

If r_A and r_B be the radial distances of points A and B respectively from the charge q, then

$$W=\int_A^B \mathbf{F}.d\mathbf{l}=-q_0\int_A^B E\,dr=-q_0\int_{r_B}^{r_A}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr$$

$$=\frac{q_0q}{4\pi\varepsilon_0}\left[\frac{1}{r_A}-\frac{1}{r_B}\right]. \qquad ...(4)$$

Thus we see that the external work required to move a test charge between two points in an electric field depends only on the radial distances (or the positions) of points A and B and is independent of path taken by the test charge between these points. The work done along a closed path (A to B along the continuous line $+B$ to A along the dotted line) is thus zero. Therefore, the field of a point charge is *conservative*. As the work done is a scalar quantity, total work done in moving a charge q_0 from point A to point B in an electric field due to a distribution of charges is just the sum of the work required to move this test charge when charges are considered individually. Thus the field due to arbitrary distribution of charges is also a *conservative field.*

As the integral is independent of path, an equivalent statement of the conservative nature of an electrostatic field is

$$\oint \mathbf{E}.d\mathbf{l}=0, \qquad \ldots(5)$$

i.e., the line integral around a closed path is zero.

or $$\int_A^B \mathbf{E}.d\mathbf{l}=-\int_B^A \mathbf{E}.d\mathbf{l}. \qquad \ldots(6)$$

There are following three cases in which the methods of integral calculus are not necessary to evaluate the line integral of **E**.

(*i*) If **E** is parallel and has the same magnitude at all points of the path of length l, then E=constant and the integral $=El$.

(*ii*) If **E** is perpendicular to the path at all points on it, then the integral is zero, as $\mathbf{E}.d\mathbf{l}=E\,dl\cos 90°=0$.

(*iii*) If **E** is zero at all points of the path, the integral is zero.

The property given by Eq. (5) of an electrostatic field can be used to verify a statement that *the electric field just outside the surface of any charged conductor is at right angles to the surface, when charges are at rest in the conductor.* For this purpose let us consider a portion of the surface of the conductor, shown in Fig. 3.3. The line integral around the path *a b c d*.

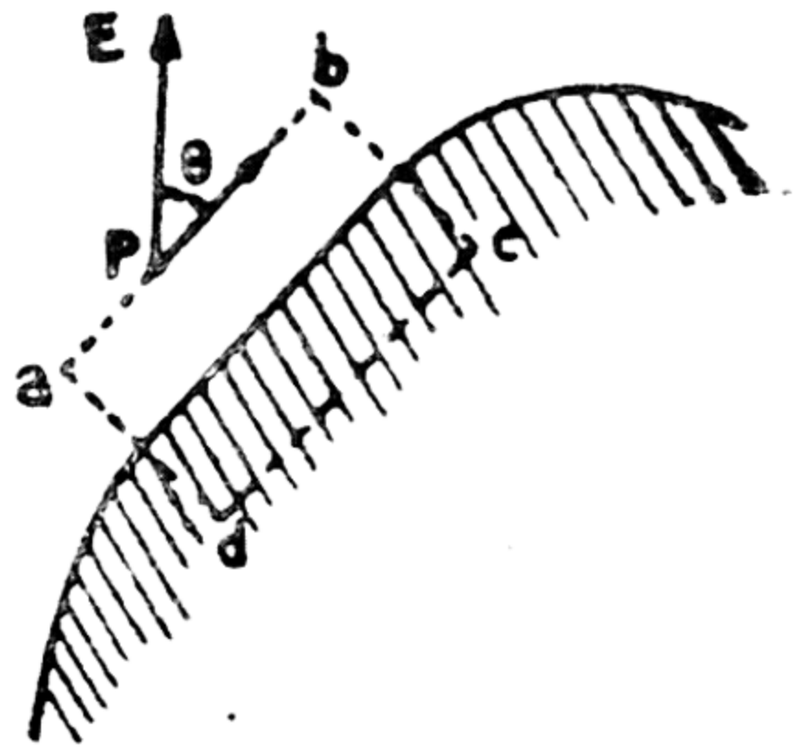


Fig. 3.3.

$$\oint \mathbf{E}.d\mathbf{l}=\int_a^b \mathbf{E}.d\mathbf{l}+\int_b^c \mathbf{E}.d\mathbf{l}+\int_c^d \mathbf{E}.d\mathbf{l}+\int_d^a \mathbf{E}.d\mathbf{l}=0 \qquad \ldots(7)$$

We know that the electric field inside the charged conductor is zero, hence the integral along *cd* is zero and the sides *ad* and *bc* are so short that they make no contribution to the line integral. Hence the integral around the closed path

$$\oint \mathbf{E}.d\mathbf{l}=\int_a^b \mathbf{E}.d\mathbf{l}=0 \qquad \ldots(8)$$

This is only possible when $\cos\theta=0$ or $\theta=90°$ (or $-90°$ if the surface has −ve charge), as **E** cannot be zero. Thus we can state that *electric lines of force meet the surface of a conductor at right angles provided the charges in the conductor are at rest.*

3.2. POTENTIAL ENERGY

We have seen that the electrostatic field is conservative. We generally associate the conservative force with the potential energy.

Thus the external work to move a test charge q_0 from A to B against the electric field **E** is equal to the difference in potential energy of the charge q_0 at point A and at point B, in an electric field **E**, *i.e.*

$$(P.E)_B-(P.E)_A=-q_0\int_A^B \mathbf{E}.d\mathbf{l}. \qquad ...(9)$$

As the potential energy depends upon the external work done which is independent of the path of the test charge moving in an electric field, hence the potential energy is a scalar quantity. It has the unit of work or energy. In SI units its unit is *joule*.

3.3. POTENTIAL DIFFERENCE AND POTENTIAL

Relation (9) gives the difference in potential energy of charge q_0 between points A and B in an electric field **E**. The difference in potential energy per unit charge

$$\underset{q_0\to 0}{\text{Limit}}\ \frac{(P.E.)_B-(P.E.)_A}{q_0}=-\int_A^B \mathbf{E}.d\mathbf{l}. \qquad ...(10)$$

This quantity is known as the *potential difference* (*pd* or voltage) between points A and B and is generally denoted by V_{AB} or V_B-V_A. As the potential energy and charge are both scalars, the ratio of these, the potential difference, is hence the *scalar quantity*. V_{AB} is independent of the path followed by the test charge in moving from point A to point B. Its unit in SI units is joule/coulomb. This unit is called volt (V), named in honour of the Italian Scientist *Alessandro Volta*, and accepted as a practical unit of potential difference. Multiplies of the volt are the kilovolt ($1\ kV=10^3\ V$), the megavolt ($1MV=10^6\ V$) and the gigavolt ($1\ GV=10^9\ V$).

The potential difference between two points A and B in an electric field due to a point charge q is thus given as

$$V_B-V_A=-\int_A^B \frac{q}{4\pi\varepsilon_0 r^2}\,dr=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_B}-\frac{1}{r_A}\right] \qquad ...(11)$$

Usually point A is chosen to be *at infinite for the reference point* for zero potential. This gives us

$$V_B=\frac{1}{4\pi\varepsilon_0}\,\frac{q}{r_B}=\frac{W}{q_0}, \qquad ...(12)$$

i.e., the potential V at a given point in an electric field is defined as the $1/q_0$ times the work that an external agent must do in moving the test charge q_0 from infinity to this point against that electric field. Potential at any point thus obtained is called *absolute potential.*

The potential is assumed to be zero at infinity as the height is assumed zero at sea level in mechanics. This reference point can

be chosen at any other value. *In many circuit problems the earth is taken as a reference of potential and assigned the value zero.* Potential V near an isolated positive charge is positive because flositive work is required to push a positive test charge in from inpnity. Similarly, the potential V is negative due to an isolated negative charge as the positive test charge is attracted by the negative charge.

3.4. EQUIPOTENTIAL SURFACE

In any region in which $\mathbf{E}=0$ at all points, such as the region very far from all charges or the interior of a charged conductor in which all the charges are at rest, the line integral of $\mathbf{E}$ is zero along any path (*i.e.*, $\int \mathbf{E}.d\mathbf{l}=0$). It means that the potential difference between any two points inside the charged conductor is zero, *i.e.*, all points in the conductor are at the same potential and the interior of a charged conductor is an *equipotential volume.*

If we consider an arc of a circle, at the centre of which a point charge is placed, the field $\mathbf{E}$ will be $\perp$ to this arc and the work done in moving the test charge along this circle is zero as $\int \mathbf{E}.d\mathbf{l}=0$. Hence each point on this circle is at the same potential and this circle constitutes *equipotential line.*

If the surface is in such a way that it is everywhere at right angles to an electric field, then any path lying in this surface is at right angles to the field and the line integral is zero. Hence all the points on this surface are at same potential. The surface is known as an *equipotential surface.* These surfaces are perpendicular to electric lines of force.

For the special case of a uniform field, where the lines of force are straight and parallel, the equipotential surfaces are the planes perpendicular to the lines of force. The equipotential surfaces are a family of concentric spheres for a sphere of charge or a point charge and are a family of concentric cylinders for a line of charge or cylinder of charge.

Equipotential surfaces in electrostatics act as wavefronts in optics. The wavefront is the locus of points which are in the same phase. The wavefronts are the planes $\perp$ to the direction of rays, similar to the equipotential surfaces which are $\perp$ to the lines of force. The equipotential surface is plane, spherical, or cylindrical corresponding to the charges at infinity, point charge (or spherical charge) or the line charge respectively, similar to the wavefronts due to source at infinity, point source or slit source.

3.5. POTENTIAL GRADIENT

We have determined how to calculate electrostatic potential V from an electric field $\mathbf{E}$. Let us now consider how to calculate

E if V is known. If we consider two points A and B at a distance $\Delta\mathbf{l}$ apart from each other in an electric field **E**, set up by any arrangement of charges. The work done by the external agency exerting the force $\mathbf{F}\ (=-q_0\mathbf{E})$ in moving a test charge q_0 along the path $\Delta\mathbf{l}$.

$$\Delta W=\mathbf{F}.\ \Delta\mathbf{l}=-q_0\,\mathbf{E}.\ \Delta\mathbf{l}.$$

$$\therefore\quad \Delta W/q_0=\Delta V=-\mathbf{E}.\ \Delta\mathbf{l}=-E\Delta l\cos(\pi-\theta)=E\Delta l\cos\theta,$$

or

$$E\cos\theta=\Delta V/\Delta l.$$

As the component $E\cos\theta$ is the component of **E** in the direction of $-\mathbf{l}$, hence $-E\cos\theta$, which we call E_l, is the component of E along **l**. Thus in the differential limit the above equation becomes

$$dV=-E_l\,dl \text{ or } E_l=-dV/dl, \qquad \ldots(13)$$

The ratio dV/dl, the rate of change of potential with distance in the direction of dl, is known as the *potential gradient. Thus the component of electric intensity in any direction is equal to the negative of the potential gradient in that direction.* Its unit is thus *volt/metre.*

We can find the components of **E** at any point in the directions of x, y and z-axes with the help of the potential gradients due to this field at that point along these axes. Thus

$$E_x=-\partial V/\partial x,\ E_y=-\partial V/\partial y$$

and

$$E_z=-\partial V/\partial z.$$

In the case of polar co-ordinates, the components are radial along radius vector and azimuthal perpendicular to the radius vector. These can be calculated as

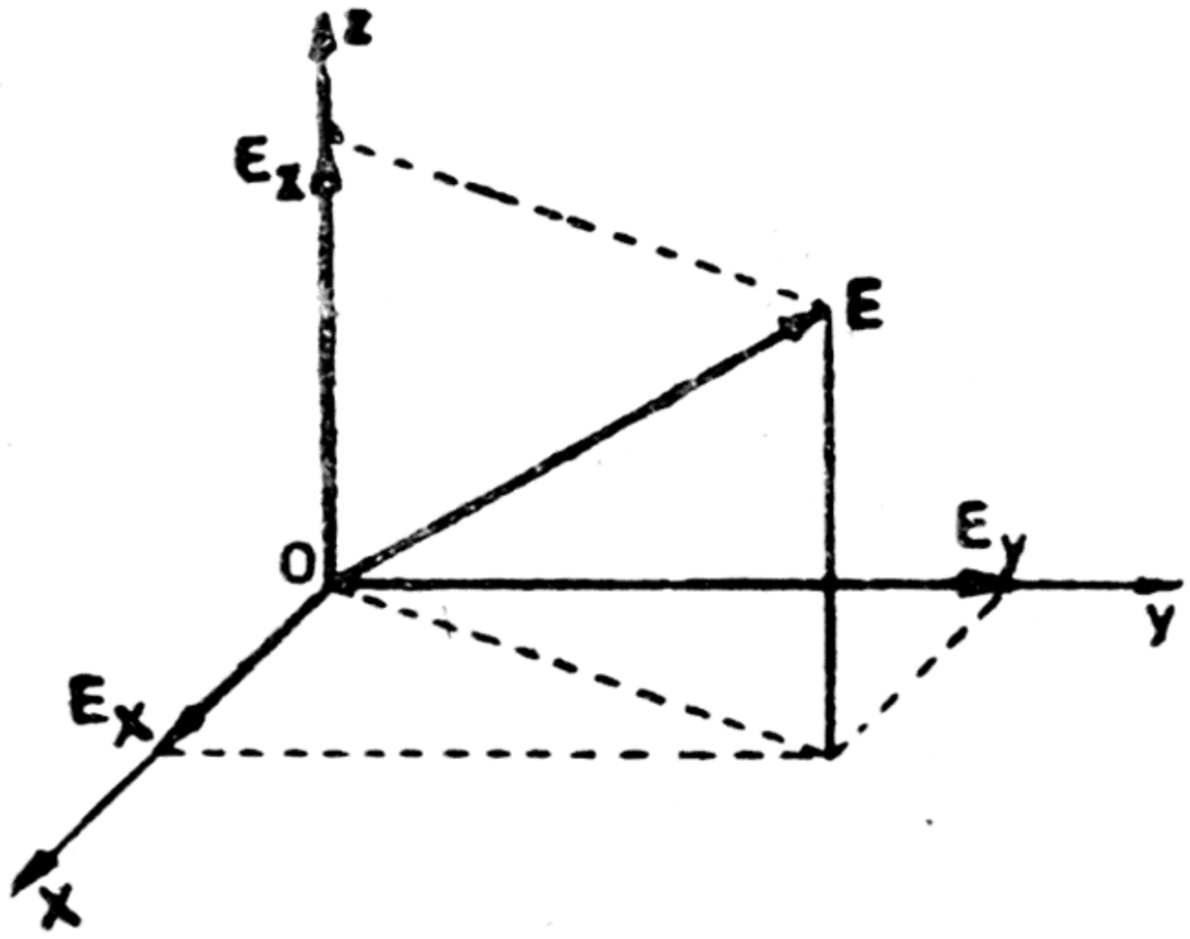


Fig. 3.4.

$$E_r=-\left(\frac{\partial V}{\partial r}\right)_\theta \text{ and } E_\theta=\frac{1}{r}\left(\frac{\partial V}{\partial\theta}\right)_r \qquad \ldots(14)$$

We know that $\mathbf{E}=E_x\mathbf{i}+E_y\mathbf{j}+E_z\mathbf{k}$.

$$\therefore\ \mathbf{E}=-\frac{\partial V}{\partial x}\,\mathbf{i}-\frac{\partial V}{\partial y}\,\mathbf{j}-\frac{\partial V}{\partial z}\,\mathbf{k}=-\left[\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right]V$$

$$=-\nabla V\equiv-\text{grad } V. \qquad \ldots(15)$$

This relation is very useful in many problems in which it is easier to calculate V rather than $\mathbf{E}$. As $\mathbf{E}$ is a vector and is equal to the gradient of a scalar field (potential), from vector calculus we thus get

$$\text{curl } \mathbf{E}=\nabla\times\mathbf{E}=\nabla\times(-\nabla V)=0. \qquad \ldots(16)$$

Thus we see that the *electrostatic field is curl free, or irrotational field.*

3.6. CALCULATIONS OF ELECTRIC POTENTIAL

(*a*) **Potential due to a point charge.** Let us consider two points A and B in an electric field due to an electric point charge q. We know that potential difference is given by the relation

$$V_B-V_A=-\int_A^B \mathbf{E}\,.\,d\mathbf{r}=-\int_A^B E\,dr. \qquad \ldots(17)$$

But E is the electric field at a distance r from the point charge q in the direction of r and is given by $E=q/4\pi\varepsilon_0 r^2$, hence

$$V_B-V_A=-\frac{q}{4\pi\varepsilon_0}\int_A^B\frac{dr}{r^2}=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_B}-\frac{1}{r_A}\right] \qquad \ldots(18)$$

Choosing point A to be at infinity, the potential at point B is thus given by

$$V=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}, \qquad \ldots(19)$$

where r is the distance of this point B from the charge q.

(*b*) **Potential due to charge distribution.** The potential at any point in the field due to a number of point charges is the algebraic sum (not the vector sum as potential is scalar) of the potentials at that point due to each charge, if the other charges are assumed to be absent. Hence

$$V=\sum_n V_n=\frac{1}{4\pi\varepsilon_0}\sum_n\frac{q_n}{r_n}, \qquad \ldots(20)$$

where q_n is the value of the n^{th} charge and r_n is the distance of the point, at which potential is required, from this charge.

If the charge distribution is assumed as continuous, then the summation may be replaced by an integration. Hence

$$V=\int dV=\frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}\ , \qquad \ldots(21)$$

where dq is the differential element of the charge distribution, and r is the distance of the point in question from this element.

If the density of charge is ρ, then above relation reduces to

$$V=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho\, dV}{r}. \qquad \ldots(22)$$

In cartesian co-ordinates, the potential at a point x, y, z is given as

$$V\ (x, y, z)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho\ (x', y', z')\ dx'\ dy'\ dz'}{r} \qquad \ldots(23)$$

(c) Potential due to a charged sphere. Let us consider a conducting sphere of radius R, having a charge q, uniformly distributed over its surface, so that the surface density of charge $\sigma=q/4\pi R^2$.

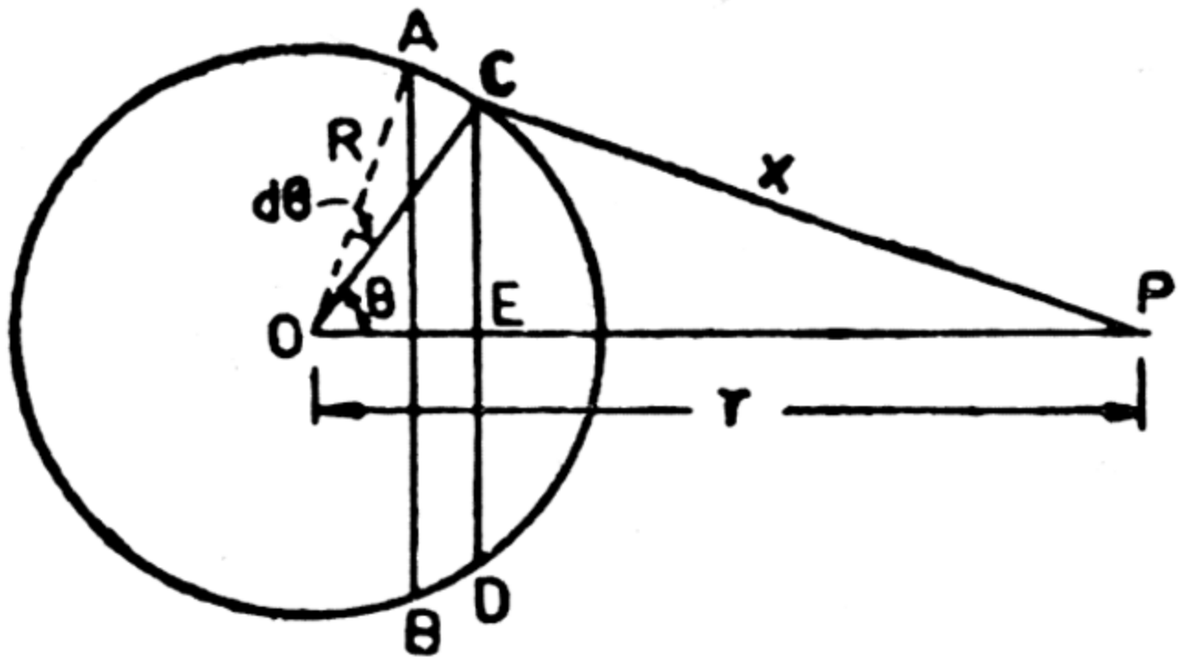


Fig. 3.5.

Cut the charged sphere by two closed parallel planes AB and CD, so as to form an annular ring $ABCD$. From Fig. 3.5 it is clear that the area of this annular ring

$$=2\pi\ (R\sin\theta)\ (R\, d\theta)=2\pi R^2\sin\theta\, d\theta.$$

$\therefore$ Charge on this ring $\Delta q=2\pi R^2\sin\theta\, d\theta\times\sigma=\frac{1}{2}\, q\sin\theta\, d\theta.$

As the ring is of small width, we can assume each point of this ring as equidistant from any point on its axis. Hence the potential at any point P at a distance r from the centre of the sphere and a distance x from the ring is given by

$$\Delta V=\frac{1}{4\pi\epsilon_0}\ \frac{\Delta q}{x}=\frac{1}{4\pi\epsilon_0}\ \frac{q\sin\theta\, d\theta}{2x}.$$

Here x and θ both are variables, let us convert them in one only. From $\triangle COP$, $x^2=R^2+r^2-2\,Rr\cos\theta$. Therefore

$$2xdx=2Rr\sin\theta\, d\theta \text{ or } \sin\theta\, d\theta=x\, dx/Rr.$$

$$\therefore \quad \Delta V = \frac{1}{4\pi\varepsilon_0}\frac{q\,dx}{2Rr}. \qquad \ldots(24)$$

Hence the potential at this point P, outside the sphere

$$V_0 = \frac{1}{4\pi\varepsilon_0}\ \frac{q}{2Rr}\int_{r-R}^{r+R} dx = \frac{1}{4\pi\epsilon_0}\ \frac{q}{r}. \qquad \ldots(25)$$

Thus the *potential at any point outside the uniformly charged sphere is the same as that when the whole charge is assumed to be concentrated at the centre of the sphere.*

Potential at the point P on the sphere

$$V_s = \frac{1}{4\pi\varepsilon_0}\int_{-R}^{+R}\ \frac{q\,dx}{2R^2} = \frac{1}{4\pi\varepsilon_0}\ \frac{q}{R}. \qquad \ldots(26)$$

Potential at the point P inside the sphere

$$V_i = \frac{1}{4\pi\varepsilon_0}\int_{R-r}^{R+r}\ \frac{q}{2Rr}\ dx = \frac{1}{4\pi\varepsilon_0}\ \frac{q}{R}. \qquad \ldots(27)$$

Thus the *potential at any point inside the charged sphere is the same as on the surface of the sphere and the charged conductor acts as an equipotential volume.*

We know that $E = -dV/dr$, hence electric intensity inside the charged sphere

$$E_i = -\frac{d}{dr}\left[\frac{1}{4\pi\varepsilon_0}\ \frac{q}{R}\right]$$

$$= 0. \qquad \ldots(28)$$

Electric intensity outside the charged sphere

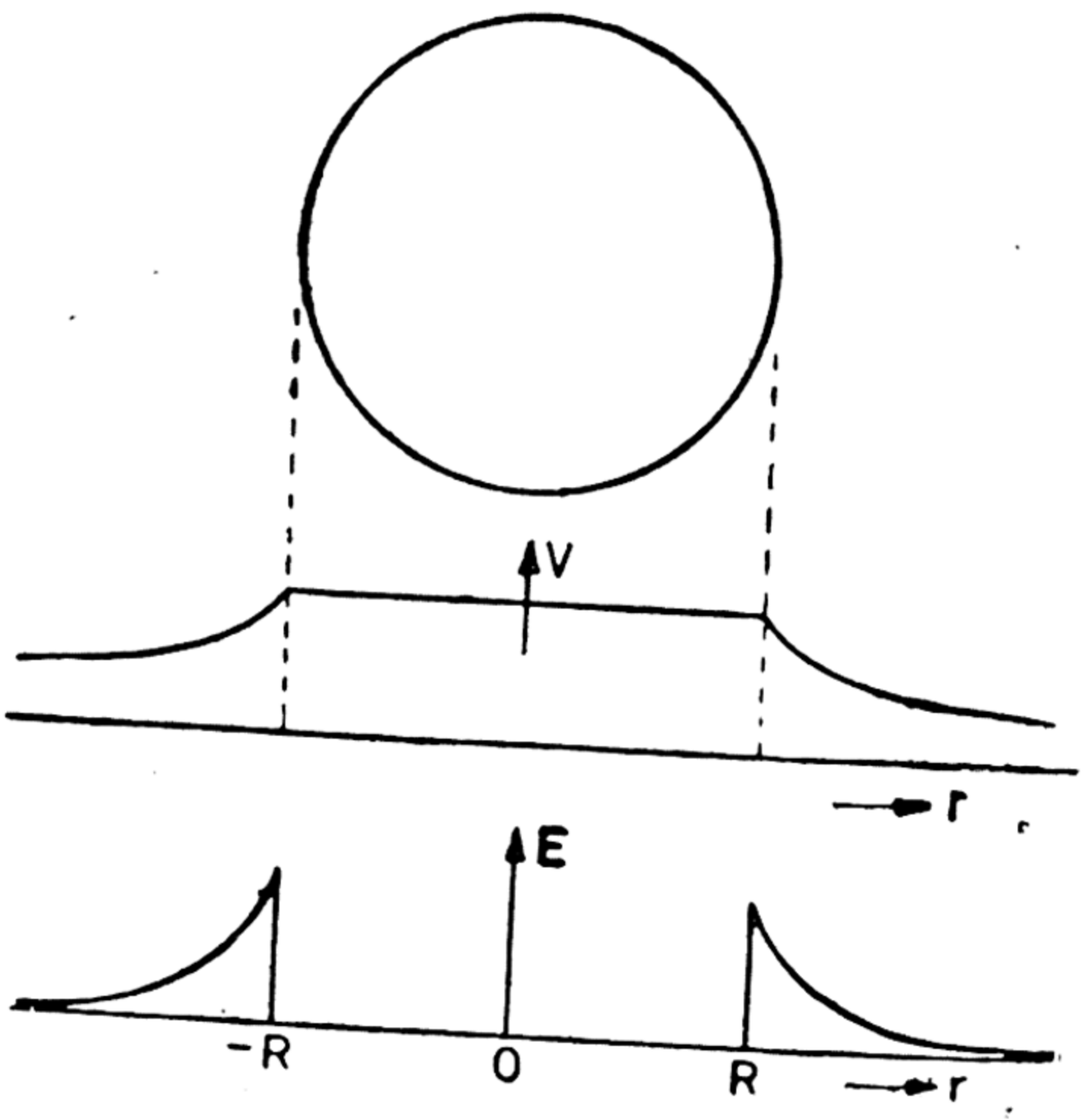


Fig. 3.6.

$$E_0 = -\frac{d}{dr}\left[\frac{1}{4\pi\varepsilon_0}\ \frac{q}{r}\right] = \frac{1}{4\pi\varepsilon_0}\ \frac{q}{r^2}. \qquad \ldots(29)$$

It will be along the direction of r. Hence

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\ \frac{q}{r^2}\hat{r}. \qquad \ldots(30)$$

Thus we see that *electric intensity is zero inside the charged sphere and decreases outside the sphere*, as shown in Fig. 3.6.

If the given sphere is non-conducting, the electric intensity outside the sphere ($r \geqslant R$) is given as

$$\mathbf{E}_0 = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}.$$

Therefore the potential (V) at the surface of the sphere

$$V_s = -\int_{\infty}^{R} \mathbf{E}.d\mathbf{r} = -\int_{\infty}^{R} \frac{q}{4\pi\varepsilon_0 r^2} \hat{r}.d\mathbf{r}$$
$$= q/4\pi\varepsilon_0 R.$$

The electric intensity inside the sphere

$$\mathbf{E}_i = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^3} r \hat{r},$$

$$\therefore \quad V_i = \frac{1}{4\pi\varepsilon_0} \int \frac{qr}{R^3} \hat{r}.d\mathbf{r} + V_{\text{constant}}.$$

Since at the surface $V_i = V_s$, hence $V_{cons} = 3q/4\pi\varepsilon_0(2R)$.

$$\therefore \quad V_i = -\frac{1}{4\pi\varepsilon_0} \frac{qr^2}{2R^3} + \frac{1}{4\pi\varepsilon_0} \frac{3q}{2R} = \frac{1}{4\pi\varepsilon_0} \left[\frac{3q}{2R} - \frac{qr^2}{2R^3} \right]$$

(*d*) **Potential due to an electric dipole.** Let us consider a dipole AB, two equal charges q of opposite signs and separated by a small distance $2a$. We want to calculate an electric potential

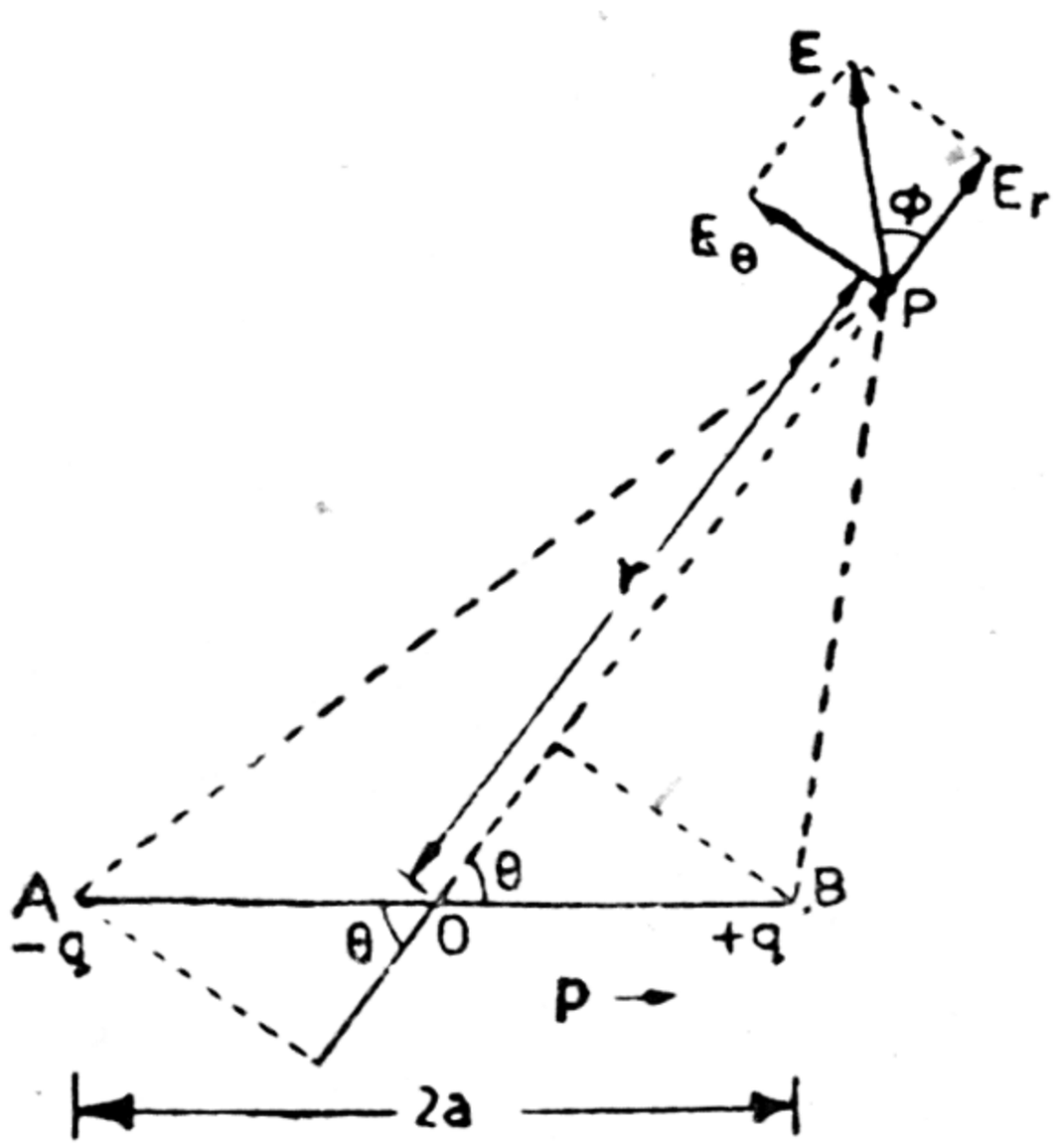


Fig. 3.7.

at a point P, which is not too close to the dipole. If r be its distance from the centre of the dipole and θ the angle between the direction of r and the dipole axis. Using the expression for the potential due to point charges, we get

$$V_P = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{PB} - \frac{q}{PA}\right].$$

Assuming that $r \gg a$, we can write $PB = r - a\cos\theta$ and $PA = r + a\cos\theta$.

$$\therefore \quad V_P = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r - a\cos\theta} - \frac{q}{r + a\cos\theta}\right] = \frac{1}{4\pi\varepsilon_0} \frac{2qa\cos\theta}{r^2 - a^2\cos^2\theta}.$$

As $r \gg a$, hence $a^2\cos^2\theta$ can be neglected in comparison to r^2.

$$\therefore \quad V_P = \frac{1}{4\pi\epsilon_0} \frac{2qa\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2} = \frac{\mathbf{p}.\mathbf{r}}{4\pi\varepsilon_0 r^3}. \quad \ldots(31)$$

$$= -\frac{1}{4\pi\varepsilon_0}\mathbf{p}.\nabla\left(\frac{1}{r}\right) = -\frac{1}{4\pi\varepsilon_0}\mathbf{p}.\,\text{grad}\left(\frac{1}{r}\right), \quad \ldots(32)$$

where **p** is the dipole moment of the electric dipole. It is directed from the negative charge to positive charge.

Relation (32) shows that the potential V is zero at all the points of which $\theta = 90°$, *i.e., no work is required to bring a test charge in from infinity along the perpendicular bisector of the dipole.*

We cannot obtain **E** from the gradient of V at once. The components of **E** in two perpendicular directions are E_r (the radial component along the direction of r) and E_θ (the transverse or azimuthal component perpendicular to the direction of r), given by the relations

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2}\right] = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

and
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{4\pi\varepsilon_0} \frac{p\cos\theta}{r^2}\right] = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$\therefore \quad E = \sqrt{(E_r^2 + E_\theta^2)} = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3}\sqrt{(1 + 3\cos^2\theta)}. \quad \ldots(33)$$

The angle ϕ which the vector **E** makes with the radius vector **r** is thus given by

$$\phi = \tan^{-1}(E_\theta/E_r) = \tan^{-1}(\tfrac{1}{2}\tan\theta). \quad \ldots(34)$$

If $\theta = 0°$, then electric field intensity **E** is in the direction of **p** and having magnitude $(1/4\pi\varepsilon_0)\ 2p/r^3$. If $\theta = 90°$, then electric field intensity **E** is in the direction of $-\mathbf{p}$ and having magnitude $(1/4\pi\varepsilon_0)\ p/r^3$.

(e) Potential due to an infinite line charge. Let us consider an infinite line of charge of constant linear charge density λ. We have proved that the electric field strength at any point P in the space at a distance y from the line is in the direction perpendicular to the direction of this line charge and is given by the relation

$$E = \lambda / 2\pi\varepsilon_0 \, y. \qquad \ldots(35)$$

The electric potential at this point is thus given by

$$V_y = -\int_{\infty}^{y} \mathbf{E}.d\mathbf{y}$$

$$= -\int_{\infty}^{y} \frac{\lambda}{2\pi\varepsilon_0 \, y} \, dy$$

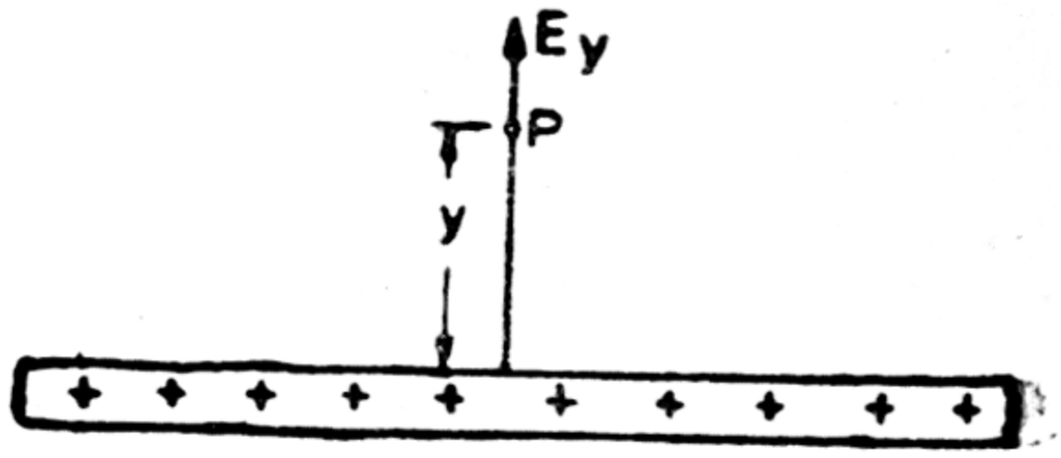


Fig. 3.8

$$= \frac{\lambda}{2\pi\varepsilon_0} \log_e \left(\frac{\infty}{y}\right) = \infty. \qquad \ldots(36)$$

Thus the electric potential due to an infinite line charge is infinity everywhere. This is correct. In practical problems we are interested in potential difference between positions with finite separation and never in absolute potentials. In this case our choice of infinity for the point of zero potential was arbitrary. We can, however, set $V=0$ at some arbitrary distance y as

$$V_y = \frac{\lambda}{2\pi\varepsilon_0} \log_e \left(\frac{y_0}{y}\right) = \frac{\lambda}{2\pi\varepsilon_0} \log_e y_0 - \frac{\lambda}{2\pi\varepsilon_0} \log_e y.$$

or
$$V_y = -\frac{\lambda}{2\pi\varepsilon_0} \log_e y + C. \qquad \ldots(37)$$

Since y_0 is constant, hence $(\lambda/2\pi\varepsilon_0) \log_e y_0$ is replaced by a constant C. Relation (37) can be used for the calculation of an electric field strength E at any point due to infinite line charge.

In the previous chapter we have proved that an electric field due to a charged cylinder of infinite length at any point outside to it is equal to that due to infinite line charge. Proceeding similarly we can therefore obtain the expression for the potential due to the charged cylinder of infinite length.

(f) Potential due to a uniformly charged disc. Let us find the electric potential at any point P on the axis of a uniformly charged circular disc, having surface charged density σ, at a distance r from the centre of the disc.

Let us divide this disc into large number of flat circular strips and consider one strip of radius x and of width Δx. Each point of

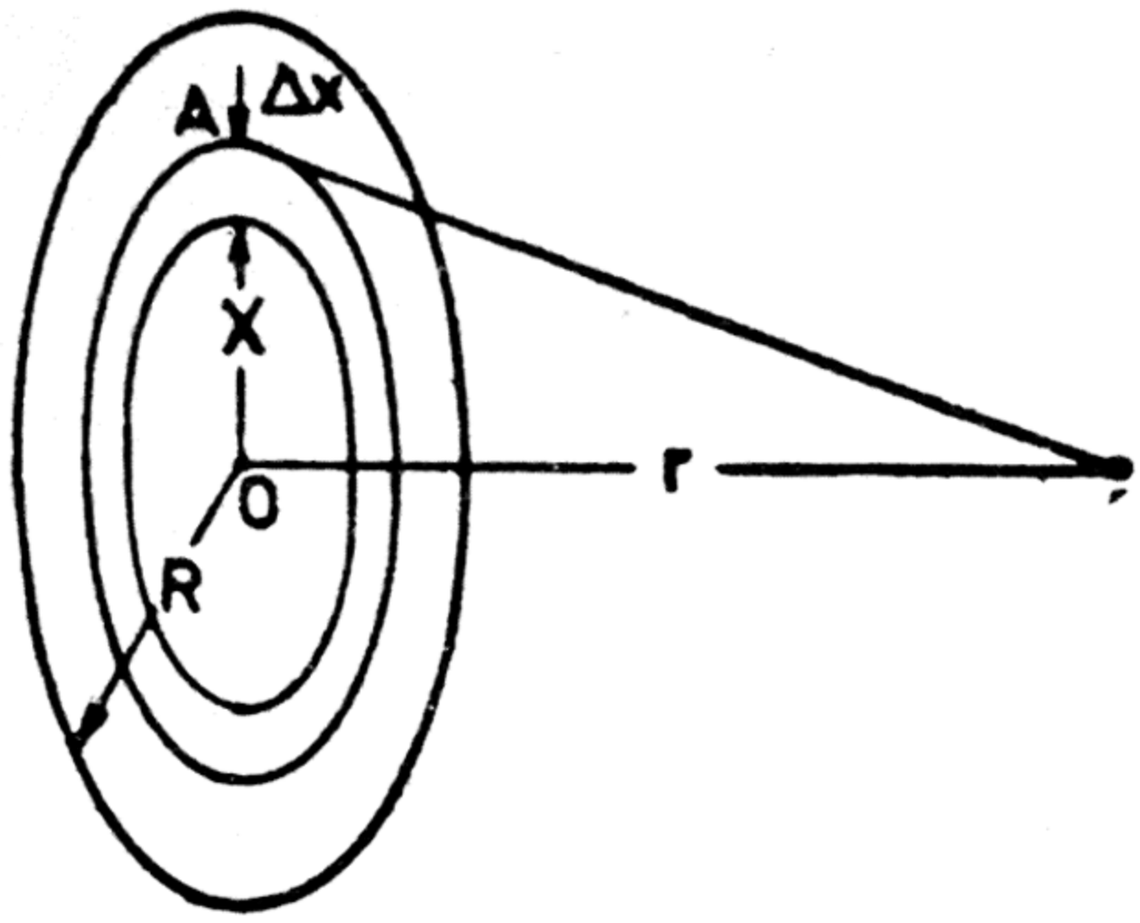


Fig. 3.9.

this strip can be assumed to be at equal distance from the point P as its width is very small. Potential at point P due to this circular strip is

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\,\frac{\Delta q}{AP} = \frac{1}{4\pi\varepsilon_0}\,\frac{2\pi x\,\Delta x\sigma}{\sqrt{(r^2+x^2)}}\,.$$

Thus the potential due to the charge on the whole disc

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{x\,dx}{\sqrt{(r^2+x^2)}}$$

$$= \frac{\sigma}{2\varepsilon_0}\,[\sqrt{(R^2+r^2)} - r]. \qquad \ldots(38)$$

At the central point O of the disc, the potential

$$V = \sigma R/2\varepsilon_0. \qquad \ldots(39)$$

Using Binomial theorem for $r \gg R$, relation (38) can be written as

$$V = \frac{\sigma}{2\varepsilon_0}\left[\, r + \frac{R^2}{2r} - r \,\right] = \frac{\sigma R^2}{4\varepsilon_0\, r} = \frac{\pi R^2\sigma}{4\pi\varepsilon_0 r}$$

$$= q/4\pi\varepsilon_0\, r, \qquad \ldots(40)$$

where q is the charge on the whole disc.

This is the relation as obtained due to a point charge. Thus at far away points the distribution of charge becomes insignificant. It is very difficult to calculate the potential at the points other than on the axis. However *potential on the edge of the disc can be calculated as under* :

To calculate the potential at point P, let us divide the disc in large number of rings with P as centre. The potential due to one segment between radii r and $r+\delta r$ is given as

$\Delta V = 2r\theta\delta r\ \sigma/4\pi\varepsilon_0 r.$

$\therefore$ Total potential due to the entire disc

$$V = \frac{\sigma}{2\pi\varepsilon_0}\int_0^{2R} \theta\ dr.$$

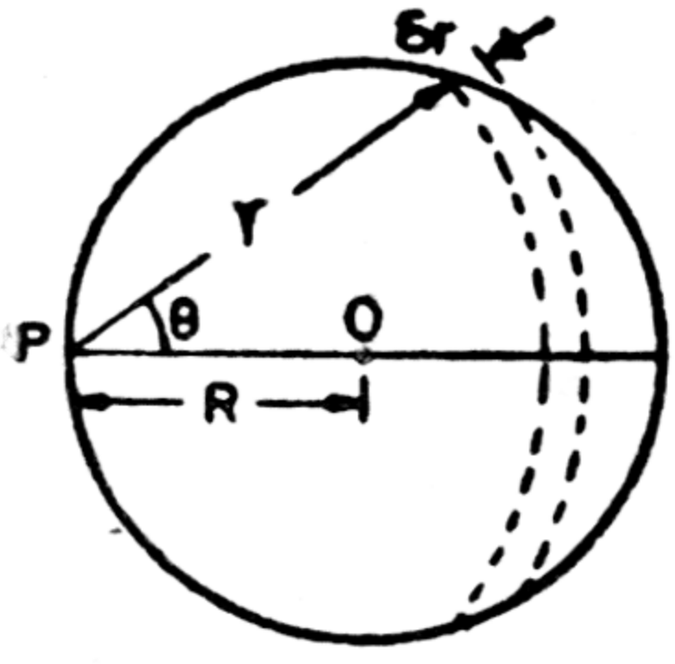


Fig. 3.10.

As $r = 2R\cos\theta$ and

$dr = -2R\sin\theta d\theta,$

hence

$$V = -\frac{\sigma}{2\pi\varepsilon_0}\int_{\pi/2}^{0} \theta 2R\sin\theta d\theta$$

$$= \frac{\sigma R}{\pi\varepsilon_0}\int_0^{\pi/2} \theta\sin\theta\ d\theta$$

$$= \sigma R/\pi\varepsilon_0. \qquad \ldots(41)$$

Comparing with Eq. (39) we see that the potential at the centre of the disc is greater than the potential at the edge. Thus a *uniformly charged disc is not an equipotential surface* and thus we may expect the electric field component in the plane of the disc in radially outward direction. The case will be different if the disc is conducting. As under static conditions a conductor behaves as an equipotential surface, hence the charge on the conductor can not be distributed uniformly. A conducting disc must have a continuous decrease in charge density from the edge to the centre, maximum at the edge and minimum at the centre.

3.7. THE DIPOLE APPROXIMATION FOR AN ARBITRARY DISTRIBUTION OF CHARGE

Let us calculate the potential at a distance large compared with the size of the set of charges. Let each charge q_i be situated at a distance $\mathbf{d}_i$ from an origin O chosen somewhere in the middle of the group of charges. The potential at a point P at a large distance R from the origin due to whole charge is given by

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}, \qquad \ldots(42)$$

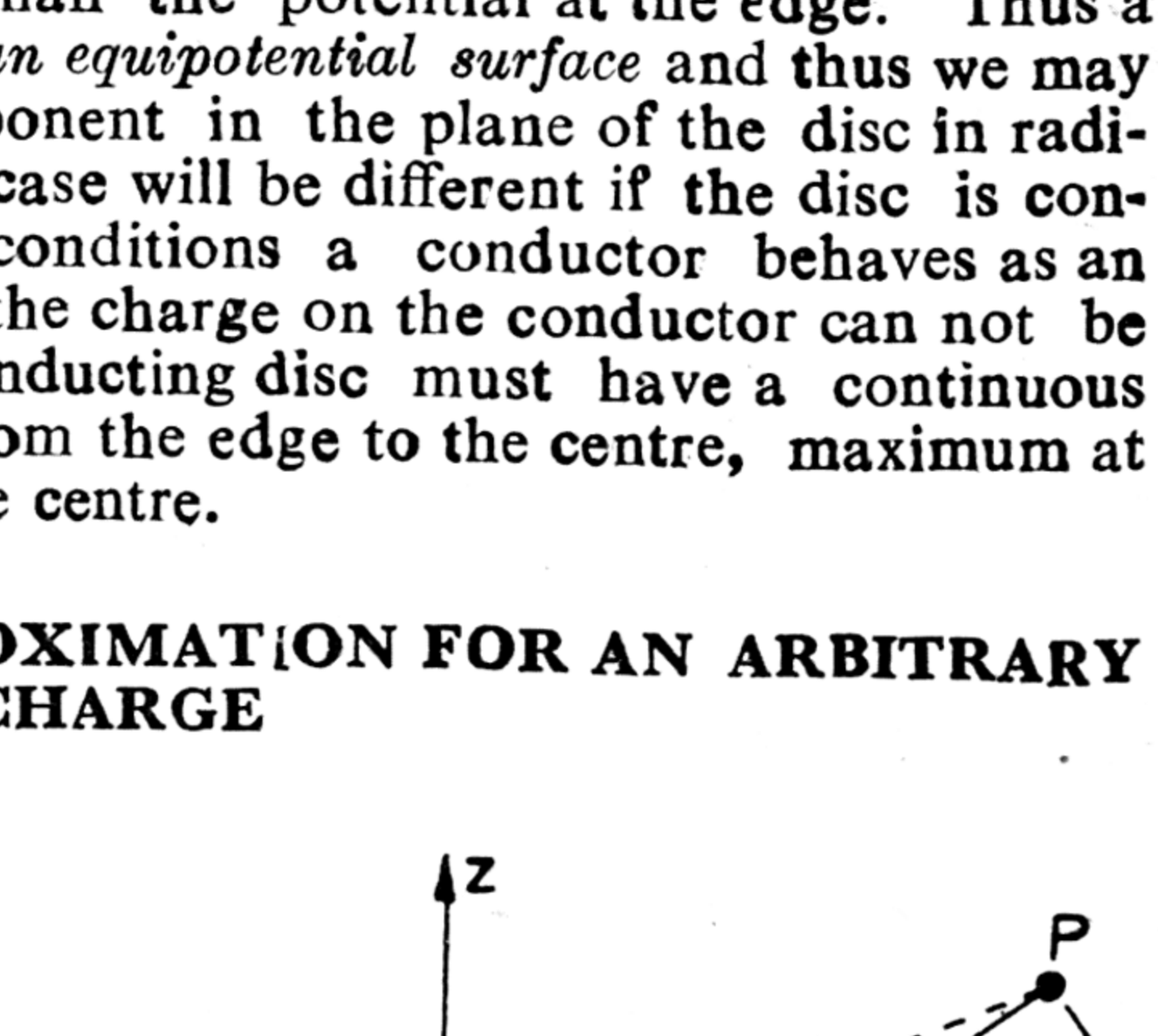


Fig. 3.11.

where r_i is the distance from P to the charge q_i.

As the point P is at a large distance, hence $\mathbf{r}_i = \mathbf{R} - \mathbf{d}_i$ can be assumed as equal to $\mathbf{R}$, thus

$$V = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\sum_i q_i = \frac{Q}{4\pi\varepsilon_0 R}, \qquad \ldots(43)$$

where Q is the total charge. Thus we see that the potential at a large distance due to set of charges (pump of charge) is same as if the whole charge was concentrated at one point, the origin.

Let us calculate the effect of the separate charges, taking most accurate expression for r_i. For the point P at a large distance we can assume r_i as given by

$$r_i = R - \mathbf{d}_i.\mathbf{e}_R, \qquad \ldots(44)$$

where $\mathbf{e}_R$ is the unit vector in the direction of $\mathbf{R}$. Since $d_i \ll R$, the above relation can be written to our approximation as

$$\frac{1}{r_i} = \frac{1}{R}\left[1 - \frac{\mathbf{d}_i.\,\mathbf{e}_R}{R}\right]^{-1} = \frac{1}{R}\left[1 + \frac{\mathbf{d}_i.\,\mathbf{e}_R}{R} + \ldots\right].$$

$$\text{Hence } V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{R}\left(1 + \frac{\mathbf{d}_i\,.\,\mathbf{e}_R}{R} + \ldots\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\sum_i q_i + \frac{1}{4\pi\varepsilon_0}\sum_i q_i \frac{\mathbf{d}_i.\,\mathbf{e}_R}{R^2} + \ldots$$

or

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} + \sum_i q_i \frac{\mathbf{d}_i.\,\mathbf{e}_R}{R^2} + \ldots\right). \qquad \ldots(45)$$

Here the dots indicate the terms of higher order in d/R and are the successive terms in the expansion of $1/r_i$. The first term is same as Eq. (43). It is the potential at very large distance, as the higher order terms are negligible and is known as monopole contribution. For molecules such as HCl, CO, CH_3Cl, $HCCl_3$, etc., the total charge is zero and the second term is dominent as the higher order terms are negligible. If we define $\Sigma q_i \mathbf{d}_i = \mathbf{p}$, then we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}.\,\mathbf{e}_R}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{R^2}. \qquad \ldots(46)$$

The quantity $\mathbf{p}$ is called the dipole moment of the distribution of charge and the potential V, the *dipole potential*. Dipole potential decreases as $1/R^2$ and varies as $\cos\theta$. Dipole fields are very important for the studies of molecular and atomic properties. For many molecules, such as CO_2, the dipole moment vanishes because of the symmetry of molecules, for them we take next term, which varies as $1/R^3$ and is called a *quadrupole potential*.

In the form of integral, relation (45) can be written as

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{R}\iiint \rho\, dV + \frac{1}{R^2}\iiint \rho\, d_i \cos\theta\, dV + \frac{1}{R^3}\iiint \rho\, d_i^2 \frac{(3\cos^2\theta - 1)}{2} dV + \ldots\right]$$

$$V=\frac{1}{4\pi\varepsilon_0}\left[\frac{A_0}{R}+\frac{A_1}{R^2}+\frac{A_2}{R^3}+\ldots\right], \qquad \ldots(47)$$

where $A_0=\iiint \rho\, dV$, $A_1=\iiint \rho\, d_i \cos\theta\, dV$,

and $A_2=\frac{1}{2}\iiint \rho\, d_i^2\,(3\cos^2\theta-1)\, dV$.

The integrals A_0, A_1, A_2, etc., depend only upon the charge distribution. A_0 represents the total charge in the distribution and is known as *monopole moment*. If the given distribution does not consist equal number of +ve and −ve charges (*i.e.*, $A_0\neq 0$), the potential at a large distance may be regarded as due to first term, as other terms are relatively negligible.

When $A_0=0$, the potential is taken as due to the second term, the coefficient $A_1=\iiint \rho\, d_i \cos\theta\, dV$ measures the relative displacement of +ve and −ve charges along the direction towards P and is known as the component of *dipole moment* along OP. The potential at a far point due to the charge distribution varies inversely as the square of the distance, as other terms are relatively negligible.

Similarly, when $A_1=0$, the potential is due to A_2, the component of quadrupole moment of the charge distribution in the direction of OP. Other higher order terms in the expansion are due to various multipole moments.

3.8. ELECTROSTATIC ENERGY OF CHARGES

We know that when we raise a stone from the earth's surface, the work we do against gravitational attraction is stored in the form of potential energy in the system (earth+stone). This energy converts in the form of kinetic energy when stone falls and finally transferred into heat energy in the system, earth+stone.

If we consider two charges q_1 and q_2 a distance r apart. There is some energy in the system because a certain amount of work was necessary to bring the charges together if the charges are of same sign. This energy is known as the electric potential energy in the system q_1+q_2 similar to the potential energy in the system earth+stone.

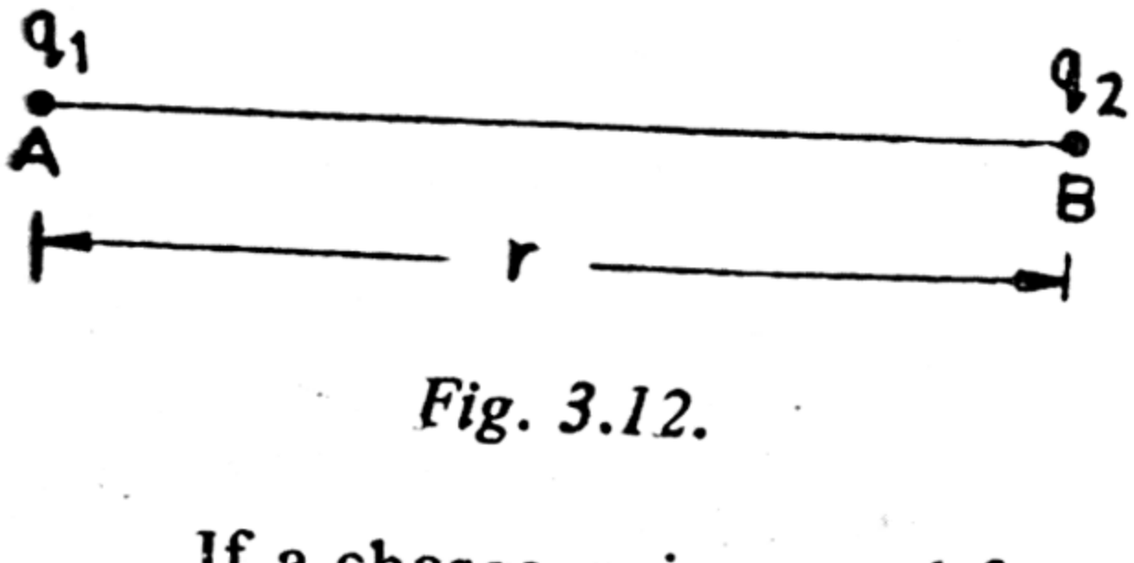


Fig. 3.12.

Suppose charge q_1 is placed at point A, the potential due to this charge at point B at a distance r is given by

$$V=\frac{1}{4\pi\varepsilon_0}\,\frac{q_1}{r}.$$

If a charge q_2 is moved from infinity to the point B, then from the definition of potential the work required is given by

$$W = Vq_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}.$$

This work is stored in the form of electrostatic energy U of the system $q_1 + q_2$, therefore we have

$$U = W = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}. \qquad \ldots(48)$$

For the calculation of potential energy of the system containing large number of particles, the procedure is to compute the potential energy for every pair of charges separately and to add these algebraically. Hence for n charges

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{\text{all pairs}} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i q_j}{r_{ij}} \quad (i \neq j) \qquad \ldots(49)$$

In this expression $\frac{1}{2}$ is introduced as each pair is counted twice here. For example, the energy contribution due to charge q_2 and q_8 is required once only but in this relation this term comes twice once when $i=2, j=3$ and other when $i=3, j=2$.

This relation can be written in different forms by noting that the final value of the potential V at the j^{th} point charge is

$$V_j = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 r_{ij}}. \quad (i \neq j) \qquad \ldots(50)$$

$\therefore$ Electrostatic energy of the system $= \frac{1}{2} \sum_{j=1}^{n} q_j V_j.$

The electrostatic energy of an arbitrary charge distribution with volume density ρ and surface density σ is given as

$$U = \frac{1}{2} \int_{volume} \rho V d\tau + \frac{1}{2} \int_{surface} \sigma V dS. \qquad \ldots(51)$$

Let us now evaluate the electrostatic energy of a charged sphere or the energy required to assemble a sphere of uniform charge density. For this let us imagine that we assemble the sphere by building up a succession of thin spherical layers of infinitesimal thickness. It Q_r is the charge of the sphere when it has been built up to the radius r, the work done in bringing a charge δQ from infinity to the layer of thickness δr is

$$\Delta W=\Delta U=(Q_r\ \delta Q/4\pi\epsilon_0\ r).$$

If ρ be the charge density in the sphere, then $Q_r=\frac{4}{3}\pi r^3\rho$

and $$\delta Q=\tfrac{4}{3}\pi\ 3r^2\rho\delta r=4\pi r^2\rho\delta r.$$

$\therefore$ $$\Delta U=4\pi\ \rho^2 r^4\ \delta r/3\epsilon_0.$$

Total energy required to assemble the sphere of radius R is

$$U=\int_0^R dU=\frac{4\pi\rho^2R^5}{15\epsilon_0}=\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R}, \qquad \ldots(52)$$

where Q is the total charge on the sphere of radius R.

3.9. ELECTROSTATIC ENERGY OF AN IONIC CRYSTAL

The chemical energies of a crystal basically depend upon the electrostatic energies and may be obtained by computing theoretically the work required to pull apart the crystal. This work is the sum of the potential energies of all the pairs of ions.

Let us consider the example of *ionic lattice*. An ionic crystal like *NaCl* consists of positive and negative ions, which can be thought of as rigid spheres. These spheres attract until they begin to touch. The ions are placed as a three dimensional checkerboard, thus forming a cubic lattice. The spacing of the ions has been determined by *X*-ray diffraction.

The easiest way of finding the work is to pick out a particular ion and calculate its potential energy with each of the other ions. The sum of these energies will give twice the energy per ion (or the energy per molecule which contains two ions). If a be the centre to centre spacing between the ions, the energy of an ion due to the presence of its neighbouring one ion is $e^2/4\pi\epsilon_0 a$. Let us first sum up the terms from the ions along a straight line. This sum is due to the ions on the left and the right of any reference ion. For a positive ion it is given by the relation

$$U_1=\frac{e^2}{4\pi\epsilon_0}\left[-\frac{2}{a}+\frac{2}{2a}-\frac{2}{3a}+\frac{2}{4a}-\ldots\ldots\right]$$

$$=\frac{-2e^2}{4\pi\epsilon_0 a}\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\right]$$

$$=\frac{-2e^2}{4\pi\epsilon_0 a}\log_e 2=-1.386\ \frac{e^2}{4\pi\epsilon_0 a}. \qquad \ldots(53)$$

Now consider the next adjacent line of ions above (or below). The nearest is negative ion, at a distance a. There are two positive ions at a distance $\sqrt{2}a$. The next pair are at the distance $\sqrt{5}\,a$, the next at $\sqrt{10}\,a$ and so on. The effect of all the ions on this line is

$$U_2 = \frac{e^2}{4\pi\varepsilon_0}\left[-\frac{1}{1} + \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{10}} \ldots\ldots\right], \qquad \ldots (54)$$

There are four such lines (above, below, infront and in back). There are four lines on the nearest diagonals. If we calculate contributions of the ions on all these lines, we get the grand total as

$$U_2 = -1.7476\ \frac{e^2}{4\pi\varepsilon_0 a} = 8.94\ eV. \qquad \ldots (55)$$

This value is about 10% greater than the experimental value. This confirms the idea that the whole lattice is held together by electrostatic attraction.

3.10. ELECTROSTATIC ENERGY IN NUCLEI

The nucleons that hold the stable nuclei are held together by strong attractive forces and therefore work must be done in separating them from each other until they are very large distance apart, *i.e., the energy must be supplied to the nucleus to separate it into the individual constituents.* This energy is called the binding energy of the nucleus. The binding energy may be regarded as being made up of a number of terms, each of which represents some general characteristic of nuclei. The only charge dependent term in the relation for binding energy is the *Coulomb repulsion* between the protons, all other terms are based on the fact that the *p-p, n-n and n-p forces are identical and are the short range forces.*

The electrostatic energy is simply the work done against electrostatic forces in assembling the Z-protons in a sphere of radius R. The uniform charge density of the charge in the nucleus is

$$\rho = Ze / \tfrac{4}{3}\pi R^3.$$

To find the electrostatic energy of the sphere, let us first calculate the work required to increase the radius r of the sphere to $r+dr$. The charge dq on the shell of thickness dr on the sphere of radius r is

$$dq = \rho \times 4\pi r^2 dr = (3Ze/R^3)\ r^2 dr.$$

Hence the work in bringing this charge from infinity to r against the charge on the sphere of radius r is

$$dU = \left(\frac{3Ze}{R^3}\right) r^2 dr \times \frac{\frac{4}{3}\pi r^3 \rho}{4\pi\varepsilon_0 r}$$

$$= \frac{3Z^2e^2}{4\pi\varepsilon_0 R^6} r^4 dr.$$

Integrating from 0 to R we have obtained the work done against Coulomb repulsion or the electrostatic energy of the nucleus.

$$\therefore \quad U=\frac{3Z^2e^2}{4\pi\varepsilon_0 R^6}\int_0^R r^4dr=\frac{3}{5}\ \frac{Z^2e^2}{4\pi\varepsilon_0 R}. \qquad ...(56)$$

This relation is not correct for small number of protons (Z small). In such cases Z^2 must be replaced by $Z(Z-1)$.

$$U=\frac{3}{5}\ \frac{Z(Z-1)e^2}{4\pi\varepsilon_0 R}. \qquad ...(57)$$

This relation may be used for calculating the nuclear radius as given below. Consider a positron emission

$$zX^A \rightarrow z_{-1}Y^A + e^+ + \nu. \qquad ...(58)$$

The difference in the electrostatic energies between these nuclei is given by

$$\Delta U=\frac{3}{5}\ \frac{e^2}{4\pi\varepsilon_0 R}[Z^2-(Z-1)^2]$$

$$=\frac{3}{5}\ \frac{e^2(2Z-1)}{4\pi\varepsilon_0 R}. \qquad ...(59)$$

The maximum positron energy, when the daughter nucleus is formed in the ground state, is given by

$$E_\beta=\frac{3}{5}\frac{e^2(2Z-1)}{4\pi\varepsilon_0 R}-m_o c^2-(m_n-m_p)c^2,$$

where m_oc^2 is the rest mass energy of the positron and $(m_n-m_p)c^2$ is the energy absorbed in transforming a proton of mass m_p into a neutron of mass m_n.

Let us use a relation $R=R_0A^{1/3}$ and $2Z-1\simeq A$, we get

$$E_\beta=\frac{3}{5}\ \frac{Ae^2}{4\pi\varepsilon_0 R_0A^{1/3}}-0.51\ MeV-1.29\ MeV. \qquad ...(60)$$

The slope of the curve between E_β and $A^{2/3}$ gives us $R_0\simeq 1.2\times10^{-15}$ m.

3.11. THE ELECTRON VOLT

When an electron or proton falls through a potential difference of V_B-V_A volts, it acquires a kinetic energy

$$K.E.=q(V_B-V_A)=(1.60\times10^{-19})\ (V_B-V_A)\ joule.$$

This amount is very small and is multiple of 1.60×10^{-19}, hence we introduce a new unit of energy, known as *electron volt* (eV). It is the energy in joules divided by the magnitude of the quantum of charge 1.60×10^{-19} coulomb.

$$\therefore \quad 1eV=1.60\times10^{-19}\ joule. \qquad ...(61)$$

The utility of this unit is obvious, as charge in atomic and nucleus cases is generally taken in terms of the electronic charge (*i.e.*, *Ze.*) and the potential difference in volts. If an α-particle, having charge $2e$, falls through a potential difference of $100V$, then it will acquire energy 200 *eV*. Its million and billion times quantities are respectively abbreviated as *MeV* and *BeV*.

3.12. HIGH VOLTAGE BREAKDOWN

Let us consider a big sphere A of radius a connected with a small sphere B of radius b by a wire. The wire will have little influence on the fields outside but will keep the spheres at the same potential. If the bigger ball carries a charge Q and the smaller ball carries a charge q, then their potentials can be written as

$$V_A = \frac{1}{4\pi\epsilon_0}\frac{Q}{a},$$

and $$V_B = \frac{1}{4\pi\epsilon_0}\frac{q}{b},$$

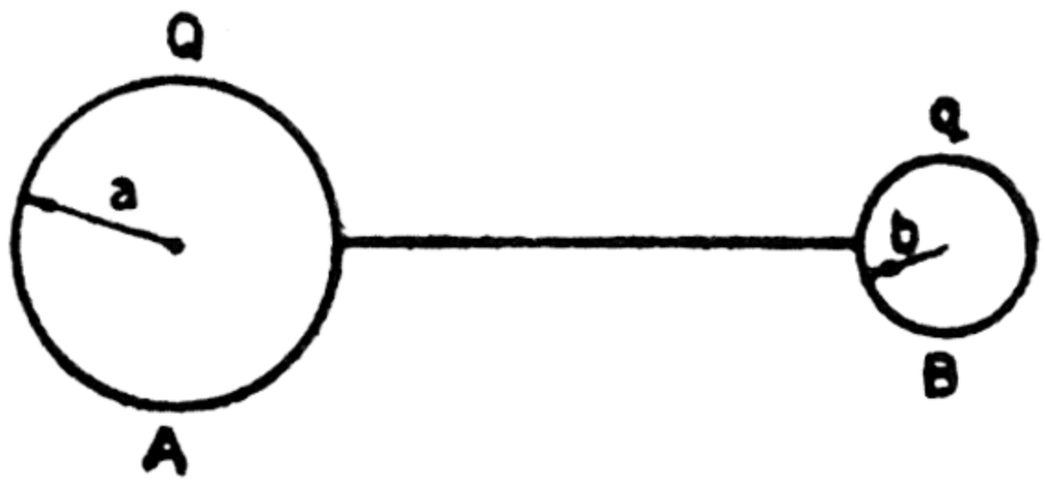


Fig. 3.13.

As $V_A = V_B$, hence $Q/a = q/b$. ...(62)

The ratio of the electric fields at the surfaces

$$\frac{E_A}{E_B} = \frac{Q/4\pi\epsilon_0 a^2}{q/4\pi\epsilon_0 b^2} = \frac{Qb^2}{qa^2} = \frac{b}{a}.$$ ...(63)

Therefore the field is higher at the surface of the small sphere.

The same can be explained with a conductor having a very sharp end. It can be shown by simple knowledge of lines of force that the field around this sharp end is much higher than the field in the other regions. The reason is that charges try to spread out on the surface of the conductor and the tip of the sharp end is as far away as it is possible from most of the surface. Some of the charges on the conductor get pushed all the way to the sharp end. A relatively small charge on the sharp end can thus produce a large surface charge density, hence the high field just outside of it.

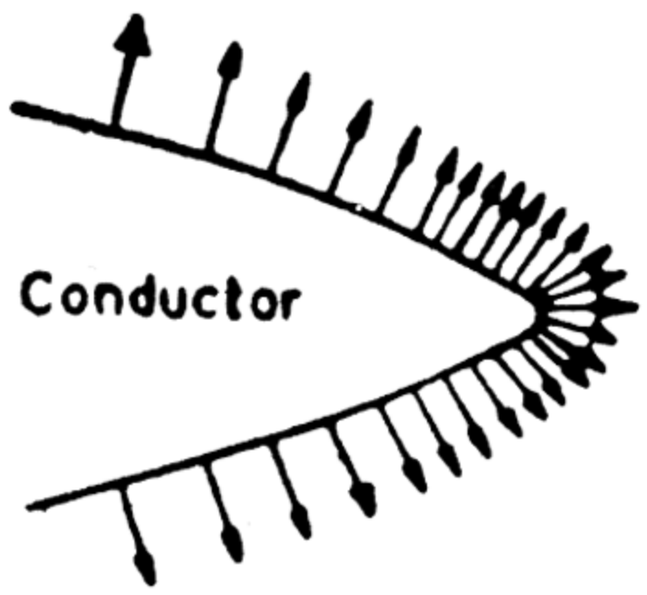


Fig. 3.14.

Due to this very high field at the sharp ends of conductors, the air near these points *breaksdown*. In this process the loose charge (electron or ion) present some where in the air is accelerated by this field and picks up enough speed before it hits another atom to be

able to knock an electron off that atom. In this way more and more ions are produced. Their motion constitutes a discharge or spark. *Hence for no spark in air while charging an object, we must require the smooth surface.*

3.13. ELECTROSTATIC GENERATOR

Let us consider large hollow metal sphere B with an original positive charge q_B, indicated by +ve signs with circles around them and having inner radius r_B. If it has small opening so that a small metal sphere A of radius r_A with a positive charge q_A can be placed inside it. When sphere A is at the centre of B and the effect due to opening in B is very small and can be neglected, the positive charge on sphere A will induce negative charge on the inner surface of B and the electric field between these two charges is symmetric and radial. From Gauss's law, the field between spheres A and B is due to the charge on A only and at a distance r from the centre of sphere A is given by

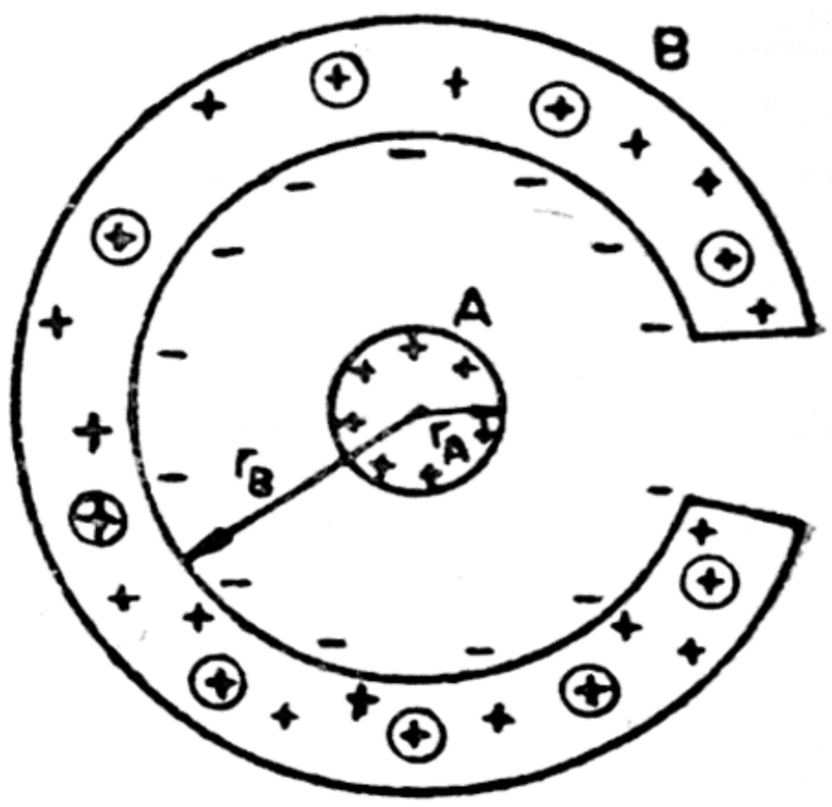


Fig. 3.15.

$$E=\frac{1}{4\pi\epsilon_0}\ \frac{q_A}{r^2},$$

$\therefore$ Potential difference between spheres A and B is

$$V_A-V_B=\frac{q_A}{4\pi\epsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right). \qquad ...(64)$$

Thus the *pd* is independent of the original charge q_B on the sphere B. For q_A positive, the $V_A > V_B$, or the inner sphere A will always be at higher potential than the outer hollow sphere B.

If the spheres are connected by a fine wire (outer surface of A with the inner surface of B), the charge will flow to the outer sphere B as it was on the lower potential, till the potential difference $V_A-V_B=0$ or from above relation until q_A becomes zero. This leads to the conclusion that *all the charge q_A of the sphere A is transferred to the hollow sphere B, irrespective of the initial value of charge and potential of the hollow sphere B.*

3.14. VAN DE GRAAFF GENERATOR

It was invented by R.J. Van de Graaff in the United States in 1931. It is an important instrument for producing high voltages in the million volts region. Its design is based on the principle that *if a charged conductor is brought into internal contact with a hollow conductor, all of its charge transfers to the hollow conductor no matter how high the potential of the latter may be.*

It consists of a belt, made of paper, silk, rayon or other flexible non-conducting material, which is run by means of a motor at high speed over two pulleys. The positive pole of a dc power supply at a few kilovolts is connected to a pointed comb like conductor. Because of the large electrostatic field in the air near the spray points, positive and negative ions are formed and a spray of positive ions is repelled from the points. These positive ions attach themselves to the surface of a moving belt and are carried mechanically by the belt to the terminal where they are removed by another corona discharge to another set of needle points. This set of needle points is connected with the internal surface of the hollow hemispherical conductor mounted on insulating supports. The upper pulley is within this large metal hemisphere. The charge is thus transferred to this hemisphere and increases its potential.

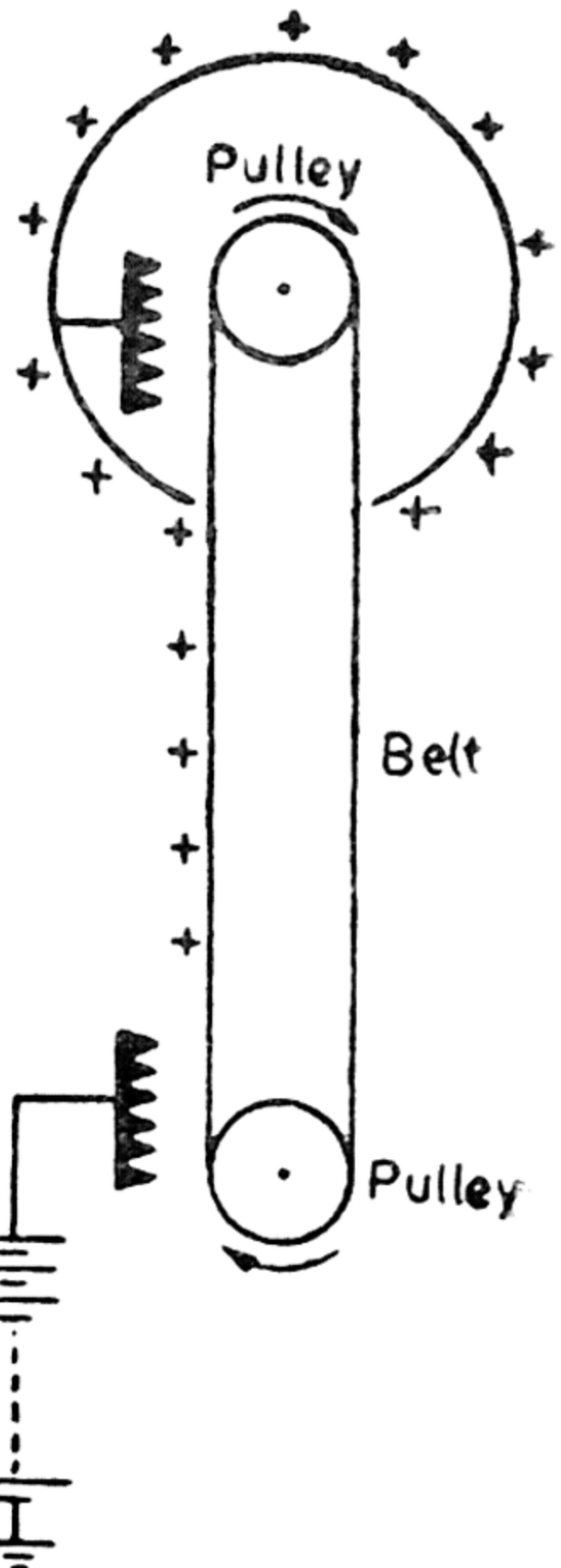


Fig. 3.16.

In the latter forms of the generator another pointed conductor, also connected to the sphere, is placed at the top on the down side of the belt. A discharge of negative electricity thus takes place from this pointed conductor, so that electrons are collected and carried downward by the belt. This negative charge ultimately passes to the dc source. Although it does not alter the potential, it increases the current which the apparatus can supply.

The potential V of the sphere at any moment is equal to Q/C, where Q is the charge on the sphere at that instant and C its capacity. The charge transfer process is repeated and the potential of the hollow sphere rises until the rate of loss of charge by leakage equals the rate at which charge is introduced. The leakage can be minimized and the maximum potential is limited by electrical discharge of the surrounding atmosphere. The electric field at the surface of the hollow hemisphere of radius R is V/R, where V is the potential of the hemisphere. Experimentally it is observed that the sparking field or the *dielectric strength* in dry air at atmospheric pressure is about 3 mega volts/m. Thus theoretically the maximum voltage attainable is 3 mega volts for a sphere of radius one meter. In practice, the steady operating potentials of the high voltage terminal in machines operating in atmospheric air is about one-third of the theoretical values for isolated spheres.

The maximum electric field that can exist outside of the sphere without breakdown goes to larger values if the pressure of the gas is increased, or even better, if the space is completely evacuated.

Satisfactory vacuum is not easy. If highly pressed air alone is used a definite fire hazard will exist by virtue of the compressed oxygen. Modern machines often use high pressure nitrogen. The addition of one or two per cent of an electronegative gases such as freon (CCl_2F_2) or sulphur hexafluoride increases the electrical breakdown strength by a factor of perhaps two.

Exercises

Example 1. *Find potential difference between the centre and one corner C of a square, if on the other three corners A, B, and D, charges -5, $+2$ and $+3\mu C$ respectively be placed. Given that the length of each side of the square is $3\sqrt{2}$ meter.*

Potential at the centre of square

$$V_0=\frac{1}{4\pi\varepsilon_0}\sum\frac{q_i}{r_i}=\frac{1}{4\pi\varepsilon_0}\left[\frac{-5\times10^{-6}}{3}+\frac{2\times10^{-6}}{3}+\frac{3\times10^{-6}}{3}\right]$$

$=0.$

Thus the central point O is at the same electrical level as infinity and no work will be needed to bring a test charge from infinity up to this point.

Similarly potential at the corner C of the square

$$V_c=\frac{1}{4\pi\varepsilon_0}\left[\frac{-5\times10^{-6}}{6}+\frac{2\times10^{-6}}{3\sqrt{2}}+\frac{3\times10^{-6}}{3\sqrt{2}}\right]$$

$$=9\times10^9\times(5\sqrt{2}-5)\times10^{-6}/6=3106.6 \text{ volts.}$$

$\therefore\ V_c-V_0=3106.6$ volts.

Example 2. *Determine the law of variation of potential with distance in an electric field produced between two parallel infinite planes, bearing equal and opposite charges.*

In chapter 2, it has been established that the electric field **E** is the same at any point between such plates. The potential difference between plate A and an arbitrary point at distance x from the plate is given by

$$V_{A\infty}=Ex=\phi_A-\phi_x.$$

If the potential of the negatively charged plate B is assumed zero and the separation of the plates is d, then we have

$$\phi_A=V_{AB}=Ed.$$

$\therefore$ Potential at point x is

$$\phi_x=V_{AB}-V_{AB}\,(x/d)=V_{AB}\,(1-x/d).$$

Thus we see that the potential between two infinite planes bearing charges of opposite sign and equal charge density varies with distance according to linear law and vanishes on moving from positive to negative plate.

Example 3. *Find out the points where* (*i*) $V=0$ *and* (*ii*) $E=0$ *in the following figure* :

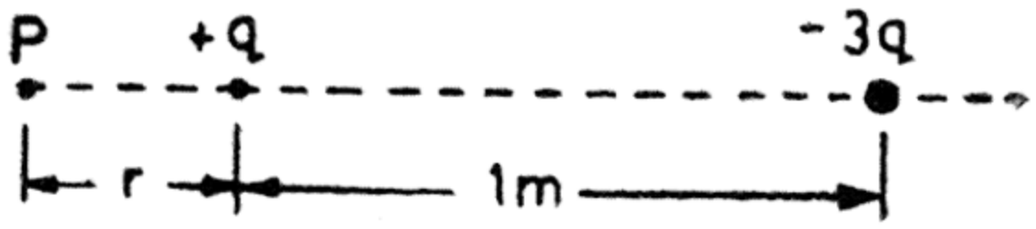


Fig. 3.17.

(*i*) Let P be the point on the axis, either on the left or the right of charge $+q$, at a distance r. Therefore the potential due to both the charges at this point will be

$$V_P=\frac{q}{4\pi\varepsilon_0 r}+\frac{-3q}{4\pi\varepsilon_0(1\pm r)}=0.$$

$$\therefore \quad r=-0.5\text{m or } 0.25\text{m}.$$

Thus the potential will be zero at point P on the axis which is either 0.5m on the left or 0.25m on the right of charge $+q$.

(*ii*) The charges are oppositely charged. The electric fields will be in the same direction at every point in between these charges. The total field can be zero only on the left of charge $+q$. Hence

$$E_P=\frac{q}{4\pi\varepsilon_0 r^2}+\frac{(-3q)}{4\pi\varepsilon_0(1+r)^2}=0 \text{ or } \frac{1}{r^2}-\frac{3}{(1+r)^2}=0$$

$$\therefore \quad r=(1\pm\sqrt{3})/2.$$

Since the negative sign is not permissible, the only point on the axis at which total electric field E will be zero is $(1+\sqrt{3})/2$ metre on the left of charge $+q$.

Example 4. *If the electrostatic potential in a region is represented as* $\phi=2x+3y-z$, *obtain expressions for the potential gradient and electric field strength.*

In the present problem the electrostatic potential

$$\phi=2x+3y-z.$$

Therefore

$$\text{grad } \phi=\nabla\phi=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(2x+3y-z)$$

$$=\mathbf{i}\frac{\partial}{\partial x}(2x+3y-z)+\mathbf{j}\frac{\partial}{\partial y}(2x+3y-z)+\mathbf{k}\frac{\partial}{\partial z}(2x+3y-z)$$

$$=2\mathbf{i}+3\mathbf{j}-\mathbf{k}.$$

The electric field strength at any point in space is defined as the negative gradient of potential at that point. Thus

$$\mathbf{E}=-\nabla\phi=-[2\mathbf{i}+3\mathbf{j}-\mathbf{k}]$$
$$=-2\mathbf{i}-3\mathbf{j}+\mathbf{k}.$$

Example 5. *Two drops of water, each with a charge of* 3×10^{-9} *coulomb and having a surface potential of 500 volts form a single drop. Calculate the surface potential of the new drop so formed.*

Let r be the radius of the each drop, the surface potential of the drop is given by the relation

$$V=\frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{or} \quad r=\frac{1}{4\pi\varepsilon_0}\frac{q}{V}.$$

Substituting the numerical values

$$r=9\times10^9\times3\times10^{-9}\times(1/500)=0.054\text{m}.$$

The volume of the each drop is $\frac{4}{3}\pi(0.054)^3\text{m}^3$. The volume of the combined drop will thus be $\frac{8}{3}\pi(0.054)^3$. The radius of this drop R is given as

$$\tfrac{4}{3}\pi R^3=\tfrac{8}{3}\pi(0.054)^3 \quad \text{or} \quad R=2^{1/3}\times0.054=0.068\text{m}$$

and the total charge $q'=2q=6\times10^{-9}$ coul. Therefore the potential of this new drop is

$$V=\frac{1}{4\pi\varepsilon_0}\frac{q'}{R}=9\times10^9\times\frac{6\times10^{-9}}{0.068}=794 \text{ volts}.$$

Example 6. *In the Bohr model of the hydrogen atom, the electron rotates in a circle of radius 0.53 A° about the nucleus. Calculate the speed of the electron in the orbit and the amount of energy needed to tear the electron loose from the nucleus of the atom.*

Electron rotates in an orbit due to centripetal force provided by the Coulomb attraction between it and the nucleus. In hydrogen nucleus we have one proton having charge equal to the electronic charge. Therefore

$$\frac{mv^2}{r}=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}.$$

After substituting values (m of electron $=9.1\times10^{-31}$ kg., e on electron $=1.6\times10^{-19}C$, r of the orbit $=0.53\times10^{-10}$ metre and $1/4\pi\varepsilon_0=9\times10^9 Nm^2/C^2$), we get $v=2.18\times10^6$ m/sec.

To find the energy needed to tear the electron loose, we must know the work required to carry the electron from its orbit out to infinity, which will be the product of electronic charge and the potential at the surface of the orbit.

$$\therefore \qquad W=eV=e\frac{1}{4\pi\varepsilon_0}\frac{e}{r}=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}.$$

As charge on the electron is −ve, hence is attracted by the +vely charged nucleus, thus a negative work will be done in bringing this electron from infinity to surface. Therefore the work done in tearing the electron loose (*i.e.*, out to infinity) from the nucleus will be positive. Substituting numerical values, we get

$$W = 4.35 \times 10^{-18} \text{ joule.}$$

Actually, all this amount of work is not required, as electron has its kinetic energy

$$E = \tfrac{1}{2}mv^2 = 2.17 \times 10^{-18} \text{ joule.}$$

$\therefore$ External energy needed to tear the electron loose

$$= W - E = 2.17 \times 10^{-18} \text{ joule.}$$

This is known as the *ionization energy* of the hydrogen atom.

Example 7. *A proton, which is at a distance 10^{-14} m from the centre of a mercury nucleus, is released from nucleus. Calculate the K.E. and its speed when it gets far away from the nucleus.*

Mercury nucleus has a charge $80 \times 1.6 \times 10^{-19}$ C (as its atomic number is 80), proton will be repelled and caused to move radially toward infinity by this positive charge. Potential of the proton at its initial position is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{(9 \times 10^9)\,(80 \times 1.6 \times 10^{-19})}{10^{-14}} = 11.5 \times 10^6 \text{ volts.}$$

This is the work done when we bring this proton from infinity to this point against the electric field due to the +vely charged nucleus. When this proton moves away, this work is released as K.E. and is acquired by the proton.

$\therefore$ $K.E. = 11.5 \times 10^6$ electron volts $= 11.5$ *MeV*.

As mass of proton $m = 1.67 \times 10^{-27}$ kg and 1 *MeV* $= 1.6 \times 10^{-13}$ joule, hence the proton velocity

$$v = \sqrt{(2K.E./m)} = 4.69 \times 10^7 m/\text{sec.}$$

As this velocity is very large, we must use *relativistic relation*

$$K.E. = m_0c^2\left[\frac{1}{\sqrt{(1 - v^2/c^2)}} - 1\right].$$

Substituting numerical values, the relativistic velocity v may be calculated.

Example 8. ***Calculate the potential due to a uniformly charged ring at a point on its axis and hence deduce an expression for electric intensity.***

Consider the circular ring with charge λ per unit length. To find the potential at a point P (Fig. 2.7) on the axis at a distance x, divide the ring into small elements each of length Δl. Each point of this element is at the same distance r from point P, hence potential due to the charge on this element

$$\Delta V = \frac{1}{4\pi\varepsilon_o} \quad \frac{\lambda \Delta l}{r} .$$

Therefore the potential due to the whole ring

$$V = \frac{1}{4\pi\varepsilon_0} \sum \frac{\lambda \Delta l}{r} = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda dl}{r} = \frac{1}{4\pi\varepsilon_0} \quad \frac{\lambda}{r} \int dl.$$

As charge density λ is same and each point on the ring is at the same distance from point P hence we have taken λ and r outside the integral. The integral is extended around the ring and the sum of the elements of length Δl around the ring is $2\pi a$.

$$\therefore \qquad V = \frac{\lambda}{2\varepsilon_0} \quad \frac{a}{r} = \frac{\lambda}{2\varepsilon_0} \quad \frac{a}{\sqrt{(a^2+x^2)}} .$$

This relation is true for all values of x.

The electric field along the x-axis

$$E_x = -\frac{\partial V}{\partial x} = \frac{\lambda a x}{2\varepsilon_0 (a^2+x^2)^{3/2}} .$$

Example 9. *Calculate the potential due to a thin charged rod of length L at the point along and ⊥ to its length.*

Suppose the thin rod has a uniform charge of charge density λ per unit length. To find potential at point P, (Fig. 3.18), let us divide the rod into a series of small elements, each may be assumed as point charge. The potential at this point P due to one element of length Δx is given by

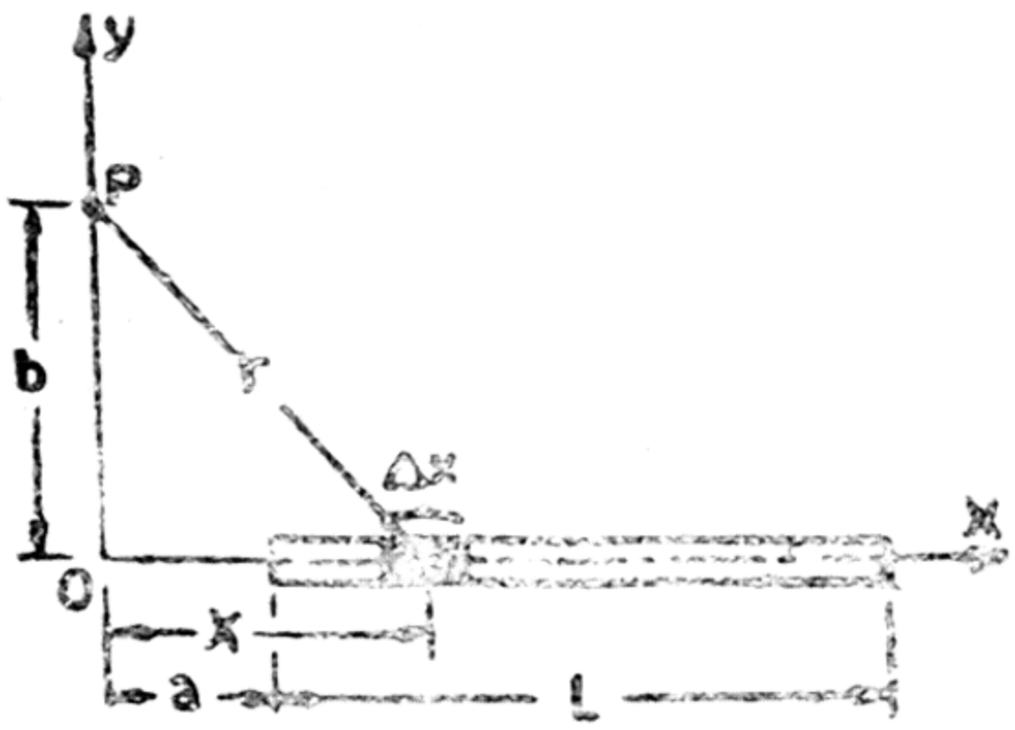


Fig. 3.18.

$$\Delta V = \frac{1}{4\pi\varepsilon_0} \quad \frac{\lambda \Delta x.}{r}$$

$\therefore$ Total potential due to whole rod $V = \Sigma \Delta V$

$$= \Sigma \lambda \Delta x / 4\pi\varepsilon_0 r.$$

Assuming $\Delta x \to 0$, summation can be replaced by an integral.

$$\therefore \qquad V = \int_a^{L+a} \frac{\lambda dx}{4\pi\varepsilon_0 (b^2+x^2)^{1/2}} = \frac{\lambda}{4\pi\varepsilon_0} \int_a^{L+a} \frac{dx}{(b^2+x^2)^{1/2}}$$

or $$V=\frac{\lambda}{4\pi\varepsilon_0}\left[\log_e\{x+\sqrt{(b^2+x^2)}\}\right]_a^{a+L}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{L+a+\sqrt{\{b^2+(L+a)^2\}}}{a+\sqrt{(b^2+a^2)}}\right].$$

Let us restrict this result to following limiting cases

(*i*) $b=0$ *and* $L\ll a$ *or at a far distance on its axis*

$$V=\frac{\lambda}{4\pi\varepsilon_0}\log_e\frac{2(L+a)}{2a}=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(1+\frac{L}{a}\right).$$

As $\log_e(1+z)=z-z^2/2+z^3/3-\ldots=z$ if z is less then 1, hence

$$V=\frac{\lambda}{4\pi\varepsilon_0}\frac{L}{a}=\frac{1}{4\pi\varepsilon_0}\frac{q}{a},$$

where $\lambda L=q$, the total charge on the rod.

This relation is same as due to a point charge. Thus at a large distance charged rod behaves as a point charge.

(*ii*) $a=0$, $L\ll b$ *or at the points on the line* $\perp$ *to axis*

$$V=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{L}{b}+\left(1+\frac{L^2}{b^2}\right)^{1/2}\right]=\frac{\lambda}{4\pi\varepsilon_0}\log_e\left(1+\frac{L}{b}\right).$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\frac{L}{b}=\frac{q}{4\pi\varepsilon_0 b},$$

which is again as due to a point charge.

Example 10. *A dielectric cylinder of radius 'a' is of infinite length and is non-uniformly charged, such that the charge volume density* ρ *varies inside directly as the distance from its axis, with* $\rho=0$ *at the axis and* ρ_0 *at the surface. Calculate the p.d. between the axis and the surface.*

To solve this problem, let us divide the charged cylinder into large number of co-axial cylindrical shells. Consider a cylindrical shell of radii x and $x+\delta x$, the electric intensity E must be calculated first as given in example 7 of Chapter 2.

We know that $\rho=cx$, $\rho_0=ca$, E inside the cylinder $=\rho_0 r^2/3a\varepsilon_0$ and E outside the cylinder $=\rho_0\ a^2/3r\varepsilon_0$. We also know that the potential difference between two points, where **E** is the electric field intensity, is given by

$$V_A-V_B=-\int_B^A \mathbf{E}.d\mathbf{r}.$$

Hence the p.d. between the axis and the surface of the cylinder

$$V_{axis}-V_{surface}=-\int_a^0\frac{\rho_0 r^2}{3a\epsilon_0}dr=\frac{\rho_0}{3a\varepsilon_0}\int_0^a r^2\,dr$$

$$=\rho_0\ a^2/9\varepsilon_0.$$

Example 11. *Calculate the electrostatic potential at the centre, inside, outside and on the surface of a charged sphere having charge distribution as*

$$\rho=\rho_0\,(1-r/a) \text{ when } r\leqslant a \text{ and } \rho=0 \text{ when } r>a.$$

To find potential due to a spherical distribution, the sphere is divided in large number of spherical shells. Let us imagine a spherical shell of radii r and $r+\delta r$. The potential at the centre due to the charge δq over this shell is given by

$$\triangle Vc=\delta q/4\pi\varepsilon_0\, r=4\pi r^2\,\delta r\rho/4\pi\varepsilon_0 r=\rho r\,\delta r/\varepsilon_0.$$

$$\therefore \quad Vc \text{ due to entire spherical charge}=\int_0^a \frac{r}{\varepsilon_0}\,\rho_0\left(1-\frac{r}{a}\right)dr$$

$$=\frac{\rho_0}{\varepsilon_0}\left[\frac{r^2}{2}-\frac{r^3}{3a}\right]_0^a=\frac{\rho_0\,a^2}{6\varepsilon_0}.$$

Potential V_0 outside this spherical charge distribution$=q/4\pi\varepsilon_0\, r_0$.

$$\therefore \quad V_0=\frac{1}{4\pi\varepsilon_0 r_0}\int dq=\frac{1}{4\pi\varepsilon_0 r_0}\int_0^a 4\pi r^2\rho_0\left(1-\frac{r}{a}\right)dr$$

$$=\frac{\rho_0}{\varepsilon_0 r_0}\left[\frac{r^3}{3}-\frac{r^4}{4a}\right]_0^a=\frac{\rho_0 a^3}{12\varepsilon_0 r_0}.$$

Potential V_i inside the charge distribution is equal to V due to the portion of sphere between radii 0 and r_i and due to the portion of sphere between radii r_i and a. Hence

$$V_i=\frac{1}{4\pi\varepsilon_0 r_i}\int dq+\int dV$$

$$=\frac{1}{4\pi\varepsilon_0 r_i}\int_0^{r_i} 4\pi r^2\rho_0\left(1-\frac{r}{a}\right)dr+\int_{r_i}^a \frac{4\pi r^2}{4\pi\varepsilon_0 r}\rho_0\left(1-\frac{r}{a}\right)dr$$

$$=\frac{\rho_0}{\varepsilon_0 r_i}\left[\frac{r^3}{3}-\frac{r^4}{4a}\right]_0^{r_i}+\frac{\rho_0}{\varepsilon_0}\left[\frac{r^2}{2}-\frac{r^3}{3a}\right]_{r_i}^a$$

$$=\frac{\rho_0}{6\varepsilon_0}\left(a^2-r_i^2+\frac{r_i^3}{2a}\right).$$

Example 12. *An electric quadrupole is formed by a charge $-2e$ at the origin and charges $+e$ at the points $(\pm a, 0)$. Show that the potential at a large distance r $(r\gg a)$ is approximately given by the expression $V=e\ a^2\,(3\cos^2\theta-1)/4\pi\varepsilon_0\, r^3$, where θ is the angle between the direction of r and the axis of x.*

Applying relation for potential due to a point charge to Fig. 3.19, assuming charge $+e$ at A, $-2e$ at O and $+e$ at B, we get total potential

$$V=\Sigma V_i=\frac{1}{4\pi\varepsilon_0}\left[\frac{e}{PB}-\frac{2e}{PO}+\frac{e}{PA}\right]$$

As $OP=r$ and $AO=OB=a$, and $\angle POA=\theta$, hence from the geometry of the figure

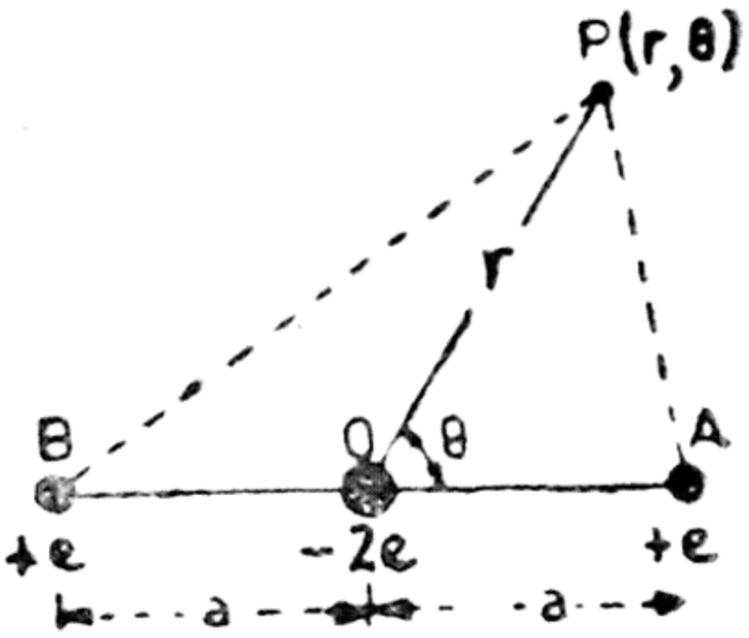


Fig. 3.19.

$$PA=(r^2+a^2-2ra\cos\theta)^{1/2} \text{ and}$$

$$PB=(r^2+a^2+2ra\cos\theta)^{1/2}.$$

Thus we have

$$V=\frac{e}{4\pi\varepsilon_0}\left[\frac{1}{(r^2+a^2+2ra\cos\theta)^{1/2}}-\frac{2}{r}+\frac{1}{(r^2+a^2-2ra\cos\theta)^{1/2}}\right]$$

$$=\frac{e}{4\pi\varepsilon_0 r}\left[\left(1+\frac{2a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{-1/2}-2+\left(1-\frac{2a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{-1/2}\right]$$

Expanding by binomial theorem, we get

$$V=\frac{e}{4\pi\varepsilon_0 r}\left[\left(1-\frac{a}{r}\cos\theta+\frac{a^2(3\cos^2\theta-1)}{2r^2}+\ldots\right)-2+\left(1+\frac{a}{r}\cos\theta+\frac{a^2(3\cos^2\theta-1)}{2r^2}+\ldots\right)\right]$$

$$=\frac{e}{4\pi\varepsilon_0 r}\cdot\frac{a^2}{r^2}(3\cos^2\theta-1)=\frac{ea^2(3\cos^2\theta-1)}{4\pi\varepsilon_0 r^3}.$$

Thus we see that the *potential due to a quadrupole varies inversely as the cube of the distance, provided* $r \gg a$.

Example 13. *The NaCl crystal consists of* Na^+ *and* Cl^- *ions. These are arranged at the corners of cubes stacked together in such a way that the ions in nearest neighbours to any ion have opposite charges. If the side of the cube is R and the electronic charge is—e, calculate the potential energy or the electrostatic energy (i) per ion, (ii) per gm of NaCl crystal.*

If the crystal is not of very small size, the number of ions on its surface will be negligible compared to the total number of ions inside the crystal. *NaCl* crystal is made up of large number of cubes. One cube is shown in Fig. 3.20. The potential energy of the ions inside the crystal will be nearly same as due to the whole

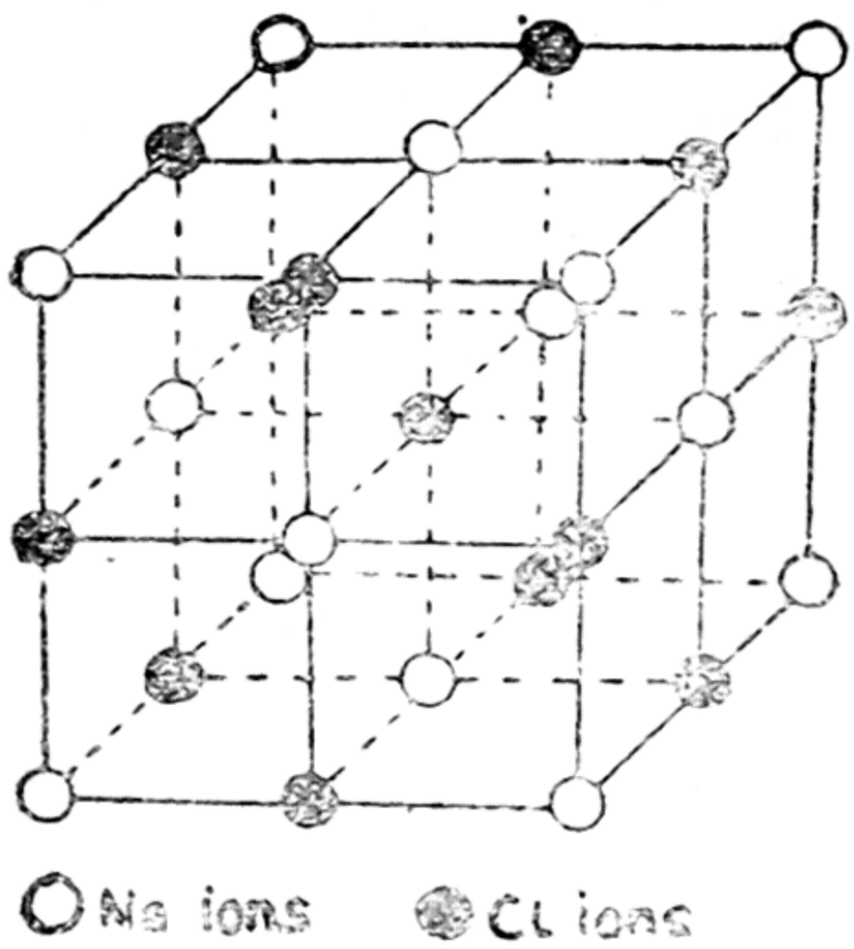


Fig 3.20.

ions. Let us now consider an ion A, say Na^+, in the interior of the crystal. When we bring it from infinity to its position, work is done. To find the electrostatic energy associated with this ion, we should consider all other ions inside the crystal. It has six nearest Cl^-, each at a distance R from it, the electrostatic energy of ion A due to these Cl^- is $-6e^2/4\pi\varepsilon_0 R$ (there are six pairs, each consists two equal and opposite charges e and $-e$ separated by a distance R). The next nearest neighbours are the twelve Na^+, each are at the diagonally opposite corners and at a distance $R\sqrt{2}$ from ion A. The energy contributed by these 12 ions is $+12\,(e^2/4\sqrt{2}\pi\varepsilon_0 R)$, the next nearest neighbours are eight Cl^-, each at a distance $R\sqrt{3}$ from ion A, thus contributing $-8(e^2/4\sqrt{3}\pi\varepsilon_0 R)$ energy. The next are six Na^+, each at a distance $R\sqrt{4}=2R$ and contributing $6(e^2/8\pi\varepsilon_0 R)$energy and so on. Proceeding in this way and adding all the terms, we get the total potential energy of ion A

$$U=\frac{1}{4\pi\varepsilon_0}\left[-\frac{6e^2}{R}+\frac{12e^2}{R\sqrt{2}}-\frac{8e^2}{R\sqrt{3}}+\frac{6e^2}{2R}-\cdots\right].$$

In this infinite series the latter positive and negative terms cancel each other. Actual addition of large number of terms gives

$$U=-1.7476\,e^2/4\pi\varepsilon_0 R.$$

It is the energy per ion of an ionic crystal. In general the factor 1.7476 is replaced by α, known as *Madelung Constant*, which depends upon the crystal structure.

Total number of ions in 1 gm of $NaCl$

$$=\frac{2\times\text{Avogadro's number}}{\text{Molecular weight of } NaCl}$$

$$=2\times 6.023\times 10^{23}/58.5.$$

$\therefore$ The electrostatic potential energy per gm of $NaCl$ crystal

$$=\frac{1}{2}\times\frac{2\times 6.023\times 10^{23}}{58.5}\times\left(-1.7476\,\frac{e^2}{4\pi\varepsilon_0 R}\right)$$

$$=-1.80\times 10^{22}\,e^2/4\pi\varepsilon_0 R.$$

Here factor $\frac{1}{2}$ is introduced as each ion-pair is counted twice while summing up the energy per ion for a large number of ions. The—ve sign in the result shows that this much amount of energy

is required to remove one ion from the crystal. The result thus obtained is not exactly correct, however it can be obtained by quantum mechanics.

Example 14. *If $2q$ and $-q$ charges are separated by a small distance a, equidistant from the origin along x-axis. Calculate the electric field and potential at point P with polar coordinates r, θ.*

The electric potential due to the system of charges at a large distance r is given by

$$V=\frac{1}{4\pi\varepsilon_0}\left[\frac{A_0}{r}+\frac{A_1}{r^2}+\frac{A_2}{r^3}+\cdots\right].$$

In our particular problem, the electric potential at point P will be entirely due to the first term (monopole strength) as other contributions may be neglected for $r \gg a$.

Hence

$$A_0=\iiint \rho\, dV=2q-q=q.$$

$\therefore$ $V=q/4\pi\varepsilon_0 r$ and the electric field $E=-\partial V/\partial r=q/4\pi\varepsilon_0 r^2$.

The direction of this field E will be along r, *i.e.*, radially away from the charges.

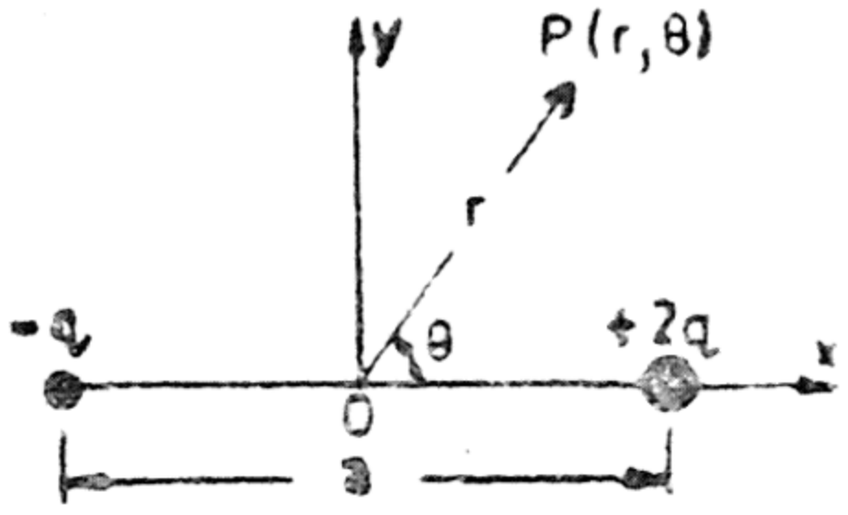


Fig. 3.21.

Example 15. *Calculate the resultant dipole moment of a system of charges, shown in Fig. 3.22. Hence determine the electric potential and intensity at a point quite far away from the charges.*

The monopole moment A_0 for this system is zero as net charge is zero. To find dipole moment, it is easy to find its x-and y-components due to individual charges.

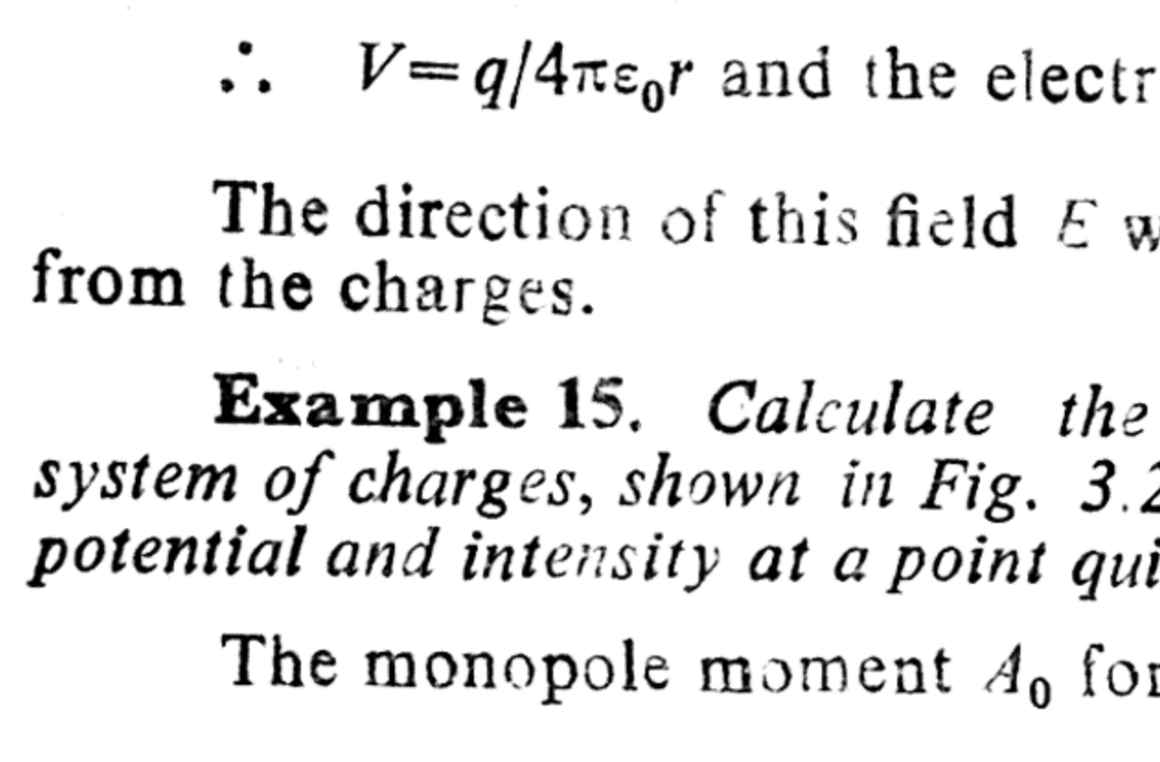


Fig. 3.22.

As each side is of length a, hence by the property of hexagonal each corner of the hexagon is at the same distance a from its centre. Hence (x, y) co-ordinates of these corners are respectively $(-a, 0)$, $(-a/2, a\sqrt{3}/2)$, $(a/2, a\sqrt{3}/2)$, $(a, 0)$, $(a/2, -a\sqrt{3}/2)$ and $(-a/2, -a\sqrt{3}/2)$.

$\therefore$ Component of dipole moment along x-axis for all charges

$$p_x=\iiint \rho x\, dV$$

$$p_x = 2q(-a) - 3q(-a/2) + 3q(a/2) - 2q(a) + 3q(a/2) - 3q(-a/2)$$
$$= 2qa.$$

Similarly its y-component $p_y = \iiint \rho y \, dy$ for all charges

$$= 2q(0) - 3q(a\sqrt{3}/2) + 3q(a\sqrt{3}/2) - 2q(0)$$
$$+ 3q(-a\sqrt{3}/2) - 3q(-a\sqrt{3}/2) = 0$$

$\therefore$ Total dipole moment $\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} = 2qa\mathbf{i}$.

Hence its component along $OP = 2qa \cos\theta$.

$\therefore$ $$V = A_1/4\pi\varepsilon_0 r^2 = 2qa \cos\theta/4\pi\varepsilon_0 r^2.$$

Thus the electric field components are

$$E_r = -\partial V/\partial r = 4qa \cos\theta/4\pi\varepsilon_0 r^3$$

and $$E_\theta = -(1/r)\partial V/\partial\theta = 2qa \sin\theta/4\pi\varepsilon_0 r^3.$$

$\therefore$ $$E = (2qa/4\pi\varepsilon_0 r^3)(1 + 3\cos^2\theta)^{1/2}$$

and the direction of this resultant with OP is given by

$$\phi = \tan^{-1}(E_\theta/E_r) = \tan^{-1}(\tfrac{1}{2}\tan\theta).$$

Example 16. *If four charges each of value q are placed at points $(\pm a, \theta)$ and $(0, \pm a)$ and a fifth charge $-4q$ is at the origin. Calculate the potential and electric intensity at a point far away from the system of charges.*

Let us now consider a point P at a distance r from the origin. It is clear that for all charges the net value of $\iiint \rho dv$ is zero, hence monopole moment is zero. Similarly for all charges the net value of $\iiint \rho r \cos\theta \, dv$ is zero, hence dipole moment is also zero. The potential at point P is thus due to quadrupole moment (A_2) of the charge distribution, where

$$A_2 = \iiint \tfrac{1}{2}\rho (3\cos^2\theta - 1) r^2 \, dv \text{ for all charges.}$$

As point P is very far from origin, the angle θ can be assumed for charges on the axis of x and $90 - \theta$ for the charges on the y-axis.

Contribution to A_2 of charge $-4q$ at $(0, 0) = 0$
" " " $+q$ at $(+a, 0) = \frac{1}{2}qa^2(3\cos^2\theta - 1)$
" " " $+q$ at $(-a, 0) = \frac{1}{2}qa^2(3\cos^2\theta - 1)$
" " " $+q$ at $(0, +a) = \frac{1}{2}qa^2(3\sin^2\theta - 1)$
" " " $+q$ at $(0, -a) = \frac{1}{2}qa^2(3\sin^2\theta - 1)$

$\therefore$ $A_2 = qa^2[3\cos^2\theta - 1 + 3\sin^2\theta - 1] = qa^2$.

Hence the potential at $P = V = A_2/4\pi\varepsilon_0 r^3 = qa^2/4\pi\varepsilon_0 r^3$.

Potential is spherically symmetric as it does not depened on θ, hence the electric intensity at point P will be in radially outward direction.

$$\therefore \quad \mathbf{E}=-\frac{\partial V}{\partial r}\hat{r}=\frac{3qa^2}{4\pi\varepsilon_0 r^4}\hat{r}.$$

Example 17. *Two large parallel metal plates are separated by a distance of 0.50 cm. If a 100 volts battery is connected across them, calculate the field between the plates and the charge surface density on them.*

As **E** is constant between the parallel large plates, the work done in carrying a unit charge from one plate to the other is (force/unit charge) × distance = Ed. Thus by the definition of potential

Potential difference between the plates $V_B - V_A = Ed$.

$$\therefore \quad 100 \text{ volt} = E \times 0.5 \times 10^{-2} \text{ metre},$$

or

$$E = 2 \times 10^4 \ V/\text{m (or } N/C).$$

To find the charge surface density on the positive plate, let us consider a Gaussian box with one end in this surface and other opposite end in between these plates. As no lines of flux emerge from the top and sides of the box and the number of lines of flux passing through the lower face of the box is EA. Thus by Gauss's law

$$\int \mathbf{E}.\, d\mathbf{A} = q/\varepsilon_0 \text{ or } EA = \sigma A/\varepsilon_0.$$

$$\therefore \quad \sigma = \varepsilon_0 E = (8.85 \times 10^{-12} C^2/Nm^2) \times (2 \times 10^4 N/C)$$

$$= 1.77 \times 10^{-7} \ C/m^2.$$

Example 18. *A conducting sphere of radius R carries a charge q. (a) Calculate the total energy stored in the surrounding space. (b) What is the radius of the spherical surface if half the stored energy lies within it ?*

At any point P at a distance r from the centre of the sphere ($r > R$), the electric intensity E is given by

$$E = q/4\pi\varepsilon_0 r^2.$$

$\therefore$ Energy density at this point $= \frac{1}{2}\varepsilon_0 E^2 = q^2/32\pi^2\varepsilon_0 r^4$.

The energy which lies in a spherical shell between the radii r and $r+dr$ is

$$dU = (4\pi r^2\, dr)\,.\,(q^2/32\pi^2\varepsilon_0 r^4).$$

$$\therefore \text{ Total energy stored } U = \int dU = \frac{q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0 R}.$$

In the second part of the question

$$\tfrac{1}{2}U = \frac{q^2}{8\pi\varepsilon_0}\int_R^{R_0}\frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{R}-\frac{1}{R_0}\right]$$

Using value of U, we get $R_0=2R$.

Example 19. *Eight negative charges each of value e, one placed at the corners of a cube of side x cm, at the centre to which a positive charge of $+2e$ is placed. Calculate the electrostatic energy of the system.*

The given assembly of charges consists of (*i*) twelve pairs each with $-e$ and $-e$ charges separated by a distance $(AB=BC=\ldots)=x$ and contributing an electrostatic energy

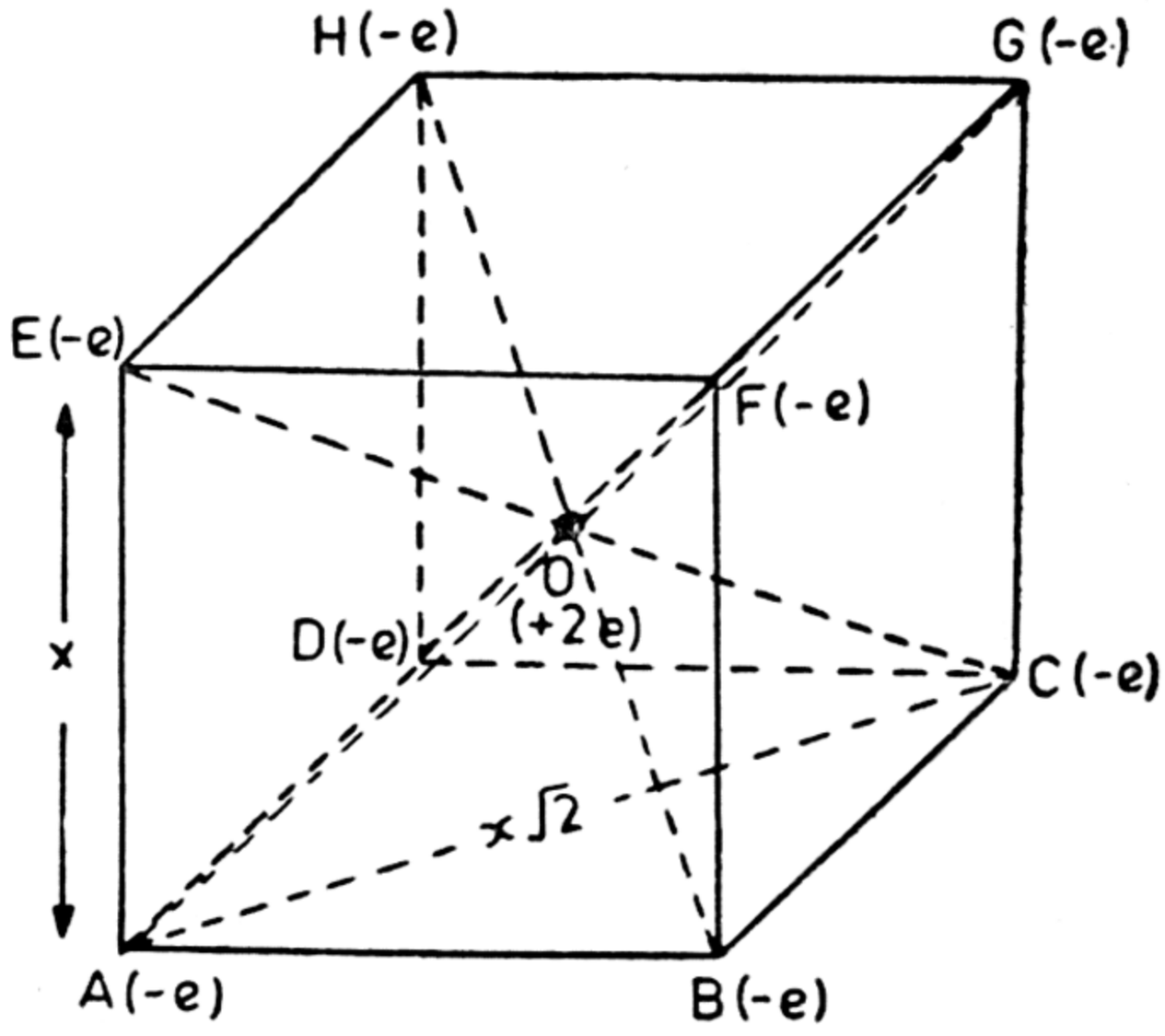


Fig. 3.23.

$$U_1 = 12\times\frac{(-e)(-e)}{4\pi\varepsilon_0 x} = \frac{12e^2}{4\pi\varepsilon_0 x},$$

(*ii*) twelve pairs each with charge $-e$ and $-e$ separated by a distance $(AF=BE=\ldots)=x\sqrt{2}$ and contribuing an electrostatic energy

$$U_2 = 12\times\frac{(-e)(-e)}{4\pi\varepsilon_0 x\sqrt{2}} = \frac{12e^2}{4\pi\varepsilon_0 x\sqrt{2}},$$

(*iii*) eight pairs each with charge $-e$ and $2e$ separated by a distance $(OA=OB=\ldots)=x\sqrt{3}/2$ and contributing an electrostatic energy

$$U_3 = 8\times\frac{(-e)\,2e}{4\pi\varepsilon_0 x\sqrt{3}/2} = -\frac{16e^2}{4\pi\varepsilon_0 x\sqrt{3}/2},$$

(*iv*) four pairs each with charge $-e$ and $-e$ separated by a distance $(AG=BH=CE=DF)=x\sqrt{3}$ and contributing an electrostatic energy

$$U_4 = 4\times\frac{(-e)(-e)}{4\pi\varepsilon_0 x\sqrt{3}} = \frac{4e^2}{4\pi\varepsilon_0 x\sqrt{3}}.$$

Therefore the total electrostatic potential energy of the given system or the work done in forming such a system of charges

$$U = \frac{12e^2}{4\pi\varepsilon_0 x} + \frac{12e^2}{4\pi\varepsilon_0 x\sqrt{2}} - \frac{16e^2}{4\pi\varepsilon_0 x\sqrt{3/2}} + \frac{4e^2}{4\pi\varepsilon_0 x\sqrt{3}}$$

$$= \frac{4e^2}{4\pi\varepsilon_0 x}\left[3 + \frac{3}{\sqrt{2}} - \frac{8}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right] = 3.88\times 10^{10}\,\frac{e^2}{x}\ \textit{joules.}$$

Oral Questions

1. Does it mean that no work is done in carrying a positive test charge from one point to the other on the equipotential surface? Does it mean that no force will have to be exerted to carry the test charge from one point to the other on it?
2. What can you say about E, (*i*) in the region where *V* is constant, (*ii*) at a point where *V* is zero?
3. What is the optical analog of equipotential surface?
4. What can be said about E, if it is defined as a gradient of scalar field?
5. Does the potential of a positively charged insulated conductor have to be positive?
6. A metal sphere suspended by a silk thread is placed in a uniform external electric field E. What is the magnitude of the electric field inside the sphere if (*i*) it is uncharged, (*ii*) charged?
7. Which of the following is a equipotential surface: (*a*) uniformly charged disc, (*b*) charged spherical conductor, (*c*) circle with a point charge at the centre, (*d*) spherical surface with a point charge at the centre?
8. How are MeV and the atomic mass unit (*mu or u*) related?
9. What is the energy associated with a doubly charged helium atom when passes through a potential difference of 1,000 volts?
10. Can we obtain infinite potential from Van de Graaff generator? If not why?

Problems

1. A potential difference of 100 *V* is applied between two parallel plates, Calculate the speed of a proton released from plate *B* just before it hits plate *A*. (1.4×10^5 *m/sec*)

2. A charge *q* is placed at each of four corners ($\pm a$, 0, 0), (0, $\pm a$, 0) of a square. Show that the potential at a point (*x*, *y*, *z*) near the origin is

$$V = \frac{q}{4\pi\varepsilon_0 a}\left[4 + \left(\frac{x^2+y^2-2z^2}{a^2}\right) + \cdots\right]$$

3. A particle of charge 3×10^{-9} C is situated in a uniform electric field directed to the left. In moving to the right a distance of 5 cm the work of an applied force is 6×10^{-5} joule and the change in K.E. of the particle is 4.5×10^{-5} Joule. Calculate the magnitude of the electric field intensity and the work of the electrical force. (10^5 N/C, -15×10^{-6} *joule*)

4. An α—particle is accelerated through a p.d of 10^6 volts in an electrostatic generator. Find the kinetic energy acquired by it. What K.E would a

proton acquire under these same circumstances ? Which particle would acquire the greater speed, if both were started from rest ?

(3.2×10^{-13} joule, 1.6×10^{-13} *joule, proton*)

5. The electric field in the atmosphere at the earth's surface is approximately 200 volts/metre, directed downward. At 1400 metre above the earth's surface the electric field in the atmosphere is only 20 volts/m, again directed downward. Calculate the average charge density in the earth's atmosphere below 1400 metre. (1.1×10^{-12} *coulomb*/m^2)

6. A soap bubble 10 cm in radius with a wall thickness of 3.3×10^{-6} cm and charged to a potential of 100 volts breaks and falls as a spherical drop. Show that the potential of the drop is 10 k*V*.

7. An α—particle ($m = 4 \times 1.67 \times 10^{-27}$ kg., $q = 2 \times 1.6 \times 10^{-19} C$) is emitted from a radon nucleus ($81 \times 1.6 \times 10^{-19} C$) with an energy 4.79 MeV. Assuming it is released at rest near the outer edge of the nucleus and then lies out into space, calculate the radius of the radium nucleus. (5.3×10^{-14} *metre*)

8. A rod of length L carries a charge $\lambda = kx^3$ *C*/m. Find the potential at the point $x = 0$, if the rod is along positive x—axis with left end at $x = b$. Also calculate the electric field.

$3k \times 10^9 [(L+b)^3 - b^3]$, $-9k \times 10^9 [(L+b)^2 - b^2]$

9. A certain very long cylinder of radius b has a charge distribution $\rho = kr^3$, where r is the distance from the cylinder axis. Find the difference in potential between the surface of the cylinder and its axis. ($kb^{5/25}\varepsilon_0$)

10. Find the potential at an arbitrary point inside a uniformly charged sphere and prove that the potential has an extreme value at the centre of the sphere. ($\rho_0/2\varepsilon_0$) ($R^2 - r^2/3$)

11. *A* spherical charge distribution has a volume density $\rho = A/r$, where r is the distance from the centre of the distribution, inside this distribution and is zero outside to it. Calculate the potential at an arbitrary point.

(A/ε_0) $(R - \frac{1}{2}r)$ for $r \ll R$ and $AR^2/2\varepsilon_0 r$ for $r \gg R$.

12. If $_{92}U^{238}$ nucleus splits in two equal fragments. Calculate the (*i*) repulsive force acting on each fragment and (*ii*) the mutual electric potential energy of the two fragments. Given that the radius of the initially spherical U^{238} nucleus is 8.0×10^{-15} metre and the fragments are equal in size and charge, spherical and just touching.

13. Show that the potential energy of two coplaner dipoles p_1 and p_2 at a distance r apart is $(p_1 p_2/4\pi\varepsilon_0 r^3)$ $(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2)$, where θ_1 and θ_2 are the angles made by p_1 and p_2 respectively with the line joining their centres.

14. Positive charge is distributed with uniform density ρ throughout the volume of a sphere. Show that the electric field at a distance r from the centre of the sphere is $E = \rho r/3\varepsilon_0$. Show that if a particle of mass m and charge $-q$ is placed in the sphere, it will execute simple harmonic vibration back and forth through the centre with a period $2\pi(2\varepsilon_0 m/\rho q)^{1/2}$.

15. Four equal charges each of value $+q$ are symmetrically arranged along the circumference of a circle of radius 'a' and a charge $-7q$ is placed at the centre. Calculate the electric field at a far distant point at a distance r from this system. ($3q/4\pi\varepsilon_0 r^2$)

16. Calculate the electric potential and hence the electric intensity at a point P at a distance r ($\gg a$) from the origin due to a system of charges, $-2q$ at the origin, $+q$ at point $(a, 0)$ and $+q$ at a point $(0, a)$.

$qa(\cos\theta+\sin\theta)/4\pi\varepsilon_0 r^2$, $qa\sqrt{(5+6\sin\theta\cos\theta)}/4\pi\varepsilon_0 r^3$

17. Six equal positive charges, q, are kept at six corners of a regular hexagon of sides 'a' If a charge $-6q$ is placed at the centre, calculate the potential and hence electric intensity at a large distance r.

$(3qa^2/8\pi\varepsilon_0 r^3,\ 9qa^2/8\pi\varepsilon_0 r^4)$

18. Two coaxial cylinders form the plates of a capacitor. The radius of the outer cylinder is fixed. Show that if the radius of the inner cylinder is $1/e$ times the radius of the outer cylinder, the field strength at the surface of the inner cylinder is least for a given potential difference between them.

19. The electric potential at any point in the x—y-plane is given by

$$\phi = 5x\,(x^2+y^2)^{1/2} + y\,(x^2+y^{2\,-1/2}.$$

Find the cartesian components of the electric field intensity at that point.

20. Two protons in a nucleus of ^{238}U are 6.0×10^{-15} meter apart. What is their mutual electric potential energy ? The charge on a proton is 1.6×10^{-19} coul,

(3.84×10^{-14} joule.)

21. Given point charges $+q$ at $(a/2, 0, 0)$ and $-q$ at $(-a/2, 0, 0)$ in empty space, show that for a point in xy plane at distance r from the origin the limiting form of electric field components when $r \gg a$ is as below

$$E_x = \frac{aq}{4\pi\varepsilon_0 r^2}\left(\frac{3x^2}{r^2}-1\right),\ E_y = \frac{aq}{4\pi\varepsilon_0 r^2}\cdot\ \frac{3xy}{r^2}$$

22. A sphere of radii 1 cm is charged to a potential of 3000 volts. Calculate the out ward pull per unit area.

4

Capacitors and Electrometers

4.1 CAPACITANCE

We know that the potential of a charged conducting sphere of radius R, assuming the sphere as completely isolated is given by

$$V'_{+} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}, \qquad \ldots(1)$$

where q is the charge on the sphere.

Let us now imagine a second sphere of radius R, carrying a negative charge $-q$ and situated at a large distance ($\geqslant R$) from the first sphere so that one can consider each sphere as electrically isolated. The potential of this sphere is given by

$$V'_{-} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{R}. \qquad \ldots(2)$$

$\therefore$ Potential difference between these spheres

$$V' = V'_{+} - V'_{-} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{R} \qquad \ldots(3)$$

or

$$q = (2\pi\varepsilon_0 R)V' = C'V'. \qquad \ldots(4)$$

This relation shows that the potential difference is proportional to the charge on each sphere. The constant of proportionality depends upon R and is known as *capacitance* of the system of two spheres. It is denoted by the symbol C'.

If these spheres are brought close together, the presence of each will now spoil the spherical symmetry of the lines of force starting from each and the above relations do not hold good. We know that the positive charge brought near to an isolated conductor raises the potential of the conductor and negative charge lowers the potential. Thus in the presence of the negatively charged sphere the potential of the positively charged sphere will decrease from V'_{+} to some lower value V_{+}. Similarly the potential of the negatively charged sphere will be raised from V_{-}' to a higher value V_{-}.

Thus the potential difference between two spheres is reduced considerably, although the charges on the spheres have not changed. The capacitance of the system of two spheres defined as $C=q/V$ is hence increased. In this way we see that the *capacitance of the system of two conductors can be increased by bringing the conductors close together.*

The capacitance of a single isolated spherical conductor, such as a sphere of radius R can be obtained by assuming the second spherical conductor carrying an equal and opposite charge and of very large (infinite) radius. As the potential of the infinitely distant sphere is zero, hence the capacitance of a single isolated conductor is given by

$$C=q/V=4\pi\varepsilon_0 R. \qquad \ldots(5)$$

As $C=q$ when $V=1$, hence the capacity of a conductor may be defined as the charge required to raise its potential by unity. The SI unit of capacitance is coulomb/volt, The term *farad* is used to represent it, after the name of Michael Faraday. The submultiples of the farad, microfarad (10^{-6} farad $=1\mu f$) and the micro-micro farad (10^{-12} farad $=1\mu\mu f=1\ pf$) are commonly used in practice as the farad is a large quantity of capacitance.

When a conductor B is brought near a charged conductor A, inductive displacement takes place, the near side of B exhibiting a negative charge and the far side a positive charge. This negative charge tends to lower the potential of A and the positive charge on B tends to raise it. These opposite charges nearly counteract each other, but the negative charge has little greater influence as it is nearer to the conductor A, thus lowers the potential of A. If the conductor B is now earthed, the positive charge on B disappears and negative charge does not go to earth as it is attracted by the +ve charge on the conductor A. Hence the potential of conductor A is much lowered due to the presence of —ve charge on the conductor B. Thus we see that *the potential of an insula ed charged conductor can be considerably decreased and hence its capacity can be considerably increased when on earthed conductor is brought near to it.* The arrangement in which one conductor is charged and other is earthed is named *as capacitor* or *condenser.* The capacitance of a capacitor depends on the geometry of each plate, their separation and the m dium in which these plates are placed. In the definition of capacitance charge q is not the net charge, which is zero but the magnitude of the charge on either plate.

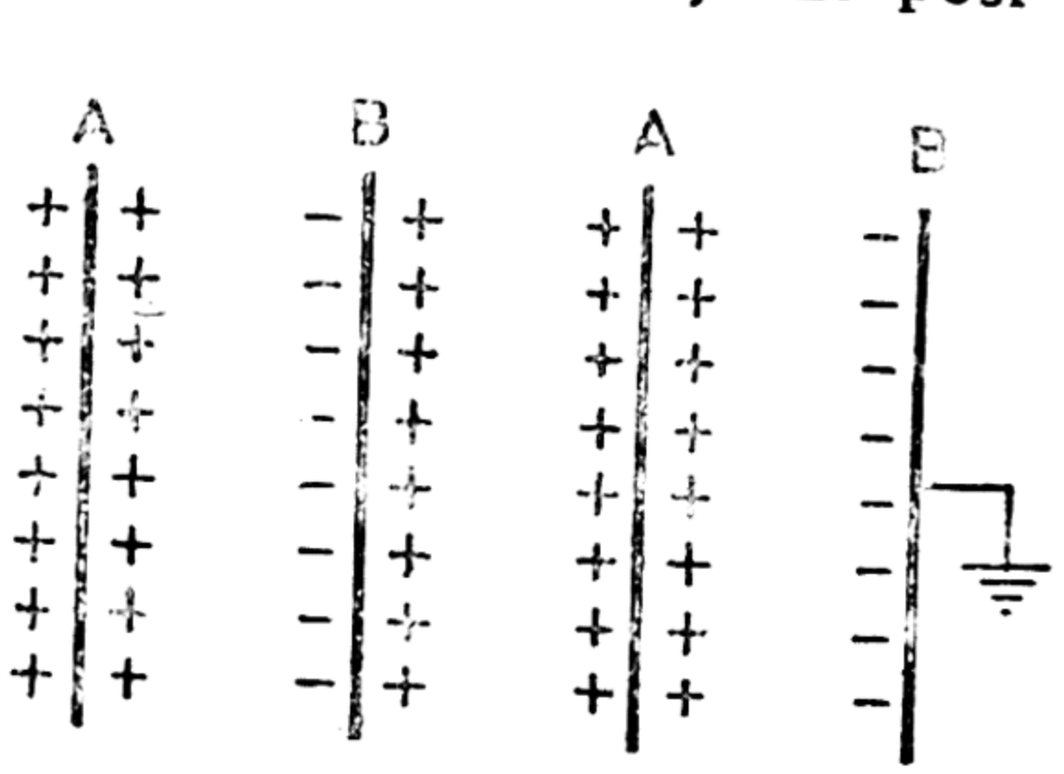


Fig. 4.1.

Charged capacitor is analogous to the rigid container of volume V containing n molecules of an ideal gas. We know that for a perfect gas

$$PV=nRT \text{ or } V=nRT/P=n/(P/RT). \qquad ...(6)$$

Its comparison with *Eq.* (5) shows that the capacity C of a capacitor, assuming a fixed temperature, is analogous to the volume V of the container. Any amount of charge can be put on the capacitor as one can put any mass (depending upon number of molecules) of gas in the container. The limits correspond to electrical break-down for the capacitor and to the break-down of the walls for the container. *Capacitors are very useful to Physicists and Engineers.* For example—

(*i*) Capacitor can be used to establish desired electric *field.* The behavior of a dielectric material can be studied by placing it in an electric field produced by a capacitor.

(*ii*) Capacitor can be used for storing *energy*, as it can confine strong electric fields to small volume. Energy accumulated in a large capacitor is used to accelerate the electrons by discharging the capacitor in a much shorter time in *electron synchrotrons* (a type of accelerator).

(*iii*) Capacitors are very useful to reduce voltage fluctuations in electronic power supplies, to transmit signals, to detect electromagnetic oscillations at radio frequencies and to other important electronic circuits.

4.2. CALCULATION OF CAPACITANCES

The method for the calculation of capacitance involves integration of the electric field between two conductors or the plates which are just equipotential surfaces to obtain the potential difference. Thus the capacitance of a capacitor

$$C=\frac{q}{V}=\frac{q}{-\int \mathbf{E}.d\mathbf{l}}. \qquad ...(7)$$

Let us now consider different cases :

(*a*) **Spherical capacitor.** Let us now consider an isolated spherical conductor of radius R. Capacitance of such a spherical conductor

$$C=\frac{q}{V}=\frac{q}{q/4\pi\varepsilon_0 R}=4\pi\varepsilon_0 R. \qquad ...(8)$$

(*b*) **Capacity of a spherical conductor enclosed by an earthed concentric spherical shell.** Let the radii of the inner sphere and the outer spherical shell be a and b respectively. If a charge $+q$ be given to the inner sphere, a charge $-q$ will be induced on the inner surface of the outer sphere. We know that the

intensity at any point inside a hollow charged conductor is zero, thus the intensity inside the region between two spheres will be that due to the charge on inner sphere only. Intensity at any point P, distant r from the centre of the inner sphere can be obtained by using Gauss's law. Considering a gaussian spherical surface through this point P, we have

$$\int \mathbf{E}.\, d\mathbf{S} = (1/\epsilon_0)\Sigma q_i$$

$$\therefore \qquad E.\, 4\pi r^2 = q/\epsilon_0 \text{ or } E = q/4\pi\epsilon_0 r^2.$$

This relation indicates that the field between two concentric spheres is independent of the charge on the outer sphere.

$\therefore$ Potential difference between the spheres

$$V = -\int_b^a \mathbf{E}\, d\mathbf{r} = \frac{-q}{4\pi\epsilon_0}\int_b^a \frac{dr}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right].$$

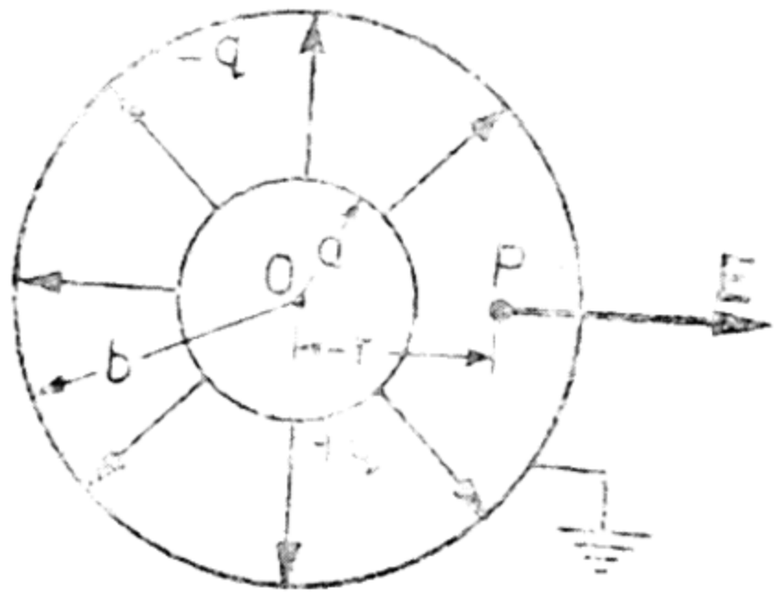


Fig. 4.2.

Hence the capacitance of this system

$$C = \frac{q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a} \quad \ldots(9)$$

Thus we see that if $b = \infty$, $C = 4\pi\epsilon_0 a$, which corresponds to that of an isolated sphere, *i.e., the charged sphere may be regarded as a condenser in which the outer surface has been removed to infinity.*

(*c*) **Capacity of an earthed sphere enclosed by a concentric spherical shell.** Let a and b be the radii of an earthed sphere and the hollow spherical shell respectively. If the shell is thick, let us assume that its outer radius is c. Suppose the total charge given to the outer shell B is q and the induced charge on the inner sphere A is $-q_1$. The charge $+q_1$ will be on the inner surface of the outer sphere and the remaining charge $q_2(=q-q_1)$ will remain on the outer surface of the outer sphere. In this way system works as the combination of two condensers: (*i*) that between sphere A and the inner surface of shell B and (*ii*) that between outer surface of shell B and the earth. Capacitance of the former is $4\pi\epsilon_0 ab/(b-a)$ while that of the latter is $4\pi\epsilon_0 c$.

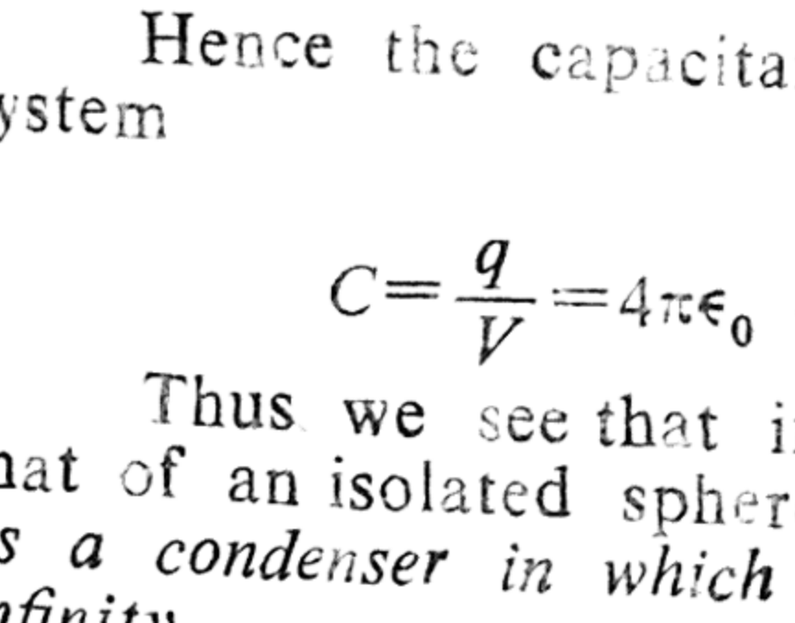


Fig. 4.3.

$\therefore$ Capacity of the system $C = 4\pi\epsilon_0\left[\frac{ab}{b-a} + c\right]$ ...(10)

If the shell is not thick, we can write $c=b$.

This relation can also be derived as follows : The potential is zero at infinity and also that of the inner sphere, the potential of the outer shell is thus given by

$$V=-\int_{\infty}^{c}\mathbf{E}.\,d\mathbf{r}=-\int_{a}^{b}\mathbf{E}.\,d\mathbf{r}$$

$$=-\int_{\infty}^{c}\frac{1}{4\pi\varepsilon_0}\frac{q_2}{r^2}\,dr=+\int_{a}^{b}\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}dr$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q_2}{c}=\frac{q_1}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right].$$

$\therefore$ $q_2=4\pi\varepsilon_0 c\,V$ and $q_1=4\pi\epsilon_0\,V\,ab/(b-a)$,

or $q=q_1+q_2=4\pi\varepsilon_0\,V[c+ab/(b-a)]$.

$$\therefore \text{Capacity } C=\frac{q}{V}=4\pi\varepsilon_0\left[c+\frac{ab}{b-a}\right]. \qquad ...(11)$$

(*d*) **Capacity of a cylindrical capacitor.** When a metallic cylinder of radius a is placed coaxially inside an earthed hollow metallic cylinder of large radius b, we get cylindrical capacitor. A submarine cable is an example of such a capacitor in which the inner conductor consists of copper cable and the sea water is the outer earthed cylinder.

If a charge q is given to the inner cylinder, induced charge $-q$ will reach to the inner surface of the outer cylinder. Assume that the capacitor is of very large length ($l \gg b$) so that the lines of force are radial and the fringing of lines of force at the end can be ignored.

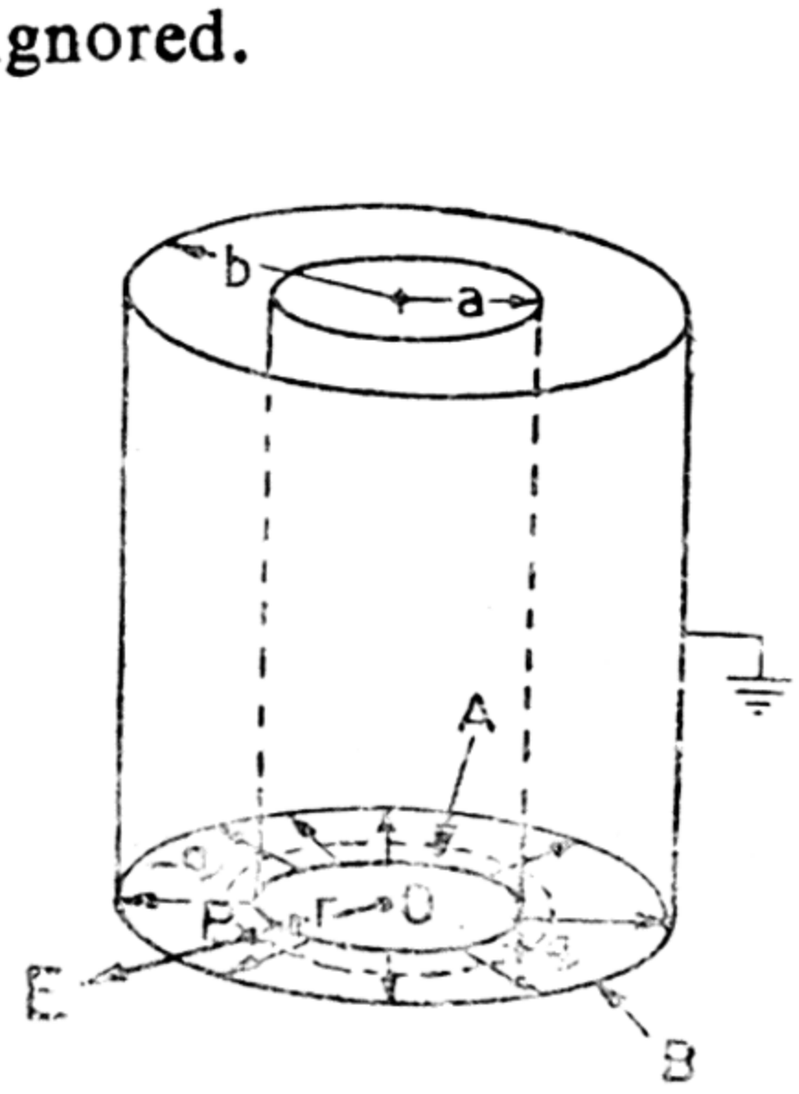


Fig. 4.4.

To find the electric field between the cylinders, let us consider a gaussian cylinder of radius r ($>a$ and $<b$) and length l'. From Gauss's law, we have

$$\int\mathbf{E}.\,d\mathbf{S}=(1/\varepsilon_0)\Sigma\,q.$$

As there is no flux through the ends of the cylinder ($\mathbf{E}\perp d\mathbf{S}$) and $\mathbf{E}$ is constant on and $\perp$ to the gaussian curved surface, hence

$$E.\,2\pi rl'=(1/\varepsilon_0)\,\lambda l' \text{ or } E=\lambda/2\pi\varepsilon_0 r,$$

where λ is the charge per unit length on the inner cylinder. Therefore, the potential difference between the cylinders

$$V=-\int_{b}^{a}\mathbf{E}.d\mathbf{r}=-\int_{b}^{a}\frac{\lambda}{2\pi\varepsilon_0 r}dr=\frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{b}{a}\right)$$

If the capacitor is of length l, its capacity

$$C_l = \frac{q}{V} = \frac{l\lambda}{(\lambda/2\pi\varepsilon_0)\log_e(b/a)} = \frac{2\pi\varepsilon_0 l}{\log_e(b/a)} \quad \ldots(11)$$

$\therefore$ Capacity per unit length $C = 2\pi\varepsilon_0/\log_e(b/a)$. $\ldots(12)$

(*c*) **Capacity of a parallel plate capacitor.** Two metallic parallel plates of any shape but of same size and separated by a small distance constitute parallel plate capacitor. If charge q is given to the plate P, it will induce a charge $-q$ to the upper surface of the earthed plate Q. Let A be the area of each plate. If the separation d is assumed to be very small compared with the plate dimensions, so that the fringing effect at the edges of the plate can be neglected. Thus electric lines of force starting from plate P and ending at the Q plate are parallel to each other and perpendicular to the plates.

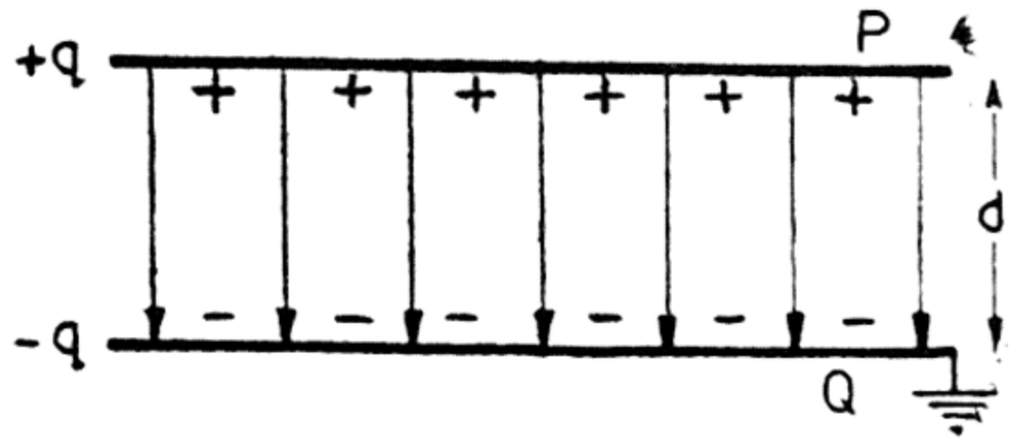


Fig. 4.5.

We know from § 2.9 (*G*) that the intensity at a point between the parallel plates is σ/ε_0, where σ is the surface density of charge. Thus the potential of plate P is given by

$$V = -\int_Q^P \mathbf{E}.\ d\mathbf{r} = \int_o^d \frac{\sigma}{\varepsilon_0} dr = \frac{\sigma d}{\varepsilon_0}.$$

Here the distance is measured from the lower plate Q in upward direction.

$$\therefore \text{ Capacity } C = \frac{q}{V} = \frac{\sigma A}{\sigma d/\varepsilon_0} = \frac{\varepsilon_0 A}{d}. \quad \ldots(13)$$

Instead of two plates, if there are n similar plates at equal distances from each other and the alternate plates are connected together, the capacitance of the arrangement is given by $C = (n-1)\varepsilon_0 A/d$.

From relations (5), (8), (9), (12) and (13). it is clear that relations for capacitance depend only on geometrical factors. These relations suggest that ε_0, the constant obtained in Coulomb's law, can be measured by building a capacitor of accurately known dimensions and determining its capacitance experimentally by measuring charge q and potential V.

4.3. EFFECT OF A DIELECTRIC

Relations obtained earlier for the capacitances of different shapes and sizes hold only when the plates are in a vacuum or air. Michael Faraday, in 1837, first investigated the effect of filling the space between the plates with nonconducting material (or *dielectric*), say mica or oil. He took two identical capacitors and placed a dielectric in one and in other an air at normal pressure. When both capacitors were charged to the same potential difference, he found experimentally that the charge on the one containing the dielectric was greater than that on the other. From the relation $C=q/V$, it follows that capacitance of a capacitor increases if a dielectric is placed between the plates.

$$\frac{\text{Capacitance with the dielectric } C_d}{\text{Capacitance without the dielectric } C} = \text{Dielectric constant } k.$$

Instead of maintaining the two capacitors at the same potential difference we can give the same charge to each capacitor. Experiment shows that the potential difference V_d between the plates of the capacitor filled with the dielectric is smaller than that without it and $V_d/V=1/k$, k has no unit, as it is just a ratio. Its value is 1 for vacuum and ∞ for metals.

The above experiment can be repeated by inserting a sheet of dielectric between the plates of a capacitor. It will be observed that the potential difference decreases as we insert the dielectric and it returns to its original value when the dielectric is removed, showing that the original charges on the plates were not affected by insertion of the dielectric. This reduction in potential difference implies a reduction in the electric intensity, which further implies a reduction in the net charge per unit area. As there is no leakage of charge, the reduction in the net charge is possible as a result of the induced charges appearing on the two surfaces of the dielectric, as shown in Fig. 4.6 (*b*).

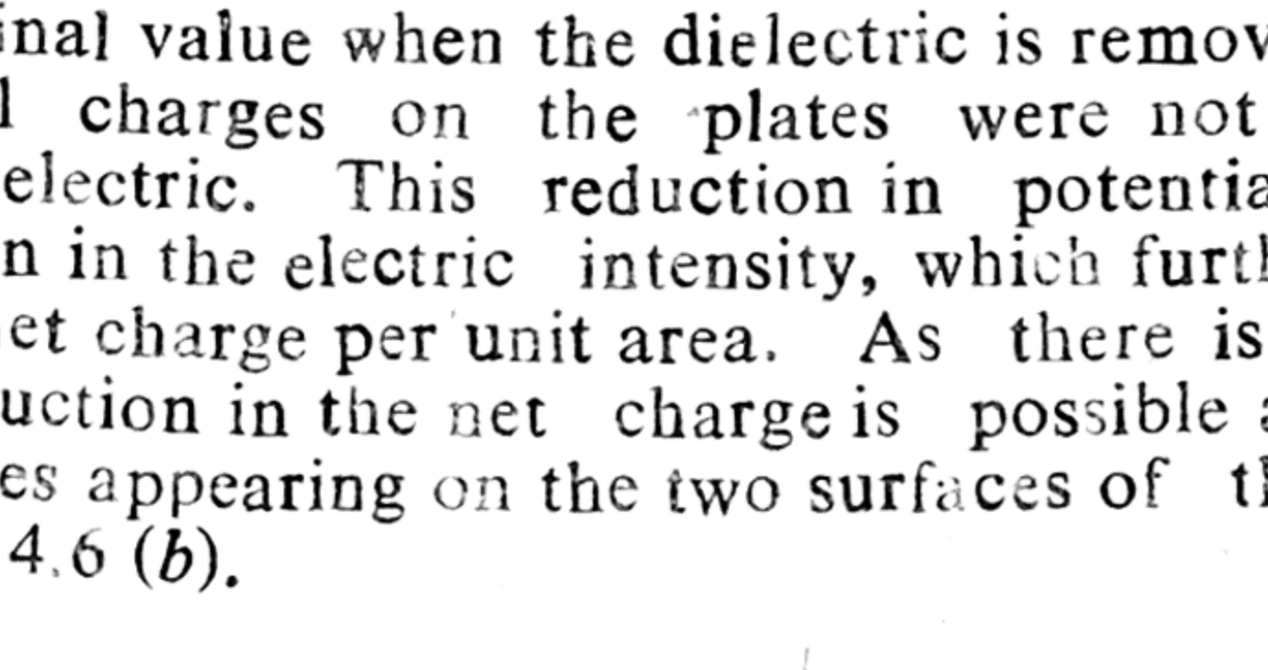


Fig. 4.6.

The dielectrics are often used to permit a higher potential difference to be applied between the plates. Thus in the presence

of the dielectric between the plates, the capacitances are thus given as :

$$C_d = kC = k\varepsilon_0 A/d \text{ for parallel plate capacitor,}$$
$$= 2\pi k\varepsilon_0 l/\log_e (b/a) \text{ for cylindrical capacitor,}$$
$$= 4\pi\varepsilon_0 kR \text{ for spherical capacitor.}$$

In general we can write the capacitance of any capacitor as

$$C_d = k\varepsilon_0 L,$$

where L depends on the geometry and has the dimensions of a length.

4.4. ENERGY STORED BY A CHARGED CONDENSER

A charged capacitor stores an electric potential energy in it, which is equal to the work required to charge it. It is identical to the potential energy stored in a compressed spring or the gravitational potential energy stored in the earth-moon system. This energy can be recovered if the capacitor is allowed to discharge. If the charging is done by a battery, electrical energy is stored at the expense of chemical energy of battery.

For the calculation of the energy stored in a charged capacitor we assume that as the capacitor gets charged to a higher and higher potential, the work to bring up each increment of charge increases, as the collected charge increases. Suppose that at a time t, a charge $q(t)$ is present on the capacitor and V is the potential of the capacitor. If dq amount of charge is brought against the forces of the field due to the charge already present on the capacitor, the additional work needed will be

$$dW = V dq = (q/C) dq.$$

∴ Total work to charge a capacitor to a charge q_0

$$W = \int dW = \int_0^{q_0} \left(\frac{q}{C}\right) dq = \frac{q_0^2}{2C}. \qquad \ldots(14)$$

∴ Energy stored by a charged capacitor $U = W = \frac{1}{2} q_0^2/C$
$= \frac{1}{2} CV^2.$

If q_0 is in coulombs, C in farads and V in volts, U will be in *joules.*

The relation (14) shows that the charged capacitor is the electrical analog of a stretched spring whose elastic potential energy is $\frac{1}{2}kx^2$. The charge q_0 is analogous to the elongation x and $1/C$, *i.e.*, the reciprocal of capacitance to the force constant k.

We may consider that the work done in charging the capacitor produces an electric field. To determine the relationship between stored energy and the electric field, let us choose a parallel plate capacitor of area A and plate separation d. The electric field is

uniform in the region between two plates and the fringing effects are negligible. Energy stored by such a capacitor is given by

$$U=\frac{1}{2}\frac{q^2}{C}=\frac{1}{2}\frac{(\sigma A)^2}{\varepsilon_0 A/d}=\frac{1}{2}\frac{\sigma^2 Ad}{\varepsilon_0}, \qquad \ldots(15)$$

or energy per unit volume $=\sigma^2/2\varepsilon_0$. ...(16)

But we know that the electric field E due to a parallel plate conductor with a surface density σ is σ/ε_0. Thus we get

$$U/\text{volume}=\sigma^2/2\varepsilon_0=\tfrac{1}{2}\varepsilon_0 E^2=\tfrac{1}{2}\varepsilon_0 \mathbf{E}.\ \mathbf{E}. \qquad \ldots(17)$$

If the capacitor is placed in a medium of *dielectric constant k*, the value of capacitance $C=k\varepsilon_0 A/d$.

$\therefore$ Stored energy per unit volume $=\tfrac{1}{2}k\varepsilon_0\ E^2$. ...(18)

Although this relation was derived for the special case of a parallel plate capacitor, it is also true for others. This relation shows that the energy stored per unit volume depends on E^2. If E is the electric field in a space of volume $d\tau$, then the total stored energy in an electrostatic field is given by

$$U=\tfrac{1}{2}\varepsilon_0\, k\int E^2 d\tau. \qquad \ldots(19)$$

This energy is known as *self energy*. Self energy does not change when we change the relative position of the charges and is not included when we calculate the energy stored in an assembly of charges.

Self Energy. Let us consider two isolated charges, which are far apart. The self energy of each charge is given by Eq. (19). The total stored energy, when the charges are brought close enough together so that there is an appreciable overlap of the field, $(\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2)$, is thus given by

$$U=\tfrac{1}{2}\varepsilon_0\int(\mathbf{E}_1+\mathbf{E}_2).(\mathbf{E}_1+\mathbf{E}_2)d\tau$$

$$=\tfrac{1}{2}\varepsilon_0\int E_1^2 d\tau+\tfrac{1}{2}\varepsilon_0\int E_2^2 d\tau+\tfrac{1}{2}\int 2\mathbf{E}_1.\ \mathbf{E}_2 d\tau. \qquad \ldots(20)$$

The first two terms are the self energies of the two separate charges and the third term is the energy resulting from the overlapping of the two fields and is equal to the *potential energy of one charge in the field of the other.*

At Constant Potential. If the potential is kept constant by connecting the capacitor with a battery, the expression for the energy stored by the capacitor of capacitance C can be obtained as under. If the distance between the plates is decreased, the capacity increases and the charge is accumulated and energy is thus stored. Work done by the battery when a charge dq increases the capacitance by dC is given by

$$dW=Vdq=V^2dC.$$

$\therefore$ Total work done by the battery

$$W=V^2\int_{C_1}^{C_2} dC=V^2\left[C_2-C_1\right].$$

If $C_1=0$, *i.e.*, the plates are initially at infinite distance apart and $C_2=C$, then

$$W=V^2C=qV \text{ joules.} \qquad \ldots(21)$$

4.5. FORCE BETWEEN CAPACITOR PLATES

Let us consider a capacitor consisted of two parallel plates P and Q. Plate P is given a charge $+q$ and plate Q is earthed. Lines of force start from plate P and end at plate Q. If the plates are assumed to be of infinite size and the end effects (curved lines at the ends) are neglected, one can assume the electric intensity of the field between these plates as equivalent to the field of an infinite charged conducting plane. There are two possible cases.

(*a*) **When the charge remains the same.** The force of attraction per unit area between two plates of a capacitor is equal to the outward electrical force per unit area on the surface of plate P and is given by

$$F=\sigma^2/2\varepsilon_0=\tfrac{1}{2}\varepsilon_0E^2 \text{ Newton/meter}^2. \qquad \ldots(22)$$

(*b*) **When the potential difference between the plates remains constant.** In this case potential is applied by connecting a battery across the plates of a capacitor. As the potential is measured, hence the expression for the force of attraction should be expressed in terms of potential difference between the plates.

$\therefore$ $$Q=CV=(\varepsilon_0A/d)\ V=\sigma A,$$

or Charge per unit area $\sigma=\varepsilon_0V/d$.

$\therefore$ $$F=\frac{1}{2\varepsilon_0}\left(\frac{\varepsilon_0V}{d}\right)^2=\frac{\varepsilon_0}{2}\ \frac{V^2}{d^2}. \qquad \ldots(23)$$

This relation is used in an attracted disc electrometer.

The expression for the force can also be derived from the expression for the energy stored. If **F** is the attractive force between the plates, the external force $-\mathbf{F}$ is required to move the plates apart, the external work required to increase the plate separation by an amount $d\mathbf{x}$ will be $-\mathbf{F}.d\mathbf{x}$. Hence the increase in internal energy $dU=-\mathbf{F}.d\mathbf{x}$. As the energy stored for the parallel plate capacitor of separation x,

$$U=\tfrac{1}{2}q^2/C=q^2x/2\varepsilon_0A. \qquad \ldots(24)$$

$$F=-\frac{d}{dx}\left[\frac{q^2x}{2\varepsilon_0A}\right]=-\frac{q^2}{2\varepsilon_0A}. \qquad \ldots(25)$$

As q^2 is positive and A is +ve, hence F is negative which indicates that **F** is in the negative **x** direction, giving *attraction* between the plates.

4-6. FORM OF CAPACITORS

The capacitors most commonly used are of the parallel plate type. Parallel plate capacitors consist of two metallic foils with

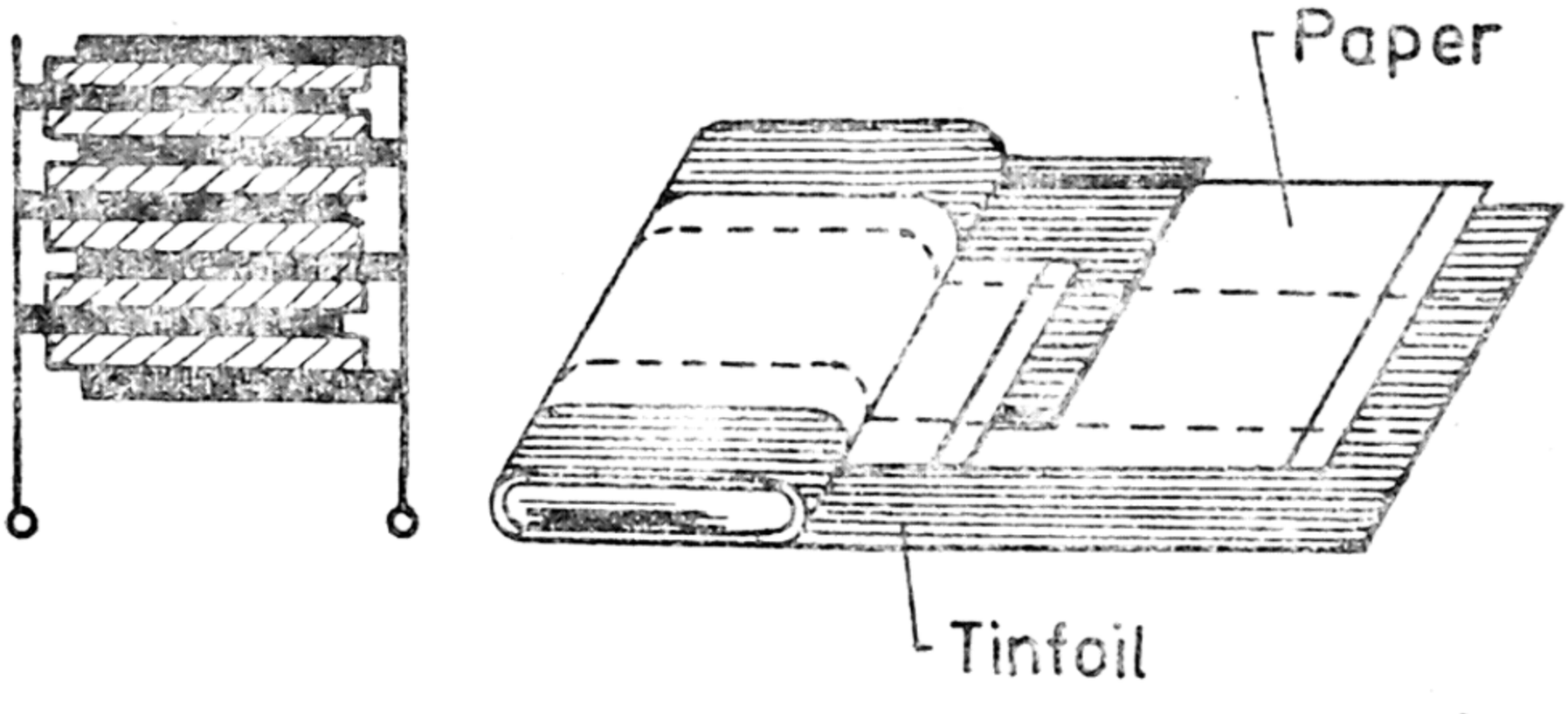


Fig. 4.7.

mica or paraffined paper as a dielectric between them. In the former case capacity can be increased by connecting alternating plates to the same terminal (*i.e.*, connecting capacitors in parallel) and in the latter case it is done by taking two strips of metal foil (tin) insulated by the dielectric (paraffined paper) and rolled up in the form of a cylinder for compactness. The paraffin-waxed paper is a cheaper dielectric and can easily by rolled, but the power absorbed in such a capacitor is large in *ac* circuits.

Mica capacitors consist of a pile of alternate parallel layers of mica and copper sheets. Since mica absorbs little power, hence mica capacitors are generally used in *ac circuits*, *e.g.*, in radio and electronic sets. Apart from the capacitance, an almost equally important factor is the *capacitor working voltage*, which depends on the thickness and dielectric strength of the dielectric. Because of energy dissipated in the capacitor, the working voltage decreases as the frequency of the source is increased, roughly as $1/\mu^{1/2}$. Mica capacitors are having capacitance range from 0.0001 to 0.05μf at working voltages up to 500 volts and paper capacitors from 0.01 to 10μf at working voltages up to 1000 volts. It is due to the reason that the thickness of dielectric is very small in the latter case.

Electrolytic Capacitors. Electrolytic capacitors are used when a very high capacitance is required. This type of capacitor consists of two plates of aluminium as electrodes and the extremely thin film of aluminium oxide (10^{-6} cm thick) as dielectric. This film is formed on the positive aluminium electrode by the electroysis

of a solution of aluminium borate. In the *wet* type, the anode is in the form of a cylinder, which is immersed in the electrolyte contained in a metal can, which works as a cathode. In the *dry*

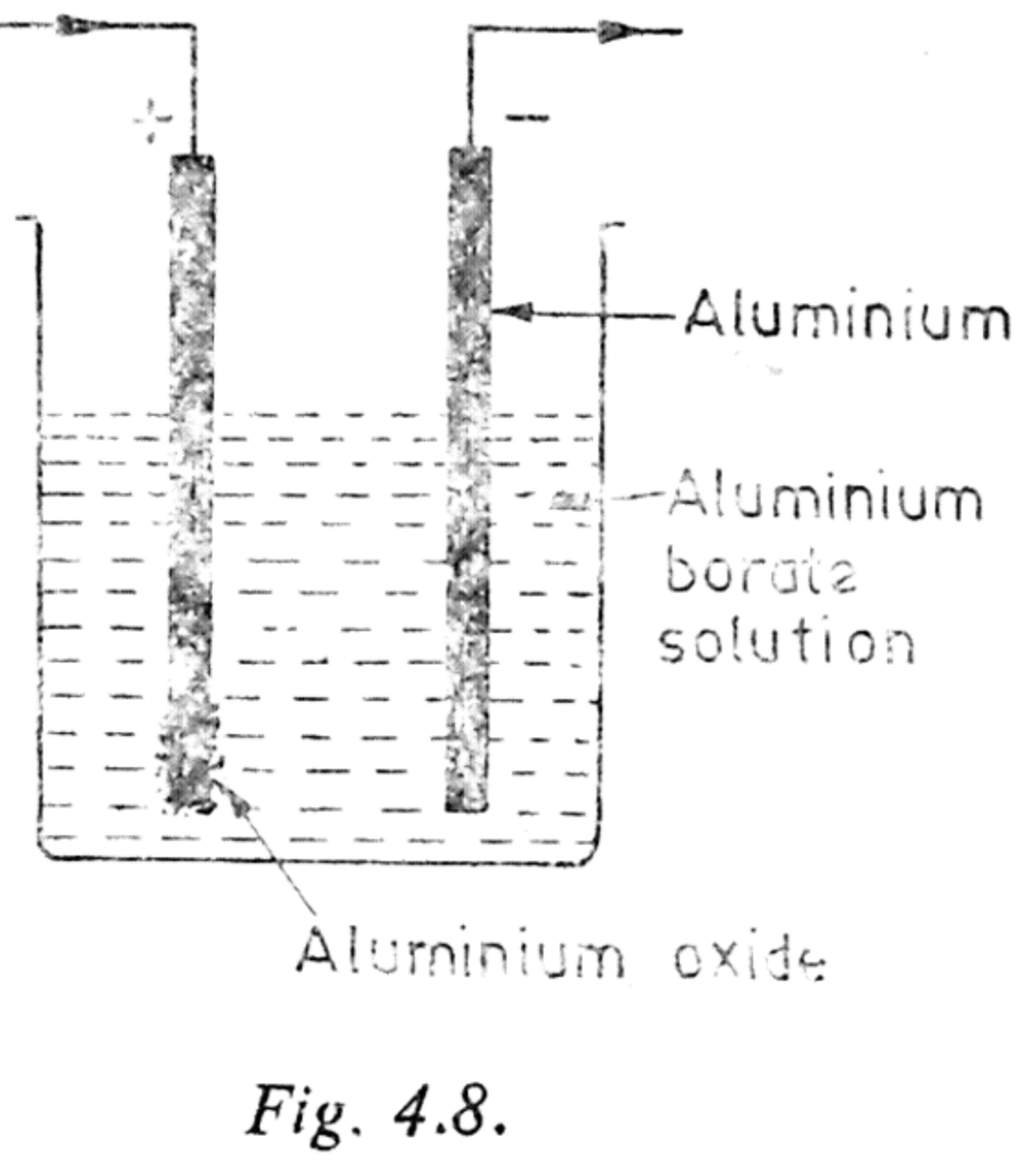


Fig. 4.8.

type, both plates consist of long strips of aluminium foil, on one of which aluminium oxide is deposited electrically and is separated by cotton or paper gauze soaked in the electrolyte and then rolled up in a cylindrical form.

Oxide film on aluminium has a low resistance to current flow in one direction and high resistance in other direction. Thus *electrolytic capacitors can only be used in uni-directional supplies* such that the potential of the oxide plate is always +ve relative to the other plate. These are widely used in radio-sets as a *smoothing capacitors.* As a capacitance varies with temperature, a stable condenser must be used in radio-sets, where accurate tuning is required. Silver mica or silver ceramic capacitor is more reliable for this purpose. It consists of a mica sheet with a silver deposite on both sides or a ceramic tube with a silver deposition on inner and outer sides. The silver deposits will work as electrodes.

Variable Capacitor. If a continuous variation of the capacitance is required, a variable air capacitor of the parallel plates or of the coaxial cylinders is used. The former consists of two sets of plates, usually semicircular, one set is fixed and the other can be rotated. These plates are insulated and parallel and are of aluminium or brass generally. When the movable part is rotated, its plates move in the gaps between the plates of the fixed part and the overlapping area between the two sets changes and thus changes the capacitance of the capacitor. The variable air capacitor of this type is used in ac bridges and in radio-sets.

The latter form consists of two coaxial cylinders of same diameter, with a small gap between them. A smaller cylinder *B* is caused to slide in or out of the cylinder *A* by a support which slides upon *C*, so that the length of *B* lying with in *A* can be varied

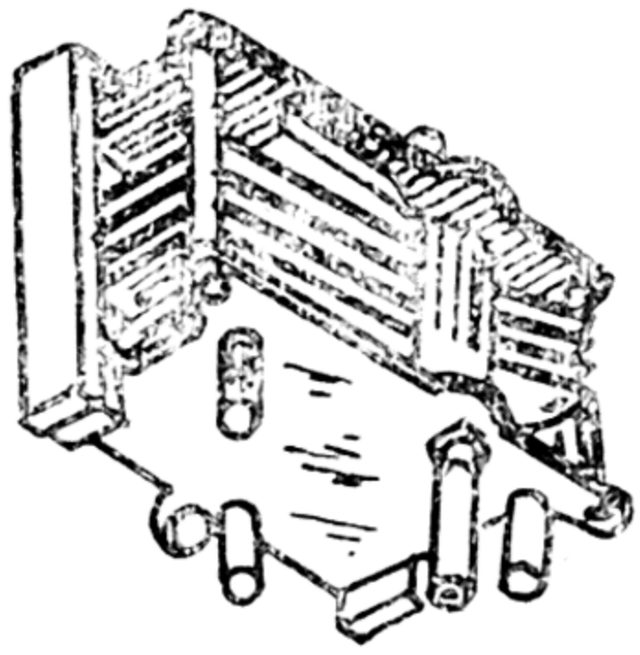


Fig. 4.9.

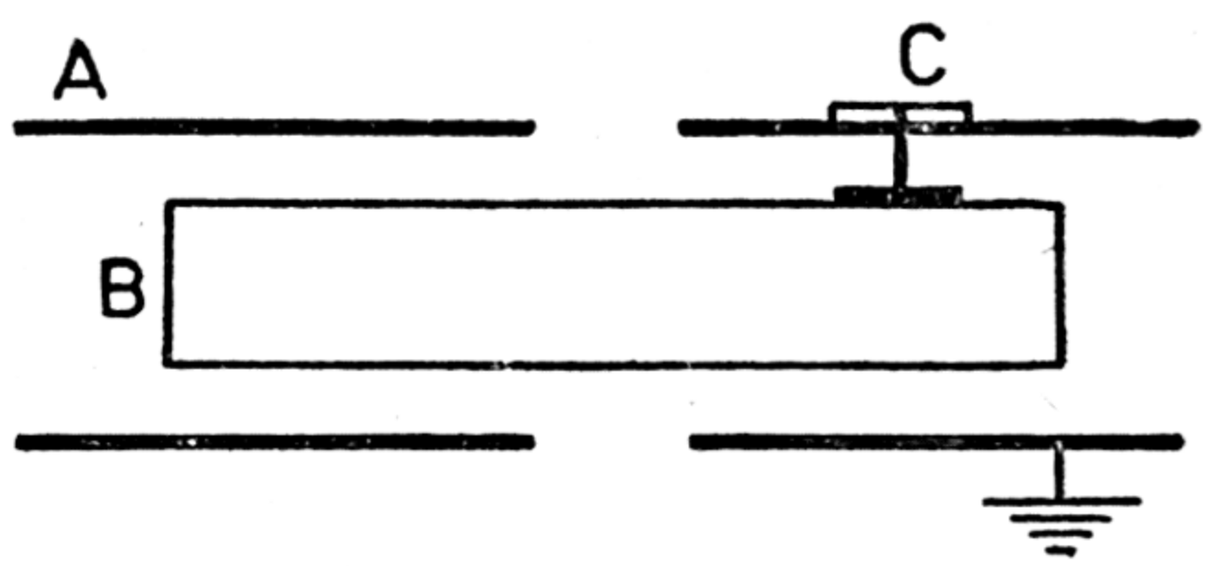


Fig. 4.10.

and measured. Cylinder *A* is insulated and *B* and *C* are earthed. The change in capacitance can be calculated by knowing the distance through which cylinder *B* moves in or out. If a scale of length is attached with a slider cylinder and the absolute capacitance is determined experimentally at one position, the capacitor can be calibrated.

Guard Ring Capacitor. In the derivation of the expression for the capacitance of a parallel plate condenser, we have assumed the field between the plates as uniform and fringing effect at the edges as negligible. But in actual practice the field is not uniform at the edges and the expression

$$C = \varepsilon_0 A/d$$

is only approximate.

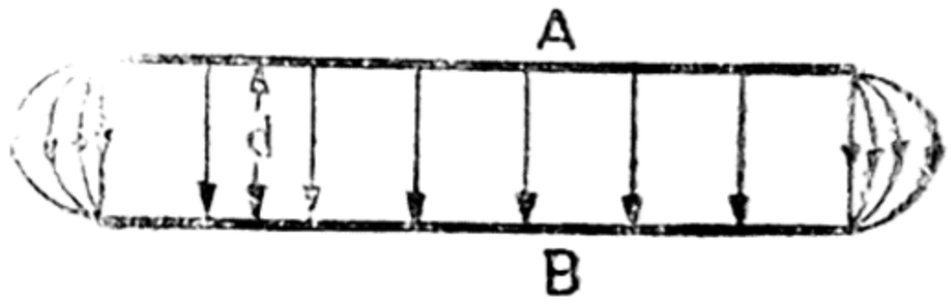


Fig. 4.11.

To overcome the end effects, Lord Kelvin devised a modified form of parallel plate capacitor by using a ring surrounding the plate *A*. The outer radius of the ring matches that of the bottom earthed plate *B* and its inner radius is slightly greater than that of

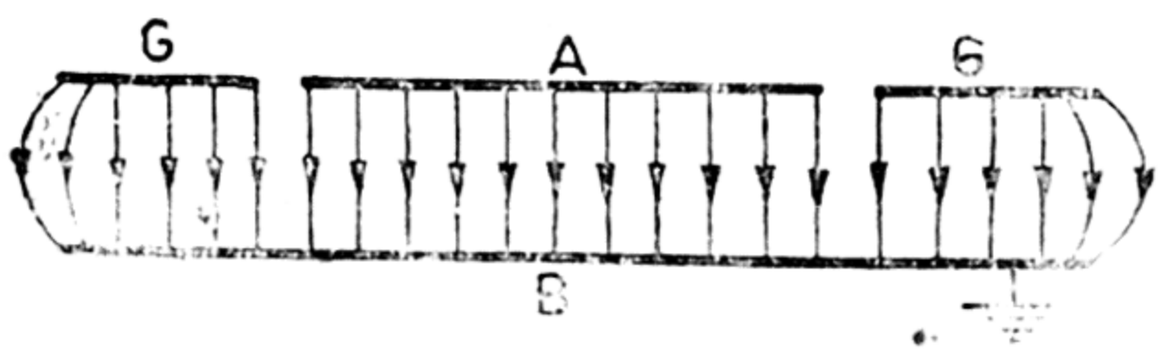


Fig. 4.12.

the plate *A*. The ring is in the plane of plate *A*, leaving a small ring shaped slot for insulation. The fringing field is now shifted to the outside edge of the ring, leaving the field uniform over the

whole area of plate *A*. It is why the ring is known as *guard ring*. Area of the guard ring is not used in calculations as its effect is to shift the fringing field. Kelvin found that the effective area of the plate $= A' =$ Area of the plate $A + \frac{1}{2}$ Area of the gap between the ring and the plate *A*. The cross-sectional view of the guard ring capacitor is given in the Fig. 4.12. A similar arrangement can be made in the cylindrical capacitor. Two coaxial cylinders constitute a capacitor.

4.7. KELVIN'S ABSOLUTE OR ATTRACTED DISC ELECTROMETER

For the measurements of potential differences the electromagnetic voltmeters are less accurate as they require some electric current for their working, where as in electrostatic cases the current is to be avoided completely. Certain instruments have been devised which measure electrostatic potentials correctly and are named as *electrometers*. The simplest type of an electrometer is *gold leaf electroscope*. On account of its low sensitiveness and the uncertainty in the value of its readings, more advanced instruments, *absolute electrometer* and the *quadrant electrometer*, are used.

The principle of absolute electrometer depends on the measurement of the force of attraction between the plates of a parallel plate capacitor, across of which the unknown potential-difference is applied, in terms of the known weights. As this pd is measured in terms of force of attraction and dimensions of the capacitor which can be expressed in terms of the fundamental units of mass, length and time, hence it is called as *absolute electrometer*. It is known as *Kelvin's absolute electrometer* as it was first devised by Lord Kelvin using a guard ring air condenser.

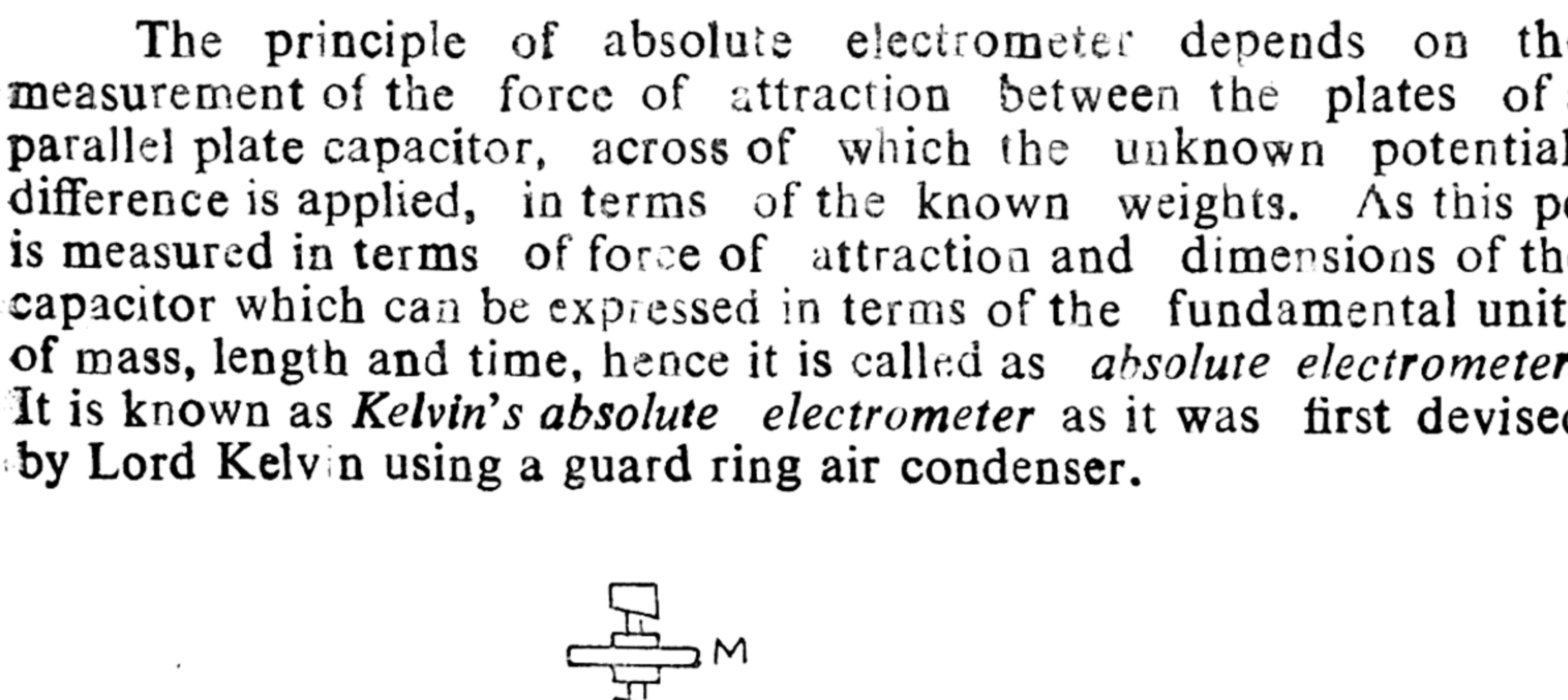


Fig. 4.13.

In this instrument, the disc *A* is surrounded by a guard ring *GG* with a small air gap. Both are in the same plane (horizontal) and are connected together by a light flexible wire and hence are

always charged to the same potential. The disc *A* is horizontally suspended by a delicate spring *S*, connected to the rod, which can be raised or lowered by a micrometer screw *M*. A fine cross is attached to the disc *A* to see by means of a telescope whether the disc *A* is in the plane of guard ring *GG* or not. Another disc *B* of radius equal to the outer radius of the guard ring is kept parallel to the disc *A* and can be raised or lowered by means of another micrometer screw *N*. The distance moved by the disc *B* can be accurately read on a graduated scale, similar to that used in spherometer.

Let V_A and V_B be the potentials of discs *A* and *B* respectively, which are at a distance *d* apart. As the use of guard ring produces a uniform electric field **E** between the discs *A* and *B*, hence

$$\int_B^A dV = -\int_B^A \mathbf{E}.d\mathbf{l} = \int_B^A Edl = E\Big[\, l \,\Big]_o^d = Ed.$$

$$V_A - V_B = Ed \quad \text{or} \quad E = (V_A - V_B)/d.$$

We know that the force of attraction exerted by this electric field is given by the relation

$$F = \tfrac{1}{2}\varepsilon_0 E^2 A = \tfrac{1}{2}\varepsilon_0 (V_A - V_B)^2 A/d^2,$$

or
$$V_A - V_B = d\sqrt{2F/\varepsilon_0 A}. \qquad \ldots(26)$$

To use the electrometer all the plates *A*, *B* and *GG* are earthed and the plate *A* is made in the plane of the guard ring *GG* with the help of the screw *M*. *A* small weight '*m*' is then placed on the plate *A*, which depresses it below the plane of *GG*. The plate is raised with the help of the screw *M*, so that it is again in the plane of *GG*. The weight '*m*' is then removed. The spring *S* pulls the plate *A* above *GG*. At this time force '*mg*' is required to bring the plate *A* in the plane of *GG*. All the earth connections are now cut off.

The unknown potential difference can be measured by connecting the +ve terminal to the plate *A* and the −ve terminal to the plate *B* and by using the above relation. *The main difficulty is the measurement of d, the distance between plates A and B. For this we use an alternate procedure given below.*

The plate *A* and guard ring *GG* are now kept at constant potential by connecting any one with the +ve terminal of an electrical machine or a bettery whose−ve terminal is earthed. The plate *B* is then connected to the first of the points, the potential

difference between which we wish to measure and its position is adjusted by means of screw N until the plate A is in the plane of GG. At this time the force of attraction due to the electrostatic field is equal to mg. If V_1 is the potential of the first point, V_0 that of the +ve terminal of battery and d_1 the separation of two plates, then

$$V_0 - V_1 = d_1\sqrt{2mg/\varepsilon_0 A}.$$

Now plate B is connected to the second point and its position is adjusted again until the plate A is in the plane of GG. If d_2 is the distance between plates A and B and V_2 the potential of the second point, then

$$V_0 - V_2 = d_2\sqrt{2mg/\varepsilon_0 A}.$$

$$\therefore \quad V_2 - V_1 = (d_1 - d_2)\sqrt{2mg/\varepsilon_0 A}. \qquad \ldots(27)$$

The difference $(d_1 - d_2)$ is the difference of the two micrometer readings say x_1 and x_2, hence

$$V_2 - V_1 = (x_1 - x_2)\sqrt{2mg/\varepsilon_0 A}. \qquad \ldots(28)$$

The potential difference is measured in terms of the difference of micrometer readings, the mass m and the effective area of the plate A, which is equal to the area of plate $A \pm \frac{1}{2}$ the area of the gap.

The *main advantage* of this electrometer over the others is that it does not require any known potential difference for the calibration. It measures the unknown pd in terms of distance, force and area. It is comparatively less sensitive than the other forms of electrometers.

Since the resistance of a voltmeter has a finite value, it always draws some current from the circuit in which it is connected and therefore shows less value of the potential difference. On the other hand, the attracted disc electrometer is an air condenser having infinite resistance and hence measures the actual potential difference. *Another use over voltmeter* is that it can be used for the determination of the dielectric constant of a material which can be taken in the form of slab as voltmeter cannot be used for this purpose.

The *main disadvantage* over the voltmeter is its cumbersome arrangement. It is not handy, like a voltmeter. It is not sensitive for measuring or comparing small pds and cannot be used for measuring ac-potential differences.

Determination of Dielectric Constant. Kelvin's attractive disc electrometer can be used for the measurement of dielectric constant of several solid substances. The substance is taken in the form of a slab of area equal to the area of the plates. It is introduced between the plates of an electrometer.

Capacity with dielectric slab of thickness t and of dielectric constant $k = A\varepsilon_0/[d - t(1 - 1/k)]$.

Thus we see that the effect of introducing the slab is the same as would be produced by diminishing d by the amount $t\,(1 - 1/k)$ without introducing the slab. Hence to bring the capacity to the same value the d should be increased by this amount when the slab is in. Calling this displacement as h, we get

$$h = t(1 - 1/k) \text{ or } k = t/(t - h).$$

Here thickness t is measured by bringing the plates in contact with and without the slab in between these two. Thus k is calculated.

4.8. DOLEZALEK QUADRANT ELECTROMETER

In 1860, Lord Kelvin invented a more sensitive electrometer known as quadrant electrometer. It was later modified by Dolezalek and is therefore known as *Dolezalek Quadrant Electrometer*. It is very sensitive and is used in number of ways.

It *consists* essentially of four hollow brass quadrants so as to form a shallow circular box with short air gaps between the quadrants. The diagonally opposite quadrants are connected in pairs and are well insulated by being supported on amber pillars. A light paddle, made of thin paper coated with silver or of aluminium is suspended by a phospher bronze strip. It is of a double sector shape and is frequently called the needle from analogy with the galvanometer needle. In the better type of instruments quartz fibre made conducting by dipping in a solution of calcium chloride is used as suspension. The upper end of the suspension is hooked on to the torsion head which can be rotated as well as raised or lowered and then clamped in any position, so that the needle is free to swing in a horizontal plane without touching the quadrants. The deflections of the needle are measured with the help of a lamp and scale arrangement, reflection taking place from a tiny mirror attached to the suspension fibre. The whole instrument is enclosed in a metal box which is earthed to protect it from stray electric field and the disturbance of air draughts. This box also provides window for the light beam. Three levelling screws are fitted to the base.

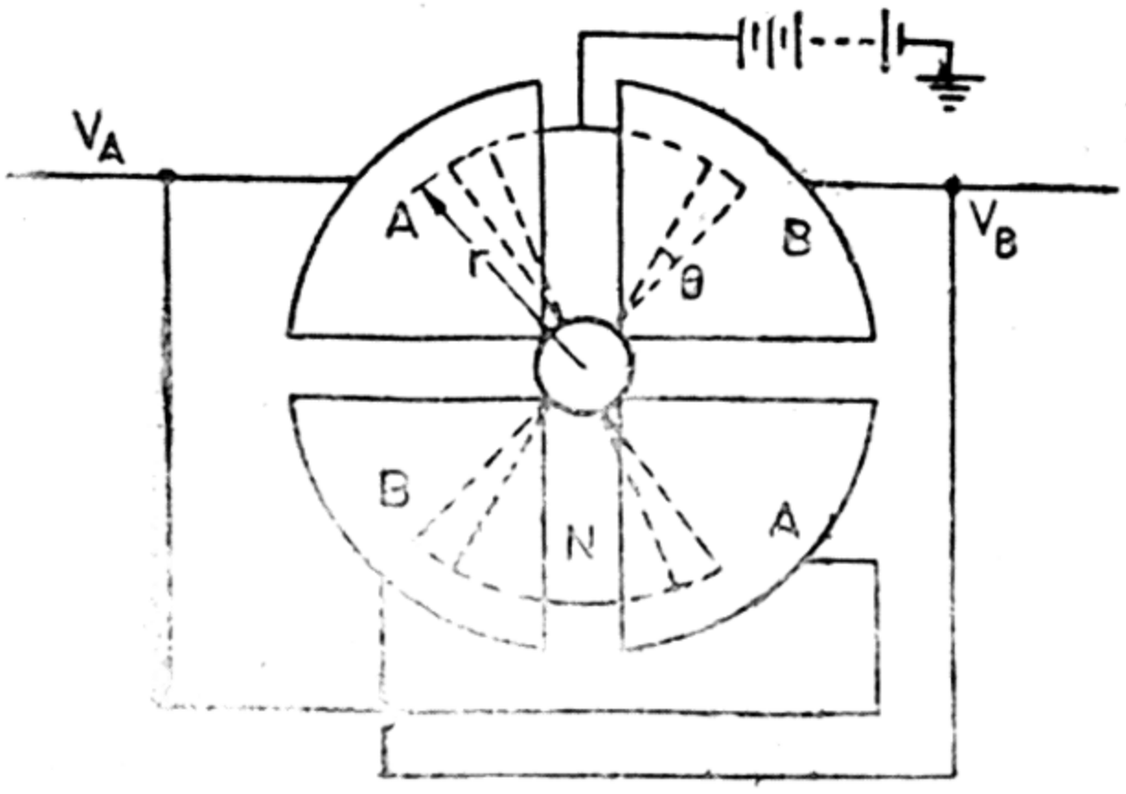


Fig. 4.14.

The needle is charged to a higher potential and the unknown potential difference is applied between the two pairs of quadrants. Since the capacity of each pair of quadrants is the same, the charge on BB is less than on AA and therefore the force of repulsion between BB and the needle is less than between AA and the needle. As a consequence the needle is *deflected towards the*

pair at lower potential BB by a couple equal to the difference between the couples due to charges on *BB* and *AA*.

The *complete theory* is somewhat difficult, but a sufficiently accurate result is obtained by an approximate method, *an energy evaluation method*. This method is based on the assumption that *one-half of the paddle is considered to be a sector of a circle of radius r and as the quadrants are edge to edge, the direct capacity between them is assumed to be negligible in comparison of the capacity between quadrants and needle*. Let θ be the angle of deflection when pairs *AA* and *BB* and the needle *N* are at potentials V_1, V_2 and V_0 respectively. The area of the needle transferred from the *AA* quadrants to the *BB* quadrants is $2\pi r^2\theta/2\pi = r^2\theta$. As there are two faces to the needle, the effective area transferred is $2r^2\theta$. Thus the capacity of the condenser formed by quadrant *B* and the needle *N* is increased and of quadrant *A* and the needle *N* is decreased by the same amount equal to $\varepsilon_0 2r^2\theta/d$, where *d* is the thickness of air space between needle and quadrants. As the potentials remain constant, *A-N* capacitor loses charge $2\varepsilon_0 r^2\theta\,(V_0-V_1)/d$ whilst the other pair *B-N* gains charge $2\varepsilon_0 r^2\theta(V_0-V_2)/d$. From the definition of potential difference it follows that the work done by the source of supply in providing this charge to *B N* is $2\varepsilon_0 r^2\theta(V_0-V_2)/^2d$ and the work done on the source of supply during the loss of charge from *A-N* capacitor is $2\varepsilon_0 r^2\theta\,(V_0-V_1)^2/d$. Thus the total energy supplied to the electrometer from the source of supply is given by

$$W=\frac{2\varepsilon_0 r^2\theta}{d}[(V_0-V_2)^2-(V_0-V_1)^2]$$

$$=\frac{4\varepsilon_0 r^2\theta}{d}[V_1-V_2]\left[V_0-\frac{V_1+V_2}{2}\right].$$

This energy is used up partly in increasing the electrostatic potential energy of the condensers formed by the two pairs of quadrants and the balance in twisting the suspension fibre. As the electrostatic potential energy of a condenser is $\frac{1}{2}CV^2$, the net increase in electrostatic potential energy of the condenser system is given by

$$\frac{1}{2}\frac{2\varepsilon_0 r^2\theta}{d}(V_0-V_2)^2-\frac{1}{2}\frac{2\varepsilon_0 r^2\theta}{d}(V_0-V_1)^2=\frac{W}{2}. \qquad \ldots(29)$$

The difference, which is half the total energy supplied from the supply, is used in twisting the fibre. It is proportional to θ, hence the deflecting couple is constant and equal to $(2\varepsilon_0 r^2/d)[V_1-V_2](V_0-\frac{1}{2}(V_1+V_2)]$. If *c* be the torsion couple per unit twist of the suspension fibre, then the restoring couple for deflection θ is $c\theta$. For equilibrium the two couples are equal, hence

$$c\theta=(2\varepsilon_0 r^2/d)\,(V_1-V_2)\,[V_0-\tfrac{1}{2}(V_1+V_2)]$$

or $$\theta=(2\varepsilon_0 r^2/cd)\,(V_1-V_2)\,[V_0-\tfrac{1}{2}(V_1+V_2)]$$
$$=K\,(V_1-V_2)\,[V_0-\tfrac{1}{2}(V_1+V_2)], \qquad \ldots(30)$$

where $K=2\varepsilon_0 r^2/cd$ is a constant for the instrument.

There are two ways of using the quadrant electrometer as a potential measurer :

(*i*) **Heterostatic use.** Here the needle is charged to a very large potential as compared to V_1 and V_2. Thus $V_0 \gg V_1$ or V_2 and

$$\theta=K[V_1-V_2]\,V_0=K'[V_1-V_2], \text{ where } K'=KV_0. \qquad \ldots(31)$$

Hence in heterostatic use, $\theta \propto (V_1-V_2)$.

(*ii*) **Idiostatic use.** Here the needle is connected to one pair of quadrants say AA. The unknown potential difference is applied across BB and the common contact of N and AA. Thus $V_0=V_1$ and

$$\theta=K(V_1-V_2)\,[V_1-\tfrac{1}{2}(V_1+V_2)]=\tfrac{1}{2}K(V_1-V_2)^2$$
$$=K''(V_1-V_2)^2, \qquad \text{where } K''=\tfrac{1}{2}K. \qquad \ldots(32)$$

Hence in idiostatic use $\theta \propto (V_1-V_2)^2$.

Relative merits of Heterostatic use and Idiostatic use. We have $K'/K''=2V_0$. Hence $K' \gg K''$ and therefore the heterostatic use is more accurate than the idiostatic use, as a small potential difference produces a much larger deflection in the former case. Another great advantage of the heterostatic use is that the deflection is proportional to the potential difference. When the instrument is used idiostatically the deflection is proportional to the square of the potential difference and independent of the direction and hence the instrument can be used for the measurement of *alternating potential differences*. As $K' \gg K''$, hence for a given deflection, the value of V_1-V_2 will be greater in idiostatic use than in heterostatic use. The range of the instrument is therefore considerably extended. In idiostatic use the needle is connected to the pair of quadrants and thus a high potential source to be connected with the needle is not required. Thus we see that to *measure small potentials, the electrometer is used heterostatically and to measure high potentials and alternating potentials it is used idiostatically.*

The electrometer must be mounted on a very rigid support such as a stone block and must be *adjusted* so that the quadrants are quite horizontal and the needle hangs symmetrically between them. The quadrants are first made horizontal with the spirit level and by opening the case the needle is adjusted by eye in such a way that it swings freely between the top and bottom of the hollow quadrant space and symmetrical between AA and BB quadrants. The final adjustment is done by the spot of light obtained on the scale, as given under :

AA and *BB* pairs of quadrants are earthed and the needle is charged to a high potential. If the spot of light on scale changes, it means that the needle is not symmetrical between the quadrants. The levelling screws are adjusted until the position of the spot is the same whether the needle is charged or not. As we know that equal and opposite values of $V_1 - V_2$ should give equal and opposite deflections, the electrometer should be adjusted, after connecting needle to about 100 volts, *AA* pair to the earth and the *BB* pair of quadrants to the +ve terminal of the battery, negative terminal of which is earthed, until equal deflections on the both sides of the zero are obtained when the battery terminals are reversed. Both these processes are repeated for the *proper adjustment*.

A small potential difference is applied between the two pairs of quadrants and the potential of the needle is gradually increased. If a graph is plotted between the deflection θ and the potential of the needle, a curve so obtained will be parabolic. The voltage of the needle corresponding to the maximum deflection gives the condition for maximum sensitivity. The needle is thus charged to this potential and the potential difference between the pairs of quadrants is now varied and the deflections are obtained corresponding to known potential differences and a graph is plotted between the potential difference and the deflection. The curve so obtained will be straight line and is known as calibration curve. With the help of this curve the unknown potential difference can be obtained by observing the deflection corresponding to that potential difference.

Quadrant electrometer is not an *absolute* instrument. In precise measurements it is usually used as a null instrument. Used heterostatically it can measure 1/3000 volt of potential. Its sensitivity can be increased by taking special precautions. Its capacitance is of the order of 80 *pf*.

Uses of a Quadrant Electrometer. *Measurement of small (steady) potential difference.* Steady small potential can be measured by using the quadrant electrometer *heterostatically, i.e.,* by connecting the needle to the +ve terminal of a high potential battery, the negative terminal of which is earthed and by applying the unknown potential difference across two pairs of quadrants. The deflection in this case is proportional to the unknown potential difference $(V_1 - V_2)$.

(2) *Measurement of Alternating Potential Difference.* In this case the electrometer is used *idiostatically, i.e.,* the needle is connected to one pair of quadrants and the unknown potential difference is applied across two pairs of quadrants. The deflection is given by

$$\theta \propto (V_1 - V_2)^2.$$

(3) *Capacitance Measurement.* To compare the capacitances of two capacitors, C_1 and C_2, connect the apparatus as shown in

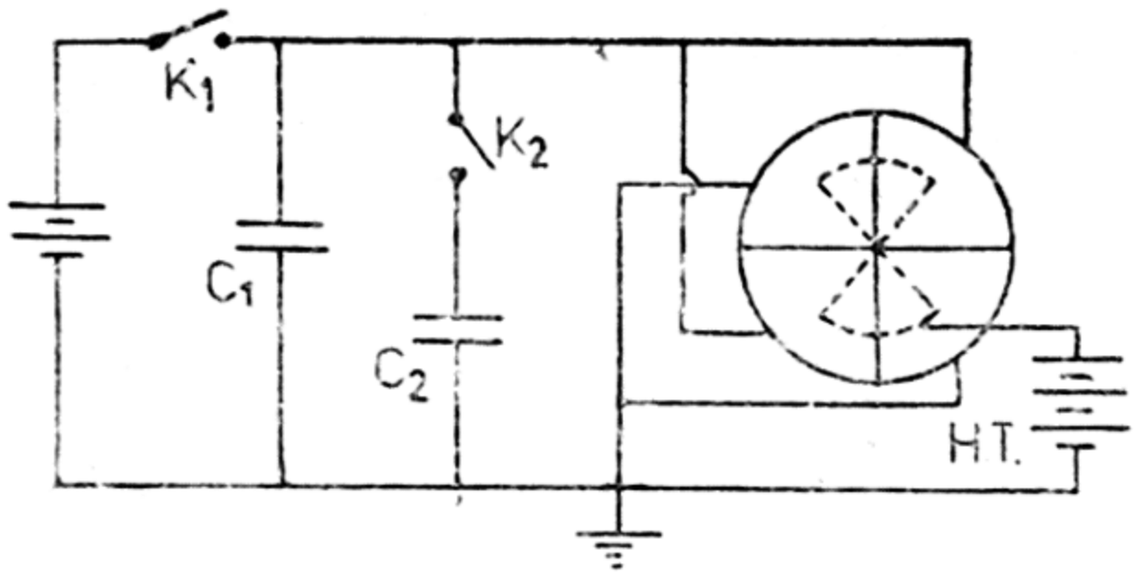


Fig. 4.15.

Fig. 4.15. Hence the electrometer is connected heterostatically by connecting the needle to high tension battery, one pair to earth and other pair to battery.

Close key K_1, thus charging condenser C_1 and the quadrant electrometer to a certain potential V_1. If θ_1 is the corresponding deflection indicated by the electrometer, then charge $q=(C+C_1)\,V_1$, where C is the capacity of the electrometer. Now open key K_1 and close key K_2, thus allowing capacitor C_1 and the electrometer to share the charges with capacitor C_2. If θ_2 is the new deflection, then

$$q=(C+C_1+C_2)V_2$$

$$\therefore \qquad q=(C+C_1)V_1=(C+C_1+C_2)V_2$$

$$\text{or} \qquad \frac{C_1+C}{C_2}=\frac{V_2}{V_1-V_2}=\frac{\theta_2}{\theta_1-\theta_2}. \qquad (\text{As } V_1 \propto \theta_1 \text{ and } V_2 \propto \theta_2)$$

If C, the capacity of the electrometer is small and is negligible compared to C_1 and C_2, then

$$C_1/C_2=\theta_2/(\theta_1-\theta_2). \qquad \ldots(33)$$

In this way two capacitances C_1 and C_2 can be compared very easily. If one *capacitance is known, other can be calculated* with the help of this equation.

The capacitance of the quadrant electrometer may be determined as follows. A standard capacitor, say guard ring air condenser of capacitor C_1 is removed from the circuit 4.15. The electrometer is first charged by closing key K_1 and opening key K_2 and the deflection θ_1 is obtained. Then key K_1 is opened and the charge is shared with the standard capacitor C_2 by closing key K_2 and the deflection θ_2 is also obtained. Hence,

$$q=CV_1=(C+C_2)\,V_2$$

$$\text{or} \qquad C=C_2\,\frac{V_2}{V_1-V_2}=C_2\,\frac{\theta_2}{\theta_1-\theta_2}. \qquad \ldots(34)$$

Thus the capacitance of the electrometer C can be calculated.

(4) *Measurement of dielectric constant of a solid in the form of a slab (Hopkinson's method).* dc methods of determining the dielectric constant give the values 10% greater than the values obtained by ac methods. It is due to dielectric absorption phenomenon.

In the Hopkinson's circuit, AB is a guard-ring condenser, C a variable condenser consisting of two coaxial cylindrical conductors

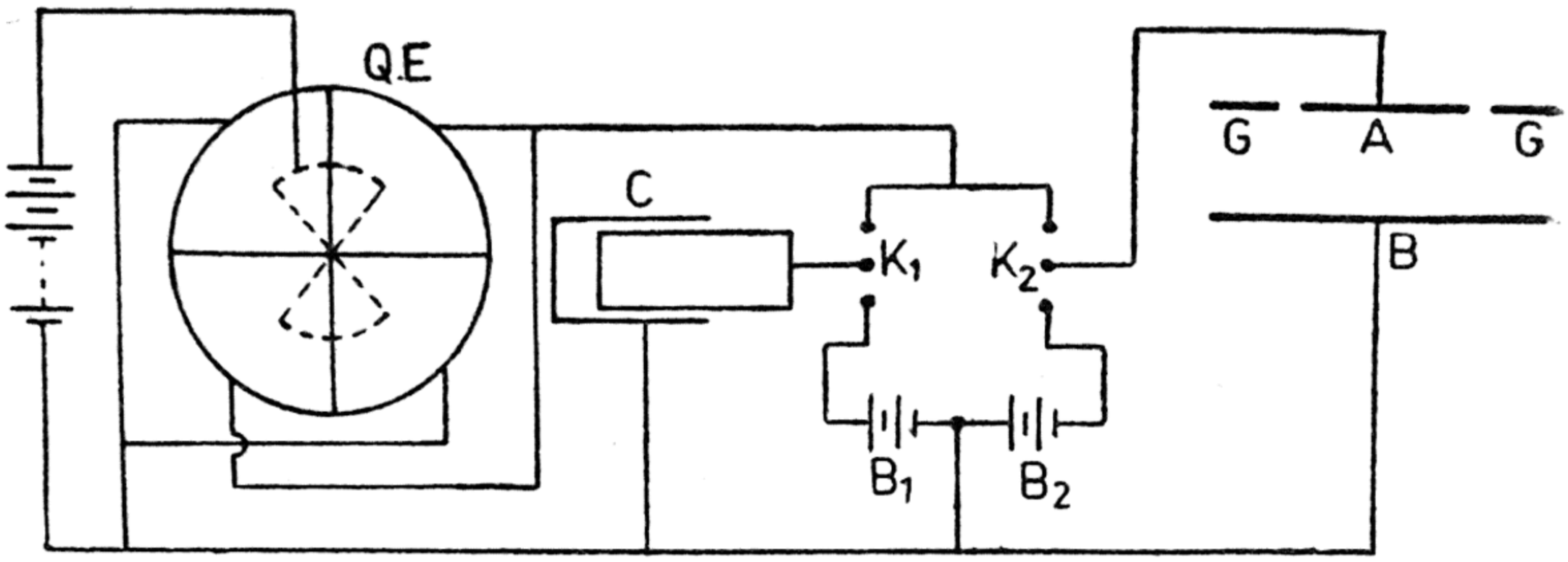


Fig. 4.16.

(inner is capable of sliding inside the outer one), Q.E. the quadrant electrometer, B_1 and B_2 the two batteries with common terminals earthed and the ends at potentials $+V$ and $-V$.

By pressing keys K_1 and K_2, condensers are charged to equal and opposite potential. If the capacitances of these capacitors are equal, then their charges will be equal and opposite. The net charge given to quadrant electrometer is zero and the needle is thus undeflected. Thus by adjusting variable capacitor C, the position of no deflection in the electrometer is obtained.

The position of plate B is noted on the scale and the variable condenser C is kept fixed. The substance of which dielectric constant is required is taken in the form of a slab and is placed on the plate B. Now this plate B is lowered until there is no deflection in the electrometer needle while pressing keys K_1 and K_2. The new position of plate B on the scale is thus noted.

If d is the distance between the plates in the first part and x is the distance by which the plate B is lowered to obtain a null deflection in the second part (when a slab of thickness t and of area equal to that of the guard-ring capacitor is placed), then

$$\frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 A}{[(d-t+x)+t/k]}$$

$$\therefore \quad d-t+x+t/k=d \quad \text{or} \quad k=t/(t-x).$$

The thickness of the dielectric slab can be determined from a micrometer screw.

(5) *Measurement of Ionisation Currents.* Quadrant electrometer can be used for measuring ionisation current as it is too small to be detected by a galvanometer. An ionization chamber, consisting of a metallic box containing two plates insulated and separated from each other, is connected in series with a resistance ($\sim 10^{10}$

ohms and a battery (30—200 volts). When the air between plates A and B is exposed to X-rays, a current flows in this circuit and a potential difference is created between ends of the resistor. Thus potential difference is applied across one pair of quadrants of elect-

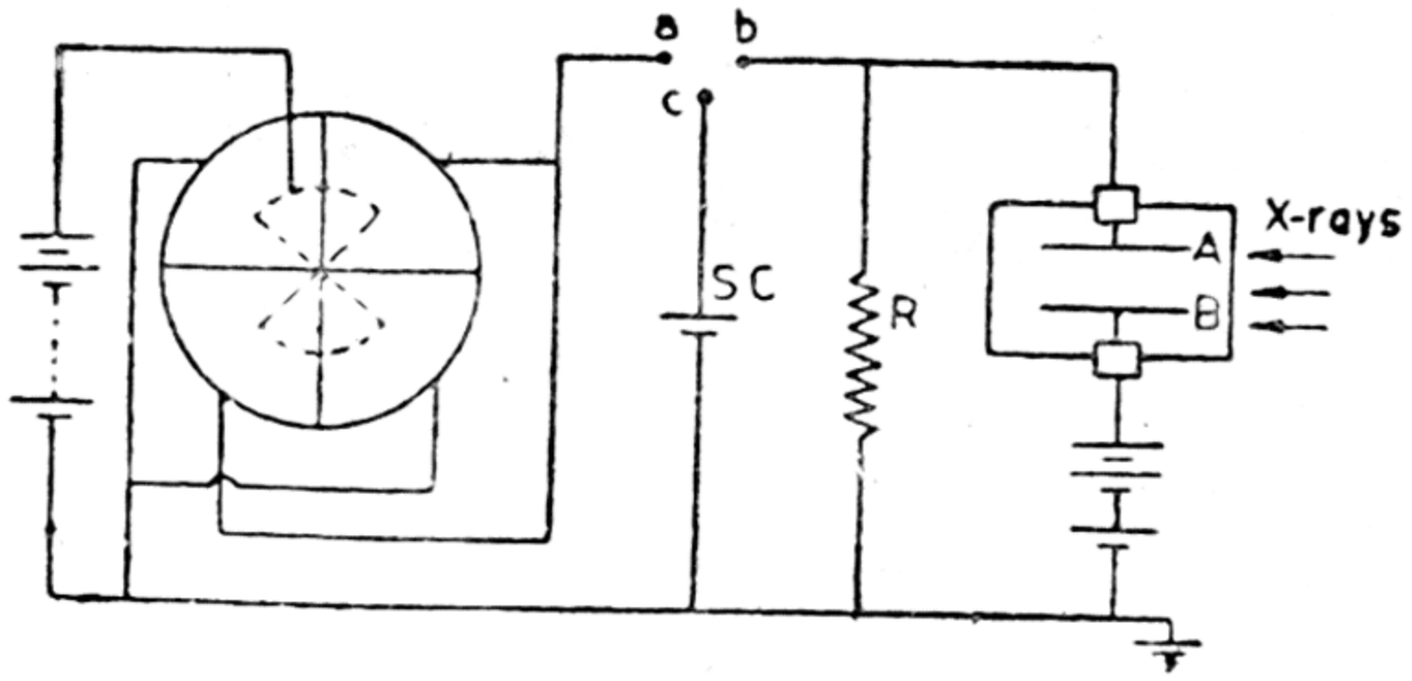


Fig. 4.17.

rometer, the other pair of which is earthed and the needle is connected to a high tension battery. The steady deflection of the needle is recorded. The electrometer is calibrated by connecting a standard cadmium cell (1.018 volts) across the pair of quadrants with a two-way key. If θ_1 and θ_2 be the deflections in the former and latter cases respectively, then we have

$$IR \propto \theta_1 \text{ and } 1.018 \propto \theta_2.$$

or $$I = 1.018\theta_1 / R\theta_2.$$

It is in amperes if R is measured in ohms.

Modifications of Dolezalek Electrometer

(*a*) *Compton Electrometer.* It is the modification of Dolezalek quadrant electrometer and was given by A.H. and K.T. Compton. In this instrument the electrostatic control is varied by tilting the paddle slightly about its long axis and by moving one of the quadrants a little above or below the horizontal plane of the other three.

This method of control gives nearly constant sensitivity over a large range of deflection and increases its sensitivity 70 times greater. It can be quickly adjusted. Its capacity is about 10 *pf*.

(*b*) *Lindemann Electrometer.* This instrument, designed in 1924, works on the principle of quadrant electrometer. It is very compact, has a stable zero, a constant sensitivity, very small capacitance (2 *pf*), does not require levelling and is very robust. For its construction and working the readers are advised to consult *Electricity and Magnetism by Fewkes and Yarwood and A Degree Physics Part V Electricity and Magnetism by C.J. Smith.*

(*c*) *Electrostatic Voltmeter.* It is the instrument used to measure potential difference and is based on the electrostatic principles. It consists of two fixed parallel brass plates each of the shape of a

double quadrant and are placed vertically. An aluminium needle is pivoted at the centre so that it can swing about a horizontal axis in a vertical plane mid way between the fixed brass plates. The upper part of the needle carries a pointer which moves over the vertical scale of the instrument and the lower part carries adjustable weights. The controlling torque may be varied by adjusting these weights and the range of the instrument may thus be altered. The needle is earthed and the potential to be measured is applied to the fixed plates. The scale is first standardized by using known voltages. It is graduated in volts.

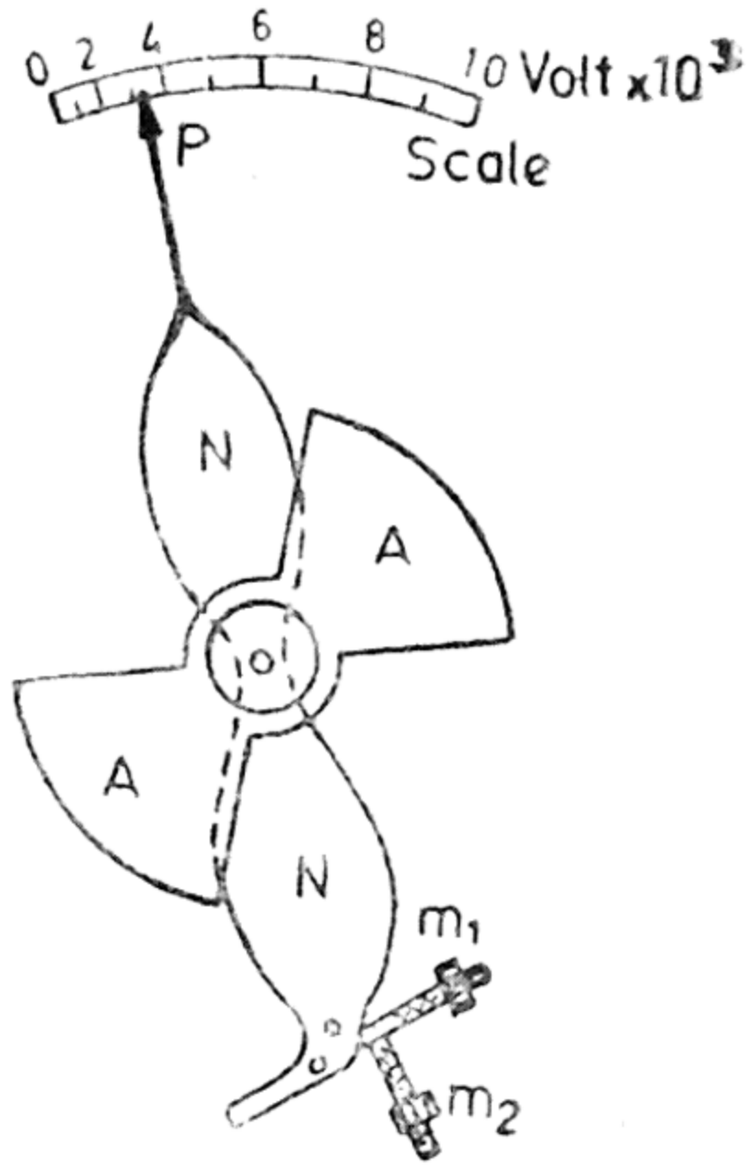


Fig. 4.13.

The instrument is gravity controlled, as the electrostatic attraction due to the potential difference applied between fixed plates is opposed by the particular weight attached to the lower end of the needle and the latter comes at rest when these two opposing influences are equal and balance each other. With different patterns a total range of about 1,000 volts to 20,000 volts can be obtained.

The next step for lower pressures is to employ the *multicellular* type. By using a vertical suspension to avoid pivot friction and by increasing the number of moving needles (or vanes) and of fixed brass plates, if can be used to measure much lower potential differences (30 to 120 volts). Higher ranges up to about 2,000 volts can be obatained by modifying the suspension or the vanes.

A great advantage of the electrostatic type is that it absorbs no power and takes only a very small capacity current. However it cannot be used for small potential differences and is costly. The scale is crowded in its lower portion.

A multiplier is used to increase the range of a low-reading electrostatic voltmeter. This consists of a high resistance with tappings at intermediate points. The potential to be measured is connected across the whole resistance and the voltmeter across part of it. The drawback of this multiplier is that the power loss is not zero and the current is not very small.

Exercises

Example 1. *A conductor is charged from an electrophorous by repeated contacts with a plate which after each contact is re-charged with a quantity q of electricity from the electrophorous. If e is the charge of the conductor after the first operation, calculate the ultimate charge.*

On the first contact, the charge taken up by the conductor of capacity C_1 is e and the charge left on the plate of capacity C_2 is $q-e$. Since the potential for either is the same, hence

$$(q-e)/C_2=e/C_1, \text{ or } e=qC_1/(C_1+C_2).$$

In the second contact, let e' be the charge taken up by the conductor, then $e+e'$ becomes the total charge over the conductor and the charge left over the plate is $q-e'$. Hence we have

$$(q-e')/C_2=(e+e')/C_1 \text{ or } e'=qC_1^2/(C_1+C_2)^2=e^2/q.$$

Similarly the charge on the conductor on the third contact is e^3/q^2 and so on. Hence the ultimate charge

$$Q=e+e^2/q+e^3/q^2+\ldots=eq/(q-e).$$

Example 2. *Three concentric thin spherical shells are of radii* a, b, c $(a<b<c)$. *The first and third are connected by a fine wire through a small hole in the second and the second is connected to earth through a small hole in the third. Show that the capacity of the condenser so formed is* $4\pi\varepsilon_0[ab/(b-a)+c^2/(c-b)]$.

If a charge q is given to the outermost sphere C, as it is connected with the outer surface of innermost sphere A, some of the charge will go to the outer surface of the sphere A, some will go to the inner surface and the rest will remain on the outer surface of this sphere C. The charge q_1 on the inner sphere will induce a charge $-q_1$ on the inner surface of the second sphere B and will induce a charge $+q_1$ on the outer surface which will go to earth. Charge q_2 on the inner surface of the sphere C will also induce a charge $-q_2$ on the outer surface of the sphere B. In this way three capacitors are formed, one by the outer surface of A and inner surface of B, second by the outer surface of B and inner surface of C and the third by the outer surface of C and the earth. The pd between the spheres A and B

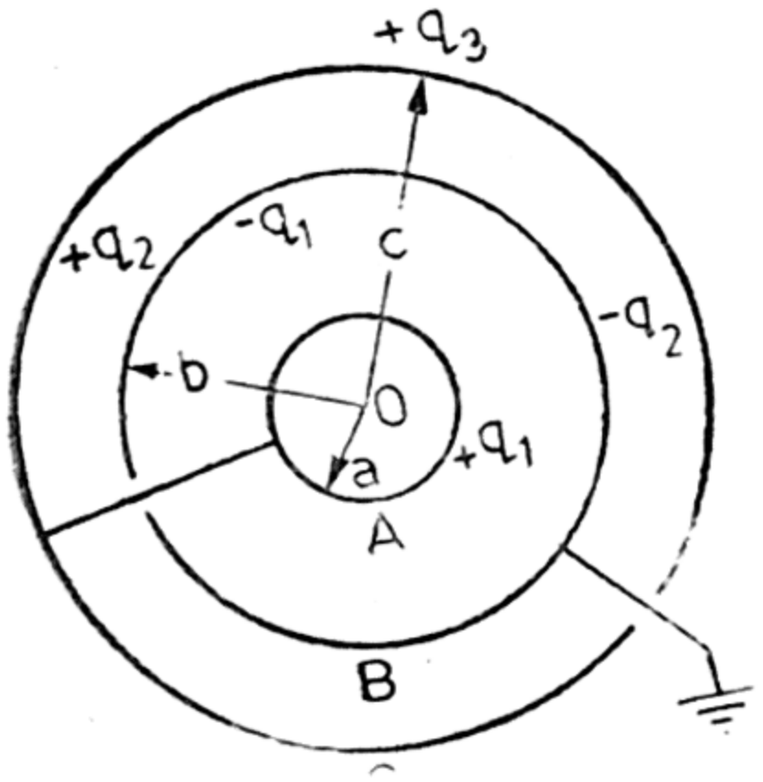


Fig. 4.19.

$$V_A-V_B=-\int_b^a \mathbf{E}.d\mathbf{r}=-\frac{q_1}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2}=\frac{q_1}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right].$$

$$\therefore \text{ Capacitance } C_1=\frac{q_1}{V_A-V_B}=4\pi\varepsilon_0\ \frac{ab}{b-a}.$$

Similarly capacitance $C_2=\dfrac{q_2}{V_B-V_C}=4\pi\varepsilon_0\dfrac{bc}{c-b}$,

and capacitance $C_3=4\pi\varepsilon_0\, c$.

$\therefore$ Combined capacity of the condenser so formed

$$=C_1+C_2+C_3$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{ab}{b-a}+\frac{bc}{c-b}+c\right]=4\pi\varepsilon_0\left[\frac{ab}{b-a}+\frac{c^2}{c-b}\right]$$

Example 3. *A one core lead sheathed cable has a conductor core of 0.5 cm diameter and the lead sheath has inside diameter 1.5 cm. The insulating material is rubber. At what voltage will the insulation break down ? Given that rubber has a dielectric strength 400 kV/cm.*

The capacity per unit length of the cable (cylindrical)

$$C=2\pi\varepsilon/\log_e (r_2/r_1).$$

$\therefore$ Charge per unit length on the cable $\lambda=2\pi\varepsilon\ V/\log_e (r_2/r_1)$.

If the voltage applied is of such a value that the insulation is just to break, the intensity of field at the inner surface of sheath will be the same as the dielectric strength, *i.e.*,

$$\text{Dielectric strength}=\text{Intensity}=\lambda/2\pi\varepsilon r_2$$

$$=\frac{2\pi\varepsilon V}{\log_e(r_2/r_1)}\cdot\frac{1}{2\pi\varepsilon r_2}=\frac{V}{r_2\log_e (r_2/r_1)}$$

$\therefore$ 400×10^3 volts/cm $=4\times10^7$ volts/m $=V/0.75\times10^{-2}\log_e (3)$

or $V=4\times10^7\times0.75\times10^{-2}\log_e 3=3.3\times10^5$ volts.

Example 4. *Prove that the capacitance of two very long, thin parallel wires in air is given by* $C=\pi\varepsilon_0/\log_e (d/a)$, *where a is the radius of each wire and d is the distance between the centres of wires.*

Let A and B are two very long parallel thin wires, separated by a distance d and are in the direction perpendicular to the paper (Fig. 4.20). Let us consider any point P in the space around the wires, such that $PA=r_1$, $PB=r_2$ and $\angle PAB=\theta$. Let the charges per unit length on A and B are $+\lambda$ and $-\lambda$ respectively. As we know that the electric intensity at any point at distance r due to the long charged cylinder is given by $E=\lambda/2\pi\varepsilon_0 r$.

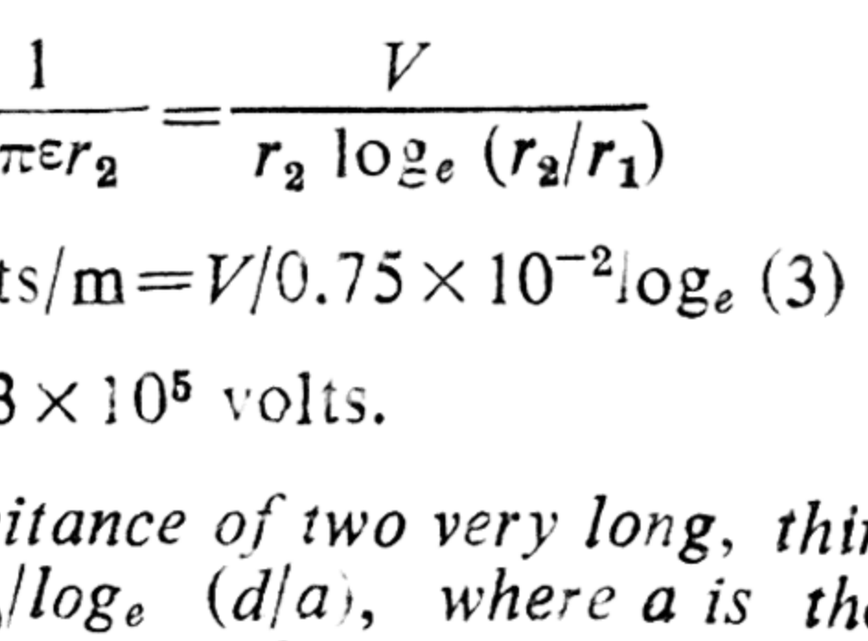


Fig. 4.20.

$\therefore$ $V=-\int \mathbf{E}.d\mathbf{r}=-\int E dr=-\int(\lambda/2\pi\varepsilon_0 r)dr.$

Hence the potential at point P due to wire A

$$=-\frac{\lambda}{2\pi\epsilon_0}\int_R^{r_1}\frac{dr}{r}=\frac{\lambda}{2\pi\epsilon_0}(\log_e R-\log_e r_1),$$

where R is the distance of the point of zero potential, say earth.

Similarly, the potential at point P due to wire B

$$=-\frac{-\lambda}{2\pi\varepsilon_0}\int_{R}^{r_2}\frac{dr}{r}=\frac{\lambda}{2\pi\varepsilon_0}(\log_e r_2-\log_e R).$$

Hence the total potential at P due to both wires

$$=(\lambda/2\pi\varepsilon_0)\log_e r_2/r_1.$$

$\therefore$ The potential V_A at wire $A=(\lambda/2\pi\varepsilon_0)\log_e (d-a)/a$
and the potential V_B at wire $B=(\lambda/2\pi\varepsilon_0)\log_e a/(d-a)$.

$\therefore$ Potential difference between the wires

$$V_A-V_B=\frac{\lambda}{2\pi\varepsilon_0}\left[\log_e\left(\frac{d-a}{a}\right)-\log_e\left(\frac{a}{d-a}\right)\right]$$

$$=\frac{\lambda}{\pi\varepsilon_0}\log_e\frac{d-a}{a}.$$

Thus the capacitance per unit length $=\lambda/(V_A-V_B)$

$$=\frac{\pi\varepsilon_0}{\log_e[(d-a)/a]}=\frac{\pi\varepsilon_0}{\log_e(d/a-1)}$$

$$=\pi\varepsilon_0/\log_e(d/a). \qquad (\text{as } d\gg a).$$

This equation is applied to two parallel telegraph wires which are far removed from the earth.

If the *single thin wire of telegraph* is at a height h above the earth, the potential difference V between the wire and earth will be half of V_A-V_B, *i.e.* $(\lambda/2\pi\varepsilon_0)\log_e(d/a)$. A single wire will have an electrical image (will be discussed in chapter 5) at a distance h below the earth level, thus the distance between two wires (one wire and other its image) $d=2h$.

$\therefore$ Potential difference $V=(\lambda/2\pi\varepsilon_0)\log_e(2h/a.)$

Therefore the capacitance per unit length

$$=(\lambda/V)=2\pi\varepsilon_0/\log_e(2h/a).$$

Example 5. *If d is the separation of the plates of a parallel plate condenser and a slab of thickness t and of dielectric constant k is inserted between the plates and parallel to them. Calculate the increase in capacitance.*

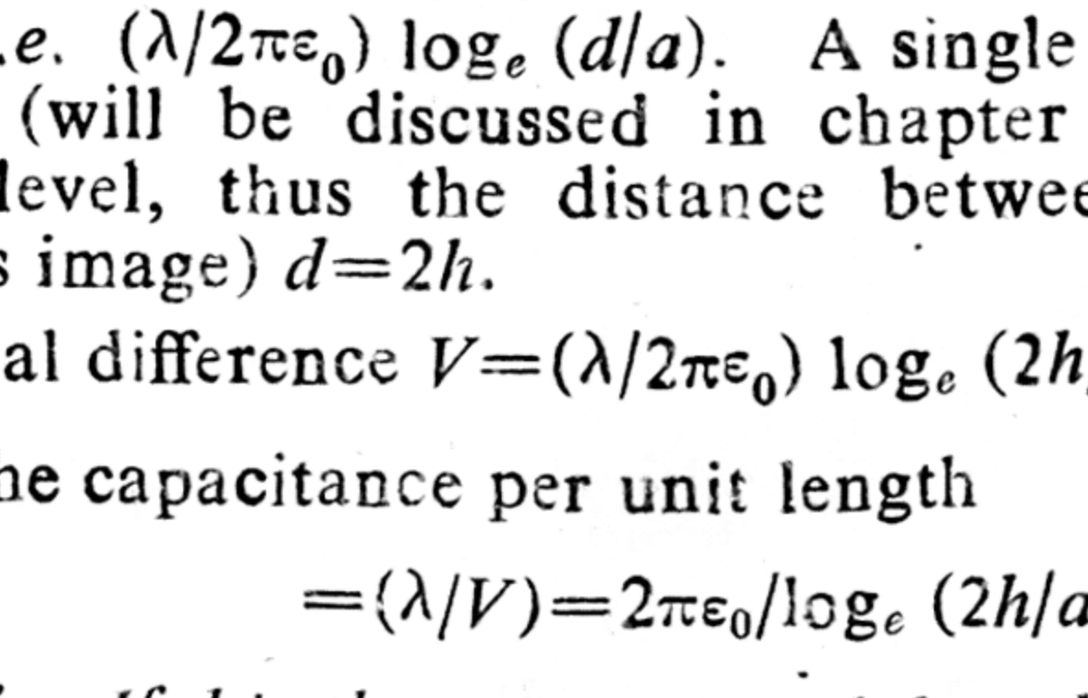


Fig. 4.21.

The capacitance of the capacitor before the slab is inserted

$$C_0=\varepsilon_0\ A/d.$$

When the dielectric slab is inserted between these plates P and Q, its potential difference decreases and is given as

$$V=-\int_Q^P \mathbf{E}.d\mathbf{r}=\int_Q^P Edr=\int_Q^S Edr+\int_S^R Edr+\int_R^P Edr$$

$$=\sigma x/\epsilon_0+\sigma t/\varepsilon_0 k+\sigma\,[d-(t+x)]/\varepsilon_0=\sigma[d-t\ (1-1/k)]\varepsilon_0.$$

$\therefore$ Capacitance of the capacitor after inserting the dielectric

$$C=\sigma A/V=A\varepsilon_0/[d-t\ (1-1/k].$$

Thus the capacitance is increased in the ratio $C : C_0$

$$=d : [d-t\ (1-1/k)]=1 : [1-(1-1/k)\ t/d].$$

If this slab is of metal, for which $k=\infty$, then we find the ratio

$$C/C_0=d/(d-t).$$

This approaches to infinity, if $t=d$ (the space is completely filled with metal).

Example 6. *The plates of a parallel plate condenser are not quite parallel, the separation at one edge being $d+a$ and at the opposite one being $d-a$. Neglecting edge effects, calculate the capacitance of this capacitor, assuming $d \gg a$.*

Let l and b be the length and breadth of each plate. If θ is the angle which plate AB makes with plate CD such that $\frac{1}{2}\theta l=a$.

Let us divide this capacitor in large number of differential capacitors. Consider a capacitor at a distance x from the middle point of the plate and of length dx. The capacitance of this differential capacitor

$$\triangle C=\varepsilon_0 b\ dx/(d+x\theta).$$

As these capacitors are parallel, hence total capicitance

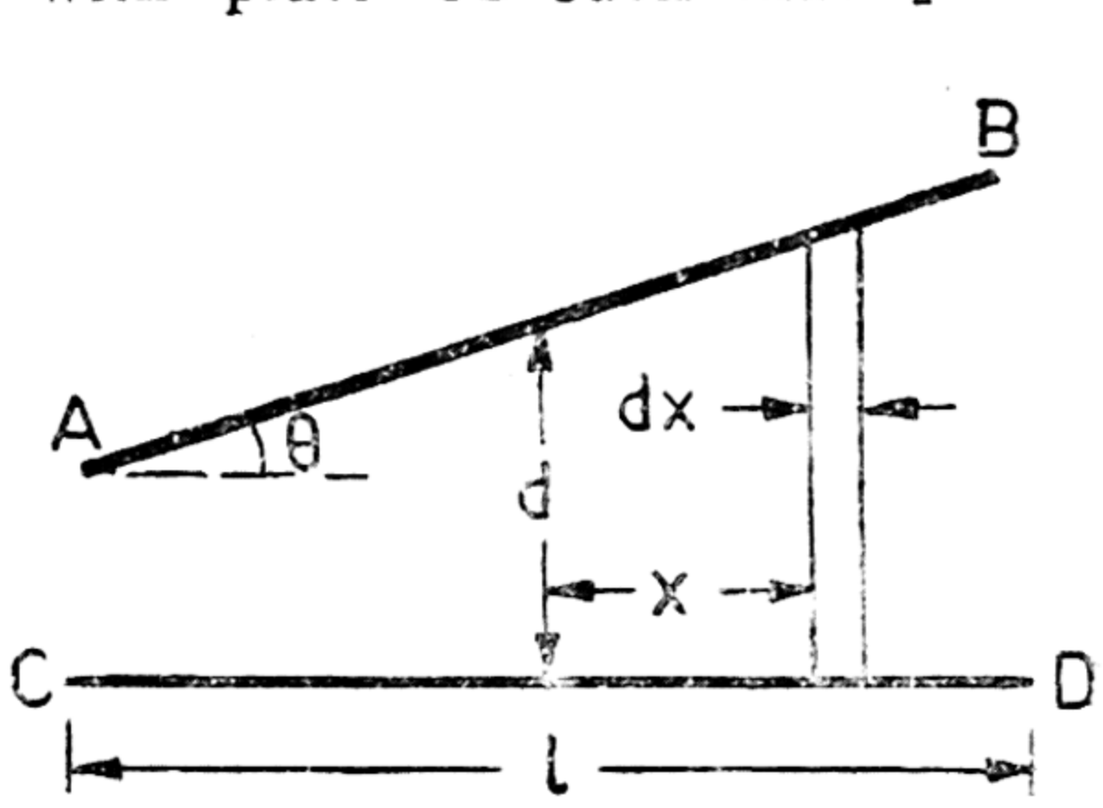


Fig. 4.22.

$$C=\Sigma\triangle C=\Sigma\ \frac{\varepsilon_0 b}{d+x\theta}dx=\varepsilon_0 b\int_{-l/2}^{+l/2}\frac{dx}{d+x\theta}$$

$$=\frac{\epsilon_0 b}{\theta}\Big[\log_e(d+x\theta)\Big]_{-l/2}^{+l/2}=\frac{\varepsilon_0 b}{\theta}\log_e\ \frac{d+\frac{1}{2}\theta l}{d-\frac{1}{2}\theta l}.$$

$$\text{As } \log_e\left(\frac{1+y}{1-y}\right)=2y\left(1+\frac{y^2}{3}\right),\ \tfrac{1}{2}\theta l=a,\ bl=A,$$

$$\therefore \qquad C=\frac{\varepsilon_0 A}{d}\left(1+\frac{1}{3}\frac{a^2}{d^2}\right).$$

Example 7. *A circular disc of radius a is placed with its plane inclined at a small angle α to another plane. The centre of the disc is at a perpendicular distance d from this plane. The disc and the plane are maintained at a potential difference V. Show that the mechanical forces acting on the disc produce a force $V^2a^2\pi\varepsilon_0/2d^2$ perpendicular to its plane and a couple $\pi\varepsilon_0V^2\alpha a^4/4d^3$.*

Let us divide the disc in large no. of strips. Consider a strip of thickness dx at a distance x from the centre of disc. If this strip is at a distance t from the plane, then the intensity E between it and the plane can be considered as constant. The potential difference

$$V=Et=\sigma t/\epsilon_0 \quad \text{or} \quad \sigma=V\varepsilon_0/t.$$

Hence the mechanical force experienced by the elementary strip per unit area $=\sigma^2/2\varepsilon_0=V^2\varepsilon_0/2t^2$.

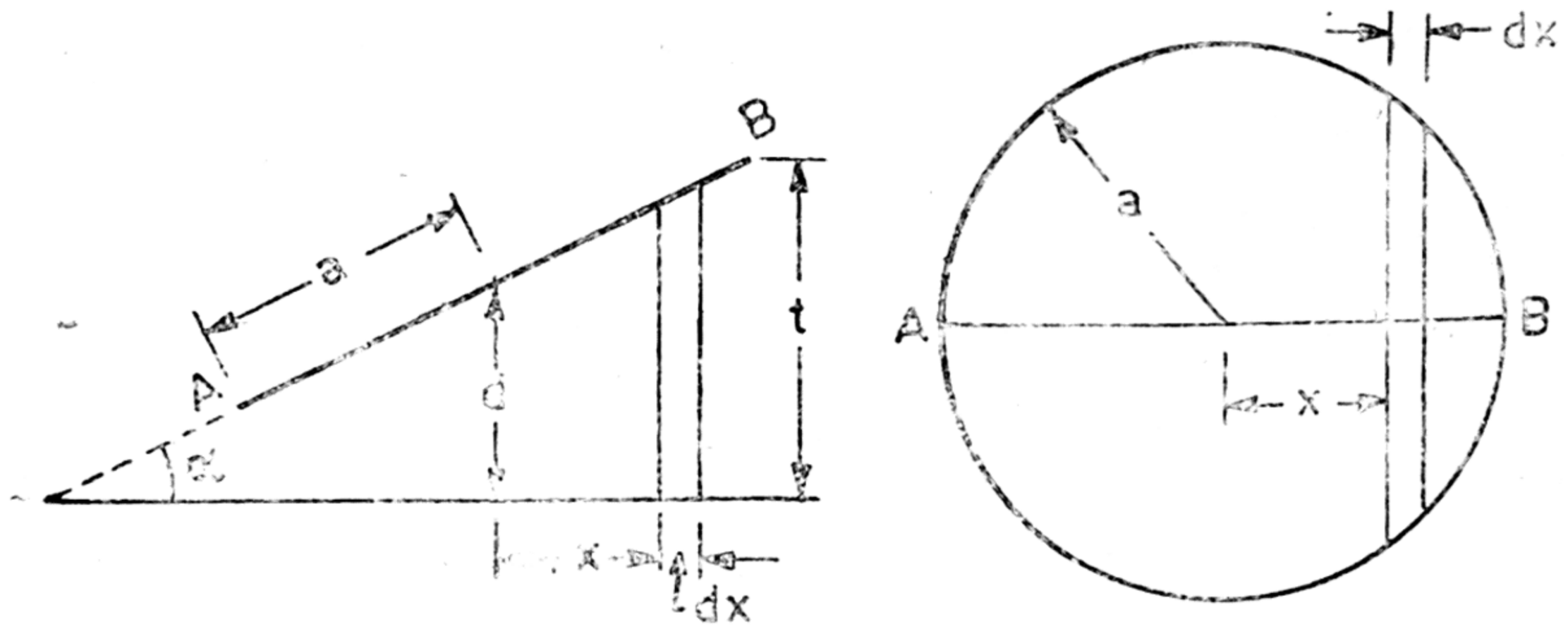


Fig. 4.23.

$\therefore$ Force experienced by this strip $=(V^2\varepsilon_0/2t^2)\ 2(a^2-x^2)^{1/2}\,dx$

$\therefore$ Total force experienced by the circular plate

$$=\int_{-a}^{a}\frac{\epsilon_0V^2\sqrt{(a^2-x^2)}}{(d+x\alpha)^2}dx. \qquad (\text{As } t=d+x\alpha)$$

$$=\frac{\varepsilon_0V^2}{d^2}\int_{-a}^{a}(a^2-x^2)^{1/2}\left(1+\frac{x\alpha}{d}\right)^{-2}dx=\frac{\varepsilon_0V^2}{d^2}\cdot\frac{a^2\pi}{2}$$

$$=\pi\epsilon_0V^2a^2/2d^2.$$

$$\text{Magnitude of the couple}=\int_{-a}^{a}\frac{\varepsilon_0V^2(a^2-x^2)^{1/2}\,dx}{(d+x\alpha)^2}\cdot x$$

$$=\pi\varepsilon_0V\alpha^2a^4/4d^3.$$

Example 8. *Show that if the equation of the rim of the movable plates of a rotary variable capacitor is $r^2\theta^3=$ const., the change in $1/\sqrt{C}$ is proportional to the change in angle of rotation.*

We know that the capacitance of the rotary variable capacitor varies directly as the overlapping area. If the movable plates are semi-circles and the axis of rotation passes through their centres. $A=\frac{1}{2}\theta r^2$, where r is the constant radius of the plates and θ is the angle through which the movable plates have entered the fixed ones. Hence the change in capacitance $dC=$constant $r \cdot r\, d\theta$.

$$\therefore \qquad dC/d\theta = \text{const. } r^2.$$

If $1/\sqrt{C}$ varies linearly with θ, then $\theta^2 C$ will be constant and hence

$$2\theta\, d\theta C+\theta^2\, dC=0 \quad \text{or} \quad dC/d\theta=-2C/\theta=\text{const.}/\theta^3.$$

Equating these values of $dC/d\theta$, we get

$$\text{const. } r^2=\text{const.}/\theta^3 \quad \text{or} \quad r^2\theta^3=\text{const.}$$

This is the required equation of the rim of the movable plates of rotary variable capacitor. This capacitor is known as a *straight-line frequency capacitor*.

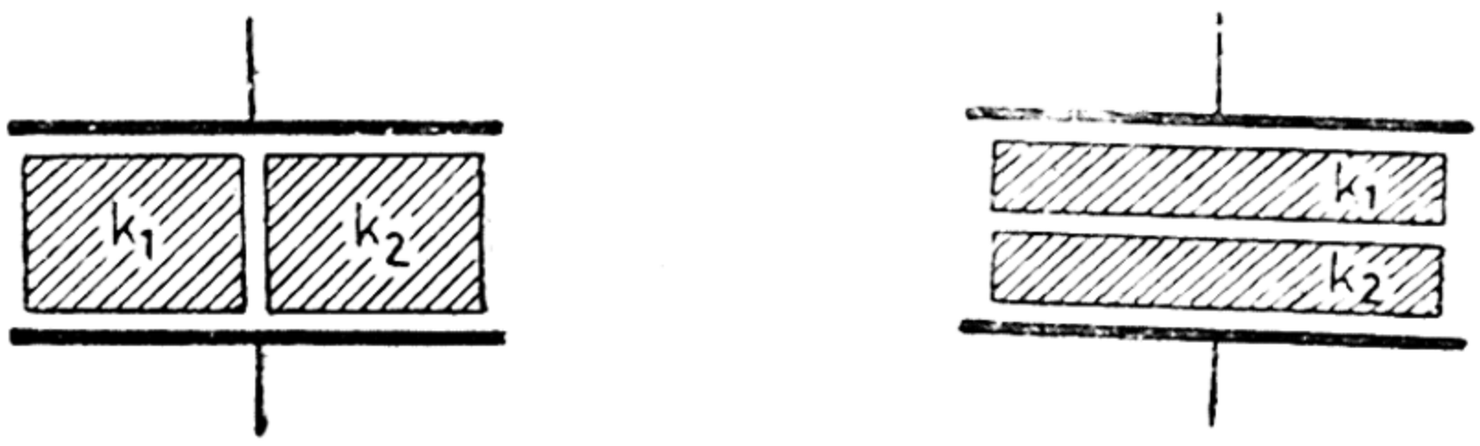
Example 9. *Calculate the capacitance of a capacitor filled with two dielectrics of same dimensions but of dielectric constant k_1 and k_2 respectively.*

Two dielectrics can be inserted in the following two ways :

The arrangement (*a*) can be assumed as two capacitors in parallel. If d is the separation and A is the area of each plate, then each capacitor will be of area $A/2$. The total capacitance

$$C_p=C_1+C_2=\frac{\varepsilon_0(A/2)k_1}{d}+\frac{\varepsilon_0(A/2)\,k_2}{d}$$

$$=\varepsilon_0 A(k_1+k_2)/2d.$$



(*a*) (*b*)

Fig. 4.24.

The arrangement (*b*) can be assumed as two capacitors in series. Each capacitor will be of area A and separation $d/2$.

$$\therefore \qquad \frac{1}{C_s}=\frac{1}{C_1}+\frac{1}{C_2}=\frac{(d/2)}{\epsilon_0 A k_1}+\frac{(d/2)}{\epsilon_0 A k_2}$$

or

$$C_s=\frac{2\varepsilon_0 A}{d}\left(\frac{k_1 k_2}{k_1+k_2}\right).$$

Hence the ratio of capacitances in these arrangements

$$C_s/C_p = 4k_1k_2/(k_1+k_2)^2.$$

Example 10. *A parallel plate capacitor has the space between the plates filled with a medium whose dielectric constant increases uniformly with distance. If d be the distance between the plates and* k_1 *and* k_2 *be the dielectric constants of the medium at the two plates respectively. show that the capacity of the capacitor per unit area is* $C=\epsilon_0(k_2-k_1)/d \log_e (k_2/k_1)$.

Electric intensity at a point between the plates of a parallel plate capacitor is given by $E=\sigma/\epsilon_0 k$.

Here k is not constant but varies from k_1 to k_2. As d is the separation of two plates, hence the increase in dielectric constant per unit length $=(k_2-k_1)/d=\alpha$ (say). Hence the dielectric constant at point at a distance x from the first plate is $k=k_1+\alpha x$ and the electric intensity at this point $=\sigma/\epsilon_0(k_1+\alpha x)$.

$\therefore$ Potential difference between the plates

$$V=-\int_d^0 \mathbf{E}\,.\,d\mathbf{x}=\int_0^d E\,dx=\int_0^d \frac{\sigma\,dx}{\epsilon_0(k_1+\alpha x)}$$

$$=\frac{\sigma}{\alpha\epsilon_0}\log_e\frac{k_1+\alpha d}{k_1}=\frac{\sigma d}{\epsilon_0(k_2-k_1)}\log_e\frac{k_2}{k_1}$$

Hence the capacitance $C=\dfrac{\sigma A}{V}=\dfrac{A\epsilon_0(k_2-k_1)}{d\log_e(k_2/k_1)}$

or the capacitance per unit area $=\epsilon_0(k_2-k_1)/d\log_e(k_2/k_1)$.

Example 11. *A 100 μf capacitor is charged to a potential of 50 volts and the charging battery is then disconnected. This condenser is then connected in parallel with second capacitor. If the potential difference drops to 35 volts, calculate the capacitance of the second capacitor.*

The charge on the first capacitor $q_1=C_1V=100\times10^{-6}\times50$ coul. When this is connected in parallel to another capacitor of capacitance C_2, the capacitance of the combination becomes C_1+C_2, while the total charge will remain q_1. Therefore the potential difference across the combination will be $V_c=q_1/(C_1+C_2)$.

In the present problem $V_c=35$ volts. Therefore

$$35=\frac{5\times10^{-3}}{100\times10^{-6}+C_2}$$

or
$$C_2=42.8\ \mu f.$$

Example 12. *A potential of 50 volts is applied between two parallel plates of a condenser which are 4 cm. apart. Obtain the force acting on charge of* 4.3×10^{-7} *coulomb placed between the plates.*

The electric field in between the parallel plate condenser due to a potential V applied across the plates separated by a distance d, is given by

$$E=V/d=50/0.04=1250 \text{ volts/meter.}$$

The force acting on a charge q placed between the plates is the force due to this field E, *i.e.*,

$$F=qE=4.3\times10^{-7}\times1250=5.375\times10^{-4} \text{ newton .}$$

Example 13. *A condenser consists of two metallic discs, each 1 metre in diameter, placed parallel to each other at a distance 4 mm apart. The potential difference between the plates is 10,000 volts. Calculate the energy stored by the condenser.*

The capacitance of a parallel plate condenser with plate area A, separated by a distance d is given by

$$C=\varepsilon_0 A/d.$$

The energy stored by the condenser

$$U=\tfrac{1}{2}CV^2=\tfrac{1}{2}(\varepsilon_0 A/d)V^2.$$

Substituting numerical values as

$$\varepsilon_0=8.85\times10^{-12} \text{ coul}^2/\text{newton-}m^2,$$
$$A=\pi r^2=3.14\times(0.5)^2 m^2,$$
$$d=0.004 m,$$
$$V=10{,}000 \text{ volts,}$$

and we get

$$U=\tfrac{1}{2}\times8.85\times10^{-12}\times3.14\times(0.5)^2\times10{,}000/0.004$$
$$=8.68\times10^{-6} \text{ joules.}$$

Example 14. *Prove that the energy loss is always positive on sharing of charges in two condensers.*

Let us consider two capacitors A and B of capacitances C_1 and C_2 and having charges q_1 and q_2 respectively. If these are connected by a wire, the charge will flow from the higher potential to lower potential. After sharing, let V be the common potential and q_1' and q_2' be the charges on A and B capacitors respectively.

$\therefore$ Total charge $q_1+q_2=C_1V_1+C_2V_2=(C_1+C_2)V$,

or
$$V=\frac{C_1V_1+C_2V_2}{C_1+C_2}=\frac{q_1'}{C_1}=\frac{q_2'}{C_2}.$$

$\therefore$
$$q_1'=C_1\ \frac{C_1V_1+C_2V_2}{C_1+C_2}$$

and
$$q_2'=C_2\ \frac{C_1V_1+C_2V_2}{C_1+C_2}.$$

Hence total energy before contact $=\frac{1}{2}C_1V_1^2+\frac{1}{2}C_2V_2^2$

and energy after contact $=\frac{1}{2}[C_1+C_2]V^2=\frac{1}{2}(C_1V_1+C_2V_2)^2/(C_1+C_2)$.

$\therefore$ Loss of energy due to sharing $=\frac{1}{2}C_1C_2(V_1-V_2)^2/(C_1+C_2)$.

It is always positive, *i.e.*, there is always a loss of energy on sharing of charges. As total charge remains same, hence this loss appears in the form of a spark and heat.

Example 15. *Two capacitors AB and CD have capacities C_1 and C_2 respectively. They are uncharged initially and A is permanently connected to earth. The two capacitors are connected in series and D is charged to a potential V. CD is then disconnected and joined in parallel with AB. Show that the loss in energy is*

$$\frac{1}{2}V^2C_1C_2(C_1-C_2)^2/(C_1+C_2)^3.$$

When the capacitors are in series, the potential of the plates B and C are each equal to V' (say) and that of the plate D is V (given). If $+q$ be the charge on plate D, then the induced charges over plates C, B and A will be $-q$, $+q$ and $-q$ respectively. From the definition of capacity

$$V-V'=q/C_2 \quad \text{and} \quad V'-o=q/C_1.$$

$\therefore$ $$q=C_1C_2\,V/(C_1+C_2)=CV.$$

The electrostatic energy of the combination $W=\frac{1}{2}CV^2$

$$=\frac{1}{2}C_1C_2V^2/(C_1+C_2).$$

When the capacitor CD is disconnected and connected with AB in parallel, the capacity of the combination $C=C_1+C_2$. The charges are $+2q$ and $-2q$. Therefore the electrostatic energy

$$W'=\frac{1}{2}\,CV^2=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}\,\frac{(2q)^2}{C_1+C_2}$$

$$=2C_1^2C_2^2V^2/(C_1+C_2)^3.$$

$\therefore$ Loss of energy $=W-W'=\frac{1}{2}\,C_1C_2(C_1-C_2)^2V^2/(C_1+C_2)^3$.

Example 16. *Outside a spherical charged conductor A, there is a concentric insulated but uncharged conducting spherical shell, which consists of two segments. Prove that the two segments will not separate if the distance of the separating plane from the centre is less than $bc/\sqrt{(b^2+c^2)}$, where b and c are the inner and outer radii of the shell respectively.*

If q is the charge on the conductor then the induce charges on the inner and outer surfaces of the shell will be $-q$ and $+q$ respectively. The surface charge densities σ_1 and σ_2 are thus given as

$$\sigma_1=-q/4\pi b^2,\ \sigma_2=q/4\pi c^2.$$

If this shell is divided into two segments by the plane RS, then by symmetry the force acting on either segment will be $\perp$ to the plane RS. Let us divide the segment RBS into annular discs by planes $\parallel$ to RS and now consider two closed discs at distances x and $x+dx$ from the centre O. If P and P' be the points of intersection of the plane at x with the inner and outer surfaces of the shell and $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit vectors along PO and OP' respectively. We know that the mechanical force per unit area is given by $\sigma^2/2\epsilon_0$, hence the forces on an element of the inner surface dS_1 at P and on an element of the outer surface dS_2 at P' are $\sigma_1^2\, dS_1\, \mathbf{n}_1/2\epsilon_0$ and $\sigma_2^2\, dS_2\, \mathbf{n}_2/2\epsilon_0$ respectively. Since the resultant force on the segment RBS is along OB. The components along OB are $-\sigma_1^2\, dS_1 \cos\theta_1/2\epsilon_0$ and $\sigma_2^2\, dS_2 \cos\theta_2/2\epsilon_0$. Summing over all such elements on the two rings through P and P', the force along OB acting on the elementary annular disc is.

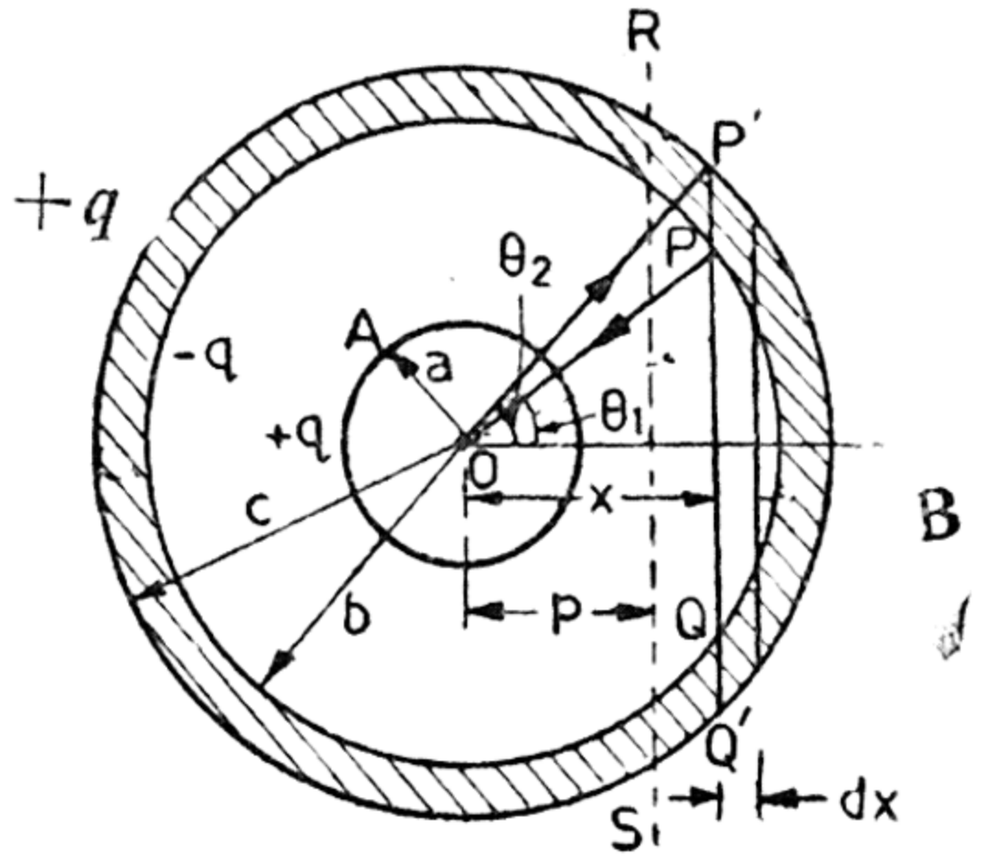


Fig. 4.25.

$$(\sigma_2^2 2\pi c^2 \sin\theta_2 \cos\theta_2\, d\theta_2 - \sigma_1^2 2\pi b^2 \sin\theta_1 \cos\theta_1\, d\theta_1)/2\epsilon_0$$

as $dS_1 = 2\pi b \sin\theta_1\, b d\theta_1$ and $dS_2 = 2\pi c \sin\theta_2\, c\, d\theta_2$.

From the geometry of the Fig. *4.25*, we have

$$b\cos\theta_1 = c\cos\theta_2 = x$$

and hence $b \sin\theta_1\, d\theta_1 = c \sin\theta_2\, d\theta_2 = -dx$.

If p be the distance of RS from the centre O, then the total force on the segment RBS along OB is

$$\frac{1}{2\epsilon_0}\left[\int_p^c \sigma_2^2 2\pi x\, dx - \int_p^b \sigma_1^2 2\pi x\, dx\right].$$

The segment will not separate if this expression is negative, *i.e.*,

$$\sigma_2^2\int_p^c x\, dx < \sigma_1^2\int_p^b x\, dx \quad \text{or} \quad \frac{1}{c^4}(c^2-p^2) < \frac{1}{b^4}(b^2-p^2)$$

$$\therefore \quad p < bc/(b^2+c^2)^{1/2}.$$

Example 17. *An insulated spherical conductor, formed of two hemispherical shells in contact, whose inner and outer radii are b and b', has within it a concentric spherical conductor of radius a and outside of it another concentric spherical conductor of radius c. These two conductors are earthed connected and the middle one receives a charge. Show that the two shells will not separate if* $2ac > bc + b'a$.

If C_1 be the capacity between the surfaces $r=a$ and $r=b$, C_2 that between $r=b'$ and $r=c$.

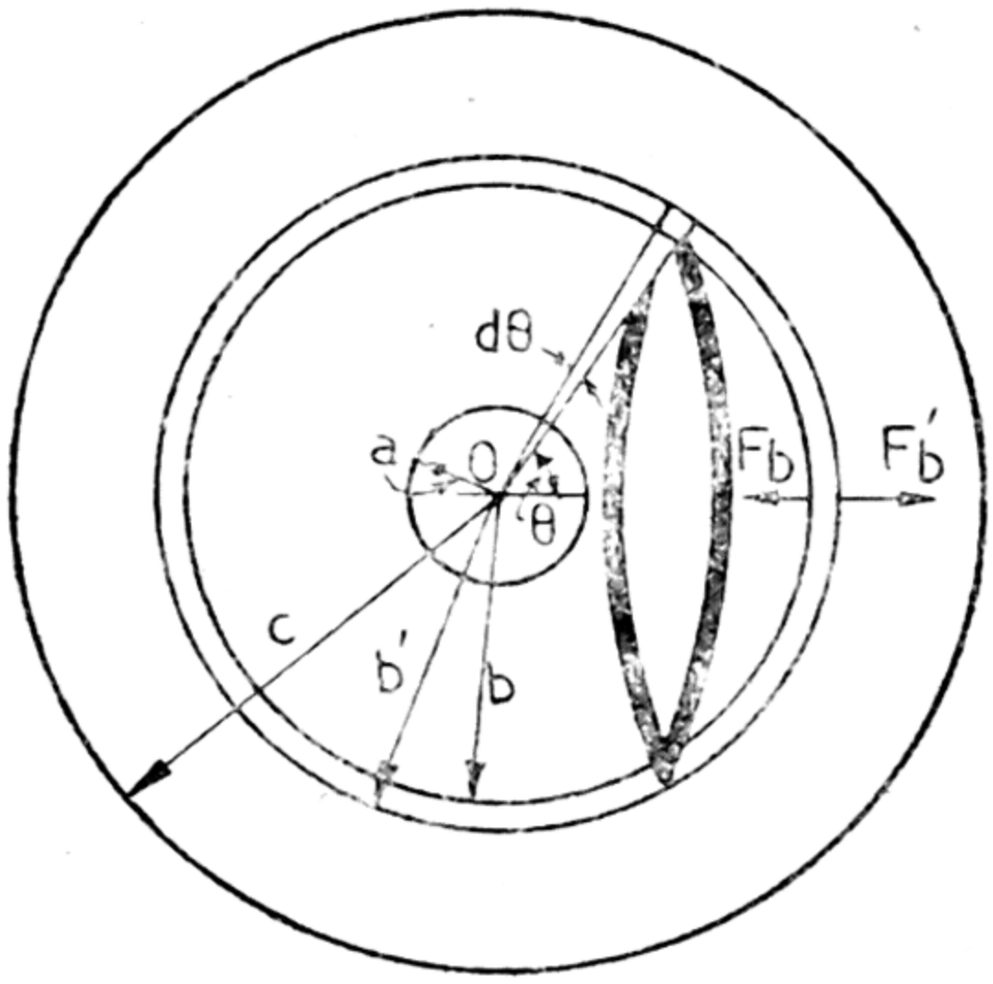


Fig. 4.26.

$$\therefore \quad C_1=4\pi\varepsilon_0\, ab/(b-a)$$

and $$C_2=4\pi\epsilon_0\, b'c/(c-b')$$

As both the surfaces of the middle conductor are at the same potential (say V). The charges shared by both surfaces are

$$q_b=C_1V=4\pi\varepsilon_0\, abV/(b-a)$$

and $$q_b'=C_2V=4\pi\varepsilon_0 b'cV/(c-b').$$

$$\therefore \quad \sigma_b=q_b/4\pi b^2=\varepsilon_0 aV/b(b-a)$$

and $$\sigma_b'=q_b'/4\pi b'^2=\varepsilon_0 cV/b'(c-b')$$

By symmetry, the force acting on either hemisphere is perpendicular to the vertical diametral plane. The force on the inner or outer hemispherical shell can be determined by breaking its surface into a large number of concentric rings by planes parallel to the plane and then summing up the mechanical force on each ring. Thus

$$F_b=\int_0^{\pi/2}\frac{\sigma_b^2}{2\varepsilon_0}(2\pi b\sin\theta)(bd\theta)\cos\theta=\frac{\pi b\sigma_b^2}{2\epsilon_0}.$$

Similarly the force on the outer surface of $F_b'=\pi b'^2\,\sigma_b'^2/2\varepsilon_0$.

The two shells will not separate if $F_b>F_b'$, *i.e.*,

$$\frac{\pi b^2\sigma_b^2}{2\varepsilon_0}>\frac{\pi b'^2\sigma_b'^2}{2\epsilon_0} \quad \text{or} \quad \frac{b^2a^2}{b^2(b-a)^2}>\frac{b'^2c^2}{b'^2(c-b')^2}.$$

$$a^2/(b-a)^2>c^2/(c-b')^2 \quad \text{or} \quad 2ac>bc+ab'.$$

Example 18. *An attracted disc electrometer has a moving plate of area 100 cm², separated by a distance of 1 mm from the fixed plate. Calculate the force between the plates when the potential difference across them is 100 V. Calculate the sensitivity at this voltage in newton's per volt.*

The force between the plates of an attracted disc electrometer is given by

$$F=\frac{\varepsilon_0V^2A}{2d^2}=\frac{8.85\times10^{-12}\times100^2\times100\times10^{-4}}{2\times(1\times10^{-3})^2}$$

$$=4.42\times10^{-4}\text{ newton.}$$

The sensitivity of this electrometer rises with the voltage V and is given by

$$\frac{dF}{dV}=\frac{\epsilon_0VA}{d^2}=\frac{8.85\times10^{-12}\times100\times100\times10^{-4}}{(1\times10^{-3})^2}$$

$$=8.85\times10^{-6}\text{ newton/volt.}$$

Example 19. *The needle of a quadrant electrometer deflects through 20 scale divisions, when it is connected idiostatically to a potential of 50 volts. It is then used heterostatically, needle is connected to 500 volts and the quadrants are maintained at smaller voltage. Find the voltage on the quadrants, if the deflection in the latter case reduces to 10 scale divisions.*

In the idiostatic use of electrometer, the needle is connected to one pair of quadrants, say, $V_0=V_1$ and

$$\theta=\tfrac{1}{2}K(V_1-V_2)^2$$

or
$$K=2\theta/(V_1-V_2)^2=2\times 20/50^2$$
$$=1.6\times 10^{-2}.$$

In the heterostatic use $V_0 \gg V_1$ (or V_2)

and
$$\theta=KV_0(V_1-V_2),$$

or the potential difference between two pairs of quadrants

$$V_1-V_2=\frac{\theta}{KV_0}=\frac{10}{1.6\times 10^{-2}\times 500}=1.25 \text{ volts.}$$

Oral Questions :

1. A capacitor is connected across a potential source. (*a*) Why does each plate receive a charge of same magnitude ? (*b*) Is it true even if the plates are of different sizes ?

2. When one grasps the two wires leading from the two plates of a charged capacitor, he feels a shock of much more strength for a capacitor of greater capacitance but charged to the same potential difference. Explain the cause of sensation of shock given to the body.

3. How do you explain the effect of (*a*) a dielectric slab, (*b*) a conducting slab, when inserted between the plates of a parallel plate condenser ?

4. For a given potential difference does a capacitor store more or less charge with a dielectric than it does without a dielectric ?

5. What will happen when a dielectric slab is inserted in one end of a charged parallel-plate capacitor ?

6. What do you expect about the change of mass of an isolated conducting sphere if given a positive charge ?

7. A potential source is connected with a capacitor and is then disconnected. What happens to the charge, the capacitance and the potential difference when a dielectric slab is inserted in between the plates of the capacitor ?

8. Keeping the capacitor in contact with a battery, a dielectric slab is slipped between the plates. Describe what happens to the charge, the capacitance, the electric field strength and the stored energy.

9. A parallel plate capacitor is charged to a charge Q. It is then disconnected from the battery, after which the plates are slowly pulled apart until the plate separation is doubled. Does Q change? Does the energy stored in the capacitor change ?

10. How can you increase the sensitivity of the quadrant electrometer ?

Problems :

1. A certain diode vacuum tube consists of a 0.2 mm diameter wire (the filament) on the axis of a 1.00 cm inner diameter hollow cylindrical plate. Both the plate and filament are 5.0 cm long. Find the capacitance of the vacuum tube. $(0.71 \times 10^{-12} f)$

2. A $1 \mu f$ capacitor and a $2 \mu f$ capacitor are connected in parallel across a 1200 V supply line. (*a*) Find the charge on each capacitor and the voltage accross each, (*b*) The charged capacitors are disconnected from the line and from each other, and reconnected with terminals of like sign together. Find the final charge on each and the voltage across each.

[(*a*) 800 μC, 800 V ; 800 μC, 400 V
(*b*) 533 μC, 533 V ; 1066 μC, 533 V]

3. In a variable air capacitor which is a pile of n plates of alternate polarity each having an area A and separated from adjacent plates by a distance d, alternate plates are connected together, one group being fixed in position and the other group being capable of rotation. Show that the maximum capacitance is $(n-1)\ \varepsilon_0 A/d$.

4. A condenser is formed of a conducting sphere of radius r_0 and a conducting spherical shell of internal radius r, insulated from it. If n conducting conceatric spherical shells are placed between these two spherical conductors, find the capacitance of the whole arrangement.

$4\pi\varepsilon_0/[1/r_0 - \Sigma(1/a_i) - 1/r + \Sigma(1/b_i)]$, where a_i and b be the inner and outer radii of i^{th} shell.

5. A $100 \mu\mu f$ capacitor is charged to 100 volts. After the charging battery is disconnected, the capacitor is connected in parallel to another capacitor. The final voltage is 20 volts. Calculate the capacity of second capacitor. $(233.3 \mu\mu f)$

6. A metal sphere of radius a is surrounded by a spherical thick metal shell of inner and outer radii b and c. This shell is isolated with no net charge and is surrounded by a grounded spherical shell of radius d. Find the potential of the inner sphere, when a charge q is placed on it. What is the capacitance of the sphere ?

$$C = 4\pi\varepsilon_0/(1/a + 1/c - 1/b - 1/d).$$

7. A capacitor is formed of two concentric spherical conducting shells of radii a and b. If the medium between the spheres has a dielectric constant k_1 from a to r and k_2 from r to b, show that the capacity of the condenser is given by

$$C = 4\pi\varepsilon_0 \Big/ \left[\frac{1}{k_1 a} + \frac{1}{k_2 b} + \frac{1}{r}\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\right].$$

8. A capacitor has square plates each of side a making an angle of θ with each other. Show that the capacitance is given by

$$C = \frac{\varepsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right),$$

where d is the minimum distance between the plates at one end.

9. Find the capacity of a sphere of 15 cm diameter inside which there is an earthed concentric sphere of 10 cm diameter. $(25 pf)$

10. At what distance should the circular plates of a parallel plate condenser, 40 mm in diameter, be placed in order to have the same capacity as a sphere of 100 cm. (0.1 mm)

11. A parallel plate condenser has circular plates of radius 8 cm and plate separation 1mm. What will be the charge on the plates if a potential difference of 100 volts is applied ? (1.78×10^{-8} coul.)

12. Two capacitors ($2\mu f$ and $4\mu f$) are connected in parallel across a 300 volts potential difference. Calculate the total stored energy in the system and the total charge on it. (0.27 joule ; 18×10^{-4} coul)

13. Calculate the force between the plates of parallel plate capacitor, when the area of the plate is 300 cm^2 each, the separation is 0.5 cm and they are charged to p. d. 1000 volts. (5.31×10^{-3} newton)

14. Show that if the equation of the rim of the movable plates of a rotary variable condenser is $\theta r^{-2} = \text{const.}$, the change in $\sqrt{C}$ is proportional to the change in angle of rotation.

15. A parallel plate capacitor, with area A and separation d, is connected with a battery of V_0 volt. The battery is then disconnected and a slab of dielectric constant k and of thickness d is introduced. Calculate the energy stored before and after the slab is introduced. ($\frac{1}{2}C_0V_0^2, \frac{1}{2}C_0V_0^2/k$)

16. Two capacitors A and B have capacities C_1 and C_2. Capacitor A is charged by a battery and the battery is removed, it is discharged by allowing a spark to pass between the plates. Capacitor A is then charged as before, the battery is removed and the capacitor B is connected in parallel with A, a spark is appeared as the connection is made. These capacitors are then separated and each is discharged by a spark. Show that the energies of the four sparks are in the ratio.

$$(C_1+C_2)^2 : C_2(C_1+C_2) : C_1^2 : C_1C_2.$$

17. A circular gold leaf of radius b is laid on the surface of a charged conducting sphere of radius a $(a \gg b)$. Prove that the loss of electrical energy in removing the leaf from the conductor, assuming that it carries away its whole charge, is approximately $\pi\varepsilon_0 q^2 b^2/a^3$ where q is the charge of the conductor and the capacity of the leaf is comparable to b.

18. A cylindrical capacitor has radii a and b. Show that half the stored electrostatic energy lies within a cylinder whose radius is $\sqrt{(ab)}$.

19. Two capacitors of capacity C_1 and C_2 possessing charges q_1 and q_2 respectively are connected in parallel. Show that there is a loss of electrostatic energy amounting to $(C_2q_1 - C_1q_2)^2/2C_1C_2(C_1+C_2)$.

20. A line charge of linear density λ is placed in a medium of dielectric constant k_1 parallel to and at a distance a from the plane boundary with another medium of dielectric constant k_2. Find the potentials in the two media and show that the force per unit length on the line charge is given by

$$F = \frac{\lambda^2}{4\pi\varepsilon_0 a}\left(\frac{k_1-k_2}{k_1+k_2}\right)\frac{1}{k_1}.$$

21. A conducting sphere of radius r is electrified to a potential V. If the sphere consists of two separate hemispheres, show that the force between them is $\pi\varepsilon_0 V^2/2$ and if the whole be surrounded by an uninsulated concentric

spherical conductor of internal radius R and the potential of the solid sphere is still V, prove that the force between the hemispheres is $\pi\varepsilon_0 V^2 R^2/2\,(R-r)^2$.

22. A capacitor is formed by two coaxial cylinders of radii a and b. The axes of the cylinders are vertical and the inner cylinder is suspended from a balance so that it hangs only partly within the outer cylinder. Find an expression for the mass which must be added to the other pan of the balance to maintain equilibrium when a voltage V is connected between the two cylinders. $mg=\pi\varepsilon_0\ V^2/\log_e(b/a)$

23. The upper disc of an attracted disc electrometer is suspended by a spring. In equilibrium, the separation between the two discs is x when a voltage V is applied and 'a' when $V=0$. Show that the equilibrium in the former case is stable provided that $x > 2a/3$.

5

Dielectrics

5.1. DIELECTRICS

Upto this chapter we have considered the problems of electrostatics in the absence of matter. We must now consider the phenomena in a medium other than empty space (vacuum) such as solid or liquid *insulator, alternatively called dielectric.* The theory of dielectric was begun by Faraday and subsequently developed by Maxwell. From Faraday's experiments (*article* 4.3) we see that (*i*) if two capacitors are charged to the same potential, the charge on the one containing the dielectric is greater than that of the other, and (*ii*) if the capacitors are given the same charge, the *pd* between the plates with the dielectric is smaller than that without the dielectric. These results show that the capacitance of the condenser is increased by a factor k which is greater than one. This factor is known as *dielectric constant or specific inductive capacitance* and is independent of the shape and size of the capacitor, but varies widely for different materials (medium). The following table gives the value of k for different materials.

Table 5.1. Dielectric Constant k

Material	*k*	*Material*	*k*
Vacuum	1.000	Mica (25°C)	3—6
Air (1 atm., 20°C)	1.006	Glass (25°C)	5—10
Air (100 atm., 20°C)	1.0548	Bakelite (27°C)	5.5
Paraffin (20°C)	2.1	(88°C)	18.2
Petroleum oil (20°C)	2.2	Germanium (20°C)	16
Benzene (20°C)	2.284	Liq. Ammonia (—78°)	25
Polystyrene (20°C)	2.6	Glycerin (25°C)	42.5
Ice (—5°C)	2.9	Water (25°C)	78.54
Rubber (27°C)	2.94	Titanium dioxide	100
Paper (25°C)	3.5	Metal	∞

The comparatively high value for water suggests that water is a poor insulator, water is in fact a *semiconductor.*

The function of a solid dielectric between the plates of a capacitor is threefold. (*a*) *It helps in maintaining two large metal plates at a very small separation.* (*b*) *It increases the maximum potential difference which the capacitor can withstand without breakdown.* (*c*) *It increases the capacitance of a capacitor of given dimensions.*

5.2. DIELECTRIC AN ATOMIC VIEW

In any molecule there will be a distribution of protons and a distribution of electrons. The protons can be considered to act as the equivalent positive electric charge at some specific point in the molecule, the *centre of gravity* of the protons. Similarly there is a point at which a negative charge is equivalent to the distributed electrons, the *centre of gravity* of electrons. If these two points coincide then the molecule is called *non-polar* and if they are separated by a short distance then the molecule is called *polar* molecule. Symmetrical molecules (*e.g* , H_2, N_2 and O_2) are *non-polar.* On the other hand molecules H_2O, N_2O, etc., are *polar*, as both hydrogen atoms or both nitrogen atoms lie on the same side of the oxygen atom. Thus the polar molecules work as dipoles and have dipole moments. If the dielectric (with non-polar molecules) is placed in an electric field, the charge centres of a non-polar molecule become displaced (Fig. 5.1). The molecules are then said to become *polarized* by the

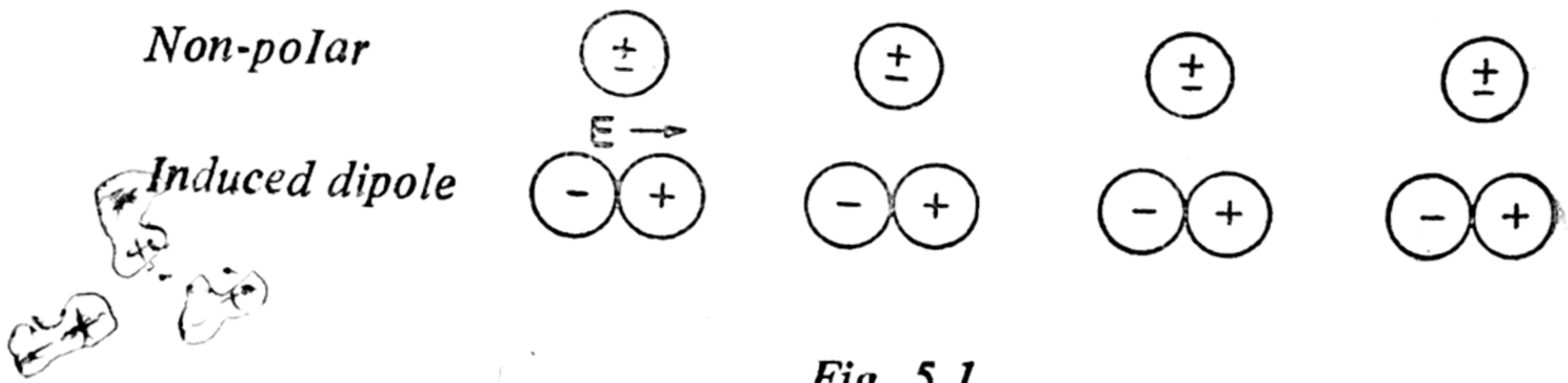


Fig. 5.1.

field and are called *induced dipoles.* The displacement is however limited by strong restoring force produced by the charge configuration in the molecule.

In a dielectric consists of polar molecules (permanent dipoles), the dipoles are oriented at random in the absence of an external

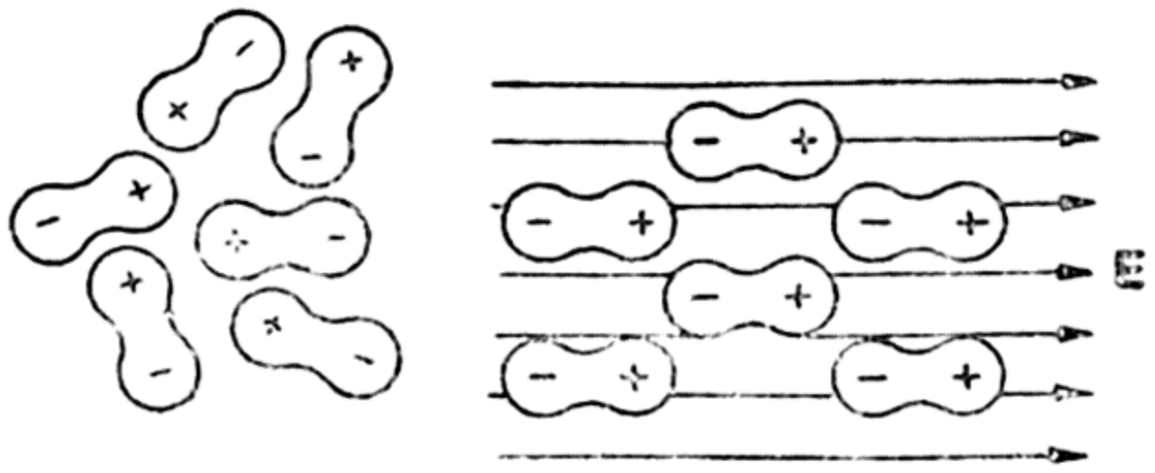


Fig. 5.2.

electric field. When an electric field is applied, the forces on a dipole give rise to a couple, whose effect is to orient the dipole *along the direction of electric field* (Fig. 5.2). The stronger the field, the greater is the aligning effect. Thus non-polar molecules become induced dipoles, whereas polar molecules are oriented by the field and therefore have their dipole moments increased. The orientation of the induced dipoles or of the permanent dipoles in an external electrostatic field is such as to set the axis of the dipole along the field. This phenomenon is known as *electric polarization*.

The main difference in above two mechanisms is the temperature dependence. The polarization of non-polar molecules is independent of temperature. As polar molecules are undergoing thermal motion, hence are randomly oriented. Thus the polar molecules can be aligned perfectly with the smallest external electric field at about absolute zero.

Few Examples of Non-Polar Molecules. In a molecule of CO_2 the oxygen ions are symmetrically placed with respect to the carbon ion, hence the net dipole moment is zero (Fig. 5.3*a*). If

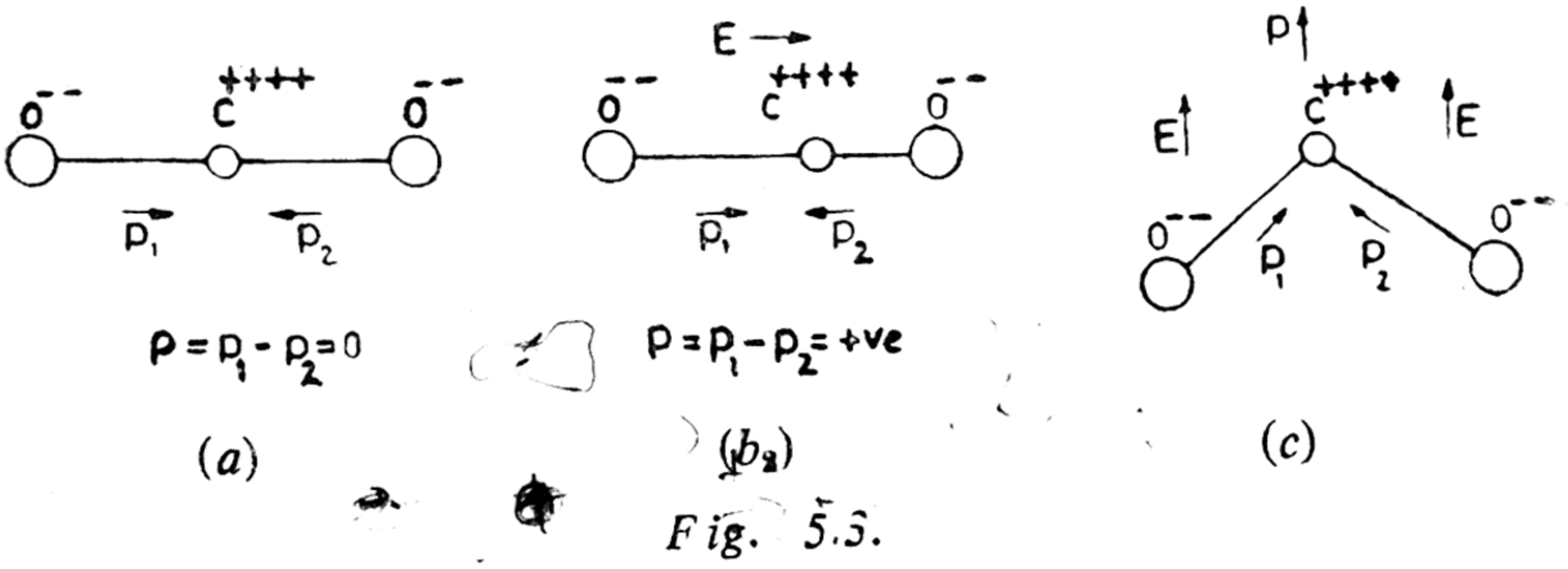


Fig. 5.3.

this molecule is placed in an electric field **E** along the line joining the ions, the oxygen ions get displaced with respect to the carbon ions and the net dipole moment (induced) is along the direction of **E** (Fig. 5.3*b*). If the electric field **E** is applied perpendicular to the line joining the ions the direction of induced dipole moment is again along the field **E** (Fig. 5.3*c*).

Carbon tetrachloride (CCl_4) is another example of non-polar molecules. The external electric field changes the orientation of the $C—Cl$ bond and thus produces induced dipole moment. Similarly H_2, O_2 and N_2 are non-polar molecules.

Few Examples of Polar Molecules. In a molecule of HCl, there is an excess of positive charge on the H-ion and an equal negative charge on the Cl-ion. This molecule, therefore, has a dipole moment at every instant of time and is a polar molecule.

Another interesting example of polar molecules is H_2O. In the water molecule two $O—H$ bonds are not opposite to each

other, unlike the case of CO_2 molecule, but are inclined at an angle of about 105°. As shown in Fig. 5.4, hydrogen ion forms a dipole moment with each of the oxygen ion, and there is net dipole moment ($\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2$).

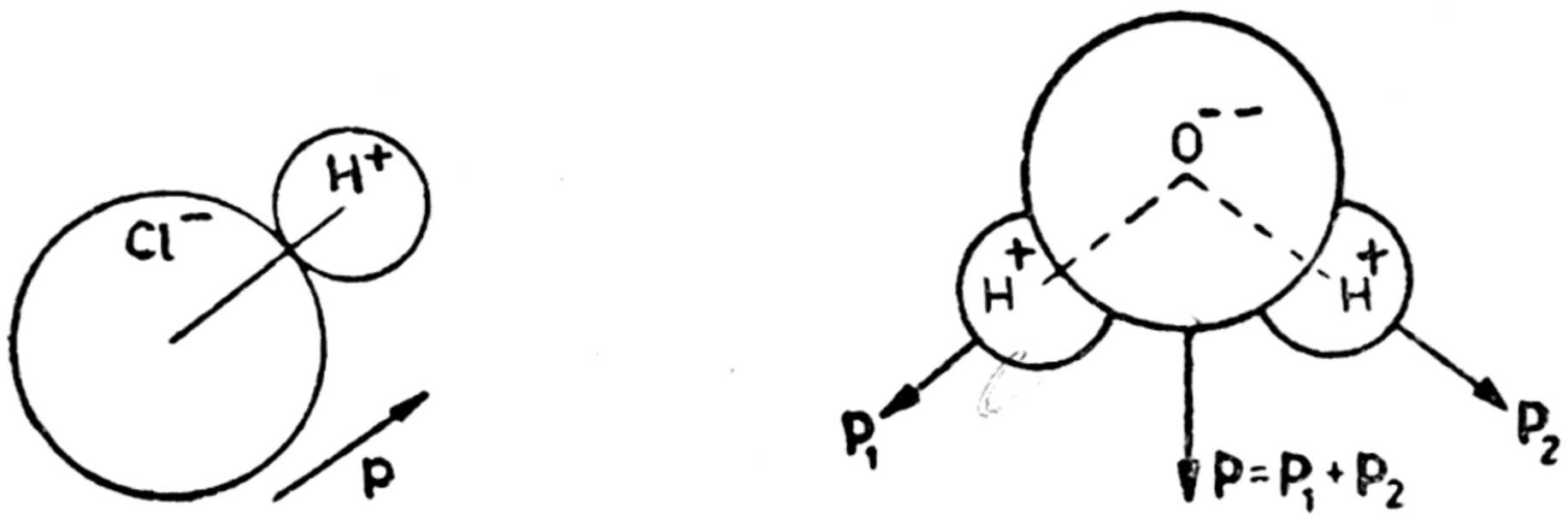


Fig. 5.4.

If a thin slab of dielectric, molecules of which are polar or non-polar, is placed in an electric field the slab as a whole becomes polarized. Due to the polarization, molecules are oriented so that the negative charges are in the left and positive charges are in the right, thus leaving the net positive charge at the right surface and the net negative charge at the left surface. Within the remainder of the dielectric net charge is zero. As the slab as a whole remains electrically neutral, the positive induced surface charge must be equal in magnitude to the negative induced surface charge. The charges so obtained on the surfaces, perpendicular to the direction of electric field **E**, are known as *polarization charges*. These charges are not free, but each is bound to a molecule lying in or near surface, that is why these are also named as *bound charges* or *fictitious charges*.

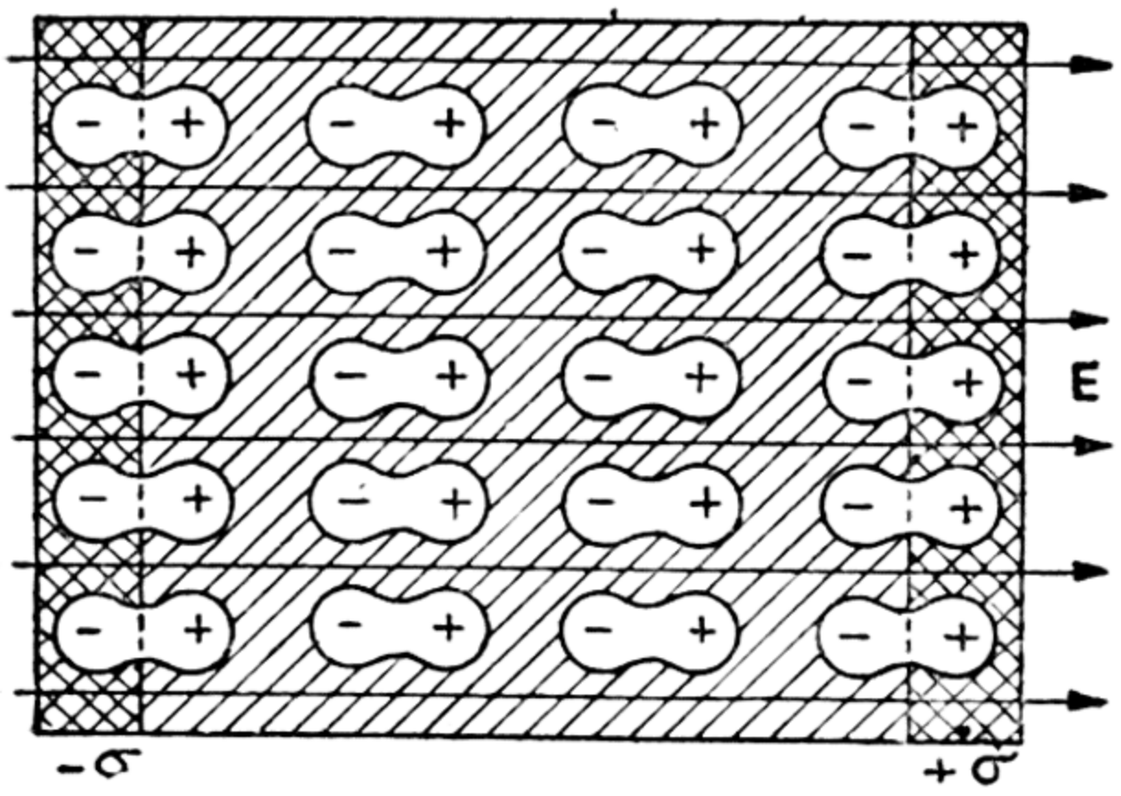


Fig. 5.5.

Fig. 5.5. shows that the induced charges always appear in such a way that the electric field set up by these charges is in opposite direction to the external electric field **E**. Thus when a dielectric is placed in an electric field, the induced surface charges weaken the original field within the dielectric.

5.3. FORCES AND TORQUES ON DIPOLES

Before proceeding to polarized molecules in bulk, let us calculate expressions for the force **F** and the torque τ on a dipole in an external field. Let a dipole of length a with charges $+q$ and q

is placed in a uniform external electric field **E**. If this dipole makes an angle θ with the direction of field and field of the dipole is negligible then the force acting on the poles of the dipole

$$\mathbf{F}=q\mathbf{E}$$

and the torque acting on the dipole

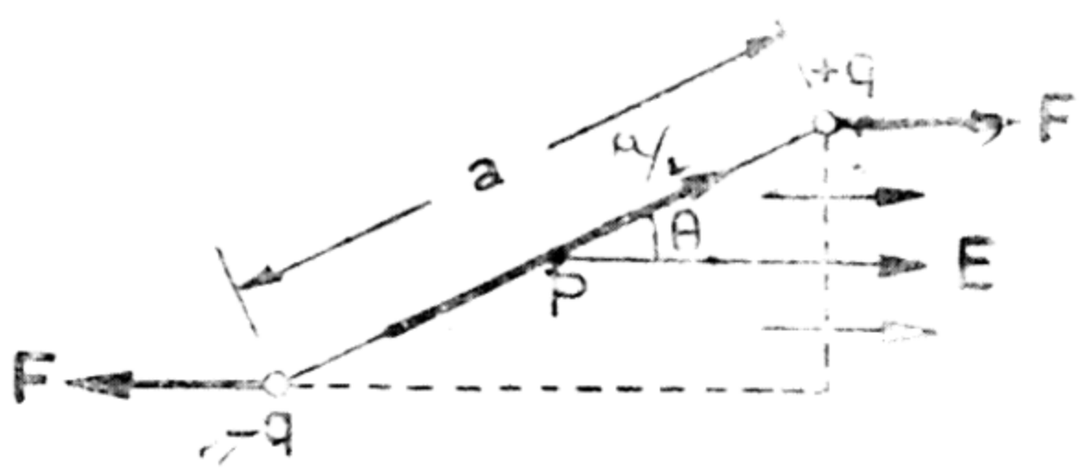


Fig. 5.6.

$$\tau=\mathbf{r}\times\mathbf{F}=\mathbf{r}\times\mathbf{F}_{+}+(-\mathbf{r})\times\mathbf{F}_{-}.$$

Here the origin is taken at the centre of the dipole. τ is a vector perpendicular to the figure in the dowaward direction and of magnitude

$$\tau=\tfrac{1}{2}aqE\sin\theta+\tfrac{1}{2}aqE\sin\theta=aqE\sin\theta=pE\sin\theta, \qquad ...(1)$$

or $$\tau=\mathbf{p}\times\mathbf{E}, \qquad ...(2)$$

where $p\ (=aq)$ is the dipole moment.

One can readily show that the torque τ is the same if any other point is chosen as the centre. If the field is not uniform in the neighbourhood of the dipole, so that its value at $+q$ is different from that at $-q$ an average value taken at the centre will be a good approximation. In true dipole (zero length) the torque is independent of any variation of **E**. The work done in rotating this dipole through an angle $d\theta$ in this field

$$dW=\tau\, d\theta.$$

$\therefore$ Total work done in rotating it through an angle θ

$$W=\int_0^{\theta}\tau d\theta=\int_0^{\theta}pE\sin\theta\, d\theta=pE\,(1-\cos\theta). \qquad ...(3)$$

The net force on the dipole is zero if the field is uniform, as in this case we have two equal and opposite forces. Thus the force must depend on the variation of the field, *i.e.*, on the gradients of the various components of the field. More specially, the net force in the x—direction $F_x=[qE_x]_{at\ q+}-[qE_x]_{at\ q-}$. The difference can be calculated by the differential formula involving the gradient. Thus

$$F_x=q\mathbf{a}.\nabla E_x=\mathbf{p}.\nabla E_x.$$

Similarly $F_y=\mathbf{p}.\nabla E_y$ and $F_z=\mathbf{p}.\nabla E_z$.

$$\therefore \qquad \mathbf{F}=\mathbf{p}.\nabla E_x\mathbf{i}+\mathbf{p}.\nabla E_y\mathbf{j}+\mathbf{p}.\nabla E_z\mathbf{k}=(\mathbf{p}.\nabla)\mathbf{E}. \qquad ...(4)$$

Using vector relations this equation can also be written as

$$\mathbf{F}=\nabla(\mathbf{p}.\mathbf{E}). \qquad ...(5)$$

We know that the conservative force can be defined as the negative gradient of the potential energy. Thus the *potential energy of the dipole*.

$$U=-\mathbf{p}.\mathbf{E}. \qquad ...(6)$$

This relation can also be obtained from relation (3) as the potential energy is equal to the work required in bringing the

dipole from infinity to the point under consideration in such a way that $+q$ and $-q$ are at all times on the same equipotential surface, *i.e.*, $\mathbf{p} \perp \mathbf{E}$ always.

Let us now consider a dipole of moment $\mathbf{p}_1$ located at the origin of coordinates and a dipole of moment $\mathbf{p}_2$ located at a point P (r distance) from the origin. The potential and field at the origin due to dipole $\mathbf{p}_2$ are given by

$$\phi = -\frac{1}{4\pi\varepsilon}\mathbf{p}_2 . \nabla\left(\frac{1}{r}\right) \text{ and } \mathbf{E} = -\nabla\phi = \frac{1}{4\pi\varepsilon}\nabla\left[\mathbf{p}_2 . \nabla\left(\frac{1}{r}\right)\right]. \quad \ldots(7)$$

Hence the force on dipole $\mathbf{p}_1$ due to dipole $\mathbf{p}_2$

$$\mathbf{F} = (\mathbf{p}_1 . \nabla)\,\mathbf{E} = \frac{1}{4\pi\varepsilon}(\mathbf{p}_1 . \nabla)\left\{\nabla\left[\mathbf{p}_2 . \nabla\left(\frac{1}{r}\right)\right]\right\}. \quad \ldots(8)$$

The interaction energy between the two dipoles, *i.e.*, U_{12}, the energy of dipole $\mathbf{p}_1$ in the field of dipole $\mathbf{p}_2$ is given with the Eqn. (6) as

$$U_{12} = -\mathbf{p}_1 . \mathbf{E} = -\frac{(\mathbf{p}_1 . \nabla)}{4\pi\varepsilon}\left[\mathbf{p}_2 . \nabla\left(\frac{1}{r}\right)\right]$$

$$= -\frac{1}{4\pi\varepsilon}\mathbf{p}_1 . \nabla\left[\mathbf{p}_2 . \left(-\frac{1}{r^3}\mathbf{r}\right)\right]$$

$$= -\frac{1}{4\pi\varepsilon}\mathbf{p}_1 . \nabla\left[-r^{-3}(\mathbf{p}_2 . \mathbf{r})\right]$$

$$= \frac{1}{4\pi\varepsilon}\mathbf{p}_1 . \left[\nabla(r^{-3})(\mathbf{p}_2 . \mathbf{r}) + r^{-3}\nabla(\mathbf{p}_2 . \mathbf{r})\right]$$

$$= \frac{1}{4\pi\varepsilon}\mathbf{p}_1 . \left[-3r^{-5}\mathbf{r}(\mathbf{p}_2 . \mathbf{r}) + r^{-3}\mathbf{p}_2\right]$$

$$= \frac{1}{4\pi\varepsilon}\left[\frac{\mathbf{p}_1 . \mathbf{p}_2}{r^3} - \frac{3}{r^5}(\mathbf{p} . \mathbf{r})(\mathbf{p}_2 . \mathbf{r})\right]. \quad \ldots(9)$$

5.4. POLARIZATION AND CHARGE DENSITY

When a dielectric is placed in an electric field its molecules become polarized by this field, or oriented in the direction of field. This effect is known as *electric polarization* and is defined as the product of $\mathbf{p}$, the component of the vector dipole moment of each molecule in the field direction and n the number of molecules per unit volume,

$$\mathbf{P} = n\mathbf{p}. \quad \ldots(10)$$

In the general form it can be written as

$$\Sigma\mathbf{p}_i = \mathbf{P}dV, \quad \ldots(11)$$

where summation is taken over all individual dipoles in the volume dV Thus polarization $\mathbf{P}$ is defined as the dipole moment per unit volume. It has the same direction as that of $\mathbf{P}$ and hence of electric field $\mathbf{E}$. In SI units, unit of $\mathbf{P}$ is coulomb meter/meter^3 or *coulomb/meter*^{2}. Actually the term polarization has two meanings : *a qualitative one*, giving an idea about relative displacement of positive and

negative charges and *a quantitative one*, giving resulting dipole moment of a unit volume of a polarized matter. Since the dipole moment per unit volume refers to the arrangement on an atomic scale, hence is not measurable directly by a macroscopic method. The simple method of measuring **P** is based upon its relationship with the polarization charge (q_p) on the surface of a dielectric slab placed in the field.

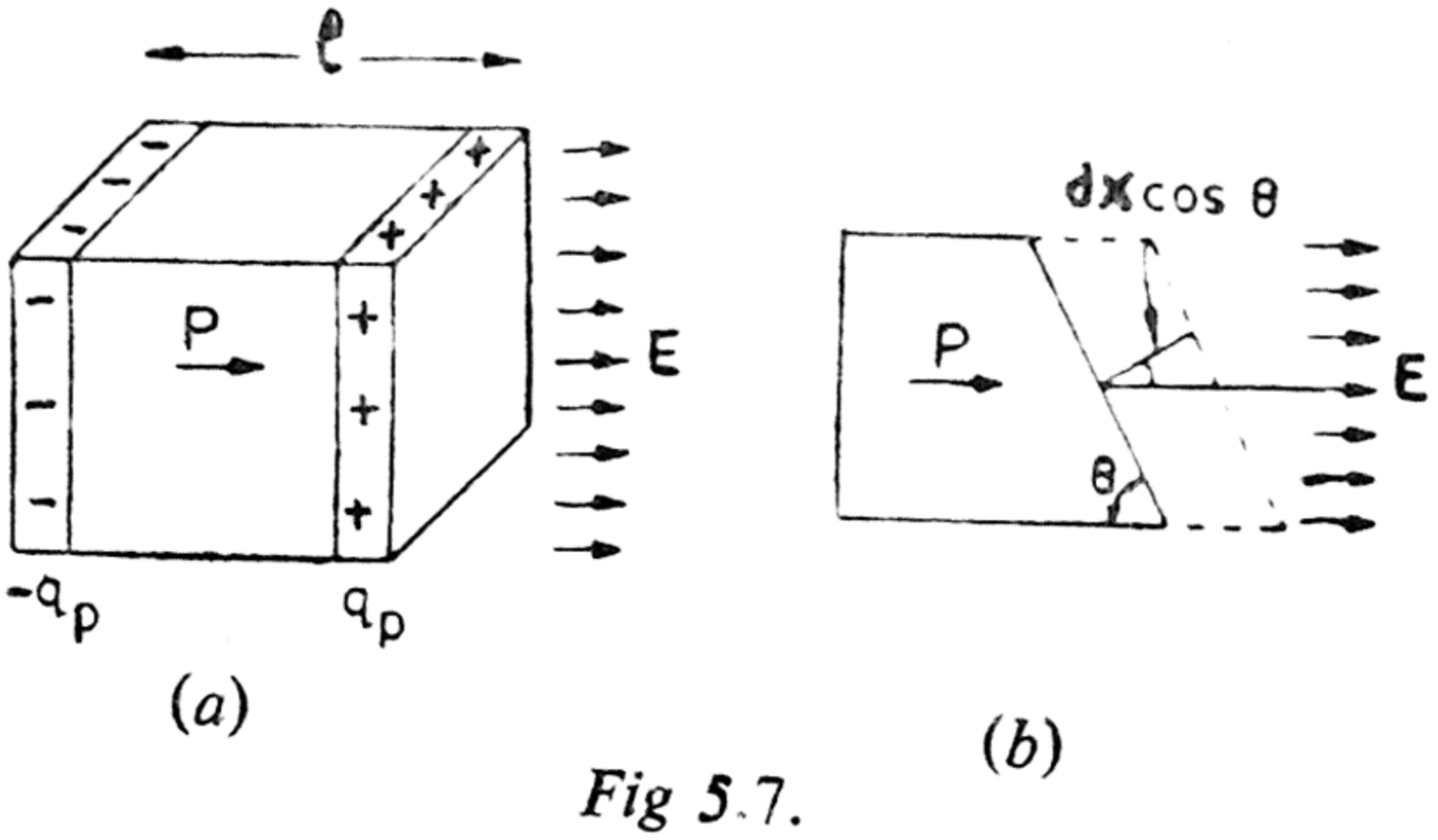


Fig 5.7.

Fig. 5.7. (*a*) shows a polarized block of dielectric placed in a uniform electric field **E**. The net charges on the surfaces are also shown. If σ_p is the density of polarization charge on the surfaces, **P** the polarization vector, A the area of cross-section and l the length of the slab, then

dipole moment of this block $= PAl$.

The second way is to consider this block as a single large dipole with surface charges $q_p (= \sigma_p A)$ separated by a distance l, *i.e.*,

dipole moment = charge × length $= \sigma_p Al$.

Comparison of the two results gives

$$P = |\sigma_p| . \qquad \ldots(12)$$

The sign of the polarization surface charge depends on the direction of polarization. The magnitude of **P** depends on the charge density and on the relative displacement of the +ve and −ve charge arrays. Its direction is along the displacement direction.

It is the special case in which **P** is normal to the surface. If the dielectric surface is not perpendicular to **P**, but the normal to the surface makes an angle θ with the direction of **P**. In this case the actual displacement of the array of positive charges within the dielectric is dx. Its effective displacement is $dx \cos\theta$. Thus for surface parallel to **P**, no surface polarization charge is induced and in general

P_n = component of **P** normal to the surface $= |\sigma_p|$. $\qquad \ldots(13)$

The polarization **P** is a vector field quantity having mathematical properties similar to **E**. For uniformly polarized matter we can apply Gauss's law to the lines originating from and ending on surface polarization charges. Thus the surface integral

$$\oint \mathbf{P}.d\mathbf{S} = -q_p. \qquad \ldots(14)$$

Here —ve sign is included because the direction of lines of **P** is from negative to positive charge, lines of **P** emerge from a volume containing negative polarization charge and end in a volume containing positive polarization charge.

If the polarization of the medium is not uniform, the surface charge is accompanied by a volume distribution of charge in the interior of the dielectric and is given by

$$\rho_p = -(\partial P_x/\partial x + \partial P_y/\partial y + \partial P_z/\partial z). \qquad ...(15)$$

If the polarization is constant throughout the medium the charge per unit volume ρ_p in its interior is zero. To find the mean electric field in the vicinity of a dielectric (inside or outside the medium) we must add eletric intensities due to outside charges, charge on the surface of the dielectric and the charge in the interior of the dielectric.

5.5. GAUSS'S LAW IN DIELECTRIC

In the previous chapters we have used Gauss's law to the problems in which no dielectric was present. Now let us apply this law to a parallel plate capacitor filled with a dielectric. Let $+q$ and $-q$ be the charges on the plates of this capacitor in both the cases, with and without the dielectric.

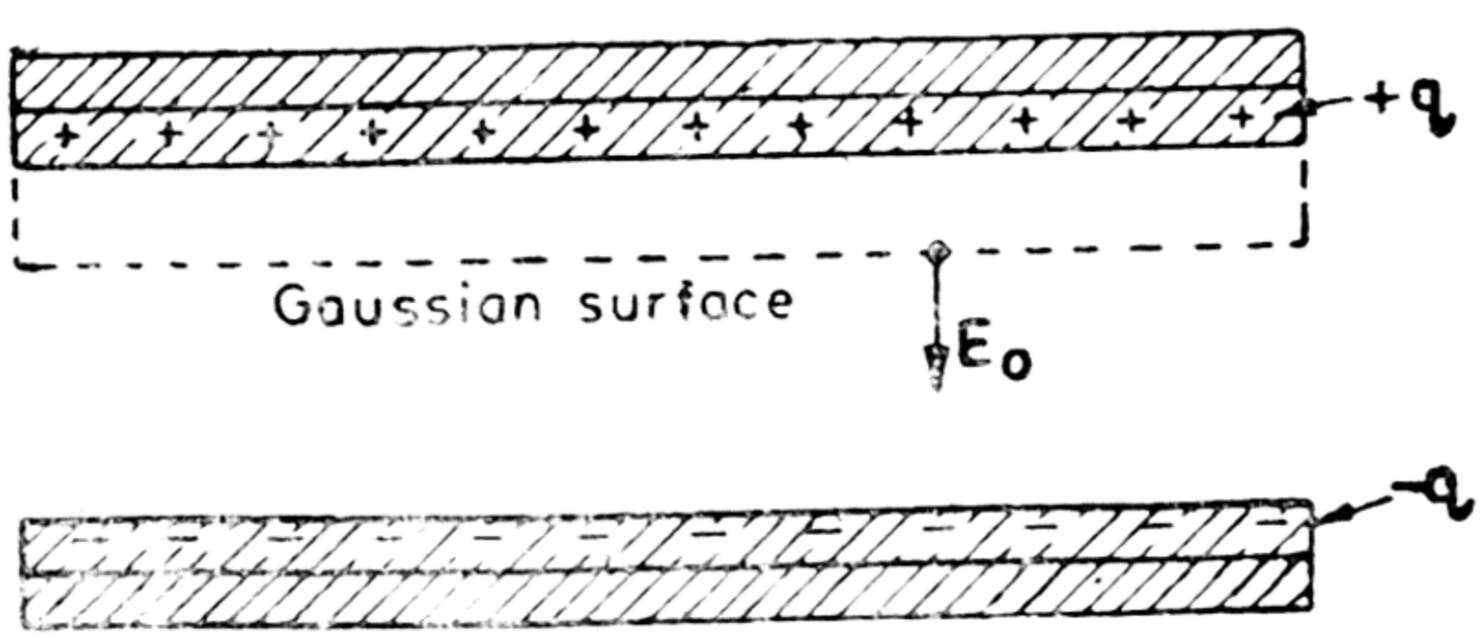


Fig. 5.8.

In the first case, when no dielectric is present (Fig. 5.8), the electric field $\mathbf{E}_0$ at any point on the gaussian surface is given by

$$\int \mathbf{E}_0 . d\mathbf{S} = q/\varepsilon_0.$$

$$\therefore \qquad E_0 A = q/\varepsilon_0 \qquad \text{or} \quad E_0 = q/\varepsilon_0 A. \qquad ...(16)$$

Since $\mathbf{E}_0 \parallel d\mathbf{S}$ and constant over the gaussian surface,

$$\therefore \qquad \int \mathbf{E}_0 . d\mathbf{S} = \int E_0 dS = E_0 A.$$

In the second case, when dielectric is present (Fig. 5.9), the net charge within the gaussian surface is $q - q'$, where q' is the induced

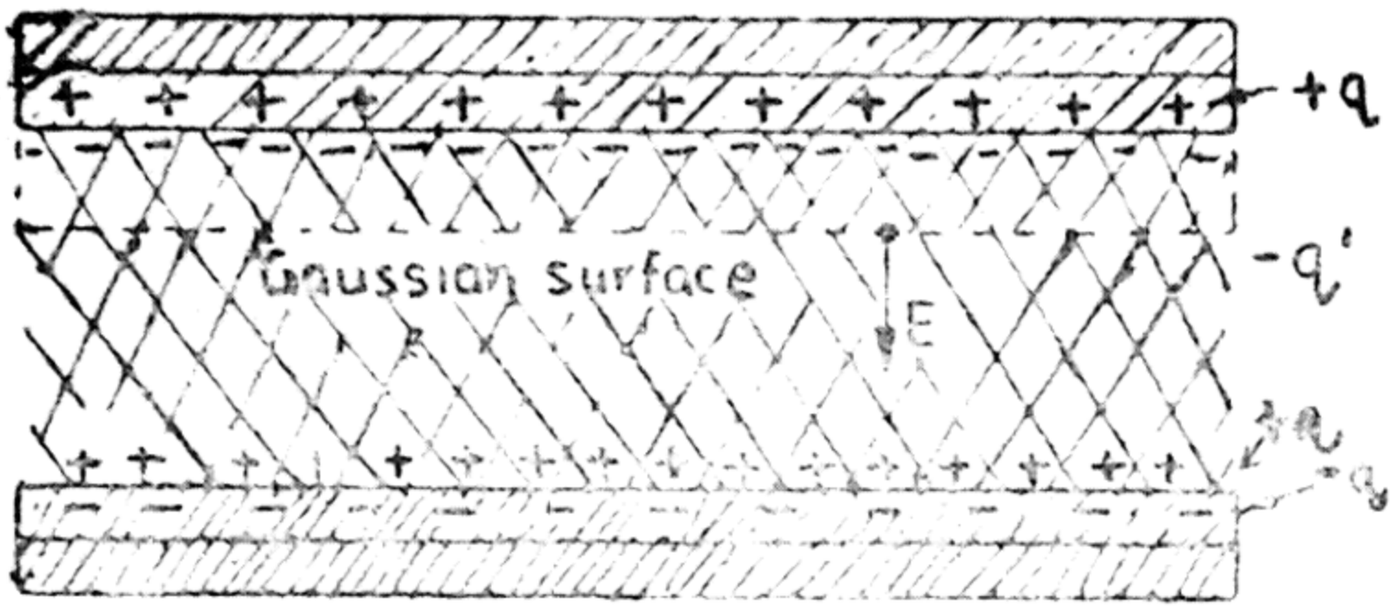


Fig. 5.9.

surface charge. The electric field **E** is give by

$$\oint \mathbf{E}.d\mathbf{S}=(q-q')/\varepsilon_0. \qquad \ldots(17)$$

$$\therefore \qquad EA=(q-q')/\epsilon_0 \quad \text{or} \quad E=q/\varepsilon_0 A-q'/\varepsilon_0 A. \qquad \ldots(18)$$

Thus the *induced charge* q' *tends to weaken the original field* (*i.e.* $E \lhd E_0$), which reveals itself as a reduction in potential difference between the plates. If V_d and V_0 are the potential differences with and without the dielectric respectively, then

$$E_0/E=V_0/V_d=k=\text{Dielectric constant.}$$

$$E=E_0/k=q/k\varepsilon_0 A. \qquad \ldots(19)$$

Combining Eqns. (18) and (19), we get

$$q/k\varepsilon_0 A=q/\varepsilon_0 A-q'/\varepsilon_0 A$$

or
$$q'=q(1-1/k). \qquad \ldots(20)$$

This shows that the induced surface charge q' is always less than the free charge q and is zero when $k=1$ or the dielectric is not present. Combining Eqns. (17) and (20), we have

$$\varepsilon_0 \oint k\mathbf{E}.d\mathbf{S}=q. \qquad \ldots(21)$$

It is the Gauss's law in the presence of a dielectric. Here we see that the flux integral contains a factor k and *the effect of the induced surface charge is ignored by taking into account* k, *the dielectric constant.*

5.6. ELECTRIC DISPLACEMENT

We know that owing to the polarization of the medium the charges are induced at the surfaces of the dielectric. The charge is negative near the positive plate of the capacitor and positive near the negative plate. If q is the charge on the plates and q' is the induced charge on the boundary of the dielectric, then we can write Eqn. (20) as

$$q/A=\varepsilon_0(q/\varepsilon_0 kA)+q'/A. \qquad \ldots(22)$$

The last term is the induced surface charge per unit area, the electric polarization P. Thus we have

$$q/A=\varepsilon_0(q/\varepsilon_0 kA)+P. \qquad \ldots(23)$$

The term P is a vector quantity having direction from the negative induced charge to the positive induced charge, as for any dipole. The quantity in the bracket $q/\varepsilon_0 kA$ is nothing but the electric intensity E at any point within the dielectric. Hence the above relation can be written as

$$q/A=\varepsilon_0 E+P. \qquad \ldots(24)$$

The quantity on the right hand side is given a special name *electric displacement* D (a name introduced by Maxwell). Thus

$$D=\epsilon_0 E+P. \qquad \ldots(25)$$

Since E and P are vectors hence D must also be vector, thus we have*

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}. \qquad \ldots(26)$$

This new quantity **D** gives added insight into problems involving fields in the presence of dielectric. It depends only on the free charges (free charge density q/A), in contrast to **E** which depends on both free and polarization charges. Even in the presence of dielectric, polarization charges are not involved in **D**. A general form of Gauss's law for **E** is written as

$$\varepsilon_0\int\mathbf{E}.d\mathbf{S}=q-q'=q-\int\mathbf{P}.d\mathbf{S}.$$

$$\int(\varepsilon_0\mathbf{E}+\mathbf{P}).d\mathbf{S}=q \quad \text{or} \quad \int\mathbf{D}.d\mathbf{S}=q. \qquad \ldots(27)$$

Thus the vector quantity **D** obeys Gauss's law. *The surface integral of the normal component of* **D** *over any charge surface, the flux of* **D**, *is equal to the total free charge only within the surface.*

5.7. THREE ELECTRIC VECTORS

E, **P** and **D** are the three electric vectors related with each other as shown by Eqn. (26). These may vary in magnitude and direction from point to point in complicated problems of electrostatics. Each has a constant magnitude for every point in the dielectric in the case of parallel plate capacitor filled with the dielectric.

Electric displacement vector **D** is connected with the free charge only. Like the electric field, the displacement field can be represented by lines known as *lines of displacement.* These lines begin and end on the free charges. The flux of **D** equals the free charge.

Polarization vector **P** is connected with the induced surface charge. It can also be represented by lines, known as lines of **P**. These lines begin and end on the polarization charges, *i.e.*, induced charges due to polarization. The flux of **P** equals the negative of the bound (*induced*) charge.

The electric field intensity **E** is connected with the charges actually present (*free and bound charges*). The lines of **E** depend upon the presence of both kinds of charges. The flux of **E** equals

* In the Gaussian system of units this relation is written a $D=E+4\pi P$,

the total enclosed charge divided by ε_0. The unit of **E** is newton/coulomb while that of **P** and **D** was coulomb/ meter^2.

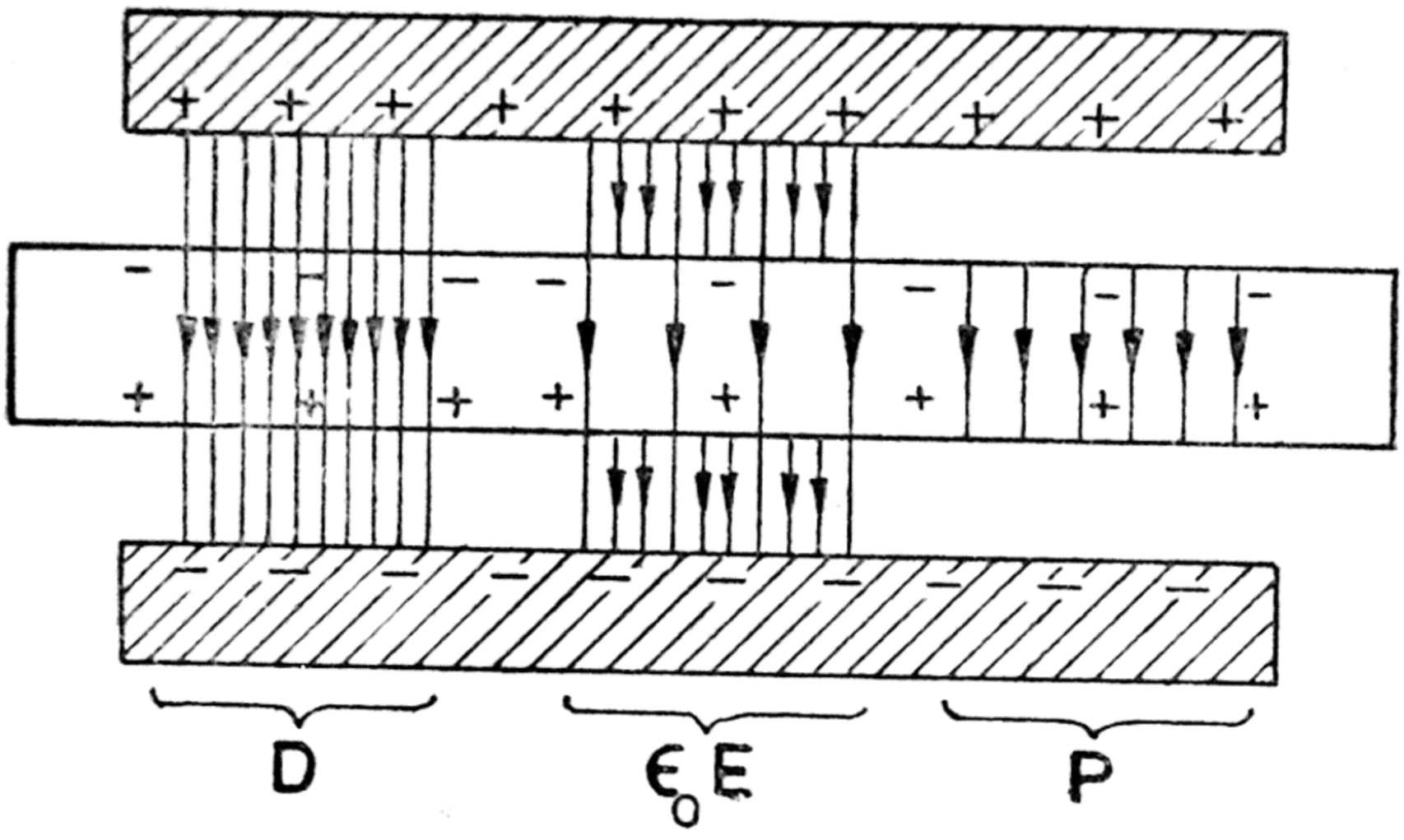


Fig. 5.10.

In free space, where there is no dielectric and thus **P**=0, we have

$$\mathbf{D}=\varepsilon_0\,\mathbf{E}. \qquad \ldots(28)$$

In the presence of a dielectric vectors **D** and **P** both can be expressed in terms of **E** alone. We can write

$$D=q/A=k\epsilon_0\,(q/k\varepsilon_0 A)=k\varepsilon_0 E$$

or
$$\mathbf{D}=k\epsilon_0\,\mathbf{E}. \qquad \ldots(29)$$

The product $\varepsilon_0 k$ is called the *permittivity* ε of the medium and hence

$$\mathbf{D}=\varepsilon\mathbf{E}. \qquad \ldots(30)$$

In a vacuum $k=1$ and hence $\varepsilon=\varepsilon_0$. For this region ϵ_0 is often referred as the *permittivity of a vacuum* (or free space). As dielectric constant k is the ratio, $\varepsilon/\varepsilon_0$, it is also named as *relative permittivity.*

Using relations (26) and (29) we can write a new relation

$$\epsilon_0\mathbf{E}+\mathbf{P}=k\varepsilon_0\mathbf{E}$$

or
$$\mathbf{P}=\varepsilon_0\,(k-1)\mathbf{E}. \qquad \ldots(31)$$

As ε_0 and k are constants for the medium, term $k-1$ is also replaced by the another property, the susceptibility χ of the dielectric medium. Thus we have

$$\mathbf{P}=\varepsilon_0\chi\mathbf{E}. \qquad \ldots(32)$$

Three quantities ε, k and χ are simply different ways of describing the same property of the dielectric. These are related as

$$\chi=k-1=\varepsilon/\varepsilon_0-1. \qquad \ldots(33)$$

5.8. BOUNDARY CONDITIONS AT THE DIELECTRIC SURFACE

Before we can solve more complicated problems, we must know how the field vectors **D** and **E** change in passing through a surface separating two media, may be two dielectrics or a dielectric and a free space. The rules governing the behaviour of **E** and **D** at the boundary between two dielectrics are embodied in *two boundary conditions.*

Let AB represents a small portion of the boundary between two media of absolute permittivities ε_1 and ε_2. The media are assumed to be homogeneous[1] and isotropic.[2] Let us consider an area dS on the boundary, so small that its curvature may be neglected. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be the electric induction vectors in the media on either side of dS, making angles θ_1 and θ_2 with the normal to dS.

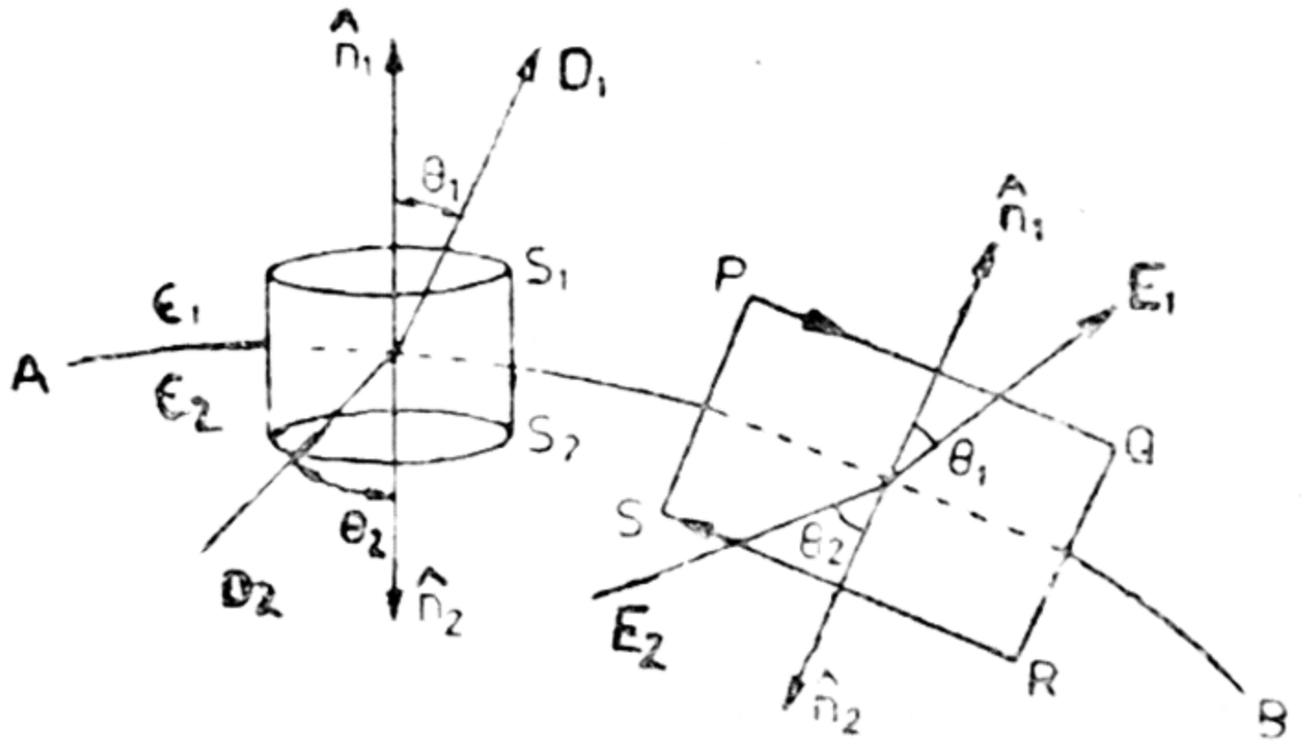


Fig. 5.11.

To find the boundary condition for **D**, let us apply Gauss's law to a small cylinder which intersects the area dS on the boundary and of height very small compared to the diameter of the base. Assuming the volume of the cylinder very small, so that the free charge enclosed by the cylinder can be taken as the free charge on dS area of the boundary. Thus we have

$$\int \mathbf{D}.d\mathbf{S}=\int_C \mathbf{D}.d\mathbf{S}+\int_{S_1} \mathbf{D}_1.d\mathbf{S}+\int_{S_2} \mathbf{D}_2.d\mathbf{S}=q$$

$$0+(\mathbf{D}_1.\mathbf{n}_1+\mathbf{D}_2.\mathbf{n}_2)\,dS=\sigma dS \quad \text{or} \quad D_1\cos\theta_1-D_2\cos\theta_2=\sigma. \qquad \ldots(34)$$

Here we have neglected the flux through the curved surface of the cylinder as this area is vanishingly small. In the majority of cases there is no free charge on the boundary, hence,

$$D_1\cos\theta_1-D_2\cos\theta_2=0 \text{ or } D_{1n}=D_{2n}. \qquad \ldots(35)$$

Therefore the *normal component of the electric displacement is the same on both sides of the boundary of two media of different dielectrics or the normal displacement is continuous across the boundary having no free charges.*

---

1 The body is said to be homogeneous, if the properties of a body are constant from point to point,

2 The body is said to be isotropic, if the physical properties of a body in the neighbourhood of some interior point are the same in all directions.

The boundary condition for **E** is found by considering the work done in taking unit charge round a small rectangle of length *dl* and of negligible height, with its longest sides parallel to the surface of separation, one in each medium. The work done in taking a unit charge around the rectangle *PQRS* must vanish, otherwise an infinite amount of work could be obtained by repeating the process indefinitely, hence

$$\oint \mathbf{E}.d\mathbf{l} = \int_P^Q \mathbf{E}_1.d\mathbf{l} + \int_Q^R \mathbf{E}.d\mathbf{l} + \int_R^S \mathbf{E}_2.d\mathbf{l} + \int_S^P \mathbf{E}.d\mathbf{l} = 0$$

$$E_1 \sin\theta_1 dl - E_2 \sin\theta_2 dl = 0 \text{ or } E_{1t} = E_{2t}. \quad \ldots(36)$$

Here we have neglected the contribution of short sides *QR* and *SP* as it is vanishingly small comparison with that of other sides *PQ* and *RS*. Therefore *the tangential components of the electric intensities are the same on the two sides of the boundary surface or the tangential component of the electric field is continuous at the boundary between two dielectrics.*

Similar boundary conditions may be applied when one of the media is a conductor. Since there is no field inside the conductor, hence the tangential component of **E** just outside the conductor must also be zero, therefore the field at the surface must be normal to the surface. Thus the displacement and electric field in a dielectric just outside of a conductor are given as

$$D_{1n} = \sigma \text{ and } E_{1t} = 0. \quad \ldots(37)$$

Refraction of the lines of force and displacement at an uncharged surface of discontinuity. We may use boundary conditions given above to show the effect of a dielectric (isotropic) boundary on the

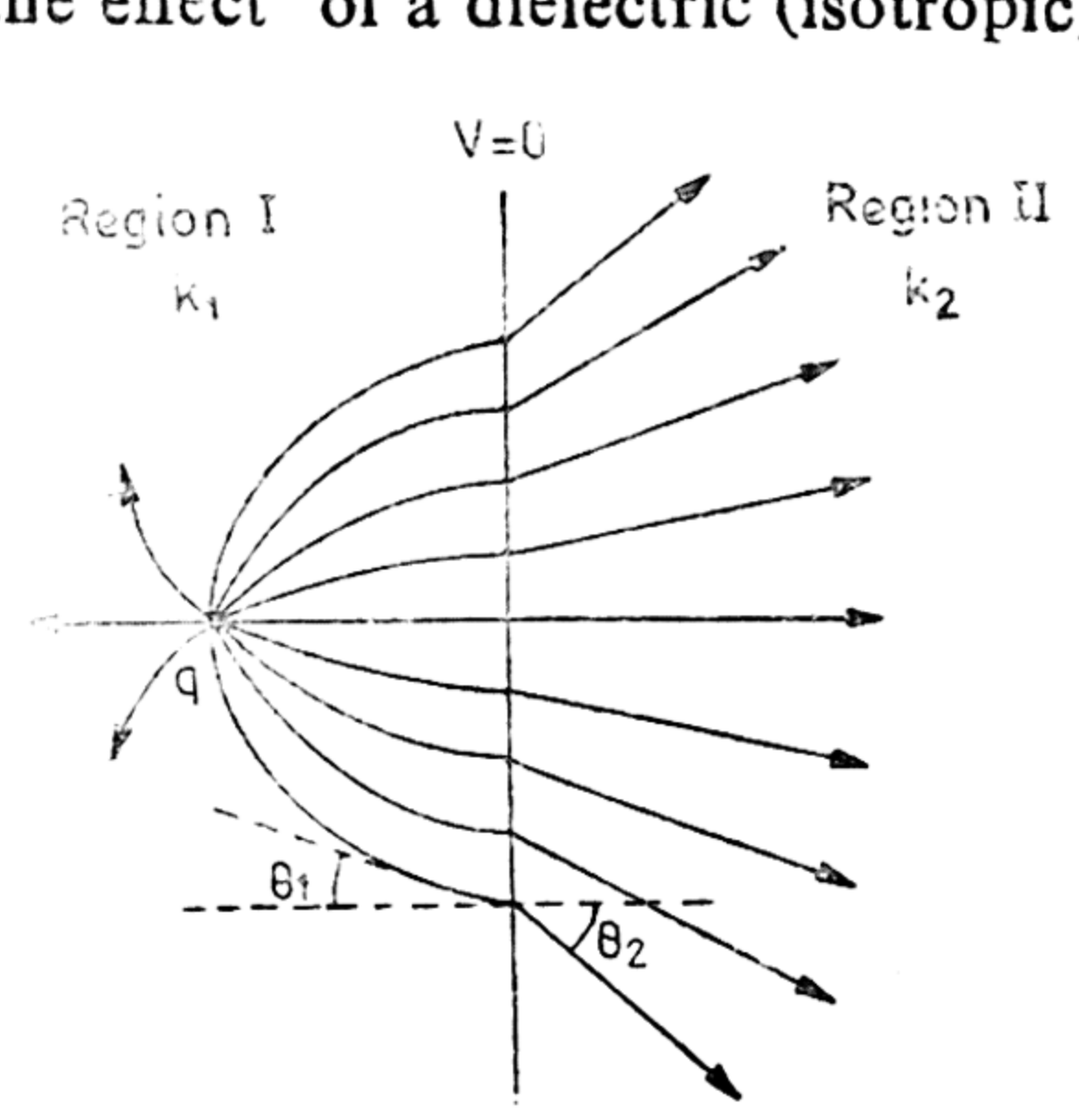


Fig. 5.12.

direction of an electric field or the electric displacement that crosses the boundary. Let the lines of force make angles θ_1 and θ_2 with the

normal to the surface in regions 1 and 2. Since in an isotropic dielectric medium $\mathbf{D}=k\varepsilon_0\mathbf{E}$, hence the displacement vector will coincide the electric field vector. Combining the conditions (35) and (36), we get

$$D_1 \cos\theta_1/E_1 \sin\theta_1 = D_2 \cos\theta_2/E_2 \sin\theta_2$$

or $$(D_1/E_1)\cot\theta_1 = (D_2/E_2)\cot\theta_2.$$

As $$D_1 = k_1\varepsilon_0 E_1 \text{ and } D_2 = k_2\varepsilon_0 E_2.$$

$\therefore$ $$k_1 \cot\theta_1 = k_2 \cot\theta_2. \qquad \ldots(38)$$

From this relation it is clear that in passing from empty space into a dielectric, the lines of force or displacement are bent away from the normal to the surface, *alike to the light rays.*

5.9. EFFECTS OF DIELECTRICS

(*a*) **Dielectric Relaxation**. Dielectric takes a finite time for the polarization to reach its maximum value. This is due to the forces between adjacent molecules which tends to prevent the alignment along the external field. The average time for *relaxation* varies from substance to substance, values from 10^{-10} sec to several thousand years.

Due to this relaxation, capacitor connected to a potential difference V acquires charge at a finite rate. Discharging process is also slow. If a capacitor is charged and discharged quickly and then left with terminals open, it will be found to have acquired a potential difference between its plates. Phenomenon of hysteresis with alternating field is also the consequence of the relaxation time.

(*b*) **Dielectric Absorption**. On account of the finite conductivity of the dielectric materials noticeable energy loss is observed in the form of heat. This loss and the hysteresis loss are referred jointly as *dielectric losses.* Similar to the relaxation phenomenon absorption effect is observed in capacitors. If an initially charged capacitor is discharged by short circuiting it, it is found that the capacitor still retains a charge when the short circuit is removed. This is due to *electric absorption.* It is when the dielectric is not perfectly homogeneous. For a perfectly homogeneous dielectric, this residual charge should not occur. Thus a gas or a perfectly homogeneous crystal like calcite shows no residual charge.

If the capacitor is repeatedly charged and discharged, dielectric absorption causes heat to be produced. If this process is rapid, as with high frequency *ac*, energy losses in the dielectric are low, *i.e.* the dielectric absorption is small as the molecules are unable to rotate to any extent in this very short time (*i.e.*, the time of the cycle).

(*c*) **Dielectric Breakdown**. It is the most important property of real dielectrics. For any material there is a maximum field intensity beyond which damage occurs that results in conduction

sparking and other breakdown phenomena. At relatively high fields, the electrons in the dielectric gain enough energy to knock other charged particles and make them available for conduction. This multiplying process occurs along certain paths in the material. Along these paths, local heating also occurs and a permanent damage results.

Electrostatic apparatus at high potentials requires a good insulation with respect to break down. Air is the most common insulating material. For a dry and at N.T.P. air, the breakdown occurs at fields of about 3×10^6 V/m. Fields somewhat less than this may produce some steady discharging currents known as *corona*. When the field is not very large over most of the region, it may come to or above the breakdown value near the point. The lines of force near the point are like those for a sphere, hence at the surface of the point $E = V/a$. Thus breakdown can occur even at 300 V at the point of 0.1 mm radius. The operation of lightning rods and brush discharge points in electrostatic generator is based on this principle.

5.10. ELECTRIC FIELDS IN CAVITIES OF A DIELECTRIC

The electric field inside a solid dielectric can be obtained if we consider a test charge placed in free space within the dielectric. According to Lord Kelvin an empty cavity is assumed within the dielectric where the test charge is placed. The result obtained depends on the shape of the cavity. Three cases are considered here, (*a*) a small *needle shaped cavity*, (*b*) a *disc-shaped cavity* and (*c*) a *spherical cavity*.

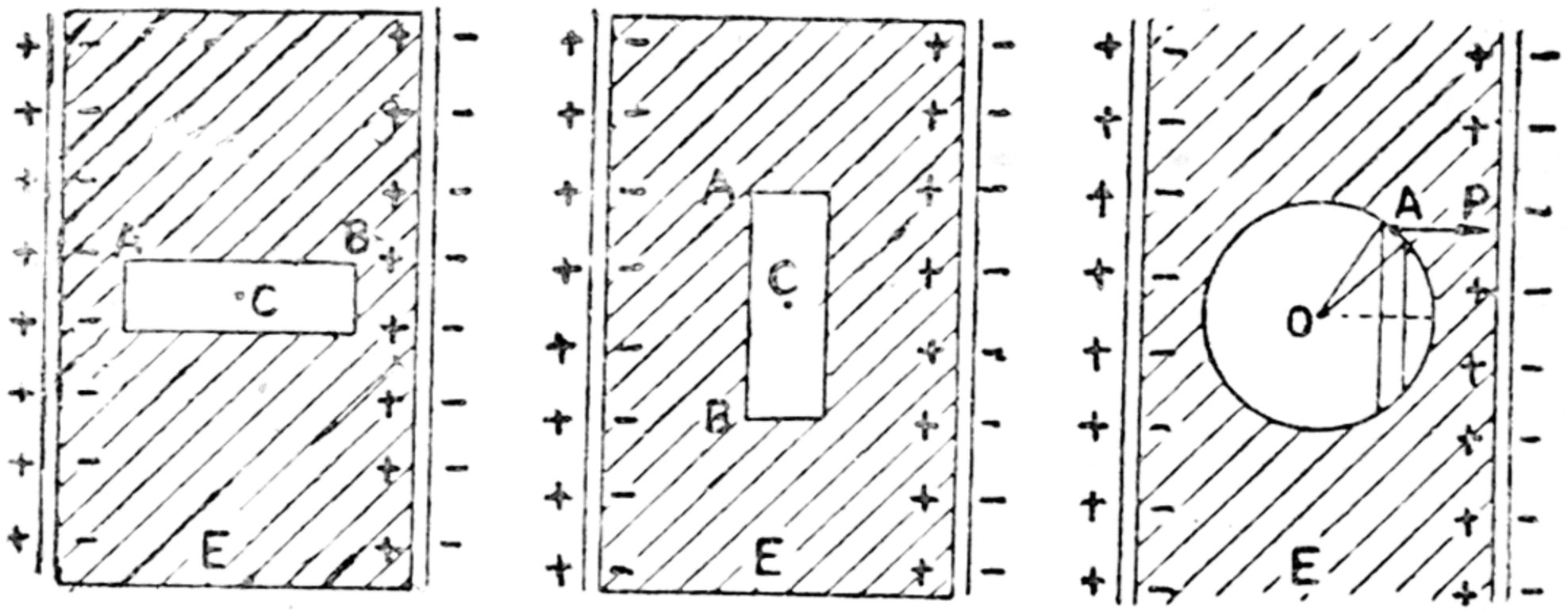
(*a*) **Needle Shaped Cavity.** Consider a dielectric placed in between the parallel charged plates producing the electric field **E**. A small portion of the dielectric is drawn so that we have a needle shaped cavity AB (length greater than its area of cross section (Fig. 5.13 *a*.) Due to the presence of electric field the polarized bound charge will appear on the surfaces at A and B. These charges will be vanishingly small, as the area of cross section of the needle is taken very small. *Therefore there will be no contribution of bound charges to the electric field at point C inside such a cavity and the force on the charge at C is only due to the applied electric field* **E**. The same result is obtained when we apply boundary condition that the tangential components of electric intensity in the two media are continuous, *i.e.*, the electric intensity inside and the outside is the same.

(*b*) **Disc Shaped Cavity.** Consider a disc shaped cavity AB (Fig. 5.13*b*) within the dielectric, where the diameter of the disc is greater than its thickness in such a way that its thickness is in the direction of electric field. Due to this field polarized bound charges appear on the side surfaces, $\perp$ to this electric field. Dissimilar to the previous case the bound charges also contribute to the electric force experienced by the test charge placed within this cavity. The electric field $\mathbf{E}_0$ inside the cavity is thus given by

$$\mathbf{E}_0=\mathbf{E}+\mathbf{P}/\varepsilon_0$$

or the electric displacement $\mathbf{D}=\varepsilon_0\mathbf{E}_0=\varepsilon_0\mathbf{E}+\mathbf{P}$.

This is the same result as for the displacement within the dielectric. This thing can also be obtained by applying boundary condition, the normal component of electric displcement **D** is continuous across the boundary.



(*a*) (*b*) (*c*)

Fig. 5.13.

In these two cases it has been assumed that the effect of the cavity in the dielectric does not make any difference to the displacement and field in immediately surrounding medium. This is only true for the small cavities.

(*c*) **Spherical Cavity**. Let the cavity in the dielectric be a sphere of a radius *r* and centre *O*, such that the radius *r* is large compared with the intermolecular distance but small compared with the dimensions of the whole dielectric. If this dielectric is placed between two charged parallel plates, the electric field experienced by a molecule of the dielectric, if assumed to be placed at the centre of the cavity, is given by

$$\mathbf{E}_m=\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_3+\mathbf{E}_4. \qquad \ldots(39)$$

Here $\mathbf{E}_1$ is the field between two parallel charged plates with no dielectric, $\mathbf{E}_2$ is the field due to polarized charges on the outer surfaces of the dielectric (*i.e.*, depolarizing field), $\mathbf{E}_3$ is the field due to polarized charges on the inner surface of the spherical cavity and $\mathbf{E}_4$ is the field due to all the dipoles inside this spherical cavity.

In this case $E_1=\sigma/\varepsilon_0$, where σ is the surface charge density, $\mathbf{E}_2=-\mathbf{P}/\varepsilon_0$, where **P** is the polarization vector due to the polarization of the dielectric when placed within the external field $\mathbf{E}_1$.

$$\therefore \qquad \mathbf{E}_m=\mathbf{E}_1-\mathbf{P}/\varepsilon_0+\mathbf{E}_3+\mathbf{E}_4. \qquad \ldots(40)$$

If all the atoms are assumed to be replaced by point dipoles parallel to each other, then the field $\mathbf{E}_4$ due to the atoms within the cavity will depend on the crystal structure. We know that the electric field at a distance *r* from the dipole of dipole moment **P** is

$$\mathbf{E}=-\frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p}}{r^3}-\frac{3(\mathbf{p}.\mathbf{r})\,\mathbf{r}}{r^5}\right]. \qquad \ldots(41)$$

Summing over all the dipoles within the sphere is equivalent to taking the spatial average of the x-component of the field

$$E_4=\Sigma E_x=-\frac{1}{4\pi\varepsilon_0}\left[\frac{p_x}{r^3}-\frac{3(p_x x+p_y y+p_z z)x}{r^5}\right]. \quad \ldots(42)$$

For an isotropic dielectric, the x, y and z-directions are equivalent, *i.e.*,

$$\bar{x}^2=\bar{y}^2=\bar{z}^2=r^2/3 \text{ and } \overline{xy}=\overline{yz}=\overline{zx}=0, \quad \ldots(43)$$

hence the field $\mathbf{E}_4$ due to the dipoles within the sphere vanishes.

Let $\mathbf{E}$ be the electric field in the dielectric when it is placed in an external electric field $\mathbf{E}_1$. From second boundary condition, we know that the normal component of electric displacement $\mathbf{D}$ is continuous, across the vacuum-dielectric interface. As $\mathbf{D}$ is $\perp$ to the surface, hence $\mathbf{D}$ in the vacuum just oustide the dielectric is equal to that in the dielectric, *i.e.*,

$$\varepsilon_0\mathbf{E}_1=\varepsilon_0\mathbf{E}+\mathbf{P}. \quad \ldots(44)$$

Combining Eqns. (40) and (44) and using $\mathbf{E}_4=0$, we get

$$\mathbf{E}_m=\mathbf{E}+\mathbf{E}_3. \quad \ldots(45)$$

For $\mathbf{E}_3$, let us consider a small element dS of the surface to the cavity at point A with polar co-ordinates r, θ. The component of the polarization normal to the surface of the sphere at this point is equal to $P\cos\theta$ and therefore equal to charge per unit area over the element dS. Hence the electric field at the centre O due to the charge over area dS is given by

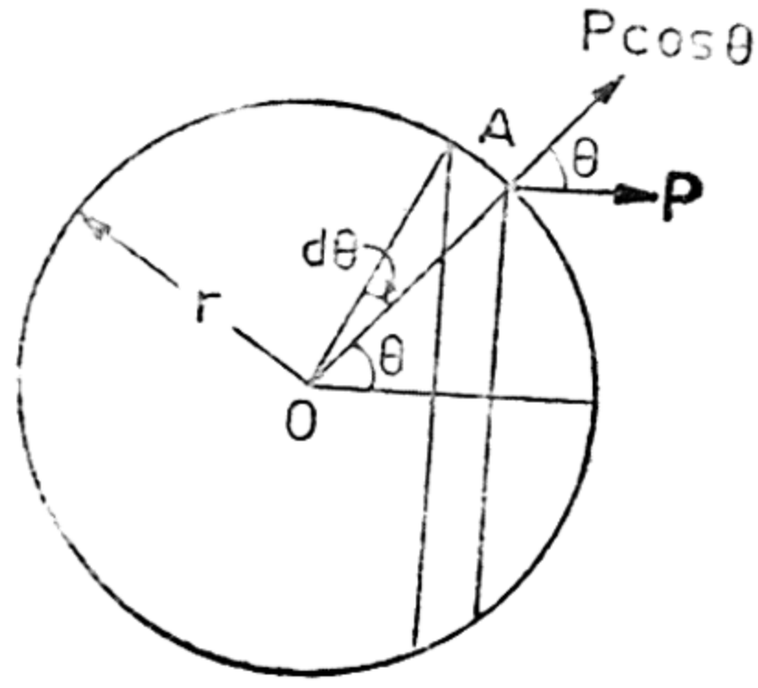


Fig. 5.14.

$$d\mathbf{E}'=\frac{1}{4\pi\varepsilon_0}\frac{(P\cos\theta)\,dS}{r^2}\hat{r}.$$

It is directed along $\mathbf{r}$ or OA. From symmetry it is clear that only the component of $d\mathbf{E}'$ along the direction of $\mathbf{p}$ will contribute to the total field $\mathbf{E}_3$. Let us assume dS as a ring shaped element of area described on the spherical surface by an arc at A between angles θ and $\theta+d\theta$. The area of such a ring $dS=2\pi r\sin\theta\,(r\,d\theta)$.

$$\therefore \quad E_3=\Sigma dE'\cos\theta$$

$$=\frac{1}{4\pi\varepsilon_0}\int\frac{P\cos\theta.\,2\pi r^2\sin\theta\,d\theta}{r^3}\cos\theta$$

$$=\frac{P}{2\varepsilon_0}\int_0^{\pi}\cos^2\theta\,\sin\theta\,d\theta=\frac{P}{3\varepsilon_0}, \quad \ldots(46)$$

or vectorially $\mathbf{E}_s = \mathbf{P}/3\varepsilon_0$. Thus Eqn. (45) reduces to

$$\mathbf{E}_m = \mathbf{E} + \mathbf{P}/3\varepsilon_0. \qquad \ldots(47)$$

For ideal dielectrics $\mathbf{P} = (k-1)\,\varepsilon_0\mathbf{E}$, hence

$$\mathbf{E}_m = \mathbf{E} + \tfrac{1}{3}(k-1)\,\mathbf{E} = \tfrac{1}{3}(k+2)\,\mathbf{E}$$

or
$$\mathbf{E} = [3/(k+2)]\,\mathbf{E}_m. \qquad \ldots(48)$$

Substituting this value of **E** in the relation for the polarization **P**, we get

$$\mathbf{P} = (k-1)\;\varepsilon_0 \frac{3}{k+2}\mathbf{E}_m = \frac{3(k-1)}{k+2}\,\varepsilon_0\mathbf{E}_m. \qquad \ldots(49)$$

The dipole moment of a molecule per unit polarizing field is called the *polarizability* or the *molecular* (*or atomic, if we are dealing with atoms rather than molecules*) *polarizability, i.e.,*

$$\alpha = p_m/E_m. \qquad \ldots(50)$$

If there are n molecules per unit volume, then polarization

$$\mathbf{P} = n\mathbf{p}_m = \alpha n\mathbf{E}_m. \qquad \ldots(51)$$

Comparing Eqns. (49) and (51), we get

$$\alpha = \frac{3\varepsilon_0}{n}\,\frac{k-1}{k+2}. \qquad \ldots(52)$$

This is called the *Clausius Mossotti relation.* It gives us the dielectric constant of a fluid in terms of α, the polarizability.

Atomic Polarizability. When an atom with atomic number Z is placed in polarising field **E**, the +ve charge will be displaced through a distance x from the centre of electron cloud in the direction of **E**. The force $Ze\mathbf{E}$ acts in the direction of **E**, while the electrostatic force tends to restore it. Hence in equilibrium

$$(Ze)\,(Zex^3/R^3)/4\pi\varepsilon_0 x^2 = ZeE \quad \text{or} \quad Zex = 4\pi\varepsilon_0\,R^3E.$$

The atomic dipole created in this process has a dipole moment $p = Zex$. Hence the *atomic polarizability*

$$\alpha = p_m/E_m = Zex/E = 4\pi\epsilon_0 R^3. \qquad \ldots(53)$$

Combining this equation with the Clausius Mossotti relation and eliminating α we get

$$\frac{k-1}{k+2} = \frac{n}{3\varepsilon_0}\cdot\; 4\pi\varepsilon_0 R^3 = \frac{4}{3}\pi R^3 n = V, \qquad \ldots(54)$$

where V is the volume occupied by the atoms (or molecules) per unit volume of the dielectric. With the help of this relation atomic

radius can be calculated. The values so obtained agree reasonably well with the results from other methods. This confirms the *Clauses Mossotti* relation.

5.11. POLARIZABILITY

The total polarizability α may usually be separated into three parts : electronic, ionic and dipolar orientational.

(*a*) **Electronic Polarizability.** The electronic contribution arises from the displacement of electronic cloud relative to the nucleus. The displacement x of the electron of mass m by the application of a field **E** is given by

$$-eE = m\omega_0^2 x = \beta x,$$

where β is the force constant and ω_0 is the resonance frequency of the electron bound harmonically to an atom.

$$\therefore \quad \text{Electronic Polarizability } \alpha_e = \frac{p}{E} = -\frac{ex}{E} = \frac{e^2}{m\omega_0^2} \quad \ldots(55)$$

The electronic polarizability depends on frequency. An electron having the resonance frequency ω_0 may be treated as a simple harmonic oscillator. The equation of motion in the field $E \sin \omega t$ is

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = -eE \sin \omega t.$$

Therefore for $x = x_0 \sin \omega t$, we get

$$m(-\omega^2 + \omega_0^2)x_0 = -eE$$

$$\therefore \quad \text{Dipole moment } p_0 = -ex_0 = \frac{e^2E}{m(\omega_0^2 - \omega^2)}$$

and electronic polarizability $\alpha_e = \dfrac{e^2/m}{\omega_0^2 - \omega^2}$. $\quad \ldots(56)$

In the visible region the frequency dependence is not usually very important in most transparent materials.

(*b*) **Ionic Polarizability.** It is produced by opposite displacements of positive and negative ions in the material due to the applied field **E**. Similar to the case of electronic polarizability the ionic polarizability is given by

$$\alpha_i = \frac{P}{E} = \frac{qx}{E} = \frac{q^2}{\omega_T^2}\left(\frac{1}{M_1} + \frac{1}{M_2}\right), \quad \ldots(57)$$

where ω_T is the transverse optical phonon frequency, q the charge on an ion and M_1 and M_2 are the masses of atoms *lie* on the odd numbered planes and on the even-numbered planes respectively.

(*c*) **Orientational Polarizability**. The orientational contribution arises when the substance is built up of molecules possessing a permanent electric field.

Debye explained the dielectric constants of some substances consisting molecules possessing a permanent electric dipole moment. In the absence of an applied electric field these molecules point in random direction and the net polarization of the dielectric is zero

$$\mathbf{P}=-\frac{1}{\Delta V}\,\Sigma\mathbf{p}_m=0, \qquad \ldots(58)$$

where the summation extends over all molecules in the volume element ΔV.

If an electric field is applied, the expression for the polarization can be obtained from *Langevin—Debye theory*, which is as under :

When the polar dielectric is subjected to an electric field the individual dipoles experience torques which tend to align them with the field. If the field is very strong, we may assume all dipoles to be aligned completely and the polarization may achieve the saturation value.

$$\mathbf{P}_s=N\mathbf{p}_m, \qquad \ldots(59)$$

where N is the number of molecules per unit volume. *This electric field will also produce induced dipole effects. For the moment, we shall ignore this contribution.*

At fields generally applied, the polarization of a polar dielectric is very small in comparison to its saturated value. It decreases with increase in temperature. This deviation from saturated value is due to the thermal energy of the molecules, which produces random dipole orientations. Statistical mechanics gives that the probability of finding a molecule with particular molecular energy U at temperature T is proportional to exp. $(-U/kT)$, where k is the Boltzmann constant, equal to 1.38×10^{-28} joule/degree.

If the dipole makes an angle θ with the direction of electric field **E**, then the potential energy of a dipole is given as $U=-\mathbf{p}.\mathbf{E}=-pE\cos\theta$. Hence the number of dipoles per unit volume per unit solid angle making an angle θ with the direction of the electric field **E** is $Ce^{(pE/kT\cos\theta)}$. Thus the number of dipoles per unit volume with in the solid angle $d\Omega$ between directions θ and $\theta+d\theta$ with **E** is given by

$$dN=Ce^{(pE/kT)\cos\theta}.\ d\Omega=Ce^{(pE/kT)\cos\theta}\,2\pi\sin\theta d\theta$$

$$=C'e^{(pE/kT)\cos\theta}\sin\theta d\theta.$$

where $C'=2\pi C$.

Hence the number of dipoles per unit volume

$$N = C' \int_0^{\pi} e^{(pE/kT)\cos\theta} \sin\theta d\theta.$$

Each dipole contributes a dipole moment $p\cos\theta$ parallel to **E** and $p\sin\theta$ perpendicular to **E**. It is clear from the spherical symmetry around **E**, that the contribution of dipole moment in the direction $\perp$ to **E** must be zero, hence the totol dipole moment per unit volume is along the direction of **E**, and **P** is given by

$$P = \Sigma p\cos\theta.\, dN = C'p \int_0^{\pi} e^{(pE/kT)\cos\theta} \cos\theta \sin\theta\, d\theta$$

$$= Np \frac{\int_0^{\pi} e^{(pE/kT)\cos\theta} \sin\theta\cos\theta\, d\theta}{\int_0^{\pi} e^{(pE/kT)\cos\theta} \sin\theta\, d\theta}$$

Substituting $pE/kT = u$, $u\cos\theta = x$ and hence $u\sin\theta\, d\theta = -dx$

$$\therefore \quad P = \frac{Np}{u} \frac{\int_{-u}^{u} e^x x dx}{\int_{-u}^{u} e^x dx} = \frac{Np}{u} \frac{\left[xe^x - e^x\right]_{-u}^{u}}{\left[e^x\right]_{-u}^{u}}$$

$$= Np\,[\coth u - 1/u], \qquad \ldots(60)$$

which is known as *Langevin equation* and the bracket term as Langevin function $L(x)$. A plot of the function shows that the polarization P tends to Np, its saturation value, when $u \to \infty$. Thus high field E and low temperature T can give this saturation value. At smaller values of u, the curve is linear. Dipole moments are of the order of 3×10^{-30} coul m, so for $E = 10^7$ volts/meter we have $pE = 3 \times 10^{-23}$. At room temperature $kT = 4 \times 10^{-21}$ joule, so that $u \approx 10^{-2}$. Thus we can write Langevin's function as

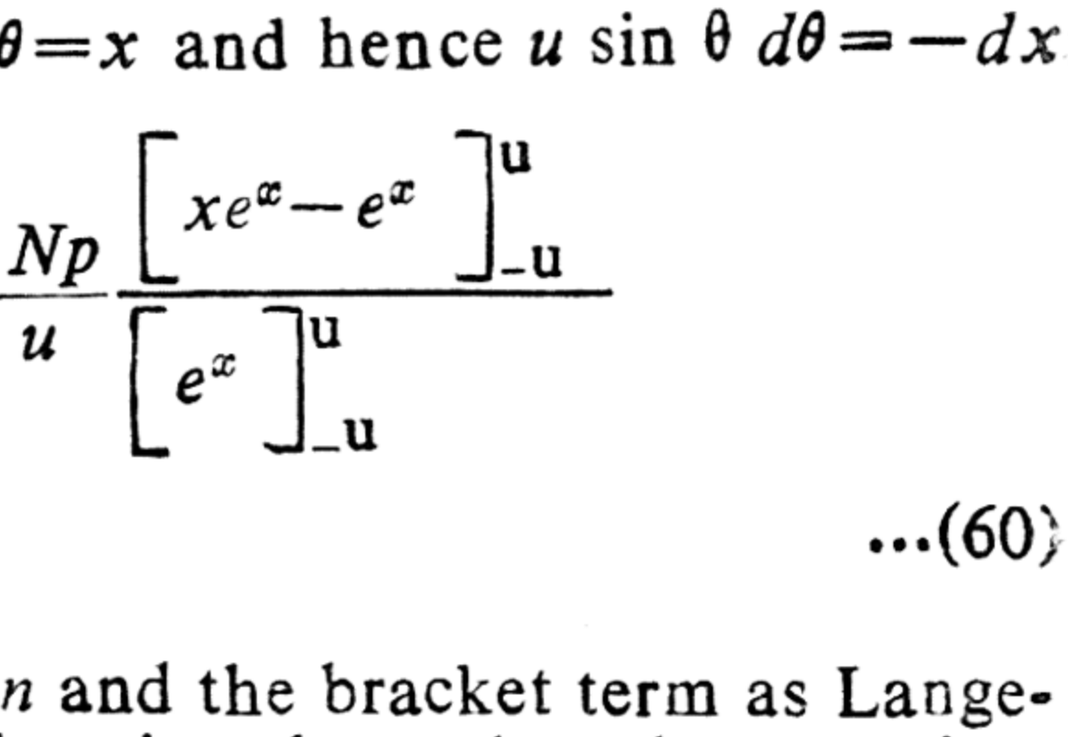


Fig. 5.15.

$$\coth u - \frac{1}{u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} - \frac{1}{u} = \frac{u}{3}$$

Hence $$P = Np\ u/3 = Np^2E/3kT, \qquad \ldots(61)$$

or $$\mathbf{P} = (Np^2/3kT)\mathbf{E}.$$

As polarizability is defined as the dipole moment of molecule per unit electric field, hence the orientational polarizability per molecule

$$\alpha_{0r}=P/NE=p^2/3kT, \qquad \ldots(62)$$

and the electric susceptibility

$$\chi=P/E=Np^2/3kT. \qquad \ldots(63)$$

At room temperature, α_{0r} is of the same order of magnitude as the electronic polarizability. Hence the total polarizability becomes

$$\alpha=\alpha_0+p^2/3kT, \qquad \ldots(64)$$

where α_0 denotes the sum of the electronic and ionic contributions. This is the *Langevin-Debye equation* and is of great importance in interpreting molecular structures.

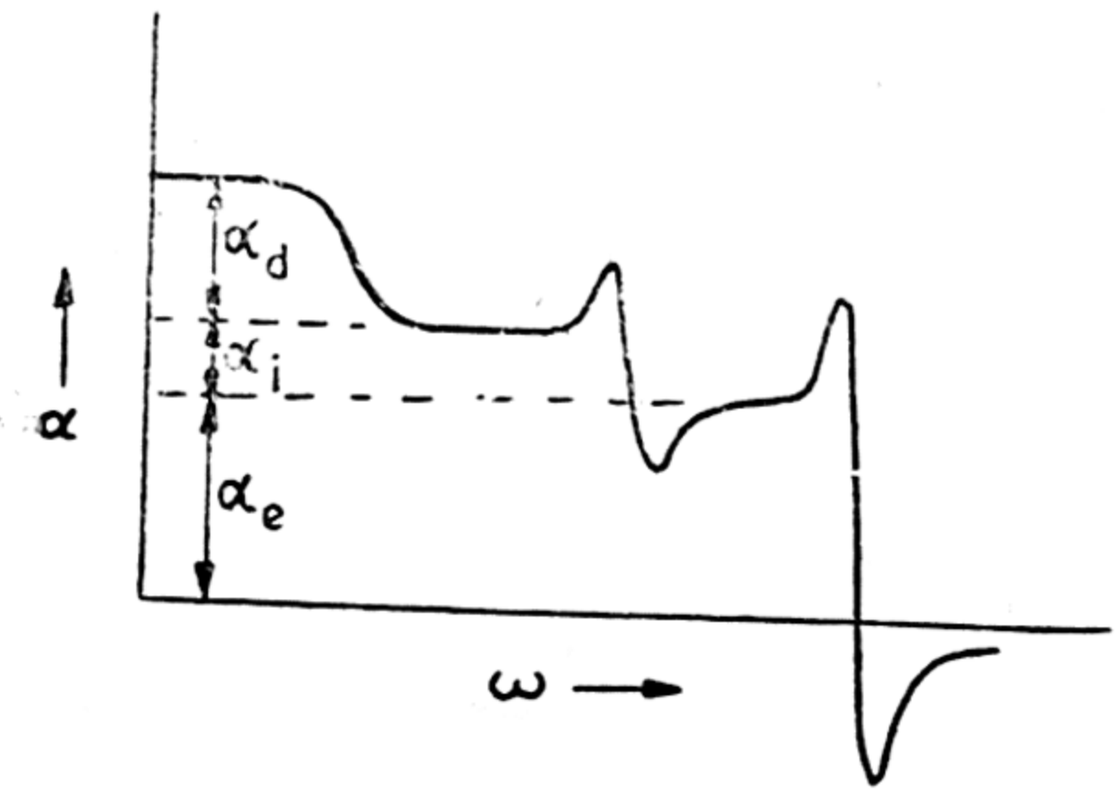


Fig. 5.16.

Frequency dependence of the several contributions to the polarizability is shown in Fig. 5.16. It shows that at optical frequencies the contribution is almost electronic.

5.12. PIEZOELECTRICITY

If a mechanical stress is applied to the crystal, the atoms are slightly displaced. The ionic displacements are symmetrical about the symmetry centres in the centrosymmetric crystals and the charge distribution inside the crystal is not changed appreciably by the applied stress. In the case of acentric crystal, the ionic displacements are asymmetric and the *electrical dipoles are produced in the crystal. This effect is known as piezoelectric effect and these crystals are called piezoelectric crystals.* Conversely when an electric field is applied to a piezoelectric crystal dipoles are induced and the atomic displacements produce a mechanical strain. This is called the *inverse piezoelectric effect.* The displacement varies periodically when the alternating field is applied to a piezoelectric crystal. At a certain frequency called *resonance frequency* these displacements are in phase with the applied field and hence are maximum. The frequency range at which resonance occurs is mostly very sharp hence such crystals are very much used for frequency controls in radio transmitters and in electronic clocks.

To obtain a relation between the mechanical stress and the electric polarization produced in a piezoelectric crystal, consider a crystal of which d placed between two metal plates A and B. If the crystal is compressed by an applied mechanical stress p, a mechanical strain s is produced in the crystal.

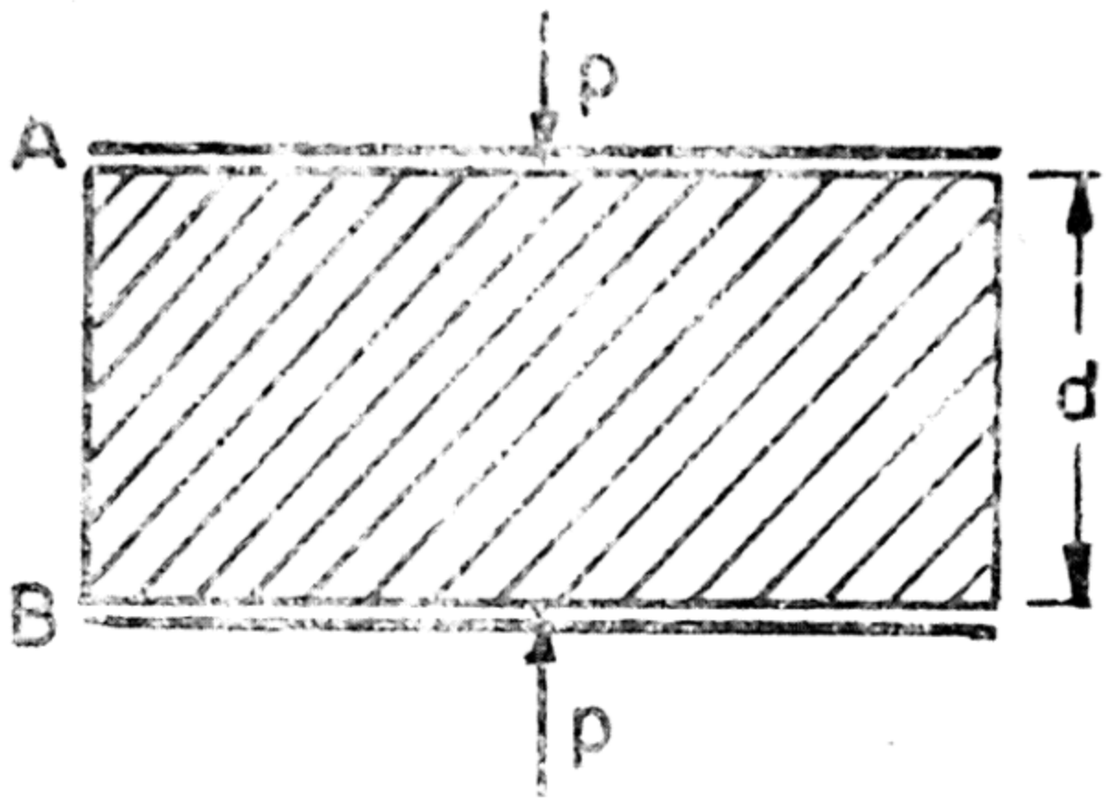


Fig. 5·17.

$$s=\frac{\text{Mechanical stress } p}{\text{Young's modulus } Y}=\frac{\Delta d}{d} \qquad ...(65)$$

The polarization density P in the crystal is proportional to the applied mechanical stress, therefore we have

$$P=\eta p, \qquad ...(66)$$

where η is the *piezoelectric constant*.

On the other hand if an electric field **E** is applied to the crystal then the induced strain is proportional to **E**, *i.e.*

$$s_{in}=\eta' E. \qquad ...(67)$$

When both an external field **E** and a stress p are applied to the crystal, the electric displacement is given by

$$D=\varepsilon_0 E+P=\varepsilon_0 E+\eta p \qquad ...(68)$$

and the internal strain

$$s=\eta' E+p/Y. \qquad ...(69)$$

5.13. ELECTRETS

When an external electric field is applied to a polar dielectric, the dipole moments get partly lined up. In general this induced polarization disappears when the external electric field is switched off. There are certain dielectrics (*organic waxes, plastic materials, etc.*) in which the polarization remains when the field is removed. Such materials are called *electrets*. The duration for which these induced polarization persist depends on the nature of the material, its polarization condition and its surroundings. When an external electric field is applied to a dielectric in the *molten* state the molecular dipoles are partly aligned in the field direction. If this dielectric is allowed to solidify, the dipoles are frozen in their new orientations and do not return to their original orientations when the external field is removed. Electrets prepared by this method (applying field at the time of molten state and removing field when it is solidify) are called *thermoelectrets*. Electrets are also prepared by

the simultaneous application of light and an electric field. Such electrets are called *photoelectrets*.

The electrets are very used in instruments such as microphones, high voltage generators, computers, radio and television receivers and other electronic devices. The photoelectrets are of very much use in technique for the reproduction of printed matter (xerography).

5.14. FERRO-ELECTRICITY

Certain types of dielectric materials are asymmetric in such a way that they are spontaneously polarized in the absence of external fields. This self polarization results from the displacement of ions due to local electric fields. The polarization due to permanent dipoles present in the dielectric is temperature dependent. A change in temperature of the dielectric changes its polarization. This effect is called *pyroelectric effect*. Nearly fifty per cent of piezoelectric crystal classes exhibit this property. There are certain pyroelectric crystals in which one can reverse the direction of polarization by applying a sufficiently intense external field. Such crystals are called *ferroelectric* and this effect is called the *ferroelectric eflect*. It should be noted that *all the pyroelectric crystals are piezoelectric crystals but the reverse is not true and similarly all the ferroelectric crystals are pyroelectric crystals but the reverse is not true.*

The term ferroelectricity is applied to the ferroelectric phenomenon. A ferroelectric crystal generally consists of regions called *domains* within each of which all the molecular depoles are parallel. In adjacent domains the polarization is in different directions. In the crystal as a whole the dipole moments of these domains are randomly oriented resulting in zero dipole moment. When an external electric field is applied these domains tend to align themselves in the field direction. The polarization induced does not vary linearly with the applied electric field **E**. The ferroelectric materials exhibit hysteresis phenomenon in a manner similar to ferromagnetic materials. They often exhibit large susceptibility. As the electric field is applied the total polarization increases rapidly until the saturation value is reached. It is the position where all the domains are parallel to the applied field. When the field decreases to zero, the polarization does not return to zero. The polarization left is called remanent polarization. The coercive field required to reverse the polarization direction is not so large as to cause electric breakdown of the crystal.

Many ferroelectric crystals undergo a transformation at a certain temperature called the *Curie point*. Above this temperature they are no longer polar even though they may still have noncentrosymmetric structure. The dielectric constant k of these crystals above the critical temperature (*transformation temperature, Curie point*) is given by the relation,

$$k = \frac{C}{T - T_c} + k_0, \qquad \ldots(70)$$

where C is the constant called the *Curie constant* and k_0 is the contribution due to the electronic polarization. This relation shows that just above the critical temperature k is very large.

One of the best known ferroelectric materials is barium titanate ($BaTiO_3$), which is often used in capacitors where small size and large capacitance are required. Because of their low spontaneous polarizabilities ferroelectric crystals are frequently used in computers as memory cells. There are several groups of ferroelectric crystals. The examples are :

Rochelle salt $NaK\ (C_4H_4O_6)\ 4H_2O$; *Potassium dihydrogen phosphate* KH_2PO_4 ; *Barium titanate* $Ba\ Ti\ O_3$; Guanidine compounds such as $C(NH_2)_3\ Al(SO_4)_2.6H_2O$.

5.15. POISSON AND LAPLACE EQUATIONS

Gauss's law in the differential form in which the space rate of variation of the electric induction components are related to the distributed space charge, can be written as

$$\text{div } \mathbf{D} = \rho.$$

We know that $\mathbf{E} = -\text{grad } V$ and in an isotropic medium

$$\mathbf{D} = \varepsilon \mathbf{E},$$

$$\therefore \text{ div } \mathbf{D} = \text{div } \varepsilon \mathbf{E} = \varepsilon \text{ div } (-\text{grad } V) = -\varepsilon \nabla^2 V = \rho$$

or
$$\nabla^2 V = -\rho/\varepsilon. \qquad \ldots(71)$$

This is known as *Poisson's equation*. If there is no free charge present, $\rho = 0$ and we get

$$\nabla^2 V = 0. \qquad \ldots(72)$$

This is known as *Laplace's equation*.

In cartesian coordinates, Poisson's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}. \qquad \ldots(73)$$

In spherical polar coordinates, we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho}{\varepsilon} \qquad \ldots(74)$$

In cylindrical polar coordinates, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}. \qquad \ldots(75)$$

Similarly Eq. (72) can be written in different system of coordinates. *These equations are very useful in electrostatic problems.*

5.16. THE UNIQUENESS THEOREM

If potential V can be found we can at once get **E** from it. Everywhere outside the conductor V has to satisfy the Laplace equation $\nabla^2 V = 0$. Thus the problem of finding the electric field **E** due to a given set of charged conductors, is just of solving the Laplace's equation. Let in cartesian coordinates, $V_1(x, y, z)$ and $V_2(x, y, z)$ are two solutions of Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$, meeting the same boundary conditions. As Laplace's equation is linear hence the linear combinations of these two solutions $\alpha_1 V_1 + \alpha_2 V_2$ should also be the another solution, where α_1 and α_2 are constants. In particular, the difference between our two solutions $V_1 - V_2$ must also be the correct solution. Call this difference as

$$W(x, y, z) = V_1(x, y, z) - V_2(x, y, z). \qquad \ldots(76)$$

Thus we find that $W = 0$ at the boundary of all the conductors, as V_1 and V_2 are equal everywhere on them. W is also a solution of another electrostatic problem, one with the same conductors but with all conductors held at zero potential. This must also be zero at all points in space. We also know that potential is zero at infinity as well as on all the conductors. Hence we see that W must be zero everywhere. It follows that $V_1(x, y, z) = V_2(x, y, z)$ everywhere, *i.e., there can be only one solution of Laplace's equation that satisfies the prescribed boundary conditions.* This is the *Uniqueness theorem.*

5.17. APPLICATIONS OF LAPLACE'S EQUATION

Solution of Laplace's equation is very tedious. It requires the knowledge of Legendre's differential equation, which is beyond the scope of this text. The readers are advised to consult any standard book on differential equations. Following two simple cases are only discussed here.

(*a*) **Conducting Sphere in a Uniform Field.** Consider an uncharged conducting sphere of radius a placed in a uniform electric field $\mathbf{E}_0$. There is an equipotential plane cutting through the middle of the sphere and perpendicular to the direction of $\mathbf{E}_0$. We define this plane to be at zero potential. Since net charge on the sphere is zero, hence $\nabla^2 V = 0$ and sphere is an equipotential surface with $V = 0$. The sphere distorts the field in its neighbourhood, but the field at a great distance remains unchanged. If the origin is taken at the centre of the sphere and the x-axis in the direction of original electric field $\mathbf{E}_0$, potential at large distances is that of the field $\mathbf{E}_0$ alone. Hence at points far from the sphere $V = -E_0 x = -E_0 r \cos\theta$. Thus we must look for a solution of Laplace's equation which satisfies the two boundary conditions

$$V = -E_0 r \cos\theta \qquad \text{for } r \to \infty$$

$$V = 0 \qquad \text{for } r = a.$$

To satisfy these boundary conditions, let us choose the solution as

$$V = -E_0 r \cos\theta + (A/r^2)\cos\theta. \qquad \ldots(77)$$

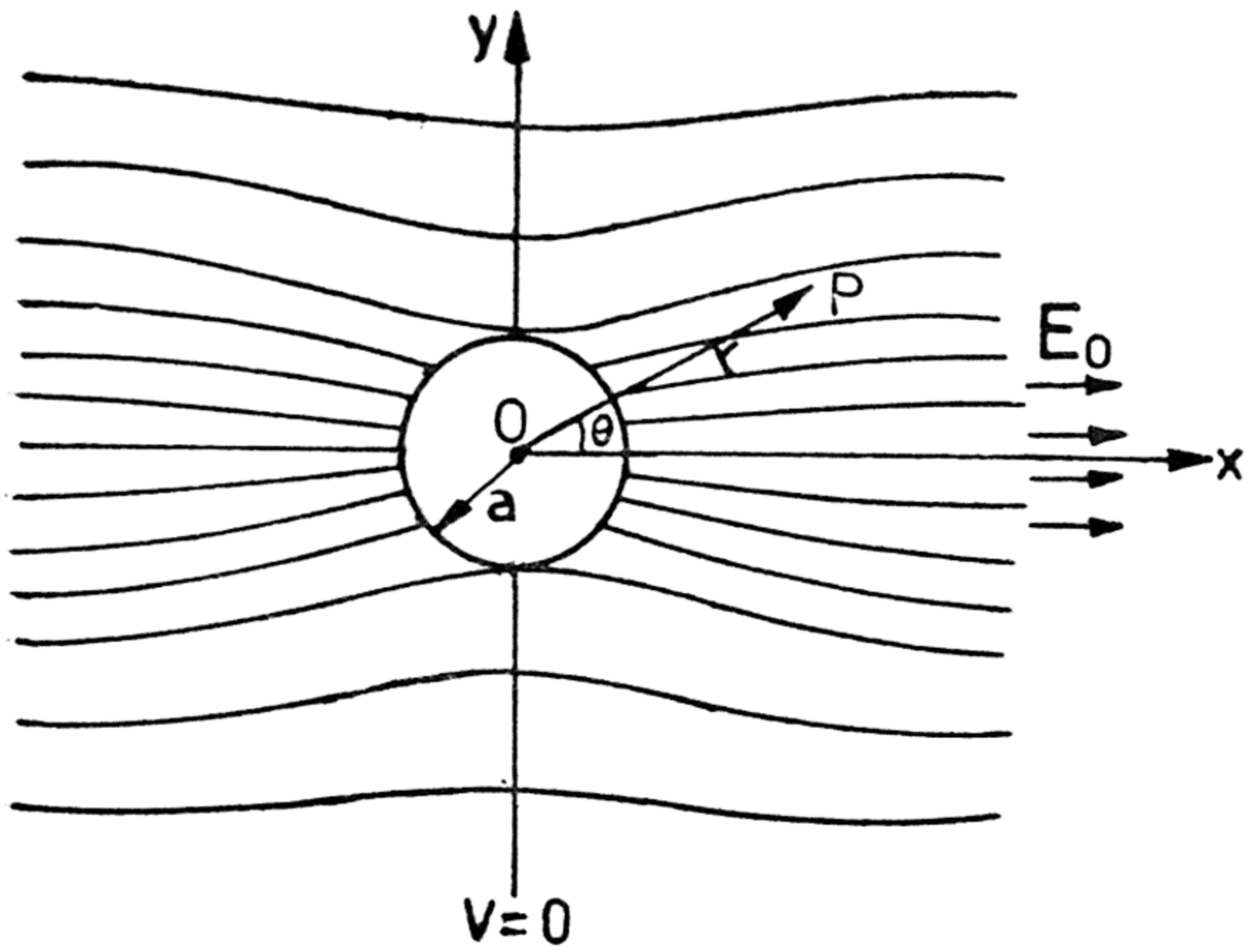


Fig. 5.18.

Both the terms are solutions of Laplace's equation $\nabla^2 V = 0$, the second term tends to zero for $r \to \infty$. From the boundary condition, $V = 0$ at $r = a$, we get $A = E_0 a^3$, hence the general solution of Laplace's equation for our particular case reduces to

$$V = -E_0 r \cos\theta + E_0 a^3/r^2)\cos\theta = -E_0 r \cos\theta\,[1 - a^3/r^3]. \qquad \ldots(78)$$

This expression is valid for all $r > a$. Inside the sphere the potential is verywhere zero.

The second term, $(E_0 a^3/r^2)\cos\theta$, amounts to the perturbation of the original uniform field by the conducting sphere. It corresponds to the effect of a dipole of dipole moment $p = E_0 a^3$ placed at the origin, with **p** along x-axis.

The components of electric intensity outside the sphere are :

$$\textit{along radius vector } E_r = -\frac{\partial V}{\partial r} = \left(1 + \frac{2a^3}{r^3}\right) E_0 \cos\theta, \qquad \ldots(79)$$

$$\perp \textit{ to radius vector } E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\left(1 - \frac{a^3}{r^3}\right) E_0 \sin\theta. \qquad \ldots(80)$$

At $r = a$, *i.e.*, the surface of conducting sphere, $E_r = 3E_0 \cos\theta$ and $E_\theta = 0$. Along the x-axis the field at the surface is thus $3E_0$ and at points on the surface along y-axis, both V and E are zero.

We know that the surface charge density on the surface of a conductor immersed in a medium of dielectric constant k is given by

$$\sigma = D = k\varepsilon_0 (Er)_{r=a} = 3k\varepsilon_0 E_0 \cos\theta. \qquad \ldots(81)$$

It is +ve on the right hand hemisphere and negative on the left.

If the sphere has a charge q, then we have

$$V = -\left(1 - \frac{a^3}{r^3}\right) E_0 r \cos\theta + \frac{a}{4\pi\varepsilon_0 r}. \qquad \ldots(82)$$

(*b*) **Dielectric Sphere in a Uniform Field.** Consider an uncharged dielectric sphere of radius a and dielectric constant k_1, surrounded by a medium of dielectric constant k_2 and placed in a uniform electric field $\mathbf{E}_0$. Let V_1 and V_2 be the potential functions inside and outside the sphere respectively. The solution to this problem must satisfy the following requirements.

(*i*) $\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$, both inside and outside respectively, as the net charge on the sphere is zero.

(*ii*) V_1 must remain finite for all $r \leqslant a$. V_2 must also remain finite at infinity.

(*iii*) $V_1 = V_2$ for $r = a$ at all angles θ, otherwise discontinuity would give an infinite electric field there.

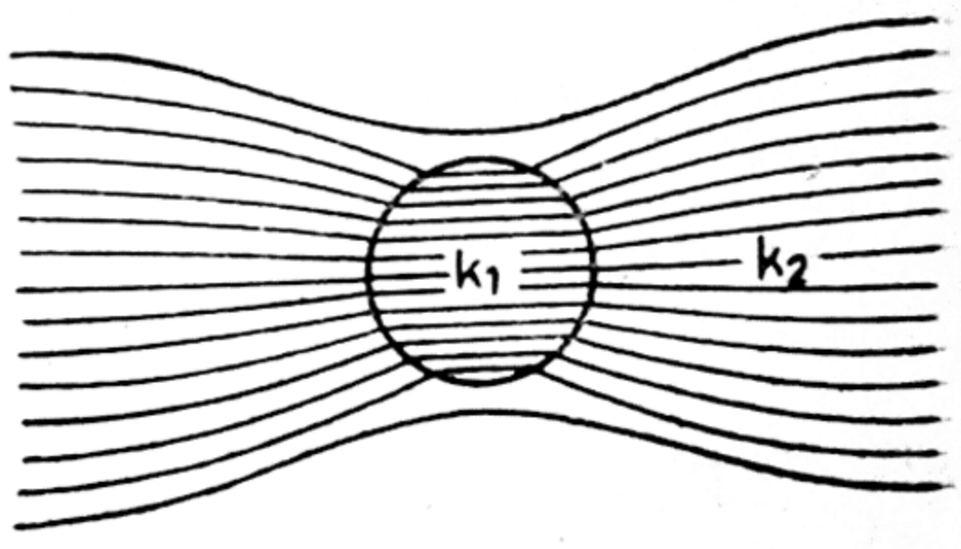


Fig. 5.19.

(*iv*) Both boundary conditions must satisfy at the surface of the sphere. $D_{1n} = D_{2n}$ and $E_{1p} = E_{2p}$.

As in the problem of the conducting sphere

$$V_2 = -E_0 r \cos\theta + (A/r^2)\cos\theta. \qquad \ldots(83)$$

As there is no volume distribution of free charge inside the sphere the potential function inside the sphere may be assumed of the same form as that used outside, hence

$$V_1 = B r \cos\theta + (C/r^2)\cos\theta. \qquad \ldots(84)$$

Evidently, C must be zero, otherwise the potential would become infinite at the origin ($r = 0$), hence

$$V_1 = Br\cos\theta. \qquad \ldots(85)$$

At the surface of the sphere ($r = a$), we have $V_1 = V_2$.

$$\therefore \quad Ba\cos\theta = -E_0 a\cos\theta + (A/a^2)\cos\theta \text{ or } B = -E_0 + A/a^3. \qquad \ldots(86)$$

The boundary condition, that the normal components of the electric displacement at the surface must be continuous, gives

$$D_{1n} = D_{2n} \text{ or } k_1\epsilon_0 E_{1n} = k_2\epsilon_0 E_{2n}.$$

As $E_n = -\partial V/\partial r$, hence $-k_1(\partial V_1/\partial r)_{r=a} = -k_2(\partial V_2/\partial r)_{r=a}$

$$-k_1 B\cos\theta = -k_2[-E_0\cos\theta - (2A/a^3)\cos\theta]$$

or
$$k_1 B = -k_2(E_0 + 2A/a^3). \qquad \ldots(87)$$

Solving Eqns. (86) and (87) we get

$$A = \frac{k_1 - k_2}{k_1 + 2k_2} E_0 a^3 \text{ and } B = \frac{-3k_2}{k_1 + 2k_2} E_0.$$

Thus the potential functions inside and outside the sphere are :

$$V_1 = -\frac{3k_2}{k_1 + 2k_2} E_0 r\cos\theta \qquad \ldots(88)$$

$$V_2 = -\left(1 - \frac{k_1 - k_2}{k_1 + 2k_2}\cdot\frac{a^3}{r^3}\right) E_0 r\cos\theta. \qquad \ldots(89)$$

As the potential inside the sphere can be written as $V_1 = Bx$, hence the field inside the sphere is uniform. It is parallel to x-axis, *i.e.*, along $\mathbf{E}_0$ and is of magnitude

$$E_1 = -\frac{\partial V_1}{\partial x} = \frac{3k_2}{k_1 + 2k_2} E_0. \qquad \ldots(90)$$

For the dielectric sphere in a vacuum ($k_2 = 1$), the original uniform field E_0 has been reduced to a value $[3/(k_1+2)]E_0$. This reduction is due to the field of the polarization charges on the surface of the sphere acting in opposite direction. This reveise field is known as the *depolarizing field* E_{dep} and is given by

$$E_{dep} = E_0 - E_1 = \left(1 - \frac{3}{k_1 + 2}\right) E_0 = \left(\frac{k_1 - 1}{k_1 + 2}\right) E_0. \qquad \ldots(91)$$

It may also be expressed in terms of a depolarizing factor L as

$$\epsilon_0 E_{dep} = LP = L\epsilon_0(k_1 - 1)E_1. \qquad \ldots(92)$$

Substitution of values gives $L = \frac{1}{3}$. $\qquad \ldots(93)$

The potential distribution outside the sphere is that of a dipole of magnitude $4\pi k_2\epsilon_0 a^3 E_0(k_1 - k_2)/(k_1 + 2k_2)$, situated at the centre of the sphere, superimposed on that due to the uniform field.

If $k_1 \to \infty$, the solutions tend to those obtained for the conducting sphere.

5.18. ELECTRICAL IMAGES

The distribution of induced charge on a surface, hence the field intensity and potential distribution may be found in many cases by the method of electrical images, developed originally by Lord Kelvin. If two equal point charges of opposite signs are separated by a certain distance $2a$, the plane passing through the mid-point of

the line joining them and perpendicular to this line is an equipotential surface (at zero potential). Now if we replace this equipotential surface by a conducting surface, and transfer one of the charges to this surface, the field between the other charge and the surface remains unaltered although that between the first charge and the surface is wiped out. Conversely, if a point charge is placed in front of an infinite conducting plane, the resultant electric field in front of the plane will be the same as that produced by the original charge plus an equal charge of opposite sign if placed at an equal distance from the plane on the opposite side. This charge is the *electric image* of the original charge in the plane, analogous to the theory of optical images. Thus an *electrical image is a set of one or more point charges on one side of a surface which produces on the other side of the surface the same electric field as is produced by the actual electrification of the surface.*

The method of images thus consists in replacing a conductor by a point charge such that the conducting surface is still an equipotential surface ($V=0$). Thus the Laplace's equation is still satisfied at all points outside the conductor and by the principle of uniqueness, the problem of a point charge and a conducting surface can be replaced by the problem of a point charge and its image. Let us apply this method for the following cases.

(1) **Point charge and infinite conducting plane.** To find the field due to a point charge q placed before a conducting plane AB at a distance d from it, we consider the field due to this charge q and its image, the charge $-q$ placed on the perpendicular dropped from q to the plane on the left at a distance d from the plane. The potential at an arbitrary point $P(r, \theta)$ is thus given by

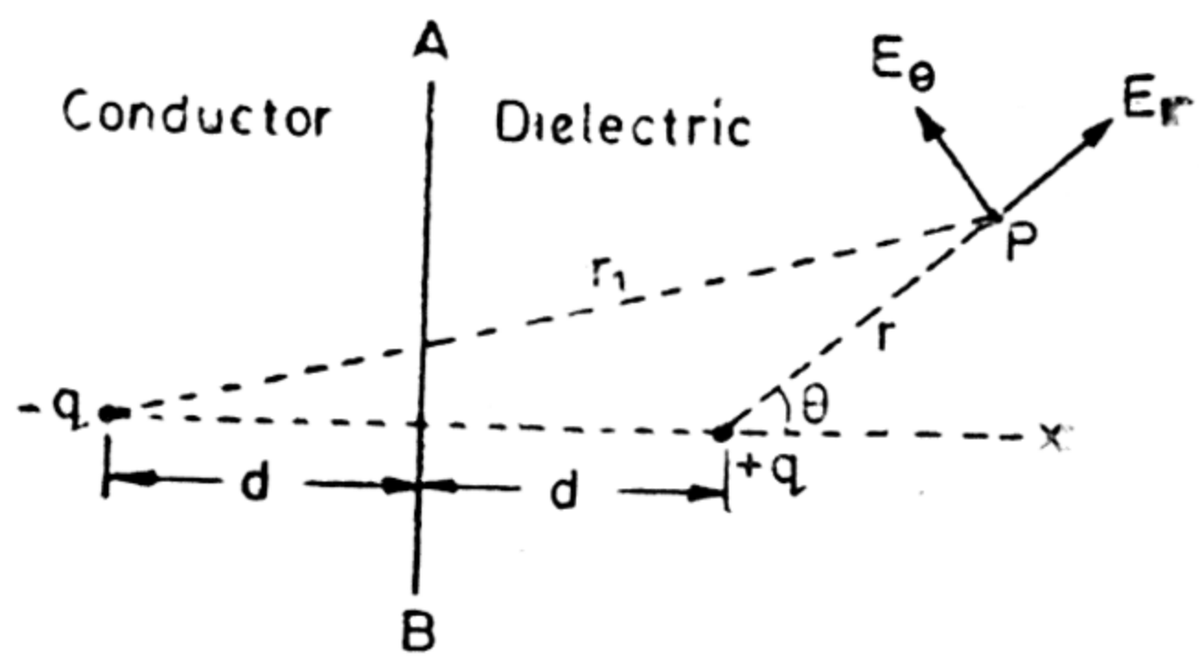


Fig. 5.20.

$$V = \frac{1}{4\pi\varepsilon}\frac{q}{r} - \frac{1}{4\pi\varepsilon}\frac{q}{r_1}$$

$$= \frac{q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{\sqrt{(r^2+4d^2+4rd\cos\theta}}\right], \quad \ldots(94)$$

where ε is the permittivity of the medium outside the conductor.

The electric intensity at point P in the direction of $\mathbf{r} = E_r = -\partial V/\partial r$ and in the direction perpendicular to $\mathbf{r} = E_\theta = -(1/\mathbf{r})(\partial V/\partial\theta)$, therefore the component of electric intensity at point P normal to the plane

$$E_x = Er\cos\theta - E_\theta\sin\theta = -\frac{\partial V}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial V}{\partial\theta}\sin\theta$$

$$E_x = \frac{q}{4\pi\varepsilon}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2d}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\right]. \quad \ldots(95)$$

If the point P is on the plane, then $r\cos\theta = -d$ and the field

$$E_x = -qd/2\pi\varepsilon r^3. \quad \ldots(96)$$

The induced charge per unit area on the plane is then $\sigma = \varepsilon E_x = -qd/2\pi r^3$. Hence the surface density of the induced charge on the plane varies inversely as the cube of the distance of the position on the plane from the original charge $+q$ and is also opposite in sign to $+q$.

The force exerted on the point charge by the induced charge on the plane = Force exerted on the charge by its image

$$= -q^2/16\pi\varepsilon d^2. \quad \ldots(97)$$

Here the $-$ve *sign indicates that the charge is attracted towards the plane.*

(2) **Point Charge and Conducting Sphere.** *Case 1 – When the sphere is at zero potential.* Let a charge q be placed at a point A at a distance f from the centre of an earthed spherical conductor of radius a. To solve the problem we have to determine the magnitude and position of a point charge q' inside the sphere, which together with the charge q makes the sphere a equipotential surface (zero potential). As an approach to the solution, consider the charge q' at A' inside the conductor such that $OA\ OA' = a^2$. Then $OA/a = a/OA'$, thus the triangles $A'OP$ and POA are similar, where P is any point on the surface of the sphere.

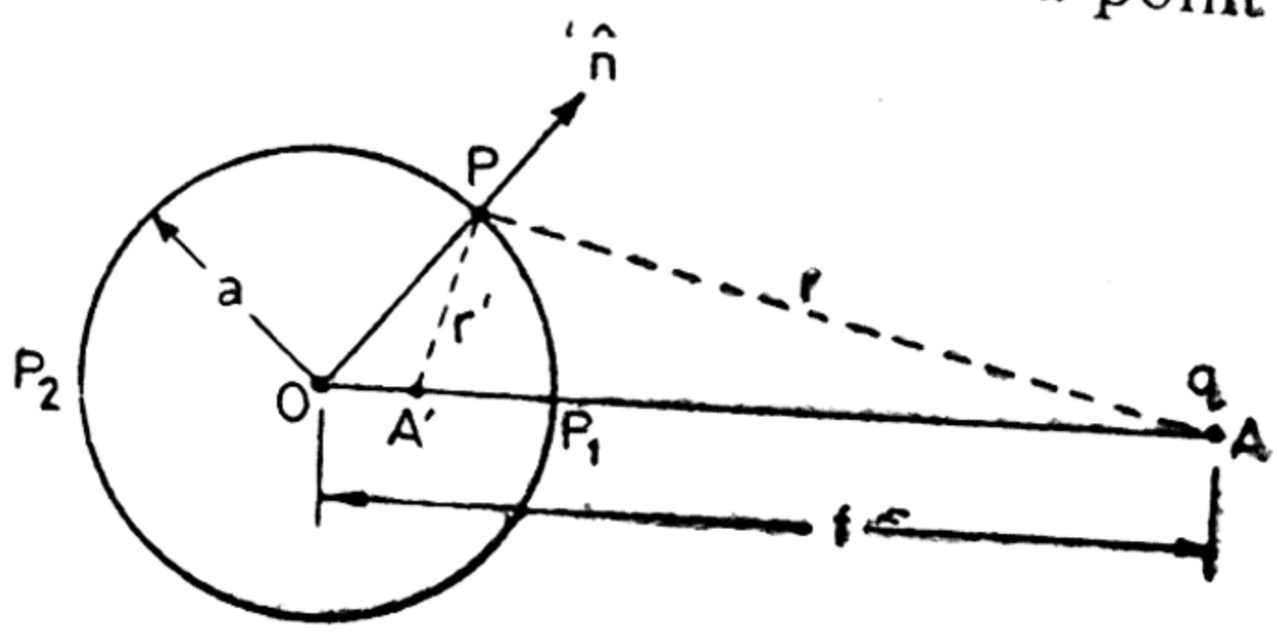


Fig. 5.21.

$$\therefore \quad OA/a = a/OA' = PA/PA', \text{ or } PA/PA' = f/a.$$

The sphere will be an equipotential surface at zero potential, hence

$$\frac{q}{4\pi\varepsilon(PA)} + \frac{q'}{4\pi\varepsilon_0(PA')} = 0, \text{ or } q' = -q\frac{PA'}{PA} = -q\frac{a}{f}. \quad \ldots(98)$$

The charge induced on the surface is clearly the charge of image, *i.e.* $-qa/f$. For investigating the field distribution this image may be used instead of the induced charge on the sphere. The surface density of induced charge σ may be found by using Coulomb's law, *i.e.*,

$$\frac{\sigma}{\varepsilon_0}\hat{n} = \text{Electric field at point } P = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\mathbf{r} - \frac{1}{4\pi\varepsilon_0}\frac{qa}{fr'^3}\mathbf{r}',$$

where **r** and **r'** are the vectors **AP** and **A'P** respectively and n denotes a unit vector along the normal **OP** to the sphere at P. Now we have

$$\mathbf{r} = \mathbf{AO} + \mathbf{OP} = \mathbf{AO} + a\hat{n} \text{ and } \mathbf{r'} = \mathbf{A'O} + \mathbf{OP} = \mathbf{A'O} + a\hat{n}.$$

Remembering that **E** is radial (along n) at P, the terms involving **AO** and **A'O** cancel and we have

$$\frac{\sigma}{\varepsilon_0}\hat{n} = \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{qa}{r^3}\hat{n} - \frac{1}{4\pi\varepsilon_0}\frac{qa}{f}\cdot\frac{a}{r'^3}\hat{n}.$$

Using relation $PA/PA' = r/r' = f/a$, we get

$$\frac{\sigma}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qa}{r^3}\left(1 - \frac{f^2}{a^2}\right) \quad \text{or} \quad \sigma = -\frac{1}{4\pi}\frac{q}{ar^3}(f^2 - a^2). \quad \ldots(99)$$

The force between charge q and the sphere is given by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(AA')^2} = \frac{1}{4\pi\varepsilon_0}\frac{q(-qa/f)}{(f - a^2/f)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2af}{(f^2 - a^2)^2}. \quad \ldots(100)$$

This $-$ve sign indicates that the charge is attracted towards the sphere.

Case II. When the sphere is insulated and without charge. When a point charge q is placed near a insulated conducting sphere, the conditions are : (*i*) The gaussian conditions in the region outside the sphere remains the same. (*ii*) The total charge induced on the sphere is zero, as the sphere is insulated, (*iii*) The potential of the sphere is uniform all over the surface and is not zero.

The charge $+q$ at an external point A and its image charge $-aq/f$ at point A' satisfy the gaussian conditions. An additional image charge of qa/f must be distributed uniformly over the surface of the sphere, in order to satisfy condition (*ii*). It is assumed at the centre O.

As regards the potential of the surface of the sphere, the two charge $+q$ at A and $-aq/f$ at A' together produce zero potential on the sphere and the net potential on the sphere is due to the charge aq/f at the centre O, which is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Charge}{Radius} = \frac{1}{4\pi\varepsilon_0}\frac{aq/f}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q}{f}, \quad \ldots(101)$$

which is uniform as required by the condition (*iii*).

Surface density of charge at point P = surface density due to charge $(q, -qa/f)$ + due to the charge (aq/f) at O,

$$\text{or} \quad \sigma = -\frac{1}{4\pi}\frac{q}{ar^3}(f^2 - a^2) + \frac{qa/f}{4\pi a^2} = -\frac{1}{4\pi}\frac{q}{a}\left[\frac{f^2 - a^2}{r^3} - \frac{1}{f}\right]. \quad \ldots(102)$$

From this result we see that when point P is at P_1, for which $r = f - a$, we get

$$\sigma_{P_1} = -\frac{1}{4\pi}\, q\, \frac{3f-a}{f\,(f-a)^2}, \qquad \ldots(103)$$

which is negative, and when point P is at P_2, for which $r=f+a$, we get

$$\sigma_{P_2} = \frac{1}{4\pi}\, q \frac{3f+a}{f\,(f+a)^2}, \qquad \ldots(104)$$

which is positive.

Thus there is a line between these positions P_1 and P_2 through point P' say, on which surface charge density $\sigma=0$, such as

$$\frac{f^2-a^2}{(P'A)^3} - \frac{1}{f} = 0, \text{ or } (P'A)^3 = f\,(f^2-a^2) = f^2\left[f - \frac{a^2}{f}\right]$$

or $$(P'A)^3 = OA^2 . AA' \qquad \ldots(105)$$

The force of attraction between the point charge $+q$ and the sphere is thus given as

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2 a}{f\,(AA')^2} - \frac{1}{4\pi\varepsilon_0}\frac{q^2 a}{f\,(AO)^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2 a}{f}\left[\frac{f^2}{(f^2-a^2)^2} - \frac{1}{f^2}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2 a^3}{f^3}\frac{(2f^2-a^2)}{(f^2-a^2)^2} \qquad \ldots(106)$$

(3) **Point Charge and a Semi-infinite Slab of Dielectric.** Let a point charge $+q$ is situated in air in front of a semi-infinite slab of dielectric of permittivity ε. This charge induces a charge on the dielectric slab. If this induced charge is replaced by an image charge (or charges) situated in the dielectric, then it must satisfy the following conditions : (*a*) Gaussian conditions hold good both for the dielectric and air medium, (*b*) Boundary conditions are also valid, (*c*) The potential at infinitely distant points is zero.

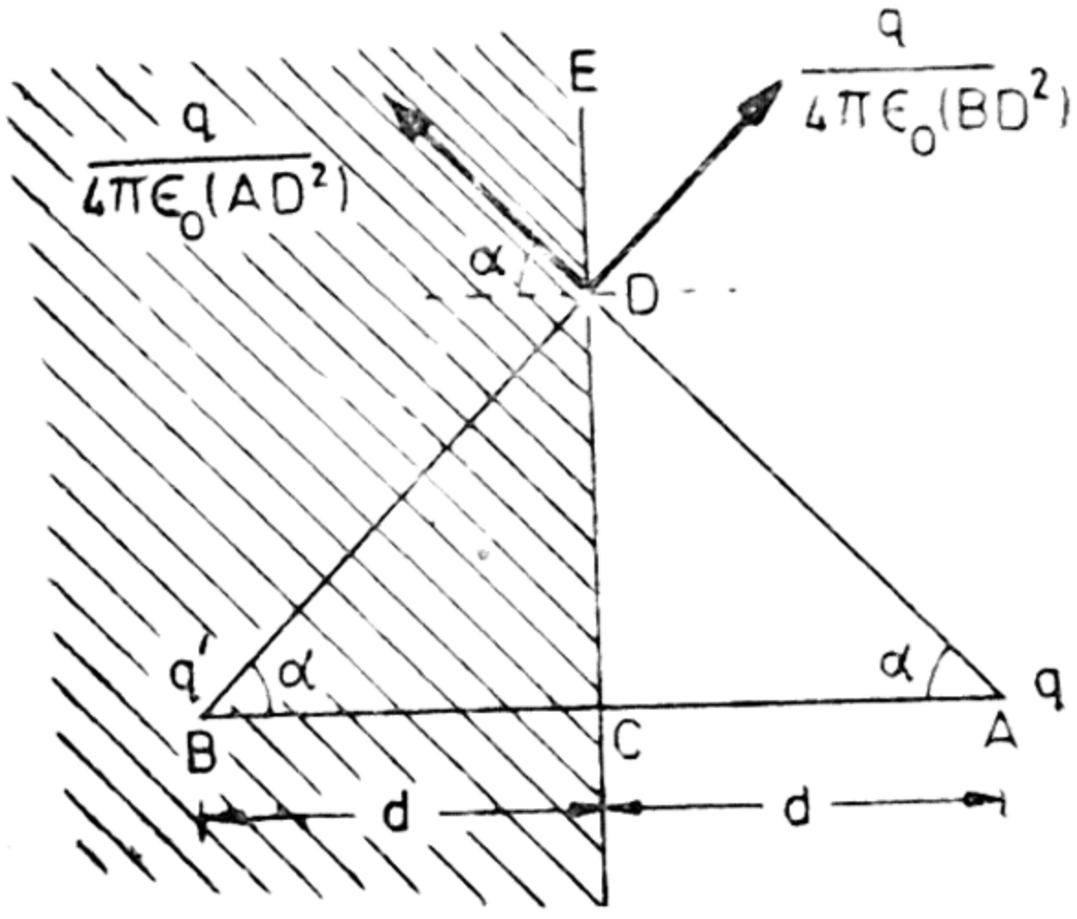


Fig. 5.22.

To solve this problem let us consider a charge q' placed at point B in the dielectric such that $AC=BC$. The value of q' is so chosen that the combination of q and q' gives the correct field distribution on the R.H.S. of the boundary and fulfils the condition (*c*). For the correct field distribution within the dielectric, *i.e.*, on the L.H.S of the boundary only the charge q is replaced by the another charge q''. The act of placing charge q'' does not conflict with gaussian condition in dielectric and condition (*c*).

The charges q' and q'' are so chosen that boundary conditions are also satisfied. For this purpose we have to assume charges q and q' when we deal with R.H.S. of the boundary and charge q'' alone when with L.H.S. The electric field at point D on the boundary due to charge q is $q/4\pi\varepsilon_0(AD)^2$ along **AD** and due to charge q' is $q'/4\pi\varepsilon_0\,(BD)^2$ along **BD**. In the second case *we have taken* ε_0 *instead of* ε *as* q' *is an image assumed in air, not in dielectric.* Hence the total tangential component of electric intensity due to the presence of charges q and q'

$$=\frac{q}{4\pi\varepsilon_0\,(AD)^2}\sin\alpha+\frac{q'}{4\pi\varepsilon_0\,(BD)^2}\sin\alpha=\frac{\sin\alpha}{4\pi\varepsilon_0\,(AD)^2}(q+q').$$

The tangential component of the field due to the charge q'', supposed to be placed at point A, giving field distribution on L.H.S.

$$=\frac{q''}{4\pi\varepsilon_0(AD)^2}\sin\alpha.$$

Thus for the continuity of the tangential components of electric intensity, we have

$$\frac{(q+q')\sin\alpha}{4\pi\varepsilon_0\,(AD)^2}=\frac{q''}{4\pi\varepsilon_0(AD)^2}\sin\alpha,$$

or

$$q+q'=q''. \qquad \ldots(107)$$

The normal components of the displacements should also be con inuous, hence

$$\frac{\varepsilon_0 q}{4\pi\varepsilon_0(AD)^2}\cos\alpha-\frac{\varepsilon_0 q'}{4\pi\varepsilon_0(BD)^2}\cos\alpha=\frac{\varepsilon q''}{4\pi\varepsilon_0(AD)^2}\cos\alpha$$

or

$$q-q'=(\varepsilon/\varepsilon_0)q''.=k\,q''. \qquad \ldots(108)$$

Solving Eqs. (107) and (108), we get

$$q'=-q\,\frac{k-1}{k+1}\quad\text{and}\quad q''=\frac{2q}{k+1}. \qquad \ldots(109)$$

From this relation we see that q' is negative if q is positive and q'' is positive. These two charges are the image charges each less than q and satisfying boundary conditions.

The force of attraction between the original charge q and the dielectric

$$=\frac{1}{4\pi\varepsilon_0}\frac{q.q[(k-1)/(k+1)]}{(2d)^2}=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d^2}\frac{k-1}{k+1}.$$

$$=q^2/4\pi\varepsilon_0\,(4d^2)\qquad(\text{for } k=\infty). \qquad \ldots(110)$$

This is same as obtained for the conductor and a point charge, given by Eq. (97). Thus we see that a *conductor behaves like a dielectric with a permittivity of infinity.*

5.16. SYSTEM OF CONDUCTORS

In this article we shall discuss the general theory of an electrostatic field due to large number of conductors. Let us consider a system consisting of n independent isolated conductors in any medium. Any charge placed on the one conductor will not only increase the potential of this conductor but also of the other $(n-1)$ conductors. If the potentials of 1st, 2nd, 3rd...nth conductors due to a unit charge on the 1st conductor are designated as p_{11}, p_{12}, p_{13},.....p_{1n} respectively. If a charge q_1 is placed on the 1st conductor, then potentials become $p_{11}\,q_1$, $p_{12}\,q_1$, p_{13}, q_1...$p_{1n}\,q_1$. If charge q_2 is placed upon the second conductor, instead of first, then potentials will be $p_{21}\,q_2$, $p_{22}\,q_2$, $p_{23}\,q_2$,.....$p_{2n}\,q_2$. From this discussion it is clear that the first number of the suffix represents the conductor upon which the charge has been placed and the second number represents the conductor upon which the resultant potential has been produced.

Thus the contribution to potential due to the charge q_n on the n^{th} conductor is $p_{n1}\,q_n$, $p_{n2}\,q_n$, $p_{n3}\,q_n$,.....,$p_{nn}\,q_n$. The actual potential of any one conductor will be given by an addition of the terms corresponding to the charges on all conductors. Thus the potential of the 1st conductor is given by

$$V_1 = p_{11}q_1 + p_{21}q_2 + p_{31}q_3 + \ldots\ldots p_{n1}q_n$$

for 2nd conductor $\quad V_2 = p_{12}q_1 + p_{22}q_2 + p_{32}q_3 + \ldots\ldots p_{n2}q_n$

$\vdots$ $\qquad\qquad \vdots \qquad\qquad \vdots$

for nth conductor $\quad V_n = p_{1n}q_1 + p_{2n}q_2 + p_{3n}q_3 + \ldots\ldots p_{nn}q_n$

These n equations can be written as

$$V_i = \sum_{i=1}^{n} p_{ij}\, q_i. \qquad \ldots(111)$$

These coefficients p_{ij} are called *coefficients of potential.*

Above linear equations can be solved for the charges as linear functions of potentials. Suppose these are

$$q_j = \sum_{i=1}^{n} q_{ij}\, V_i, \qquad \ldots(112)$$

where the coefficients q_{ij} are called the *coefficients of induction.*

If all the conductors other than the r^{th} are earthed, then $q_r = q_{rr}\, V_r$ and all other V terms are zero. Hence the term q_{rr} will be nothing but the capacitance. Thus the coefficients of the type q_{ij} are called the *coefficients of capacity.*

The reciprocity relation for the coeffieients of potential. The potential of one conductor due to a unit charge on another is equal to the potential of the second due to a unit charge on the first, *i.e.*, $p_{12} = p_{21}$, or in the generalised form

$$p_{lj}=p_{ji}. \qquad \ldots(113)$$

This is known as the reciprocity relation for the coefficients of potential.

To prove it, let us consider a simple case of three conductors, labelled as 1, 2, 3. Let charges Q_i give rise to potentials V_i and charges Q_i' give rise to potentials V_i' $(i=1, 2, 3)$. The potential V_1 on the first conductor depends upon the charge on these three conductors. In terms of charge densities on their surfaces, the potential on the conductor 1 is given by

$$V_1=\frac{1}{4\pi\epsilon_0}\left[\int_{s_1}\frac{\sigma_1 dS_1}{r_1}+\int_{s_2}\frac{\sigma_2 dS_2}{r_2}+\int_{s_3}\frac{\sigma_3 dS_3}{r_3}=\frac{1}{4\pi\epsilon_0}\int_{s_{123}}\frac{\sigma ds}{r}\right].$$

Since $Q_1'=\int s_1\ \sigma' dS'$, hence

$$Q_1'V_1=\frac{1}{4\pi\epsilon_0}\int_{s_1}\sigma' dS'\int_{s_{123}}\frac{\sigma dS}{r}=\frac{1}{4\pi\epsilon_0}\int_{s_{123}}\int_{s_1}\frac{\sigma' dS'\sigma dS}{r} \qquad \ldots(114)$$

Here r is the distance from a chosen point on the surface dS' to the surface element dS.

The above equation may be written for other conductors, and we may have

$$Q_1'V_1+Q_2'V_2+Q_3'V_3=\frac{1}{4\pi\epsilon_0}\int_{s_{123}}\int_{s_{123}}\frac{\sigma' dS'\sigma dS}{r}. \qquad \ldots(115)$$

Similarly, we have

$$Q_1V_1'+Q_2V_2'+Q_3V_3'=\frac{1}{4\pi\epsilon_0}\int_{s_{123}}\int_{s_{123}}\frac{\sigma dS\sigma' dS'}{r}. \qquad \ldots(116)$$

Thus we have

$$Q_1'V_1+Q_2'V_2+Q_3'V_3=Q_1V_1'+Q_2V_2'+Q_3V_3' \qquad \ldots(117)$$

This relation is known as Green's Reciprocal Theorem.
As a particular case, let us consider varous charges

$$Q_1=1,\ Q_2=0,\ Q_3=0\ ;\ Q_1'=0,\ Q_2'=1,\ Q_3=0.$$

$$\therefore\quad V_2=V_1'. \qquad \ldots(118)$$

Its comparison with equation (111) gives

$$V_2=p_{12}\text{ and }V_1'=p_{21}$$

$$\therefore\qquad p_{12}=p_{21}$$

In the generalised form, it may be written as

$$p_{ij}=p_{ji}. \qquad \ldots(119)$$

Hence is the required relation.

The Screening Theorem. Let us consider three conductors ; conductor 1 is surrounded by an earthed conductor 2 and the conductor 3 is in the neighbourhood. As conductor 2 is earthed, $V_2=0$ and thus the charges on these conductors are given as

$$q_1=q_{11}V_1+q_{31}V_3$$

$$q_2=q_{12}V_1+q_{32}V_3$$

$$q_3=q_{13}V_1+q_{33}V_3. \qquad \ldots(120)$$

Let us assume $q_1=1$. Since the conductor 1 is enclosed within an earthed conductor 2, potential V_1 of the conductor 1 will be same as that of conductor 2 (*i.e.*, zero). Therefore

$$q_1=q_{11}\times 0+q_{31}V_3=0.$$

This must be true for all values of V_3. It will be possible for $q_{31}=0$. Thus we see that the conductor 3 has no electric influence on conductor 1 and vice-versa, *i.e.*, the earthed conductor 2 around conductor 1 has electrically screened it from external charges (charge on the external conductor 3). Thus to protect an electrostatic apparatus from electrical disturbances, it should be enclosed in a grounded metal conductor. This is known as the *screening theorem*.

5.20. FIELD DUE TO A SET OF COLLINEAR CHARGES

Let point charges $q_1, q_2, q_3,\ldots$be situated at points $A_1, A_2, A_3\ldots$ on a line LM. Let P be any point having polar coordinates (r_1, θ_1), (r_2, θ_2), (r_3, θ_3).....with respect to points $A_1, A_2, A_3,\ldots$respectively.

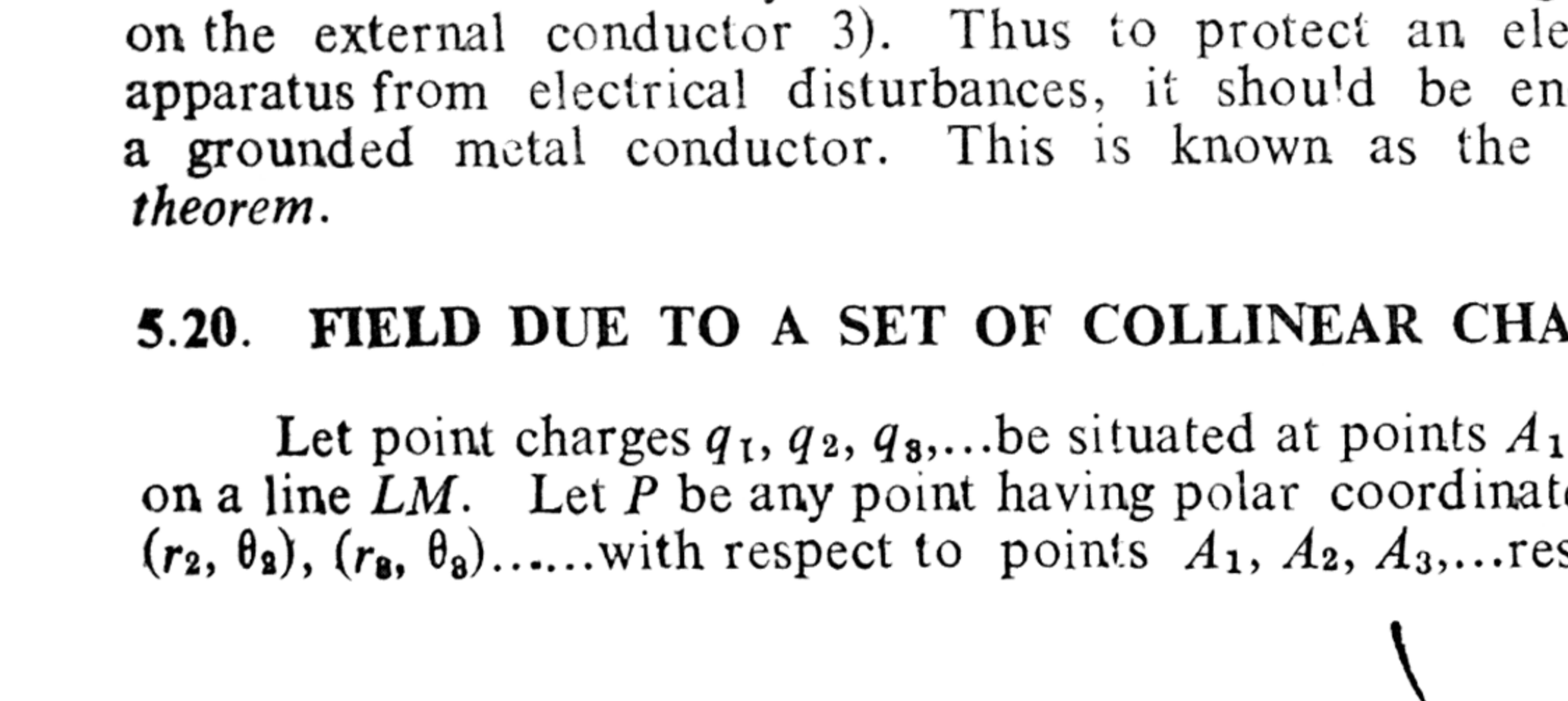


Fig. 5.23.

The electric field intensities at point P due to these charges are respectively $q_1\mathbf{r}_1/4\pi\varepsilon_0 r_1^3$, $q_2\mathbf{r}_2/4\pi\varepsilon_0 r_2^3$, $q_3\mathbf{r}_3/4\pi\varepsilon_0 r_3^3,\ldots$ The sum of their resolutes along the normal to the line of force at P must vanish, or

$$\Sigma\,(q_i \sin\phi_i/4\pi\varepsilon_0\, r_i^2)=0,$$

where ϕ_i is the angle which the tangent to the line of force at P makes with the radius vector r_i. As $\sin \phi_i = r_i \, d\theta_i/ds$, hence

$$\Sigma \frac{q_i}{4\pi\epsilon_0 r_i} \frac{d\theta_i}{ds} = 0 \quad \text{or} \quad \Sigma \frac{q_i d\theta_i}{r_i} = 0.$$

From the geometry of figure, we have $r_1 \sin\theta_1 = r_2 \sin\theta_2 = r_3 \sin\theta_3 \ldots = \lambda$ (say). Therefore

$$(1/\lambda)\, \Sigma q_i \sin\theta_i \, d\theta_i = 0 \text{ or } \Sigma q_i \sin\theta_i \, d\theta_i = 0.$$

Which gives an integration $\Sigma q_i \cos\theta_i = \text{constant}$

or
$$q_1 \cos\theta_1 + q_2 \cos\theta_2 + q_3 \cos\theta_3 + \ldots = \text{const.} \qquad \ldots(121)$$

5.21. FIELD DUE TO A NUMBER OF LONG CHARGED PARALLEL FINE WIRES

Let there be a number of long charged parallel fine wires carrying charges (per unit length) $\lambda_1, \lambda_2, \lambda_3 \ldots$ and cut the plane of paper at right angles at points $A_1, A_2, A_3 \ldots$ The electric field at any point P due to these current carrying wires are

$\lambda_1 \, r_1/2\pi\epsilon_0 r_1$, $\lambda_2 r_2/2\pi\epsilon_0 r_2$, $\lambda_3 \, r_3/2\pi\epsilon_0 r_3 \ldots$ respectively.

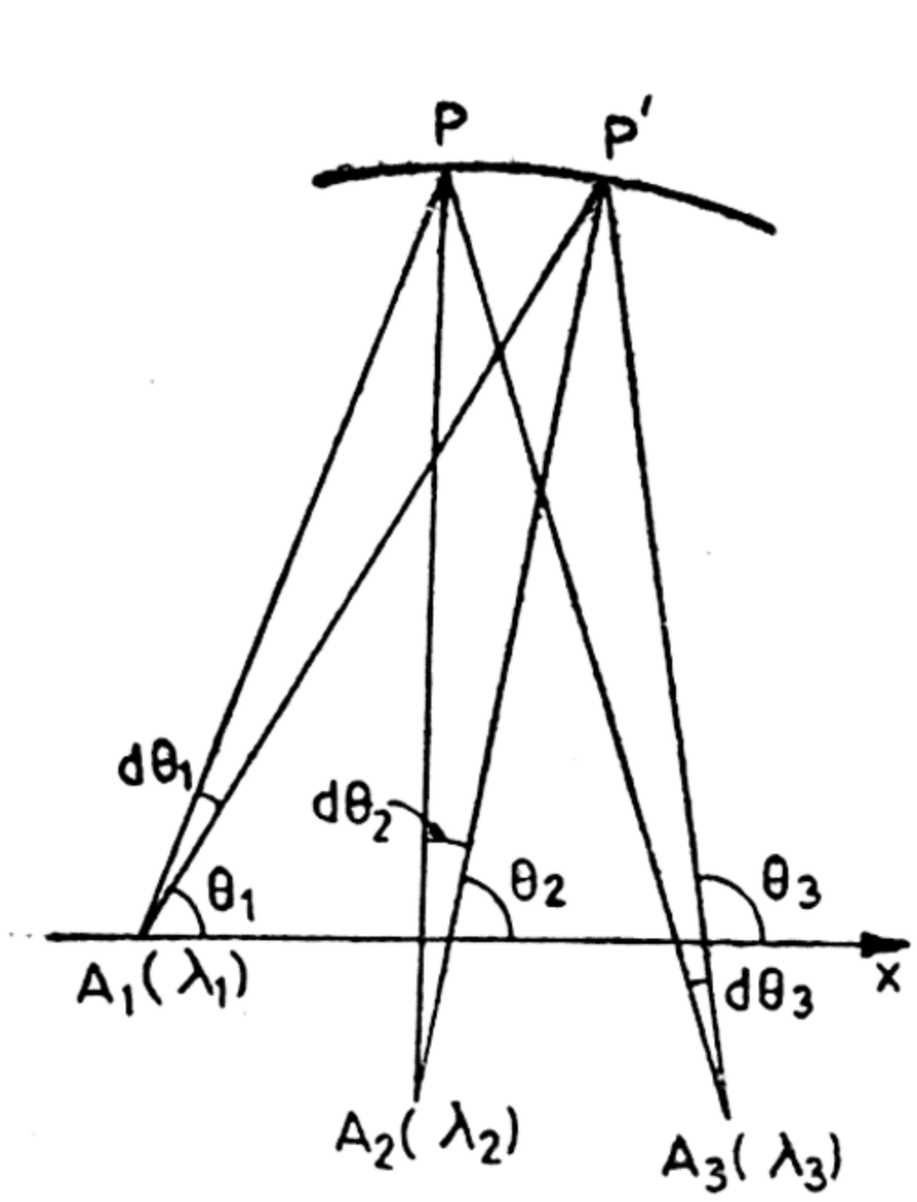


Fig. 5.24.

The sum of their resolutes along the normal to the line of force PP' must vanish.

or $\Sigma (\lambda_i/2\pi\epsilon_0 r_i) \sin\phi_i = 0,$

where ϕ_i, the angle which the tangent to the line of force PP' makes with the radius vector r_i, is given by

$$\sin\phi_i = r_i \, d\theta_i/PP'.$$

$$\therefore \quad \Sigma \lambda_i \, d\theta_i/2\pi\epsilon_0 \, PP' = 0$$

or $\Sigma \lambda_i d\theta_i = \lambda_1 d\theta_1 + \lambda_2 d\theta_2 + \lambda_3 \, d\theta_3 + \ldots$
$= 0$ $\qquad \ldots(122)$

On integration, we get

$\lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3 + \ldots = \text{const.}$

It must be noted that if any charge lies on the opposite side of the line of force, the sign of the term due to this must be changed.

Exercises

Example 1. *Calculate the torque and force experienced by an electric dipole placed in the field of a point charge. The dipole axis makes an angle α with the radius vector from the point charge.*

Let charge q be placed at the origin O of the coordinate system and the dipole is assumed to be placed in $x-y$ plane. If r be the distance of the dipole from O and θ be the angle which radius vector makes with the x-axis. The torque on the dipole

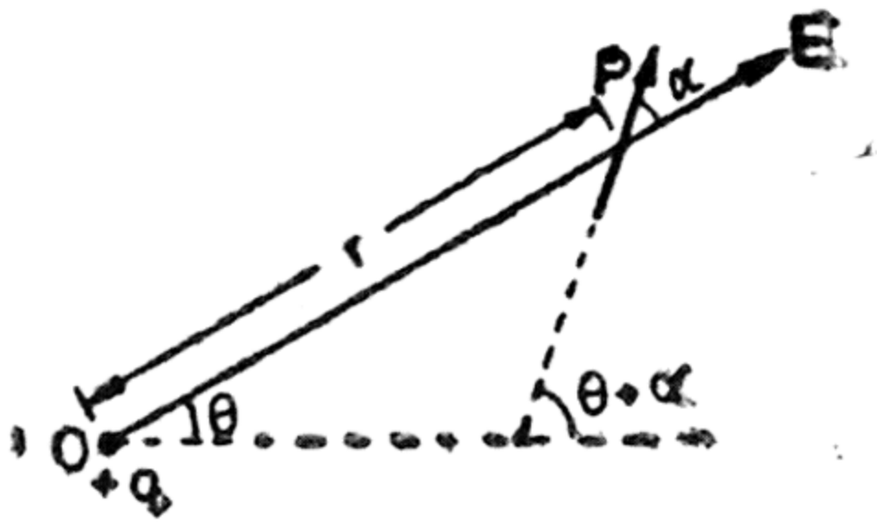


Fig. 5.25.

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} = \mathbf{p} \times \left(\frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \hat{r} \right)$$

$$= \frac{q}{4\pi\varepsilon_0 r^3} \mathbf{p} \times \mathbf{r}$$

or $$\tau = \frac{pq}{4\pi\varepsilon_0 r^2} \sin \alpha.$$

Its direction will be $\perp$ to the plane consisting **p** and **r**.

The total force on the dipole due to non-uniform field **E** is given by

$$\mathbf{F} = (\mathbf{p}.\nabla)\mathbf{E} = (\mathbf{p}.\nabla)\frac{q}{4\pi\varepsilon_0 r^3}\mathbf{r}.$$

Component of **F** along x-axis $F_x = (p_x\, \partial/\partial x + p_y\, \partial/\partial y)\, E_x$

,, ,, ,, ,, y-axis $F_y = (p_x\, \partial/\partial x + p_y\, \partial/\partial y)\, E_y$

Components of **E** along x and y-directions are

$$E_x = (q/4\pi\varepsilon_0 r^2) \cos\theta,\ E_y = (q/4\pi\varepsilon_0 r^2) \sin\theta$$

and that of **p** are $p_x = p \cos(\alpha+\theta)$ and $p_y = p \sin(\alpha+\theta)$, therefore

$$F_x = p \cos(\alpha+\theta)\frac{\partial}{\partial x}\left[\frac{qx}{4\pi\varepsilon_0 (x^2+y^2)^{3/2}}\right]$$

$$+ p \sin(\alpha+\theta)\frac{\partial}{\partial y}\left[\frac{qx}{4\pi\varepsilon_0 (x^2+y^2)^{3/2}}\right]$$

$$F_x = \frac{pq}{4\pi\varepsilon_0}\left[\frac{y^2-2x^2}{(x^2+y^2)^{5/2}} \cos(\alpha+\theta) - \frac{3xy}{(x^2+y^2)^{5/2}} \sin(\alpha+\theta)\right]$$

Similarly we have

$$F_y = \frac{pq}{4\pi\varepsilon_0}\left[\frac{-3xy}{(x^2+y^2)^{5/2}} \cos(\alpha+\theta) + \frac{x^2-2y^2}{(x^2+y^2)^{5/2}} \sin(\alpha+\theta)\right]$$

In a particular case, if the dipole is on the axis of x at a distance a, then

$F_x = -(2pq/4\pi\varepsilon_0 a^3) \cos\alpha$ and $F_y = (pq/4\pi\varepsilon_0 a^3) \sin\alpha$.

−ve sign shows that the dipole experiences an attractive radial field.

As a special case, if $\alpha = 0$, *i.e.*, the dipole is along x-axis, Fy the transverse component of force becomes zero. For $\alpha = \pi/2$, $F_x = 0$.

Example 2. *A short electric dipole of dipole moment* $\mathbf{p}_1$ *is placed in an electric field of another dipole of dipole moment* $\mathbf{p}_2$ *at a distance d from it. Calculate the force and torque that* $\mathbf{p}_1$ *experiences, when* (*a*) $\mathbf{p}_1 \parallel \mathbf{p}_2$ *and* (*b*) $\mathbf{p}_1 \perp \mathbf{p}_2$.

The electric field along x-axis at a distance d from the dipole of dipole moment p_2 placed at the origin O is given by

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{2p_2}{d^3} \text{ and } Ey = 0, \text{ or } \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2p_2}{d^3}\,\mathbf{i} = \frac{2\mathbf{p}_2}{4\pi\epsilon_0 d^3}$$

Here we have assumed $\mathbf{p}_2$ along x-axis.

(*a*) The torque experienced by the dipole of dipole moment $\mathbf{p}_1$ placed $\parallel$ to x-axis due to the electric field of dipole placed at 0

$$\tau = \mathbf{p}_1 \times \mathbf{E} \quad \text{or} \quad \tau = p_1 E \sin 0 = 0.$$

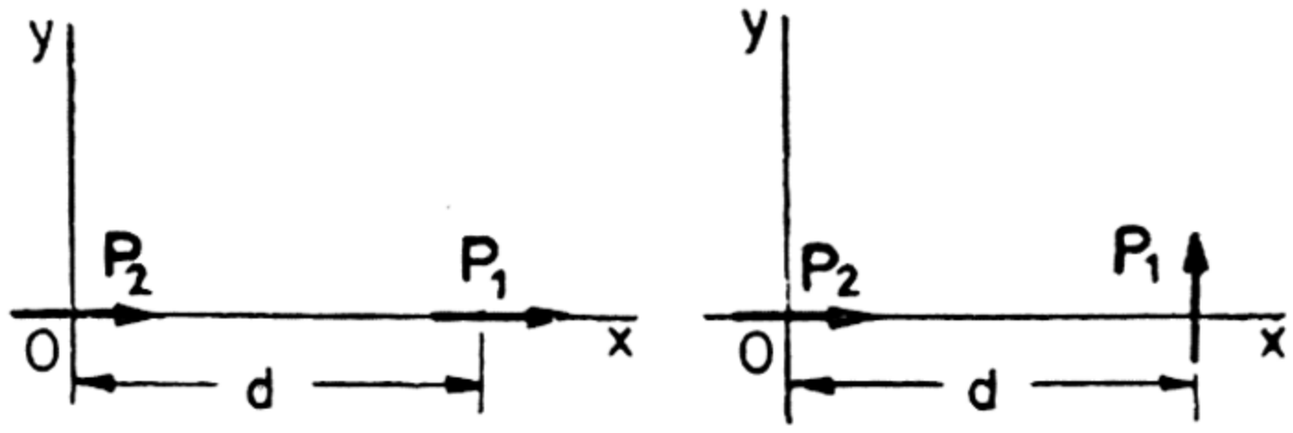


Fig. 5.26.

Force experienced by the dipole of dipole moment p_1 due to the electric field of the another dipole of dipole moment p_2 is given by

$$F_x = \left(p_x \frac{\partial}{\partial x} + py \frac{\partial}{\partial y} \right) E_x \text{ and } Fy = \left(p_x \frac{\partial}{\partial x} + py \frac{\partial}{\partial y} \right) Ey$$

or $$F_x = p_1 \frac{1}{4\pi\epsilon_0}\left(\frac{-6p_2}{d^4}\right) = \frac{-3p_1p_2}{2\pi\epsilon_0 d^4} \text{ and } Fy = 0.$$

The $-$ve sign indicates that the force is attractive and along the line joining p_1 to p_2.

(*b*) If another dipole of dipole moment p_1 is placed in such a way that $\mathbf{p}_1 \perp \mathbf{p}_2$, the torque experienced by it is given as

$$\tau = \mathbf{p}_1 \times \mathbf{E}.$$

In this case $\mathbf{p}_1 = p_1\,\mathbf{j}$ and $\mathbf{E} = E\mathbf{i}$. Therefore

$$\tau = p_1 E \mathbf{j} \times \mathbf{i} = -p_1 E\,\mathbf{k} = -(2p_1p_2/4\pi\epsilon_0 d^3)\,\mathbf{k}.$$

Thus $\mathbf{p}_1$ experiences a clock wise torque of magnitude $2p_1p_2/4\pi\epsilon_0 d^3$.

Electric field at dipole of dipole moment p_2, due to dipole of dipole moment p_1 is given by

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{p_1}{d^3}\,\mathbf{j} \qquad \text{(As } p_2 \text{ is in broad-on position)}$$

$\therefore$ Torque on $\mathbf{p}_2$, $\tau = \mathbf{p}_2 \times \mathbf{E} = p_2 E\mathbf{i} \times \mathbf{j} = -(p_1 p_2/4\pi\varepsilon_0 d^3)\mathbf{k}$.

This torque is again in clockwise direction. If the system consisting of two dipoles is free, it will experience a net clockwise torque of magnitude $3p_1p_2/4\pi\varepsilon_0 d^3$.

Example 3. *Two parallel plates have equal and opposite charges and are separated by a dielectric 5 mm thick, of dielectric constant 3. If the electric intensity in the dielectric is 10^6 V/m, calculate : (a) The free charge per unit area on the conducting plate, (b) The bound charge per unit area on the surfaces of the dielectric, (c) The polarization P in the dielectric, (d) The displacement* **D** *in the dielectric and (e) The energy density in the dielectric.*

Resultant intensity **E** in the dielectric $= E_0 - E_b = (\sigma - \sigma')/\varepsilon_0$.

Polarization vector $\mathbf{P} = \varepsilon_0(k-1)\,\mathbf{E}$

$$= 8.85 \times 10^{-12} \times (3-1)10^6 = 17.7\ \mu\ C/m^2,$$

Displacement vector $\mathbf{D} = k\varepsilon_0\mathbf{E} = 3 \times 8.85 \times 10^{-12} \times 10^6$

$$= 26.55\mu\ C/m^2.$$

As polarization vector **P** is along **E** and thus $\perp$ to the surface of the plate. Hence the bound charge per unit area on the surfaces of the dielectric

$$|\ \sigma'\ | = P_n = P = 17.7\ \mu\ C/m^2.$$

Similarly free charge per unit area $|\ \sigma\ | = D = 26.55\ \mu\ C/m^2$.

Energy density in the dielectric $= \frac{1}{2}k\varepsilon_0 E^2$

$$= \tfrac{1}{2} \times 3 \times 8.85 \times 10^{-12} \times 10^{12}$$

$$= 13.28 \text{ Joules}/m^3.$$

Example 4. *If a dielectric is introduced between the plates of a parallel plate capacitor. Show that the induced charge varies with the dielectric as $k = (1 - \sigma'/\sigma)^{-1}$. Also show that $k \to \infty$ for a metal dielectric.*

Starting with the Eq. (20) of article 5.5, we have

$$\sigma' = \sigma(1 - 1/k)$$

or

$$k = 1/(1 - \sigma'/\sigma).$$

In the metal dielectric, induce charge is same as that of the free charge but of opposite sign, *i.e.*, $\sigma' = \sigma$, hence

$$k = 1/0 = \infty.$$

Example 5. *A dielectric slab of thickness b and dielectric constant k is placed between the plates of a parallel plate capacitor of plate area A and separation d. A potential difference V_0 is applied with no dielectric present. The battery is then disconnected and the dielectric slab is inserted. Assume that $A=100$ cm², $d=1.0$ cm, $b=0.50$ cm, $k=7.0$ and $V_0=100$ volts, then calculate: (a) The capacitance C_0 before the slab inserted, (b) The capacitance C with slab inserted, (c) The free charge q, (d) The electric field in the gap, (e) The electric field in the dielectric, (f) The potential difference between the plates, (g) The polarisation P and electric displacement D in the dielectric and (h) The polarisation P and electric displacement D in the air gap.*

(*a*) Capacitance of the parallel plate condenser before the slab is inserted

$$C_0=\frac{\varepsilon_0 A}{d}$$

$$=\frac{8.85\times10^{-12}\times100\times10^{-4}}{1\times10^{-2}}$$

$$=8.85\times10^{-12} \text{ farad}$$

$$=8.85\ pF.$$

(*b*) Capacitance with *a* slab of thickness *b* and dielectric *k*

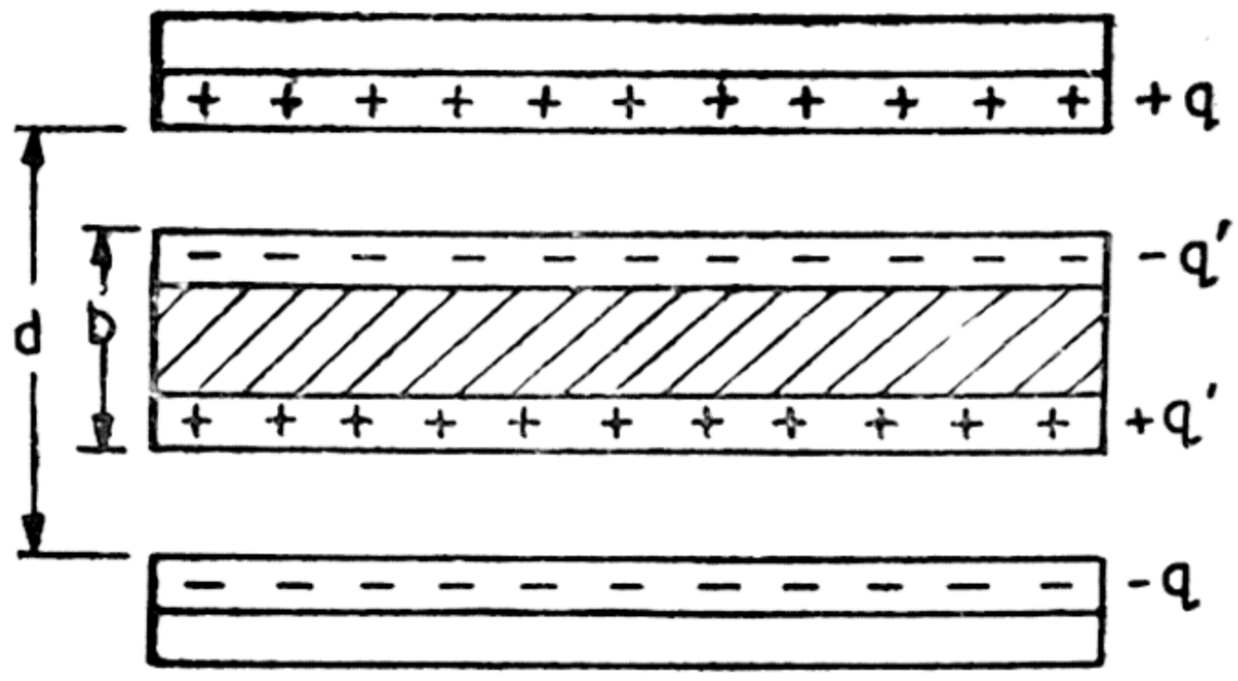


Fig 5.27.

$$C_d=\frac{\varepsilon_0 A}{(d-b+b/k)}=\frac{8.85\times10^{-12}\times100\times10^{-4}}{(1\times10^{-2}-0.5\times10^{-2}+0.5\times10^{-2}/7}$$

$$=15.49\times10^{-12} \text{ farad}=15.49\ pF.$$

(*c*) The free charge

$$q=C_0V_0=8.85\times10^{-12}\times100$$

$$=8.85\times10^{-10} \text{ coul.}$$

(*d*) The electric field in the gap

$$E_0=\frac{q}{\varepsilon_0 A}=\frac{8.85\times10^{-10}}{8.85\times10^{-12}\times100\times10^{-4}}$$

$$=10^4 \text{ volts/meter.}$$

(*e*) The electric field in the dielectric

$$E=\frac{E_0}{k}=\frac{10^4}{7}=1428.6 \text{ volts/meter.}$$

(*f*) The potential difference between the plates

$$V=\frac{\sigma}{\varepsilon_0}(d-b)+\frac{\sigma}{k\varepsilon_0}\,b=\frac{\sigma}{\varepsilon_0}\left[d-b+\frac{b}{k}\right]$$

$$=\frac{q}{\varepsilon_0 A}\left[d-b+\frac{b}{k}\right]=\frac{q}{C_d}=\frac{8.85\times10^{-10}}{15.49\times10^{-12}}$$

$$=57.1 \text{ volts.}$$

(*g*) The electric displacement in the dielectric

$$D=k\varepsilon_0 E=7\times8.85\times10^{-12}\times1428.6$$

$$=8.85\times10^{-8} \text{ coul/meter}^2.$$

and polarisation

$$P=\varepsilon_0(k-1)E=8.85\times10^{-12}(7-1)\times1428.6$$

$$=7.59\times10^{-8} \text{ coul/meter}^2.$$

(*h*) The polarisation is zero in the air gap and the displacement

$$D_0=k\varepsilon_0 E_0=1\times8.85\times10^{-12}\times10^4$$

$$=8.85\times10^{-8} \text{ coul/meter}^2.$$

Example 6. *An electric dipole of moment 2×10^{-8} coulomb-meter is placed in a uniform field of intensity 1.5×10^5 newton/coulomb.*

(i) What maximum torque does the field exert on the dipole?

(ii) How much work is done on turning dipole end to end?

We know that an electric dipole of moment p experiences a torque τ when placed in an electric field E. If the dipole makes an angle θ with the direction of E, then

$$\tau=p\,E\sin\theta$$

It is maximum for $\theta=90°$. Therefore

$$\tau_{max}=p\,E=2\times10^{-8}\times1.5\times10^5$$

$$=3\times10^{-3} \text{ newton meter.}$$

The total work done in rotating the dipole from θ_1 to θ_2 is

$$W=\int_{\theta_1}^{\theta_2} p\,E\sin\theta\,d\theta=-p\,E\,[\cos\theta_2-\cos\theta_1]$$

In the present problem the dipole is rotated through 180°. Thus by putting $\theta_1=0°$ and $\theta_2=180°$ in the above relation, we get

$$W = -pE[\cos 180° - \cos 0°] = 2pE$$

$$= 2 \times 2 \times 10^{-8} \times 1.5 \times 10^{5} = 6 \times 10^{-3} \text{ joule.}$$

Example 7. *If a dielectric slab is partially withdrawn from between two charged parallel plates of a parallel plate capacitor, calculate the force required to pull it.*

In order to calculate the force **F**, let us first see the effect on the stored energy of an elementary displacement dx of the slab. Since this displacement modifies the capacitance of the capacitor, there will be a change dq in the charge on the capacitor. If it is held at constant potential, a work $dW = Vdq$ is done by the battery. Thus the total change in stored energy in the capacitor

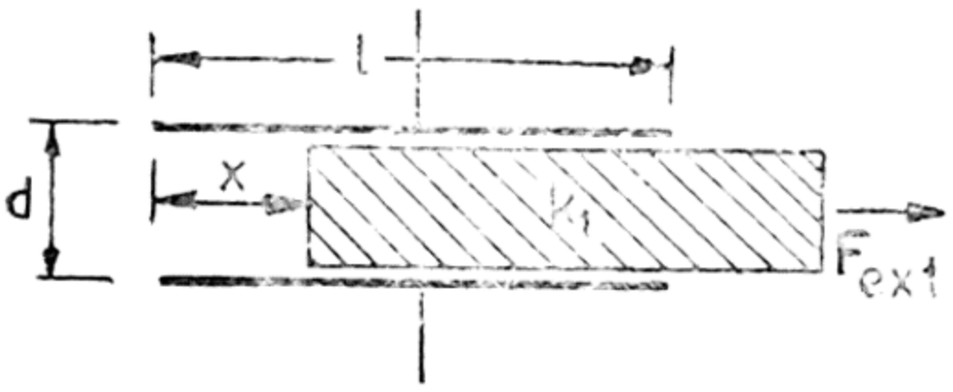


Fig. 5.28.

$$dU = -Fdx + Vdq.$$

We know that the total stored energy per unit volume in the capacitor is equal to $\frac{1}{2}\varepsilon E^2$ or $\frac{1}{2}\varepsilon_0 E^2$, if the capacitor is filled with dielectric or the medium is vacuum respectively. If b is the width of the dielectric, l the length and d the plate separation, then

$$U = \tfrac{1}{2}\varepsilon_0 E^2 bd\, x + \tfrac{1}{2}\varepsilon E^2 bd(l-x) = \tfrac{1}{2}E^2 bd[\varepsilon l - (\varepsilon - \varepsilon_0)x],$$

where x is the length of the slab outside the capacitor.

By differentiating this equation, we get the change in U produced by a small displacement dx of the slab.

$$\therefore \qquad dU = -\tfrac{1}{2}E^2 bd(\varepsilon - \varepsilon_0)dx.$$

We know that $\sigma_1 (= \varepsilon_0 E)$ and $\sigma_2 (= \varepsilon E)$ are the charge densities in the portion with no dielectric and with dielectric respectively. Hence the total charge on the capacitor plates

$$q = \sigma_1 A_1 + \sigma_2 A_2 = \varepsilon_0 Exb + \varepsilon\, Eb(l-x).$$

$\therefore$ Change in q with change in x is

$$dq = \varepsilon_0 Eb\, dx - \varepsilon Eb\, dx = -(\varepsilon - \varepsilon_0)Eb\, dx$$

Since $V = Ed$, hence

$$Fdx = -dU + Vdq = \tfrac{1}{2}E^2 bd(\varepsilon - \varepsilon_0)dx - Ed(\varepsilon - \varepsilon_0)Ebdx.$$

or

$$F = -\tfrac{1}{2}E^2 bd(\varepsilon - \varepsilon_0).$$

Here $-$ve sign indicates that the force is in the direction of decreasing x. Hence force is required to pull the slab and is independent of l and x.

Above relations also show that dU is half of the Vdq, hence half the energy contributed to the system by the battery is used in doing external work.

Example 8. *Find the electric field and polarization due to a charge in a dielectric fluid.*

Let us consider a point charge q in the homogenous isotropic medium of infinite extent and of dielectric constant k. If the point charge q was situated in a vacuum, the electric field would be a pure radial field. Since **E**, **D** and **P** are all parallel to one another at each point, hence the radial nature of the field is not changed by the presence of the medium.

Let us apply Gauss's law to a gaussian spherical surface of radius r, at the centre of which charge q is placed. Thus

$$k\varepsilon_0 \int \mathbf{E}.\, d\mathbf{S} = q.$$

$$\therefore \quad k\varepsilon_0 \int E dS = k\varepsilon_0\, 4\pi r^2 E = q \quad \text{or} \quad E = q/4\pi\varepsilon_0 k r^2$$

or vectorially $\quad \mathbf{E} = (q/4\pi\varepsilon_0 k r^3)\mathbf{r}$

and hence $$\mathbf{P} = \varepsilon_0(k-1)\mathbf{E} = \frac{(k-1)q}{4\pi k r^3}\mathbf{r}.$$

Thus the field is smaller by the factor k than would be the case if the medium was absent.

Example 9. *If the negative charge in a hydrogen atom is uniformly distributed in the spherical volume of radius a ($=0.5\times10^{-8}$ cm) around the proton. Calculate the separation between its +ve and −ve charges and polarizability α, when the atom is placed in an electric field 30 kV/cm. Assuming that the sphere of negative charge remains undistorted.*

When the electric field is applied to the atom, the +ve charge (proton) will displace from its original position (the centre of the atom). The sphere of negative charge will be the centre of the atom. As the negative charge is uniformly distributed in the sphere of radius a, hence the electric field at a point P, at a distance r from the centre ($r<a$) due to this charge distribution is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(e\,\frac{r^3}{a^3}\right) = \frac{1}{4\pi\varepsilon_0}\frac{er}{a^3}.$$

If due to electric field the +ve charge is displaced to this point P, the field acting over this charge due to distribution of −ve charge will be same as that applied by an external source.

$\therefore$ Separation $r = 4\pi\varepsilon_0 E a^3/e$

$$= \frac{4\times3.14\times8.85\times10^{-12}\times30\times10^{5}\times(0.5\times10^{-10})^3}{1.6\times10^{-19}}$$

$$= 2.6\times10^{-16} \text{ meter.}$$

This result shows that a large electric field (30×10^5 V/m) can produce a very small separation of the order of 10^{-16} m, which is very small even in comparison of nuclear radius 10^{-15} m.

The dipole moment of this arrangement

$$p=er=1.6\times10^{-19}\times2.6\times10^{-16}=4.16\times10^{-35} \text{ coul. meter}$$

The polarizability $\alpha=\dfrac{p}{E}=\dfrac{4.16\times10^{-35}}{30\times10^5}=1.39\times10^{-41}$.

Example 10. *If half the space between two concentric conducting spheres be filled with dielectric of dielectric constant k and the rest is filled with air. Show that the capacitance of the capacitor thus formed will be same as if the whole part is filled with the dielectric of dielecıric constant* $\frac{1}{2}(1+k)$.

If whole of the medium is filled with air and σ is the surface density of charge, then the potential difference between the spheres of radii a and b

$$\int_a^b dV=-\int_a^b \mathbf{E}.\,d\mathbf{r}=-\int_a^b E dr=-\int_a^b \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\,dr$$

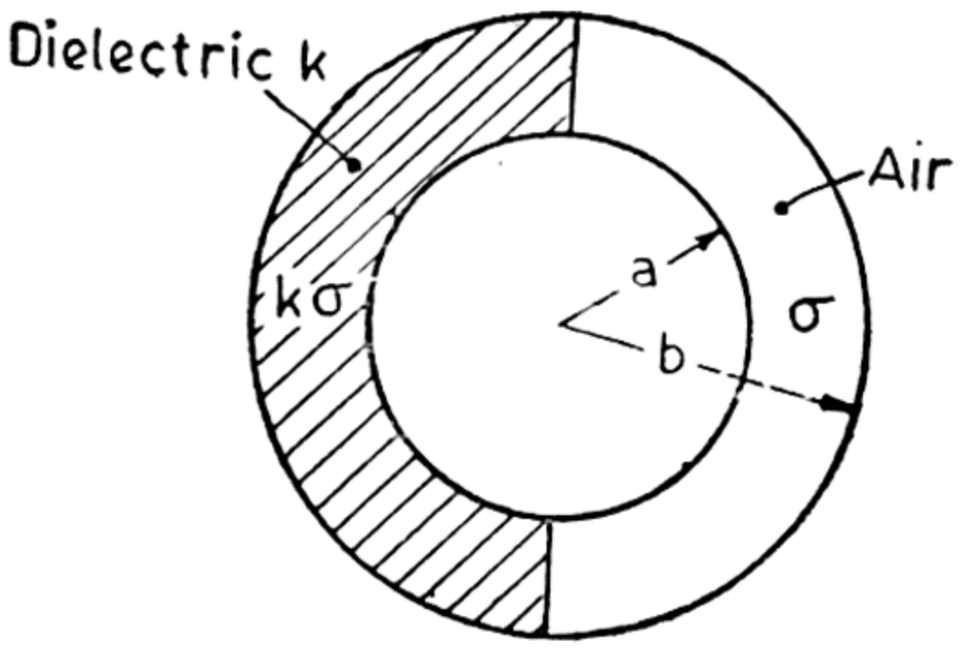


Fig. 5.29.

$$V_b-V_a=\frac{\sigma a^2}{\varepsilon_0}\left[\frac{1}{b}-\frac{1}{a}\right].$$

$\therefore$ Capacity $C=$ charge$/(V_a-V_b)$

$$=\frac{4\pi a^2\sigma}{V_a-V_b}=4\pi\varepsilon_0\,\frac{ab}{b-a}.$$

When the medium is filled with dielectric constant $\frac{1}{2}(1+k)$, then capacity increases to $C'=\frac{1}{2}(1+k)C=2\pi\varepsilon_0(1+k)ab/(b-a)$.

Now if half of the space is filled with dielectric k and half with air then in order to maintain the same potential difference between the spheres, the surface charge density over one half of the inner sphere will need to be $k\sigma$ and over the other half σ. Thus the capacity,

$$C=\frac{\text{Total charge}}{p.d.}=\frac{2\pi a^2\sigma k+2\pi a^2\sigma}{\sigma a^2(b-a)/\varepsilon_0 ab}$$

$$=4\pi\varepsilon_0\frac{1+k}{2}\frac{ab}{b-a}$$

= Capacity when the space is filled with $\frac{1}{2}(1+k)$.

Example 11. *The surface separating two dielectrics* k_1 *and* k_2 *has a surface charge density* σ. *The electric intensities on the two sides of the boundary are* $\mathbf{E}_1$ *and* $\mathbf{E}_2$, *making angles* θ_1 *and* θ_2 *with the common normal. Prove that*

$$k_2 \cot\theta_2=k_1\cot\theta_1(1-\sigma/\varepsilon_0 k_1 E_1\cos\theta_1)$$

We know that the tangential components of electric intensity remains the same while crossing the common surface of two media. Hence,

$$E_1 \sin \theta_1 = E_2 \sin \theta_2 \quad \text{or} \quad E_2 = E_1 \sin \theta_1 / \sin \theta_2. \quad \ldots(i)$$

We also know that this sum of normal components of the displacement vector is equal to σ, hence

$$D_1 \cos \theta_1 - D_2 \cos \theta_2 = \sigma \quad \text{or} \quad k_1 \varepsilon_0 E_1 \cos \theta_1 - k_2 \varepsilon_0 E_2 \cos \theta_2 = \sigma.$$

$$\therefore \quad k_2 E_2 \cos \theta_2 = k_1 E_1 \cos \theta_1 - \sigma/\varepsilon_0. \quad \ldots(ii)$$

Substituting value of E_2, in this eqn and solving, we get

$$k_2 \cot \theta_2 = k_1 \cot \theta_1 (1 - \sigma/\varepsilon_0 k_1 E_1 \cos \theta_1).$$

It is the required relation.

Example 12. *Prove by the method of boundary conditions that the field inside a spherical cavity with in an isotropic dielectric is given as* $E_i = E + P/3\varepsilon_0$.

Consider a spherical cavity of radius a existed in an isotropic dielectric of infinite extent. Let the field intensity $\mathbf{E}_i$ be constant everywhere and parallel to the original field **E**. Electric intensity at the points in the dielectric be assumed to be E plus that due to a dipole of moment **p** at the centre of the cavity. At any point on the cavity, the tangential component of the electric field intensity is continuous. Therefore.

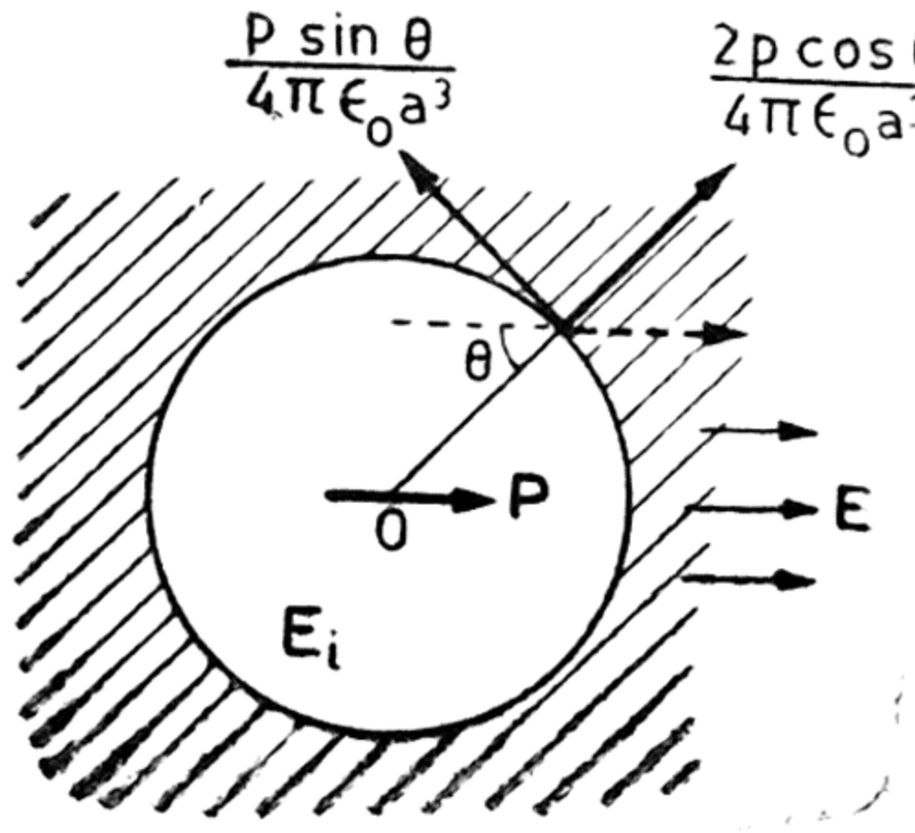


Fig. 5.30.

$$E_i \sin \theta = E \sin \theta - (p/4\pi\varepsilon_0 a^3) \sin \theta.$$

Another boundary condition that the normal component of electric displacement is continuous, gives

$$\varepsilon_0 E_i \cos \theta = \varepsilon_0 k E \cos \theta + \varepsilon_0 (2p/4\pi\varepsilon_0 a^3) \cos \theta$$

Eliminating p/a^3 from these equations, we get

$$E_i = \tfrac{1}{3}(k+2)\, E.$$

At points far away from the cavity the polarization P is constant and is given by

$$D = k\varepsilon_0 E = \varepsilon_0 E + P.$$

$$\therefore \quad E_i = \tfrac{2}{3}E + \tfrac{1}{3}(E + P/\varepsilon_0) = E + P/3\varepsilon_0.$$

It is the required relation.

Example 13. *Calculate the electric intensity at any point on the axis of a uniformly polarized bar electret.*

Let us consider a long cylindrical bar of dielectric that is uniformly and permanently polarized with the constant vector **P** parallel to the axis. The surface densities are $\sigma = P$ and $-\sigma = -P$ at the ends of the bar. The value of **E** on the axis of a uniformly charged disc of radius R at a distance x from its centre is given by

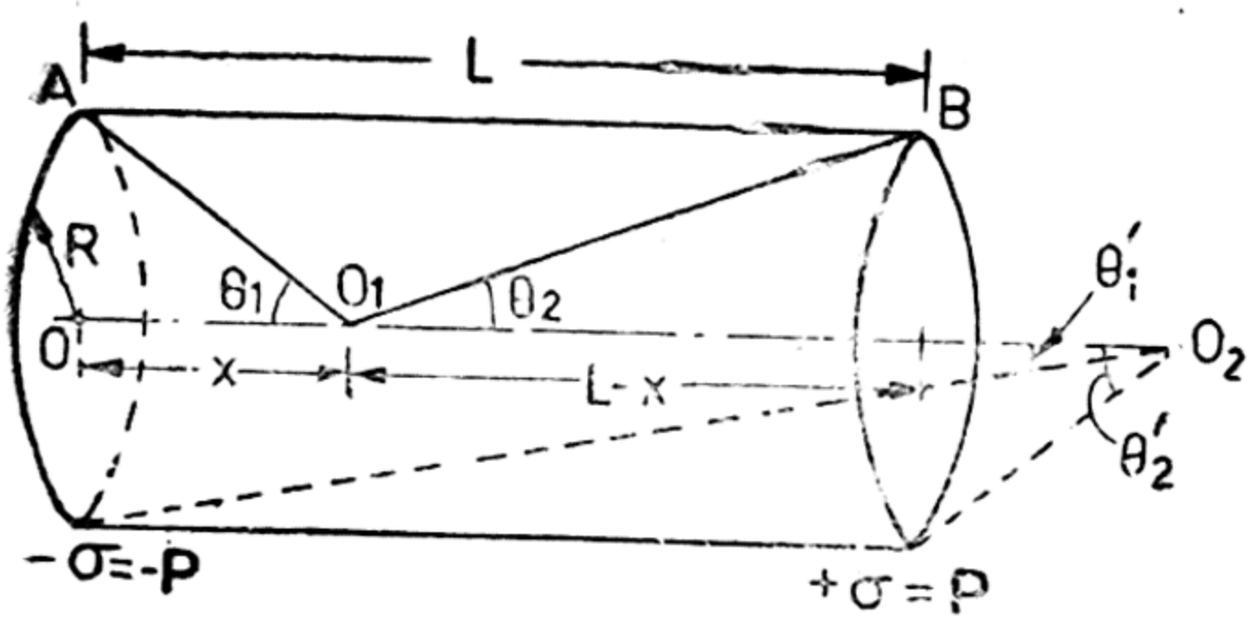


Fig. 5.31.

$$E = (\sigma/2\varepsilon_0)\ [1 - x/(x^2+R^2)^{1/2}].$$

Thus the field E due to charges on the discs A and B at a point O_1 inside the bar at a distance x from the $-$vely charged disc A is given by

$$E = \frac{-P}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{(x^2+R^2)}}\right) - \frac{P}{2\varepsilon_0}\left(1 - \frac{L-x}{[(L-x)^2+R^2]^{1/2}}\right)$$

As $\cos\theta_1 = x/(x^2+R^2)^{1/2}$ and $\cos\theta_2 = (L-x)/[(L-x)^2+R^2]^{1/2}$

$$\therefore \quad E = -P/\varepsilon_0 + (P/2\varepsilon_0)(\cos\theta_1 + \cos\theta_2).$$

When point O_1 is outside the bar, *i.e.*, at O_2 at a distance x' from O, we get

$$E = \frac{-P}{2\varepsilon_0}\left(1 - \frac{x'}{\sqrt{(x'^2+R^2)}}\right) + \frac{P}{2\varepsilon_0}\left(1 - \frac{x'-L}{\sqrt{[(x'-L)^2+R^2]}}\right)$$

$$= (P/2\varepsilon_0)(\cos\theta_1 + \cos\theta_2).$$

Field at point O, the centre of disc A, is

$$E = -\frac{P}{2\varepsilon_0} - \frac{P}{2\varepsilon_0}\left(1 - \frac{L}{\sqrt{(L^2+R^2)}}\right)$$

$$= -\frac{P}{2\varepsilon_0} - \frac{P}{2\varepsilon_0}\left[1 - \left(1 - \frac{R^2}{L^2}\right)^{-1/2}\right]$$

$$= -(P/2\varepsilon_0)(1 + R^2/2L^2).$$

Example 14. *Find expressions for electric potential and field intensity in all the regions due to a uniformly polarised spherical shell of inner and outer radii a and b respectively.*

We know that the density of polarisation **P** is equal to the dipole moment per unit volume. Hence the dipole moment of a uniformly polarized sphere of radius a is $\frac{4}{3}\pi a^3\mathbf{P}$. Electric

potential and field outside the polarized sphere are identical with those due to the short dipole of moment $\frac{4}{3}\pi a^3\,\mathbf{P}$ assumed to be placed at the centre of the sphere.

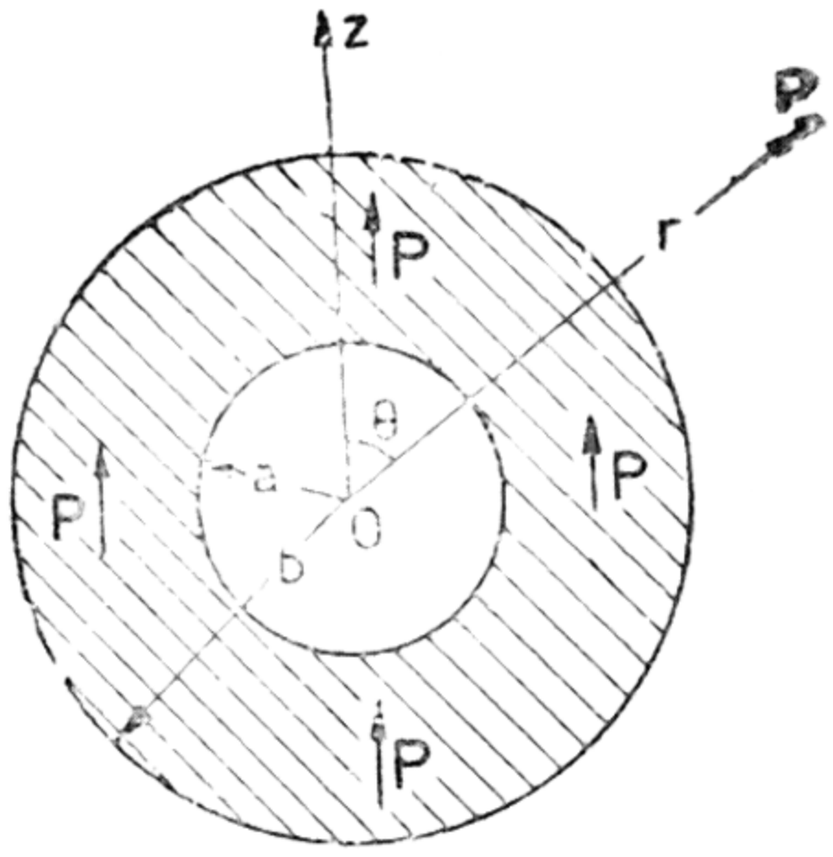


Fig. 5.32.

$\therefore$ V_0 at a piont $P(r, \theta)$

$$= \tfrac{4}{3}\pi a^3 P \cos\theta / 4\pi\varepsilon_0 r^2.$$

$$= Pa^3 \cos\theta / 3\varepsilon_0 r^2.$$

Potential at the inner point

$$= \tfrac{4}{3}\pi Pr \cos\theta / 4\pi\varepsilon_0 = Pr \cos\theta / 3\varepsilon_0.$$

In our problem $\mathbf{P}$ is directed along z-axis. The electric potential and field will be computed by subtracting the contribution due to inner sphere from the contribution due to whole sphere of radius b. Hence

(*i*) At the outer point P $(r>b)$

$$V = V_b - V_a = \frac{Pb^3 \cos\theta}{3\varepsilon_0 r^2} - \frac{Pa^3 \cos\theta}{3\varepsilon_0 r^2} = \frac{P\cos\theta}{3\varepsilon_0 r^2}(b^3 - a^3)$$

Hence $$Er = -\frac{\partial V}{\partial r} = \frac{2P\cos\theta}{3\varepsilon_0 r^3}(b^3 - a^3)$$

and $$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{P\sin\theta}{3\varepsilon_0 r^3}(b^3 - a^3).$$

(*ii*) At the point inside the dielectric $(a<r<b)$

$$V = V_b - V_a = \frac{Pr\cos\theta}{3\varepsilon_0} - \frac{Pa^3\cos\theta}{3\varepsilon_0 r^2} = \frac{P\cos\theta}{3\varepsilon_0}\left[\, r - \frac{a^3}{r^3}\,\right]$$

Hence $$E_r = -\frac{\partial V}{\partial r} = \frac{P\cos\theta}{3\varepsilon_0}\left[\, 1 + \frac{2a^3}{r^3}\,\right].$$

and $$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial r} = \frac{P\sin\theta}{3\varepsilon_0}\left[\, 1 - \frac{a^3}{r^3}\,\right]$$

(*iii*) At the point inside the inner sphere $(0<r<a)$

$$V = V_b - V_a = \frac{Pr\cos\theta}{3\varepsilon_0} - \frac{Pr\cos\theta}{3\varepsilon_0} = 0.$$

Hence $Er = 0$ and $E_\theta = 0$.

Example 15. *If all the molecular dipoles in 0.1 cm radius drop of water are pointed in the same direction, calculate (a) the intensity of polarization and (b) maximum field strength at the point 20 cm away from the drop and at the surface. Given that the dipole moment for each molecule of water is* 6×10^{-30} *coulomb-meter.*

From the hypothesis of Avogadro's 18×10^{-3} kg of water must consists 6.023×10^{23} number of water molecules. Hence the number of molecules in a drop of radius 0.1 cm and thus of weight $\frac{4}{3}\pi(0.001)^3\times10^3$ is $6.023\times10^{23}\times\frac{4}{3}\pi(0.001)^3\times10^3/18\times10^{-3}=1.4\times10^{20}$.

As all the dipoles are pointed in the same direction, hence the dipole moment of the drop

$$p=1.4\times10^{20}\times6\times10^{-30}=8.4\times10^{-10} \text{ coul meter.}$$

(*a*) Intensity of polarization = dipole moment per unit volume

$$=8.4\times10^{-10}/\tfrac{4}{3}\pi(.001)^3=0.2 \text{ coul/m}^2.$$

(*b*) Maximum field will be at the point on the extended diameter along the direction of polarization, hence

$$E=\frac{2p}{4\pi\varepsilon_0 r^3}=\frac{2\times8.4\times10^{-10}\times9\times10^9}{[20\times10^{-2}]^3}$$

$$=1.89\times10^3 \text{ newton/coul.}$$

$$\text{At the surface } E=\frac{2p}{4\pi\varepsilon_0 a^3}=\frac{2\times8.4\times10^{-10}\times9\times10^9}{[0.1\times10^{-2}]^3}$$

$$=1.51\times10^{10}\ N/C.$$

Example 16. *Calculate the average radius of the atom of an air molecule if the polarizability of atoms in the air molecules is* 9.7×10^{-41} *coul. m²/volt. Also compare this result with the relative displacement of the nitrogen nucleus and electron cloud at a field strength* $E=3\times10^6$ *volt/m.*

We know an important relation between polarizability α and the radius R,

$$\alpha=4\pi\varepsilon_0 R^3 \text{ or } R=[\alpha/4\pi\varepsilon_0]^{1/3}$$

$$\therefore \quad R=(9.7\times10^{-41}\times9\times10^9)^{1/3}=9.6\times10^{-11} \text{ meter.}$$

If x is the relative displacement of the nitrogen nucleus and electron cloud, then the force due to electric field is balanced by the electrostatic force and we have

$$ZeE=\frac{(Ze)\ (Zex^3/R^3)}{4\pi\varepsilon_0 x^2} \quad \text{or} \quad x=\frac{4\pi\varepsilon_0 R^3 E}{Ze}.$$

$$\therefore \quad x=\frac{\alpha E}{Ze}=\frac{9.7\times10^{-41}\times3\times10^6}{7\times1.6\times10^{-19}}=2.6\times10^{-16} \text{ meter.}$$

Example 17. *The polarizability of* NH_3 *molecule in gaseous state, from the measurement of dielectric constant, is found* 2.5×10^{-36} *coul² m/N and* 2.0×10^{-39} *coul² m/N at temperatures 300° K and 400°K respectively. Calculate the contribution to the polarizability because of deformation of molecules and the contribution because of permanent dipole moment at each temperature.*

From Langevin—Debye's theory the total polarizability

$\alpha = \alpha_1$ (due to deformation) $+ \alpha_2$ (due to permanent dipole)

$= \alpha_1 + p^2/3kT = \alpha_1 + \beta/T$.

$\therefore \quad 2.5 \times 10^{-39} = \alpha_1 + \beta/300$

and $2.0 \times 10^{-39} = \alpha_1 + \beta/400$.

Hence $\beta = 6 \times 10^{-37}$ and $\alpha_1 = 0.5 \times 10^{-39} C^2\ m/N$.

Polarizability due to permanent dipole at temperature $300°K$

$$\alpha_2 = \frac{\beta}{T} = \frac{6 \times 10^{-37}}{300} = 2 \times 10^{-39}\ C^2\ m/N$$

and at $400°K$ $\alpha_2 = 6 \times 10^{-37}/400 = 1.5 \times 10^{-39}\ C^2.m/N$.

Example 18. *Find the potential and hence the electric intensity due to a cylindrical conductor placed in an uniform field with the Laplace equation.*

Consider an infinitely long uncharged conducting cylinder of circular cross section placed in a uniform electric field E_0 with its axis $\perp$ to the field. If a be the radius of the cylinder and the field E_0 is along x-axis. It is clear from the symmetry that the potential is not a function of z and thus the problem is two dimensional and can be written in the form of polar coordinates r and θ.

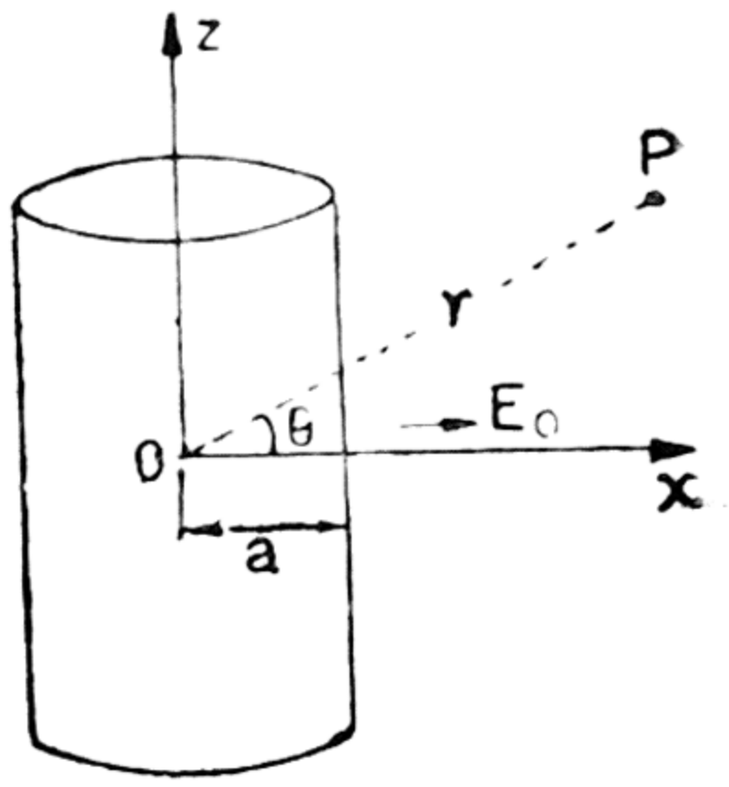


Fig. 5.33.

At a very large distance from the origin, the potential must be $-E_0 x = -E_0 r \cos\theta$.

Thus boundary conditions are

$V = -E_0 r \cos\theta$ for $r \to \infty$

$= 0$ for $r = a$.

To find a potential function at a point P outside the cylinder, we must add second term consisting of first power of $\cos\theta$ and satisfying the Laplace's equation $\nabla^2 V = 0$, or

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 V}{\partial \theta^2} = 0. \qquad \text{(in two dimensional case)}$$

Thus the potential function must be of the form

$$V = -E_0 r \cos\theta + (A/r)\cos\theta.$$

The second term becomes zero at very large distances, and $V = 0$ at the surface of the cylinder, *i.e.*, at $r = a$, hence $A = E_0 a^2$. Thus we get the relation

$$V = -E_0\, r \cos\theta + (E_0 a^2/r)\cos\theta.$$

$$= -(1 - a^2/r^2)\, E_0 r \cos\theta.$$

The first term is potential of the external field and the second term is as due to an extended dipole consisting of two parallel lines of positive and negative charges close to the z-axis,

The components of electric intensity in the directions of increasing r and θ are

$$Er = -\frac{\partial V}{\partial r} = \left(1 + \frac{a^2}{r^2}\right) E_0 \cos\theta$$

and

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\left(1 - \frac{a^2}{r^2}\right) E_0 \sin\theta.$$

The charge surface density (charge per unit area) of the conducting surface is

$$\sigma = k\varepsilon_0\, (Er)_{r=a} = 2k\varepsilon_0\, E_0 \cos\theta.$$

If the cylinder has a charge λ per unit length, then

$$V = -\left(1 - \frac{a^2}{r^2}\right) E_0 r \cos\theta - \frac{\lambda}{2\pi\varepsilon_0 k} \log_e \frac{r}{a}$$

The second term increases Er and σ. It is left as an exercise.

Eaxmple. 19. *Find the electric field due to a point charge q placed inside the sphere of zero potential.*

Let a be the radius of the earthed sphere and f be the distance from its centre to the point A at which charge q is placed. Let A' be the inverse point of A. Hence

$$OA.OA' = a^2 \quad \text{or} \quad OA/a = a/OA'.$$

Thus $\triangle\, OAP$ and $\triangle\, OA'P$ are similar and hence

$$PA/PA' = a/(a^2/f) = f/a.$$

Let q' be the charge placed at point A'. As potential at P is zero, hence

$$q/AP + q'/PA' = 0,$$

or $$q' = -q\, PA'/PA = -qa/f.$$

Thus the electrical image required is a charge $-qa/f$ at point A' and the system is equivalent to a charge $+q$ at A and $-qa/f$ at A',

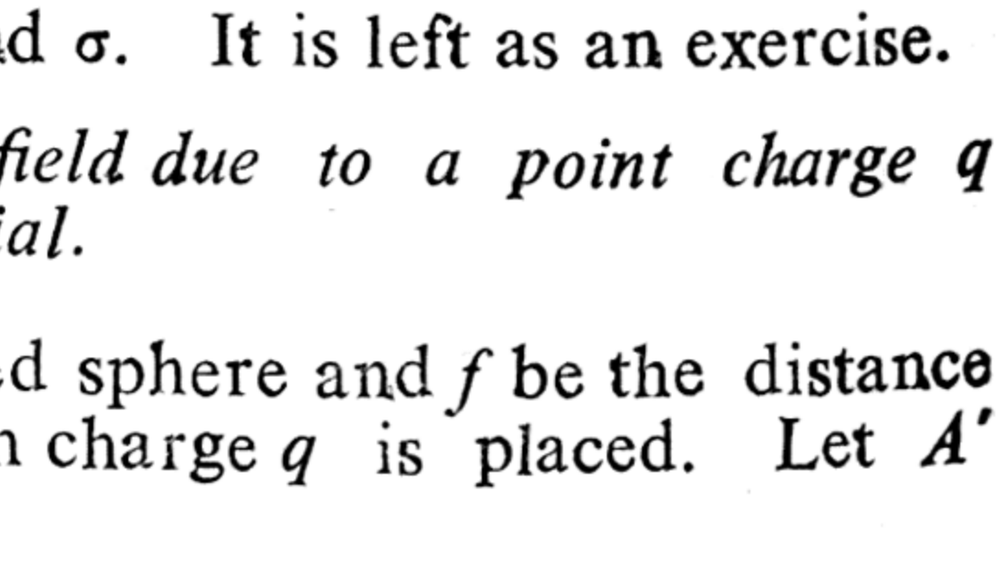


Fig. 5.34.

Electric intensity at point P due to charge q at A

$$\mathbf{E_1} = \frac{1}{4\pi\varepsilon_0}\frac{q}{AP^3}\cdot \mathbf{AP} = \frac{1}{4\pi\varepsilon_0}\frac{q}{AP^3}\,(\mathbf{AO} + \mathbf{OP})$$

and due to charge $-qa/f$ at A'

$$\mathbf{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{qa}{f}\frac{\mathbf{PA}}{(PA')^3} = \frac{1}{4\pi\varepsilon_0}\frac{qa'}{f(PA')^3}(\mathbf{PO}+\mathbf{OA}')$$

$\therefore$ Resultant intensity at point P

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{AO}}{AP^3}+\frac{\mathbf{OP}}{AP^3}+\frac{a\ \mathbf{PO}}{f\ (PA')^3}+\frac{a}{f}\frac{\mathbf{OA}'}{(PA')^3}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{OP}}{AP^3}\left\{1-\frac{a}{f}\left(\frac{AP}{PA'}\right)^3\right\}+\frac{\mathbf{AO}}{AP^3}\left\{1-\frac{a}{f}\frac{OA'}{OA}\left(\frac{AP}{PA'}\right)^3\right\}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{OP}}{AP^3}\left\{1-\frac{a}{f}\left(\frac{f}{a}\right)^3\right\}+\frac{\mathbf{AO}}{AP^3}\left\{1-\frac{a}{f}\left(\frac{a}{f}\right)^2\left(\frac{f}{a}\right)^3\right\}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left(1-\frac{f^2}{a^2}\right)\frac{\mathbf{OP}}{AP^3} = \frac{q}{4\pi\varepsilon_0}\frac{a^2-f^2}{aAP^3}\,n,$$

where n is the unit vector at point P in outward direction.

Hence the surface density of induced charge on the sphere

$$\sigma = -\frac{q}{4\pi}\frac{a^2-f^2}{a\ AP^3}.$$

Force between the charge q at A and the sphere

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{AA'^2} = \frac{1}{4\pi\varepsilon_0}\frac{q(-qa/f)}{(a^2/f-f)^2}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2af}{(a^2-f^2)^2}.$$

Here $-$ve sign shows that the charge is attracted by the sphere.

Example 20. *With in a spherical hollow earthed conductor equal point charges q are placed at equal distances f from the centre on the same diameter. Calculate the force between them.*

Let us consider points A and B at a distance f from O on both the sides. To make the potential of the sphere zero, we have to put

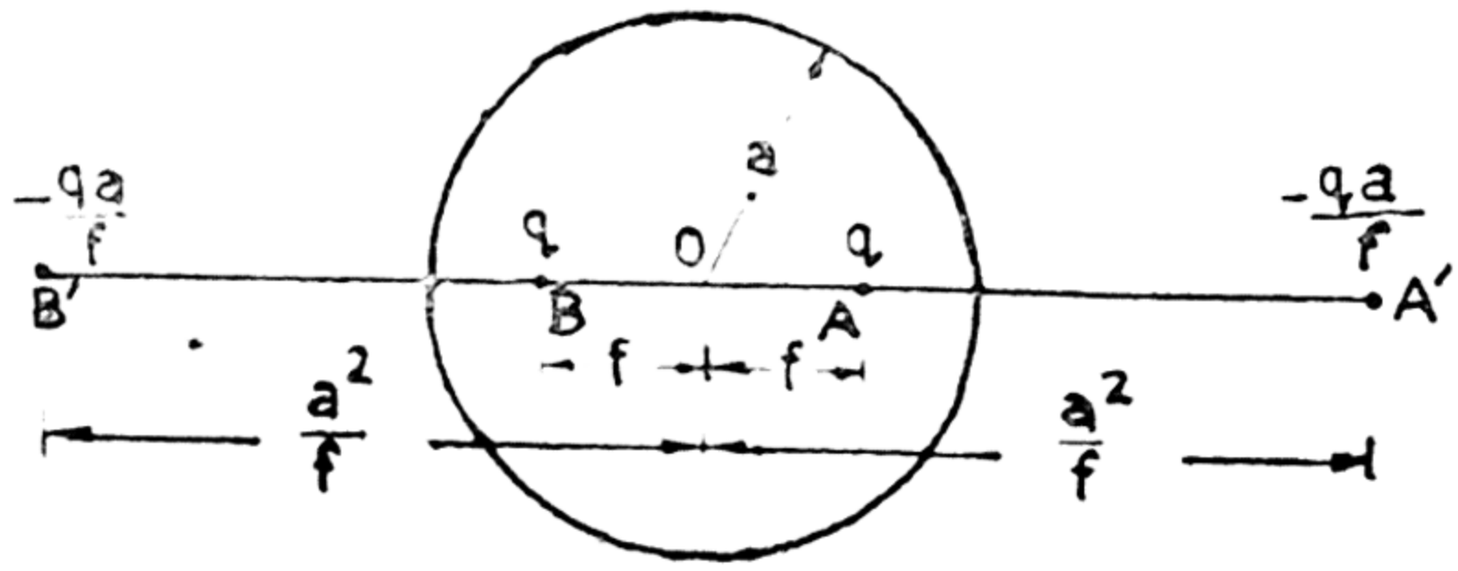


Fig. 5.35.

charges $-qa/f$ at points A' and B'. A' and B' are thus the electric images of charges at A and B respectively,

Force on A due to all charges

$$\mathbf{F}=\frac{1}{4\pi\varepsilon_0}\frac{q(qa)}{f}\frac{1}{(AA')^2}\,\mathbf{i}+\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(AB)^2}\,\mathbf{i}-\frac{1}{4\pi\varepsilon_0}\frac{q(qa)}{f}\frac{1}{(B'A)^2}\,\mathbf{i}$$

$$=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{a}{f(AA')^2}+\frac{1}{(AB)^2}-\frac{a}{f}\frac{1}{(B'A)^2}\right]\mathbf{i}$$

$$=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{a}{f}\frac{1}{(a^2/f-f)^2}+\frac{1}{(2f)^2}-\frac{a}{f}\cdot\frac{1}{(a^2/f+f)^2}\right\}\mathbf{i}$$

$$=\frac{q^2}{4\pi\varepsilon_0}\left\{\frac{4a^3f^3}{(a^4-f^4)^2}+\frac{1}{4f^2}\right\}\mathbf{i}.$$

Example 21. *A point charge q is placed at a point A at a distance f from the centre of an insulated conducting sphere of radius $a(<f)$. Calculate the charge required on the sphere in order that the line of no electrification may be inter section of the sphere by the polar plane of A and show that the numerical ratio of the greatest $-ve$ and $+ve$ surface densities of charge is $[(f+a)/(f-a)]^{3/2}$.*

We know that the polar plane of A is the plane $\perp$ to OA and passing through its image A'. When this sphere carries a total charge Q, it can be considered to be concentrated at centre O. As the sphere is insulated, the total induced charge will be zero. This condition can be satisfied by a uniform distribution of the charge qa/f over the surface which can be assumed as concentrated at centre O. Hence field at any external point can be considered as due to three point charges $Q+qa/f$, $-qa/f$ and q at points O, A' and A respectively.

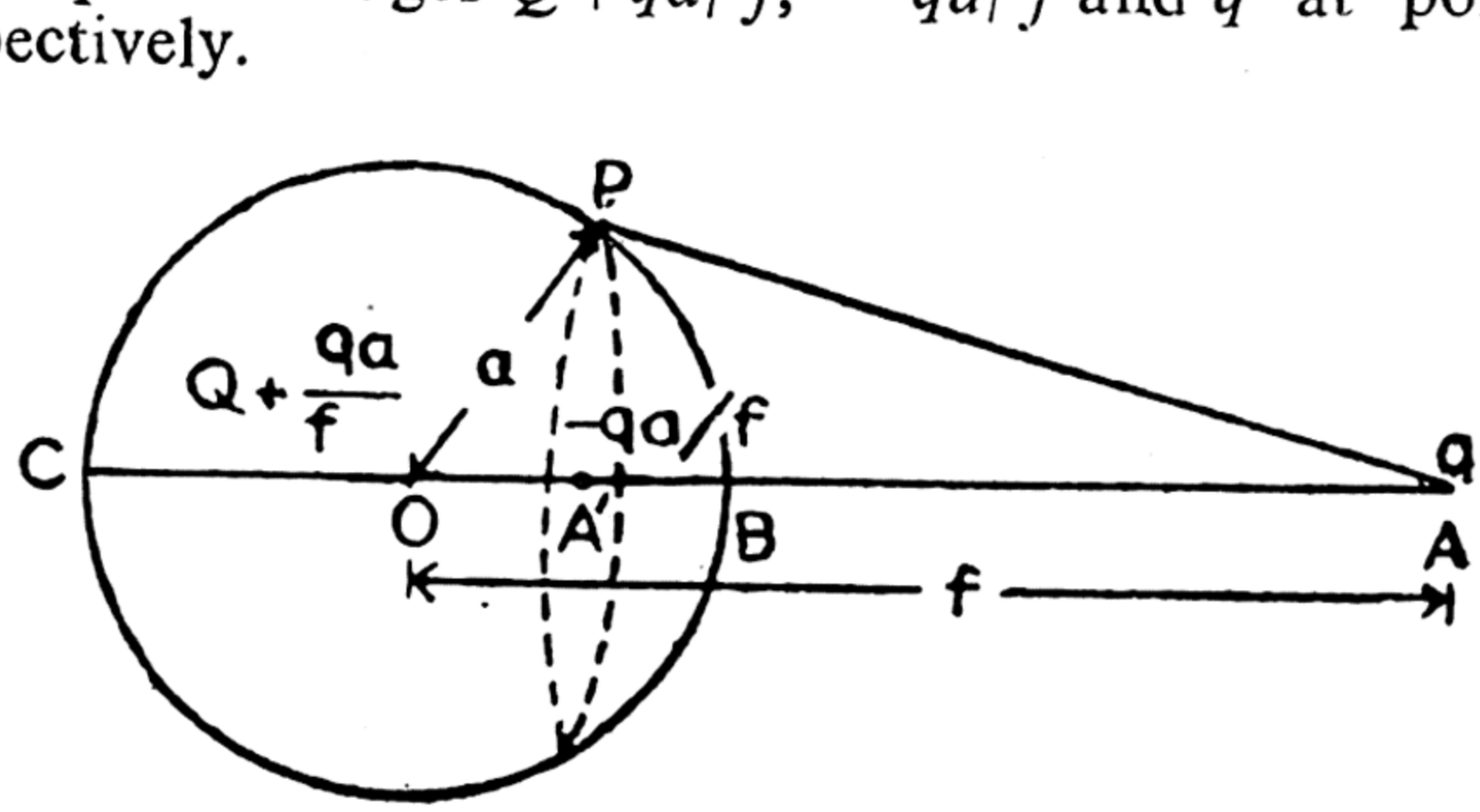


Fig. 5.36.

Similar to an example 19, one can prove that the surface charge density on the sphere due to charge q at A and $-qa/f$ at A' is $-q(f^2-a^2)/4\pi a\,AP^3$. According to the problem, charge Q is of such a value that the circle through P and $\perp$ to OA (dotted circle) becomes non-electrified. Thus

$$\sigma_P=0=\frac{q(f^2-a^2)}{4\pi aAP^3}+\frac{1}{4\pi a^2}\left[Q+\frac{qa}{f}\right]$$

or $$Q = aq[(f^2 - a^2)^{-1/2} - f^{-1}]$$

Greatest negative and positive surface densities are at points B and C respectively. The surface density of charge at B

$$\sigma_B = -\frac{q(f^2-a^2)}{4\pi a(AB)^3} + \frac{1}{4\pi a^2}\left[Q + \frac{qa}{f}\right].$$

Substituting value of Q and solving, we get

$$\sigma_B = -\frac{q}{4\pi a}\left[\frac{(f+a)^{3/2} - (f-a)^{3/2}}{(f-a)^2\,(f+a)^{1/2}}\right].$$

Similarly $$\sigma_C = -\frac{q(f^2-a^2)}{4\pi a(AC)^3} + \frac{1}{4\pi a^2}\left[Q + \frac{qa}{f}\right]$$

$$= \frac{q}{4\pi a}\left[\frac{-(f-a)^{3/2} + (f+a)^{3/2}}{(f+a)^2(f-a)^{1/2}}\right].$$

Hence $\sigma_B/\sigma_C = -[(f+a)/(f-a)]^{3/2}$.

Example 22. *Calculate the capacity of a charged spherical conductor of radius a at a distance* $\frac{1}{2}d$ *from an earthed conducting plane of infinite extent.*

Let AB be the infinite plane in front of which a small sphere of radius a is placed. In this case we must make use of an infinite series of images. First suppose a charge q_1 placed at the centre O of the sphere. Due to this charge the surface of the sphere remains equipotential, but the surface of the plane is not equipotential. Add a charge $-q_1$, the image of charge q_1 to get the plane surface equipotential. As sphere is not equipotential now, add charge q_2, the image of $-q_1$, in the sphere. This restores the equipotential

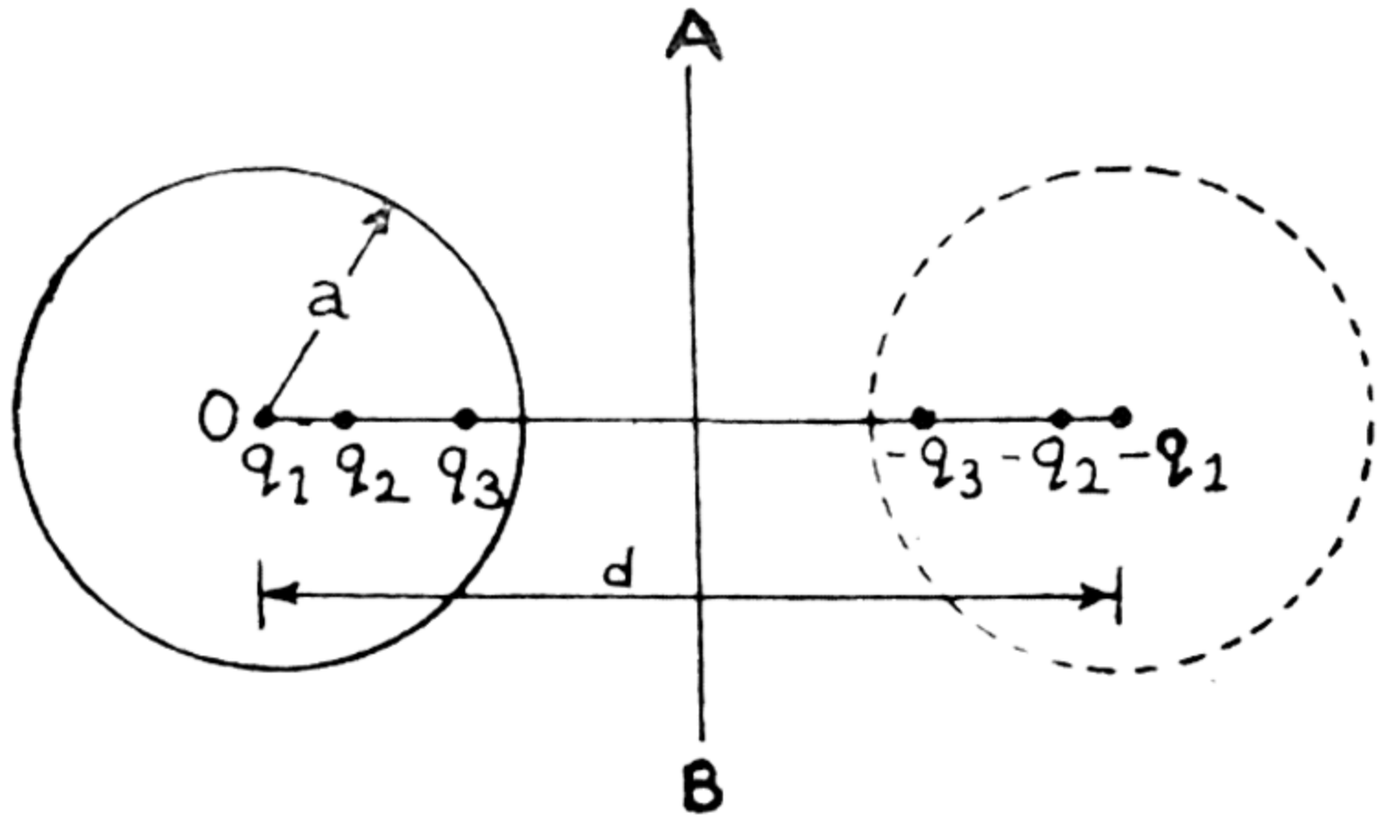


Fig. 5.37.

character of the spherical surface but disturbs that of the plane AB. Addition of the image $-q_2$ of q_2 in the plane makes the plane an equipotential surface again but disturbs the potential over the sphere. Continuation of the process gives us the desired state in which both the sphere and the plane are equipotential surfaces. Thus the total charge on the sphere is $Q = q_1 + q_2 + q_3 + \ldots$ and the charge on the plane is $-Q = -q_1 - q_2 - q_3 \ldots$

Distance of $-q_1$ is d from the centre of the sphere O. Its image charge will be $q_2=aq_1/d$ at a distance a^2/d from O. Thus its image will be a charge $-q_2$ at a distance $d(1-a^2/d^2)$ from O.

$$\therefore \quad q_3=\frac{a/d}{1-a^2/d^2}q_2=\frac{a^2/d^2}{1-a^2/d^2}\ q_1 \text{ at distance } \frac{a^2/d}{1-a^2/d^2} \text{ from } O.$$

Using $a/d=m$, one can write in general that charges q_1, mq_1, $m^2q_1/(1-m^2)$, $m^3q_1/(1-m^2)[1-m^2(1-m^2)^{-1}]$..... are placed at the distances O, ma, $ma/(1-m^2)$, $ma/[1-m^2(1-m^2)^{-1}]$... respectively from centre O.

$$\therefore \text{ Total charge } Q=q_1\left[\ 1+m+\frac{m^2}{1-m^2}\right.$$

$$\left.+\frac{m^3}{(1-m^2)\ [1-m^2/(1-m^2)]}+\ldots\right]$$

$$=q_1[1+m+m^2+m^3+2m^4+3m^5+\ldots\ldots].$$

As q_1 is the image of $-q_1$, the potential of the surface of the sphere due to this pair of charges is zero. The same is true for the pairs q_3 and $-q_2$, q_4 and $-q_3$... etc. Hence the potential of the sphere is due to the charge q_1 at the centre of the sphere alone and is given by

$$V=\frac{1}{4\pi\epsilon_0}\frac{q_1}{a}.$$

Potential of the plane AB is zero, as equal and opposite charges are symmetrically placed on its both sides. Hence the capacity of the sphere relative to the earthed plane is

$$C=Q/V=4\pi\epsilon_0 a[1+m+m^2+m^3+2m^4+\ldots\ldots]$$

Example 23. *Calculate the caapcity of two infinitely long straight parallel wires of radius a, one has charge λ per unit length and the other an equal negative charge.*

Let us consider two parallel wires intersecting the plane of paper at O and O' and having charges λ and $-\lambda$ per unit length

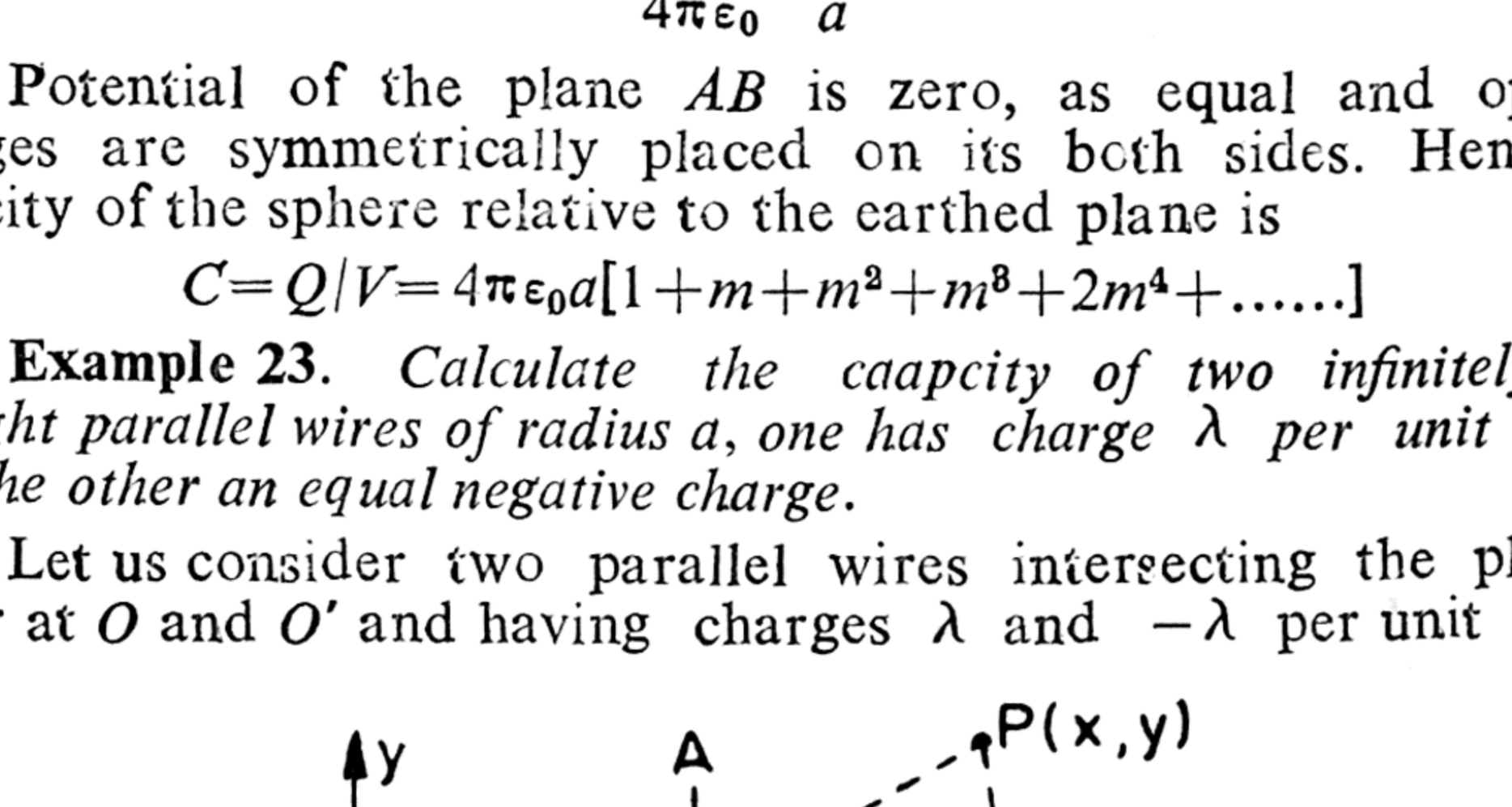


Fig. 5.38.

respectively. From symmetry it is clear that the potential at any point P depends only on the distance of P from the wires. Hence

$$V=-\frac{\lambda}{2\pi\varepsilon_0}\log_e r+\frac{\lambda}{2\pi\varepsilon_0}\log_e r'=\frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r'}{r}\right).$$

This relation makes the potential zero midway between the two wires for each point of which $r=r'$.

Equipotential surfaces are corresponding to

$$\frac{r'^2}{r^2}=\frac{r^2-2rd\cos\theta+d^2}{r^2}=m^2 \text{ (constant)}$$

or $$r^2+2r\cos\theta\, d/(m^2-1)=d^2/(m^2-1).$$

Transforming to rectangular coordinates with O as origin, $x=r\cos\theta$ and $y=r\sin\theta$, we get

$$\left(x+\frac{d}{m^2-1}\right)^2+y^2=\frac{m^2d^2}{(m^2-1)^2}.$$

This is the equation of a circle of radius $md/(m^2-1)$ with centre $[-d/(m^2-1), 0]$. Thus equipotential surfaces are cylinders of circular cross-section. We may assume surface of the wires as equipotential surfaces. Hence radius of the wire

$$a=md/(m^2-1),$$

and the distance between the axes of the two wires

$$b=d+2d/(m^2-1)=(m^2+1)d/(m^2-1).$$

Eliminating d between the above two relations, we get

$$\frac{b}{a}=\frac{(m^2+1)}{m},\ am^2-bm+a=0 \text{ or } m=\frac{b\pm\sqrt{(b^2-4a^2)}}{2a}.$$

Positive sign corresponds to +vely charged wire and −ve sign corresponds to the negatively charged wire. Hence the potential of the negatively charged wire

$$V_1=\frac{\lambda}{2\pi\varepsilon_0}\log_e m_+=\frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{b+\sqrt{(b^2-4a^2)}}{2a}$$

and that of the −vely charged wire

$$V_2=\frac{\lambda}{2\pi\varepsilon_0}\log_e m_-=\frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{b-\sqrt{b^2-4a^2)}}{2a}.$$

$\therefore$ Potential difference $V_1-V_2=\frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{b+\sqrt{(b^2-4a^2)}}{b-\sqrt{(b^2-4a^2)}}$.

$$\text{Capacity per unit length} = \frac{\lambda}{V_1 - V_2}$$

$$= 2\pi\varepsilon_0 \Big/ \log_e \left(\frac{b + \sqrt{(b^2 - 4a^2)}}{b - \sqrt{(b^2 - 4a^2)}} \right).$$

Example 24. *If conductors A, B, C, D, ..., where C completely surrounds A and B only, are charged and then A and C are connected by a thin wire, show that the charge remaining on A bears constant ratio to the charge on B.*

Let the charges on the conductors A, B, C, D, ... be q_1, q_2, q_3, q_4,....., respectively. As the conductor C encloses A and B, hence

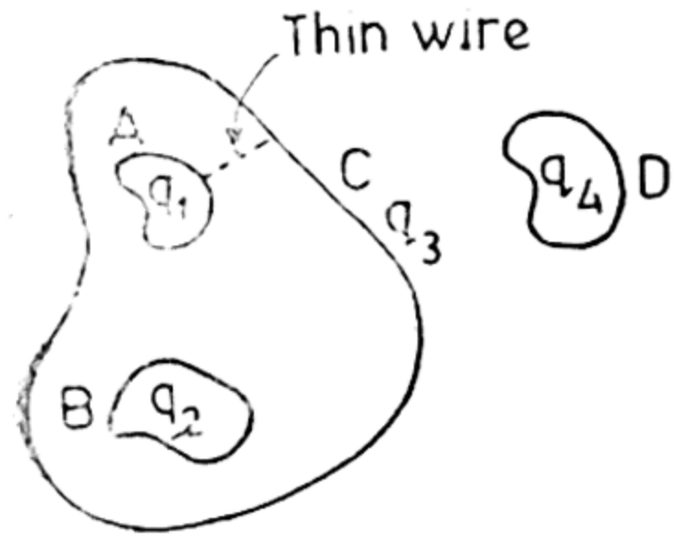


Fig. 5.39.

$$p_{33} = p_{32} = p_{31}.$$

Now suppose that there is a unit charge on D and all others have zero charge, then

$$p_{43} = p_{42} = p_{41}.$$

Similarly $p_{53} = p_{52} = p_{51}$, etc.

When a conductor A is connected to C, the charges q_1 and q_3 are distributed, let new charges be q_1' and q_3' on these conductors and the charges on other conductors remain the same.

$$\phi_A = p_{11}q_1' + p_{21}q_2 + p_{31}q_3' + p_{41}q_4 + \cdots\cdots$$

$$\phi_C = p_{13}q_1' + p_{23}q_2 + p_{33}q_3' + p_{43}q_4 + \cdots\cdots$$

Let the new potential of A or C be ϕ, hence

$$\phi_A = \phi_C = \phi \qquad \text{or} \qquad \phi_A - \phi_C = 0$$

$$\therefore \quad q_1'(p_{11} - p_{13}) + q_2(p_{21} - p_{23}) + q_3'(p_{31} - p_{33}) + q_4(p_{41} - p_{43}) + \cdots = 0$$

or

$$q_1'(p_{11} - p_{13}) + q_2(p_{21} - p_{23}) = 0.$$

$$\therefore \quad \frac{\text{charge remaining on } A}{\text{charge on } B} = \frac{p_{21} - p_{23}}{p_{13} - p_{11}} = \text{const.}$$

As we know that the coefficients of potential depend only on the shape and size and geometrical positions of the conductors, hence the above ratio is constant.

Example 25. *If three equal spherical conductors are placed at the corners of an equilateral triangle and raised to potential ϕ_1, ϕ_2, ϕ_3. Show that the energy of the system is reduced by $\frac{1}{6}[(\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_1)^2]$ W, where W is the energy of the system when the potentials are 1, −1 and 0 respectively.*

Since the three spherical conductors are equal, hence the coefficients of capacitance are equal.

$$q_{11}=q_{22}=q_{33}=a \text{ (say)}.$$

As these are placed symmetrically, hence the coefficients of inductance are equal.

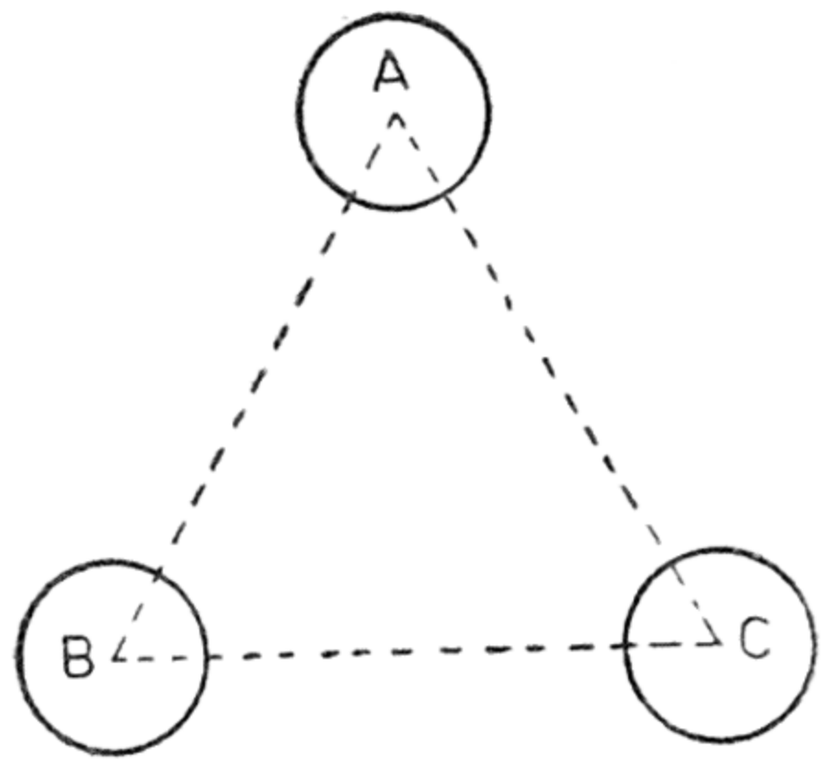


Fig. 5.40.

$$q_{12}=q_{13}=q_{23}=b \text{ (say)}.$$

Thus the relations between the charges and potentials are

$$e_1=q_{11}\phi_1+q_{21}\phi_2+q_{31}\phi_3=a\phi_1+b\phi_2+b\phi_3$$
$$e_2=q_{12}\phi_1+q_{22}\phi_2+q_{33}\phi_3=b\phi_1+a\phi_2+b\phi_3$$

and $$e_3=q_{13}\phi_1+q_{23}\phi_2+q_{33}\phi_3$$
$$=b\phi_1+b\phi_2+a\phi_3.$$

Hence total charge $e_1+e_2+e_3=(a+2b)(\phi_1+\phi_2+\phi_3)$.

$\therefore$ Initial energy of the system $W_1=\frac{1}{2}(e_1\phi_1+e_2\phi_2+e_3\phi_3)$.
$=\frac{1}{2}[a(\phi_1^2+\phi_2^2+\phi_3^2)+2b(\phi_1\phi_2+\phi_2\phi_3+\phi_3\phi_1)]$

Energy, when the potentials are $(1, -1, 0)$, is given by

$$W=\tfrac{1}{2}[a(1+1+0)+2b\{1\times(-1)+0+0\}]=a-b.$$

When the conductors are interconnected, the charge distribution changes while the total charge remains constant. Let e_1', e_2' and e_3' be the charges on the conductors A, B and C respectively and ϕ be the potential of each conductor. In this case

$$e_1'=a\phi+b\phi+b\phi=(a+2b)\phi=e_2'=e_3'.$$

$\therefore$ $$e_1'+e_2'+e_3'=3(a+2b)\phi=e_1+e_2+e_3.$$

$$3(a+2b)\phi=(a+2b)(\phi_1+\phi_2+\phi_3) \text{ or } \phi=\tfrac{1}{3}(\phi_1+\phi_2+\phi_3).$$

Energy in this case $W_2=\frac{1}{2}[e_1'\phi+e_2'\phi+e_3'\phi]=\frac{3}{2}(a+2b)\phi^2$
$=\frac{1}{6}(a+2b)(\phi_1+\phi_2+\phi_3)^2$.

$\therefore$ Loss of energy $W_1-W_2=\frac{1}{6}(a-b)[(\phi_1-\phi_2)^2+(\phi_2-\phi_3)^2$
$+(\phi_3-\phi_1)^2]$
$=\frac{1}{6}W[(\phi_1-\phi_2)^2+(\phi_2-\phi_3)^2+(\phi_3-\phi_1)^2]$

Example 26. *Positive and negative charges of same magnitude are situated at two points A and B. The line of force that leaves A at an angle α with AB meets the plane, which bisects AB at right angles at point P. Show that*

$$\sin\tfrac{1}{2}\alpha=\sqrt{2}\sin\tfrac{1}{2}(PAB).$$

Let Q be any point on the line of force, $\theta_1 = \angle QAB$ and $\theta_2 = \angle QBX$. Then as $Q \to A$, $\theta_1 \to \alpha$, and $\theta_2 \to \pi$. The equation of a line of force is then

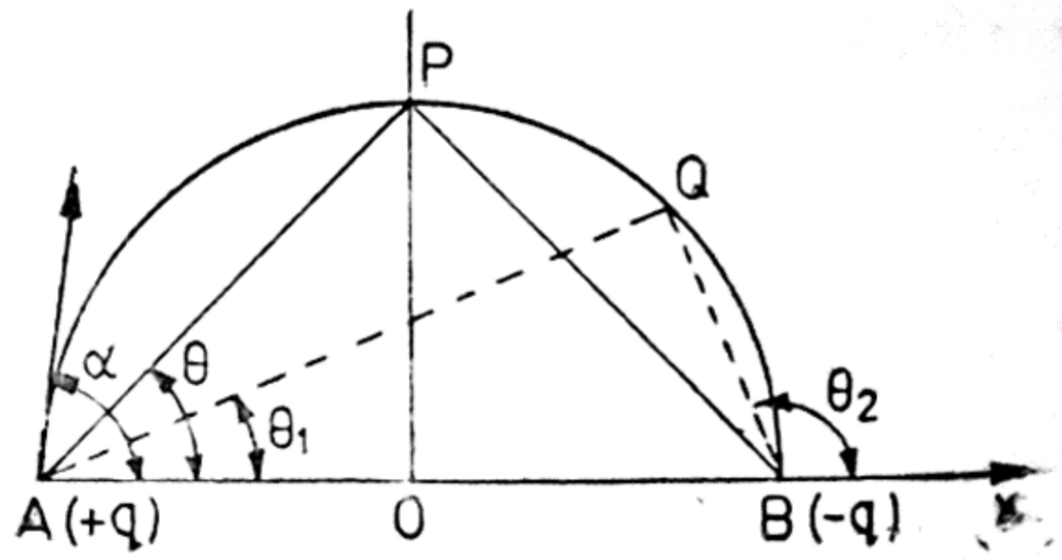


Fig. 5.41.

$$q \cos \theta_1 - q \cos \theta_2 = \text{const.}$$

$$= q \cos \alpha + q$$

$$\text{or} \quad \cos \theta_1 - \cos \theta_2 = \cos \alpha + 1$$

$$= 2 \cos^2 \tfrac{1}{2}\alpha$$

At point P on the bisecting plane

$$\theta_1 = \angle PAB = \theta \text{ and } \theta_2 = \angle PBX = \pi - \theta.$$

$$\therefore \quad \cos \theta - \cos (\pi - \theta) = 2 \cos^2 \tfrac{1}{2}\alpha, \qquad \text{or } \cos \theta = \cos^2 \tfrac{1}{2}\alpha$$

$$1 - 2 \sin^2 \tfrac{1}{2}\theta = 1 - \sin^2 \tfrac{1}{2}\alpha \qquad \text{or } \sin \tfrac{1}{2}\alpha = \sqrt{2} \sin \tfrac{1}{2}\theta.$$

Example 27. *Positive and negative unit charges are situated at two points A and B. Find at what distance from AB the plane normal to and bisecting AB is cut by the lines of force which issue from A in a direction parallel to this plane.*

Proceeding as in the above example with the difference that $\alpha = \pi/2$ and $q = 1$, we have the equation of lines of force as

$$\cos \theta_1 - \cos \theta_2 = 1$$

For point P, $\quad \cos \theta - \cos (\pi - \theta) = 1$

or $\quad \cos \theta = \tfrac{1}{2} \quad \text{or} \quad \theta = 60°.$

$$\therefore \qquad OP = OA \tan \theta = OA\sqrt{3} = (\sqrt{3}/2)\ AB.$$

Example 28. *If three infinitely long thin wires whose charges per unit length are 1, −2, 1 respectively, cut a plane at right angles in collinear points A, B, C where $AB = BC = a$. Prove that the equations of the lines of force in this plane is $r^2 = a^2 \cos (2\theta + \alpha) \sec \alpha$, where P is the origin with BC as axis and α a variable parameter.*

The equation of line of force is

$$\theta_1 - 2\theta + \theta_2 = \text{const.}$$

$$\text{or} \qquad (\theta_1 - \theta) - (\theta - \theta_2) = C.$$

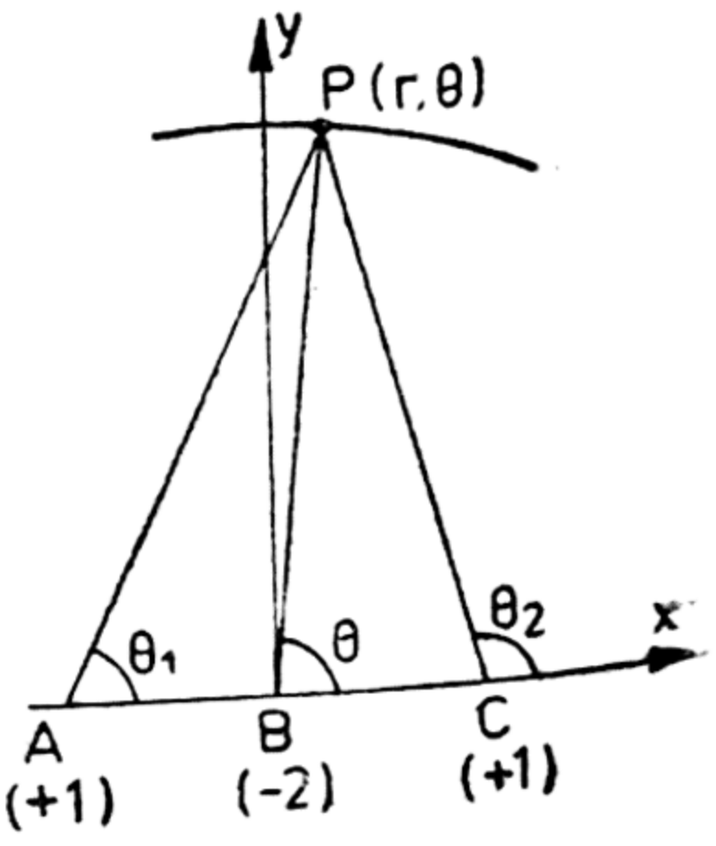


Fig. 5.42.

If (x, y) be the cartesian coordinates of point P on the line of force with charge point B as origin, then

$$\tan \theta = \frac{y}{a+x}, \quad -\tan \theta = \frac{y}{a-x}$$

and $$\tan \theta = \frac{y}{x}$$

$$\text{or} \qquad \theta_1 = \tan^{-1}\frac{y}{a+x}, \quad -\theta_2 = \tan^{-1}\frac{y}{a-x},$$

and $$\theta=\tan^{-1}\frac{y}{x}.$$

Thus equation of lines of force is given by

$$\left(\tan^{-1}\frac{y}{a+x}-\tan^{-1}\frac{y}{x}\right)-\left(\tan^{-1}\frac{y}{x}+\tan^{-1}\frac{y}{a-x}\right)=C$$

or $$\tan^{-1}\left[\frac{y/(a+x)-y/x}{1+y^2/x(a+x)}\right]-\tan^{-1}\left[\frac{y/x+y/(a-x)}{1-y^2/x(a-x)}\right]=C$$

$$\tan^{-1}\,2a^2xy/[a^2x^2-(x^2+y^2)^2-a^2y^2]=C.$$

Using relations $x=r\cos\theta$ and $y=r\sin\theta$, we get

$$\tan^{-1}\frac{a^2\sin 2\theta}{a^2\cos 2\theta-r^2}=C=\left(\frac{\pi}{2}-\alpha\right).$$

Here constant has been replaced by $(\pi/2-\alpha)$ in order to get the answer in the proper form. Above relation thus reduces to

$$\frac{a^2\sin 2\theta}{a^2\cos 2\theta-r^2}=\tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha$$

or $$r=a^2\cos(2\theta+\alpha)\sec\alpha.$$

Oral Questions

1. Which of the following are polar molecules ?

   H_2, O_2, H_2O, NO, N_2O, HCl, NH_3, CO_2.

2. Distinguish between the terms free charge, bound charge, polarization charge and fictitious charge.

3. A free atom, a dipole molecule and a small particle of dielectric, all move into the region of stronger field if placed in an inhomogeneous electric field. What will happen to their centres of mass if placed in a uniform field ?

4. A dielectric block (Fig. 5.43) is uniformly polarized. If the polarization is P, find the polarization charge density on the faces 1, 2 and 3.

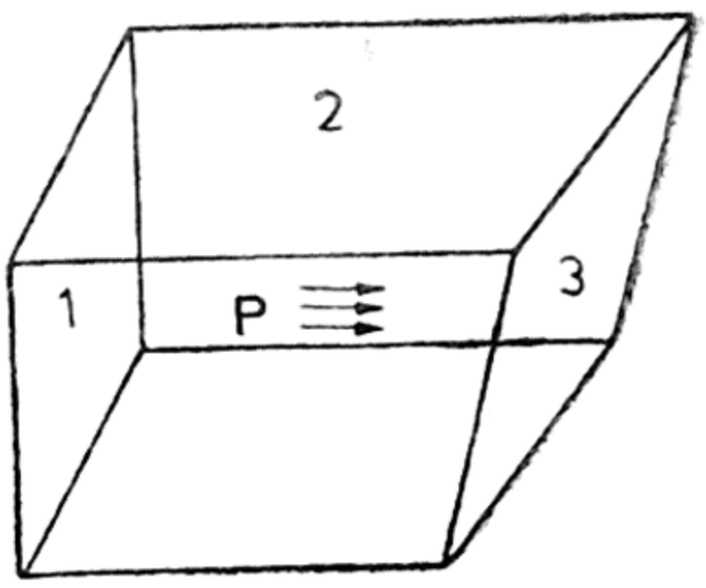


Fig. 5.43.

5. What are the values of P and χ for a metal as a polarizable body ?

6. At the boundary between two dielectric materials, show that although lines of D are continuous the magnitude of D in different media is generally different. Give the condition when magnitude of D is same in both the media.

7. Refraction of lines of force and displacement is alike to the light rays. Explain.

8. How can you check the correctness of the Classius-Mossotti relation ?

9. Should the dielectric constant of the mercury gas be close to the liquid metal ($\sim\infty$) or be close to the gas (~ 1)?

10. Why the lines of flux do not strike the dielectric normally, similar to the metal surface?

11. What do you mean by electrical image? Give its uses.

12. Correct the statement-conductor behaves like a dielectric.

Problems

1. A line of charge density λ per meter is placed along the z-axis and a short dipole of dipole moment **P** is kept at a distance d from it. If the axis of dipole lies in the plane containing the point of its location and the z-axis, calculate the couple and the translatory force on the dipole.

$2\lambda p \sin\alpha/(4\pi\varepsilon_0 d)$, $2\lambda p\cos\alpha/(4\pi\varepsilon_0 d^2)$ towards the line charge, when α is the angle which dipole makes with the direction of line charge.

2. A dipole of dipole moment 3×10^{-30} coulomb meter is placed at a distance of 10 A. U. from a nucleus of charge 5 electronic units. Calculate the force and torque experienced by the dipole, (*a*) when it is placed radially to the nucleus with its negative pole towards the nucleus and (*b*) when **p** is perpendicular to **r**.

(*a*) $F_r = 4.32\times10^{-11}$ *Nt.*, $F_\theta = 0$, $\tau = 0$

(*b*) $F_r = 0$, $F_\theta = 2.16\times10^{-11}$ *Nt*, $\tau = 2.16\times10^{-20}$ *Nt. meter.*

3. A dielectric slab of thickness 0.5 cm and dielectric constant 7.0 is placed between the plates of a parallel plate capacitor of plate area 100 cm^2 and separation 1.0 cm. A potential difference of 100 volts is applied with no dielectric slab. The battery is disconnected and the dielectric slab is inserted. Calculate **E**, **D** and **P**, (*a*) in the dielectric and (*b*) in the air gap.

(*a*) 1.43×10^3 *volts/m*, 8.9×10^{-8} C/m^2, 7.5×10^{-8} C/m^2.

(*b*) 1.00×10^4 *volts/m*, 8.9×10^{-8} C/m^2, 0.

4. A sphere of radius a is polarized along the radius vector so that $P=kr$. Find the volume and surface charge densities. Show that the total charge is zero. $(-3k, ka)$

5. A region between the plates of a parallel plate capacitor is filled with a dielectric slab of dielectric constant k and of thickness d. The capacitor is first connected with a pd V and then disconnected. If now the dielectric is partially withdrawn in the direction of length l of the plate through a distance $l-x$, calculate the pd across the capacitor. $kIV/[l+(k-1)x]$

6. A parallel plate capacitor is filled with three dielectrics having dielectric constant in the ratio 1:2:3. All the three dielectrics and plates of the capacitor have equal area. Thickness of the dielectrics is in the ratio 6:2:1. Calculate the capacitance of the capacitor thus formed, if the combined thickness of the dielectric is less than the distance between the plates of the capacitor. $\varepsilon_0 A/[(d-6t)+25t/6k]$.

7. A spherical drop of water carrying a charge of 3×10^{-11} coulomb has a potential 500 volts at its surface. What is the radius of drop? If two such drops of the same charge and radius combine to form a single drop. What is the potential at the surface of the new drop so formed?

(0.54 *mm*, 840 *volts*)

8. A conducting sphere floats half submerged in a liquid dielectric of permittivity ε_1. Above this liquid is a gas of permittivity ε_2. If the total free charge on the sphere is q, calculate a radial electric field satisfying all boundary conditions and find the free, bound and total charge densities at all points on the surface of the sphere. $E=Q/2\pi(\varepsilon_1+\varepsilon_2)r^2$.

9. A point charge q is placed at a small distance c from the centre of a spherical cavity of radius a in an infinite dielectric. Show that the charge experiences a force approximately equal to $2(k-1)q^2c/(2k+1)a^3$, away from centre.

10. If the space between concentric conducting spheres of radii a and $b(a<b)$ is filled with two dielectrics of constants k and k', their common surface being a concentric sphere of radius c, find the capacity of the capacitor.

$4\pi\varepsilon_0 kk'\,abc/[bk'\,(c-a)+ak(b-c)]$

11. The space between two concentric conducting spheres is filled on one side of a diametral plane with dielectric of constant k and on the other side with dielectric of constant k'. The inner sphere is of radius a and has a charge q. Show that the force on it perpendicular to this diametral plane is $(k-k')q^2/2(k+k')^2a^2$.

12. A flat disc type cavity has been made in a uniformly polarized sphere with its bases perpendicular to the polarization density **P**. The thickness of the removed disc may be taken small in comparison with its face area. Calculate the electric field inside this cavity. $(2P/3\varepsilon_0)$

13. Dielectric constant of helium at N.T.P. is 1,000074. What is the atomic polarizability? How much is the dipole moment per helium atom when the gas is kept in an electric field of 1500 volt/-cm? Take the Avogadro's number as 60×10^{22} per gm. mole.

$2.2\times10^{-31}\ m^3$, 1.1×10^{-24} *stat-coul. cm.*

14. An elementary dipole of strength p is placed at a point P, outside and at a distance r from the centre O of an earthed conducting sphere of radius R. The axis of the dipole is in the direction OP. Prove that its image system consists of a point charge pR/r^2 and a dipole strength pR^3/r^3, both situated at the point P′ which is inverse to P in the sphere.

15. If a dielectric sphere of dielectric constant k is placed in a uniform field E_0 in vacuum. Prove that the field outside the sphere is equal to the field E_0 plus a term due to a dipole at the centre of the sphere whose moment is equal to the total polarization of the sphere.

16. Show that the field inside a cylindrical cavity in a dielectric of constant k is $[2k/(k+1)]\,E_0$, when the axis of the cylinder is at right angles to a uniform field E_0.

17. A spherical conductor with charge q surrounded by a dielectric of constant k is subjected to a uniform external field E_0. Find the stress on the surface and the total force on the sphere.

$\frac{1}{2}k\varepsilon_0[3E_0\cos\theta+q/4\pi\varepsilon_0 ka^2]^2$, qE_0

18. A long conducting cylinder of radius a, bearing a charge λ per unit length is oriented parallel to a grounded conducting plane of infinite extent. The axis of the cylinder is at a distance d from the plane. Find the positionof the line image and find also the constant m in terms of a and d.

$m=d/a+\sqrt{[(d/a)^2-1]}$, *image at* $d(m^2-1)/(m^2+1)$

19. A dipole **p** is oriented normal to and at distance d from an infinite conducting plane. The plane is grounded. Calculate the force exerted on the plane by the dipole. $3p^2/32\pi\varepsilon_0 d^4$, *attraction.*

20. Two equal spheres each of radius a are in contact. Show that the capacity of the conductor so formed is $8\pi\varepsilon_0 a \log_e 2$.

21. A conductor is formed by the outer surfaces of two equal spheres, the angle between the radii at a point of intersection being 120°. Show that the capacity of the conductor so formed is $(5\sqrt{3}-4)\,a/2\sqrt{3}$, where a is the radius of either sphere.

22. A line charge λ per unit length is placed parallel to and at a distance d from the axis of a long circular solid cylinder of uniform dielectric of constant k. The radius of the cylinder is a ($<d$). Show that the line images can be found to determine the electric field and that there is a force of attraction between the line charge and the cylinder of amount

$$\frac{k-1}{k+1}\,\frac{8\pi\varepsilon_0\lambda^2a^2}{d(d^2-a^2)}$$ per unit length.

23. If n conductors in space be connected together by wires, prove that the capacity of the compound conductor thus formed is $\Sigma q_{rr}+2\Sigma q_{rs}$. In the first summation r has values from 1 to n and in the second r and s are from 1 to n but $r\neq s$.

24. Three equal spheres are placed at the corners of an equilateral triangle. When their potentials are ϕ, 0, 0, their charges are e, e' and e' respectively. Show that, when each sphere is at potential ϕ', each sphere has a charge $(2e'+e)\,\phi'/\phi$.

25. Three insulated spherical conductors of the same size are placed at the corners of an equilateral triangle whose sides are large in comparison with the radii of the spheres. If the conductors are then touched in turn by another small charged spherical conductor, show that the charges they receive are in geometrical progression.

26. Point charges e, $-e'$ and $-e'$ are placed at O, A and B respectively, which are collinear in such a way that $OA=OB$. If $e>2e'$, show that the greatest angle which a line of force leaving O and entering A makes with OA is α, where $e\sin^2(\frac{1}{2}\alpha)=e'$.

27. Two positive point charges q_1 and q_2 are placed at the points A and B respectively. Show that the tangent at infinity to the line of force which starts from q_1 making an angle α with BA, makes an angle $2\sin^{-1}[\sqrt{\{e_1/(e_1+e_2)\}}\sin\frac{1}{2}\alpha]$ with BA and passes through the point C in AB such that $AC/CB=e_2/e_1$.

28. Three infinite parallel straight lines, with charges q, $-q/2$, $-q/2$ per unit length, meet the plane of paper perpendicularly at A, B, C respectively. If $AB=AC$, prove that the lines of force reaching B, after having passed indefinitely close to the equilibrium point make with BC an angle BAC.

6

Steady Current

6.1 CURRENT

In electrostatics we have been concerned with stationary electric charges. If a free charge is placed in an electric field it will be acted on by a force and will move in the direction of lines of force. In an isolated metallic conductor, the free electrons present in it are in random motion like the molecules of a gas and the net rate at which electrons pass through any hypothetical plane is zero. If the ends of the conductor are connected to a battery, an electric field **E** will be set up at every point with in the conductor. This field will act on the electrons and will give them a resultant motion in the direction—**E**. This flow of electrons constitutes an *electric current and the strength of electric current also known as current is defined by the rate at which charge passes through any specified surface area* (*i.e*,, $i=q/t$, where q is the net charge passes in time t). It is a *scalar quantity*. In SI units its unit is ampere, the coulomb per second. It is also the practical unit. If the rate of flow of charge with time is not constant, *i.e.* the current varies with time then at any instant t current is defined as

$$i = dq/dt. \qquad \ldots(1)$$

The current i is the same for all cross sections of a conductor of non-uniform cross-sections. Similar to the water flow, charge flows faster where the conductor is smaller in cross-sction and slower where the conductor is larger in cross section, so that charge rate remains uncharged. The direction in which the positive charge will move in electric field is taken as the positive direction of current. Since the electrons are negatively charged, hence their direction will be opposite to that of the conventional current flow.

In metals the charge carriers are electrons. In electrolytes or in gaseous conductors the positive and negative ions travel in opposite directions. As the positive ions move slower and electrons, the negative ions, move faster, the net motion of charge is due to negatively charged electrons. This type of current is known as *conduction current*. Liquids and gases may also undergo hydrodynamic motion, which will produce currents, known as *convection*

currents. In this chapter we are not going to discuss convection currents, which are important in atmospheric electricity and in vacuum tubes.

6.2 CURRENT DENSITY, EQUATION OF CONTINUITY

Current is a characteristic of a particular conductor, like the mass, volume, etc. of an object and hence is *a macroscopic quantity.* A related *microscopic quantity* is the *current density.* It is denoted by a vector **j** and is a characteristic of a point inside the conductor rather than of the conductor as a whole. It is having direction of the flow of positive charge and a magnitude equal to the current per unit area through an infinitesimal area normal to the direction of current flow. It is measured in amperes/meter^2 or coulomb/meter^2 × sec. If the current is distributed uniformly, the magnitude of **j** for all points on the cross sectional area S is given by $j = i/S$.

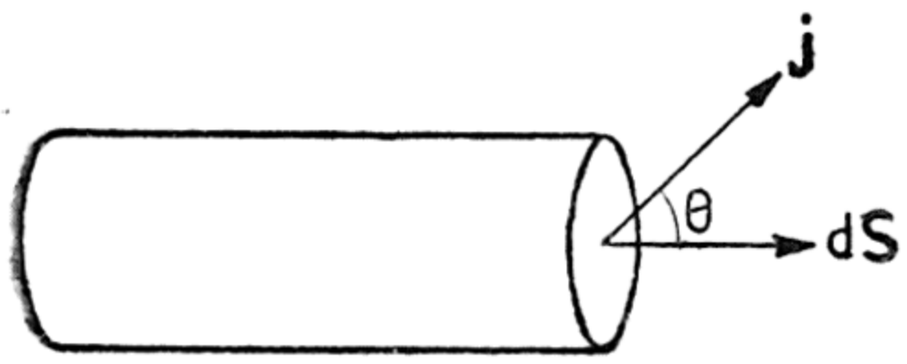


Fig. 6.1.

The current through any small area is given by $\triangle i = \mathbf{j}.\triangle\mathbf{S}$. The current through $\triangle\mathbf{S}$ is the same as through the area $\triangle S \cos\theta$ perpendicular to **j**. Thus for large areas, we can write

$$i = \int_S \mathbf{j}.d\mathbf{S}. \qquad \ldots(2)$$

The value of the integral is independent of the shape of surface taken. *The relation (2) shows that the current is the flux associated with* **j** *and is a scalar because the integral* $\mathbf{j}.d\mathbf{S}$ *is a scalar. The arrow often associated with the current in a conductor does not indicate that the current is a vector but shows only the sense of charge flow. Current is not a vector quantity as it does not obey the vector laws.* It remains unchanged if the wire is bent, tied into a knot or distorted. When **j** remains constant every where the current is known as *steady* or *stationary current.*

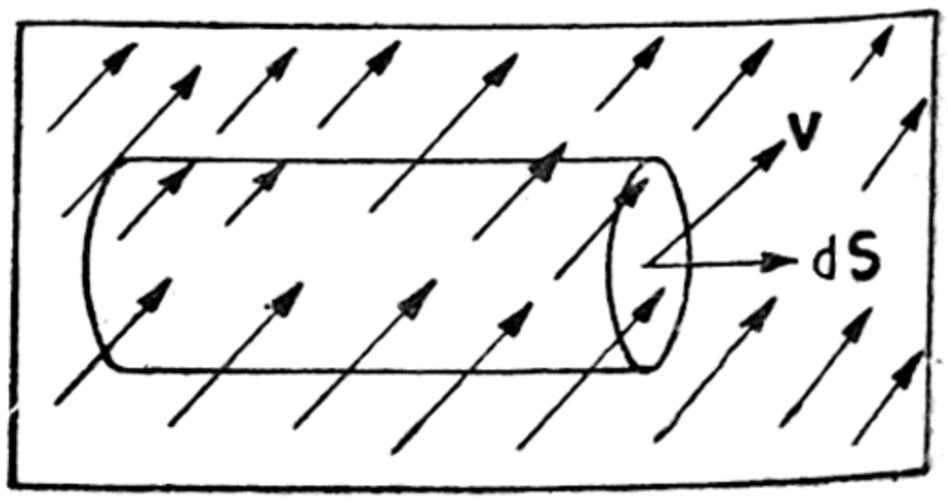


Fig. 6.2.

To derive an expression for current density, consider a special case in which all the charge carriers have the same charge q and move with the same velocity **v**. In a small time interval $\triangle t$, each charge carrier will travel a distance $\mathbf{l} = \mathbf{v}\,\triangle t$. Let us consider a cylinder (or a prism) of cross sectional area $d\mathbf{S}$ and of length **l** in the conductor. The volume of this cylinder will be $d\mathbf{S}.\mathbf{l} = d\mathbf{S}.\mathbf{v}\triangle t$. If n is the number of charge carriers per unit volume, then the charge in this cylindrical volume $= qn\,d\mathbf{S}.\mathbf{v}\triangle\mathbf{t}$. Hence the current or the rate of flow of charge $= qn\,d\mathbf{S}.\mathbf{v}$ and current density $\mathbf{j} = qn\mathbf{v}$.

If the charge carriers are having different charges and velocities, then the current through this cylinder will be the sum of the

currents due to each of the charge carrier. Let per unit volume of the conductor consists of m groups, one having n_1 particles each of charge q_1 and velocity $\mathbf{v}_1$, second having n_2 particles each of charge q_2 and velocity $\mathbf{v}_2$...and the m^{th} group having n_m particles each of charge q_m and velocity $\mathbf{v}_m$.

$\therefore$ Total number of charge carriers $n=n_1+n_2+\dots n_m$ ...(3)

and $$I=n_1q_1\,d\mathbf{S}\,.\,\mathbf{v}_1+n_2q_2\,d\mathbf{S}\,.\,\mathbf{v}_2+\dots+n_mq_m\,d\mathbf{S}\,.\,\mathbf{v}_m$$

$$=d\mathbf{S}\,.\sum_{i=1}^{m} n_iq_i\mathbf{v}_i. \qquad \dots(4)$$

Hence the charge density $\mathbf{j}=\Sigma\, n_iq_i\mathbf{v}_i$. ...(5)

In solid conductors, the charge carriers are the free electrons, hence $q_1=q_2=\dots q_m=-e$. Thus we have

$$\mathbf{j}=-e\Sigma n_i\mathbf{v}_i. \qquad \dots(6)$$

We know that the average velocity

$$v_{av}=\frac{n_1v_1+n_2v_2+\dots n_mv_m}{n_1+n_2+\dots n_m}=\frac{\Sigma n_iv_i}{n}.$$

Hence $\mathbf{j}=-ne\mathbf{v}_{av}$. ...(7)

Thus we see that the direction of $\mathbf{j}$ is opposite to $\mathbf{v}_{av}$,

The current density $\mathbf{j}$ and the charge density ρ are related at each point through a differential equation. The relation is based on the fact that electric charge can neither be created nor be destroyed and the rate of increase of the total charge inside any arbitrary volume must be equal to the net flow of charge into this volume.

$$\therefore \quad i=-\int_s \mathbf{j}\,.\,d\mathbf{S}=\frac{\partial q}{\partial t}=-\frac{\partial}{\partial t}\int_v \rho dV. \qquad \dots(8)$$

The minus sign comes here because the direction of $d\mathbf{S}$ is along outward normal and we wish current positive when the net charge is from the outside of V to within. Since we are dealing with a fixed volume V, hence we can put the RHS as $\int(\partial\rho/\partial t)\,dV$. Using divergence theorem the surface integral of Eq. (8) can be transform into a volume integral. Thus we have

$$-\int_v div\ \mathbf{j}\ dV=\int_v \frac{\partial\rho}{\partial t}dV \quad \text{or} \quad \int_v\left(\ div\ \ \mathbf{j}+\frac{\partial\rho}{\partial t}\ \right)dV=0.$$

This integral must be zero for any arbitrary volume. It is only possible when integrand in zero. Thus

$$div\ \mathbf{j}+\partial\rho/\partial t=0. \qquad \dots(9)$$

This differential equation is known as the *continuity equation.* If the region does not contain a source or sink of current, $\partial\rho/\partial t=0$ and hence for *steady current*, we have

$$div\ \mathbf{j}=0. \qquad ...(10)$$

6.3. CONDUCTIVITY AND RESISTIVITY

It is found experimentally that in a metallic conductor at constant temperature the current density **j** is linearly proportional to the electric field **E.** Thus

$$\mathbf{j}=\sigma\mathbf{E}, \qquad ...(11)$$

where σ is a constant, known as the *conductivity* of the conductor. It is a scalar quantity. The reciprocal of the conductivity is called the *resistivity* ρ. The dimensions of ρ are (volts/meter)/(ampere/meter^2)= volt-meter/ampere, or simply ohm-meter. The unit of conductivity is thus ohm^{-1} meter^{-1} or mho/meter. These constants σ and ρ are the characteristic of a material rather than of a particular specimen of a material.

Consider first a particular piece of current carrying material like the wire. If at any instant, current is directed toward an insulated boundary of the conductor, it will produce steadily increasing added field due to the piling up of the charge at the boundary. Therefore for the steady flow for which there is no source or sink of current, the **j** *must be parallel to the conductor boundaries.* If we consider two surfaces S_1 and S_2 everywhere perpendicular to **j**, each bounded by curves lying in the conductor boundaries. The currents through them $i_1=\int s_1 \mathbf{j}\,.\,d\mathbf{S}$ and $i_2=\int s_1 \mathbf{j}\,.\,d\mathbf{S}$ must be equal, otherwise charge will pile up in between. Thus the current must be same through any surface that cuts across the wire. Current behaves like electric flux and we can draw lines of flow every where parallel to **j** that do not begain or end anywhere in the conductor. Since the vectors **E** and **j** are in the same direction we can thus write

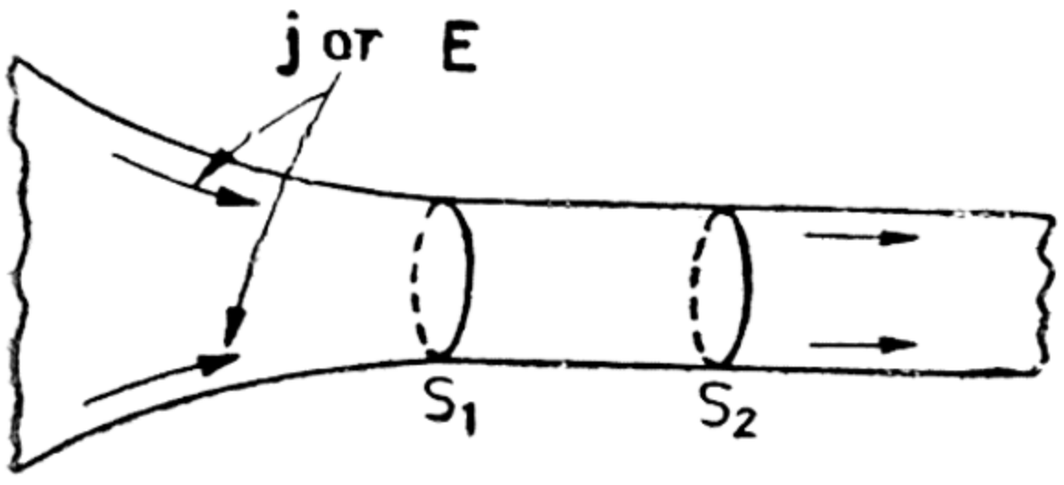


Fig. 6.3.

$$\mathbf{E}=\rho\mathbf{j}. \qquad ...(12)$$

In many conducting materials the resistivity ρ is a constant, independent of **E.** The discovery that ρ is a constant for a matallic conductor at constant temperature was made by G.S. Ohm and is known as *Ohm's law.* A material obeying Ohm's law is called an *ohmic conductor* or a *linear conductor.* Only metals and metal alloys are under this head. Thus *Ohm's law, like Boyle's law, describes a special property of certain materials and not a general property of all matter.* In non-isotropic conductors, the current may flow more easily in some directions than in others. In this case Eq. (11) can be written as

$$j_x = \sigma_1 E_x,\ j_y = \sigma_2 E_y,\ j_z = \sigma_3 E_z. \qquad \ldots(13)$$

The resistivity of all metallic conductors increases with increasing temperature. It is practically independent of temperature for alloy *manganin*. The general expression for the resistivity of a pure metal at temperature t°C is

$$\rho_t = \rho_o(1 + \alpha t + \beta t^2 + \ldots), \qquad \ldots(14)$$

where ρ_o is the resistivity at 0°C and α, β,...etc are the constants in decending order of their values. When t is not very large, higher terms are negligible and the relation is thus given as

$$\rho_t = \rho_o(1 + \alpha t) \text{ or } \alpha = (\rho_t - \rho_o)/\rho_o t, \qquad \ldots(15)$$

where the constant α is termed as the temperature coefficient of the resistivity and may be defined as the increase in resistivity per unit resistivity when the temperature is increased by 1°C.

The resistivity of a super conductor, such as alloy or compound decreases as the temperature is decreased, at first regularly like that of a metal and suddenly plunges to zero at a certain temperature, known as *critical temperature* ($\approx 0.1°K$ to $10°K$). The resistivity of a semiconductor decreases rapidly with increasing temperature.

Table 6.1 The resistitivities and the temperature conefficients of some substances.

	Material	*resistivity at 20°C* (Ω—m)	α *at 20°C* (*per °C*)
Metal	Silver	1.6×10^{-8}	3.8×10^{-3}
	Copper	1.7×10^{-8}	3.9×10^{-3}
	Aluminium	2.8×10^{-8}	3.9×10^{-9}
	Gold	2.4×10^{-8}	3.4×10^{-3}
	Iron	1.0×10^{-7}	5×10^{-3}
	Tungsten	5.6×10^{-8}	4.5×10^{-3}
Alloy	Brass	$\sim 7 \times 10^{-8}$	1×10^{-3}
	Manganin	4.4×10^{-7}	1×10^{-5} or 0
	Nichrome	1.1×10^{-8}	2×10^{-4}
Semiconductors	Carbon	3.5×10^{-5}	-5×10^{-4}
	Germanium	0.60	-4.8×10^{-2}
	Silicon	2300	-7.5×10^{-2}
Insulators	Glass	10^{10}—10^{14}	
	Mica	10^{11}—10^{15}	
	Sulphur	10^{15}	
	Wood	10^{-8}—10^{11}	

Relaxation Time. If a charge is distributed within a conducting body, the charge will distribute itself in such a way that a zero electric field exists within and tangent to the conductor surface. The time required for this process is called the *relaxation time.* For its qualitative evaluation let us proceed as :

Consider a homogeneous conducting region with a conductivity σ and permittivity ε. We have a divergence equation as

$$\nabla . \mathbf{j} = -\partial \rho / dt .$$

Since $\mathbf{j} = \sigma \mathbf{E}$ and $\nabla . \mathbf{E} = \rho/\varepsilon$, hence

$$\nabla . (\sigma \mathbf{E}) = \sigma \nabla . \mathbf{E} = \sigma \rho/\varepsilon = -\partial \rho / \partial t .$$

Thus
$$\frac{\sigma}{\varepsilon} \int_0^t dt = -\int_{\rho_0}^{\rho} \frac{d\rho}{\rho}$$

or
$$\rho = \rho_0 \, e^{-\sigma t/\epsilon}, \qquad \ldots(16)$$

where ρ_0 is the initial charge density. The relaxation time t_{re} is defined as

$$t_{re} = \varepsilon/\sigma, \qquad \ldots(17)$$

i.e. the time required for the initial charge distribution to decay to $1/e$ of its initial value. It is extremely short ($\sim 10^{-19}$ sec) for good conductors and is relatively large (few days) for insulators.

6.4. POTENTIAL AND RESISTANCE

The magnitude of the current between two surfaces S_1 and S_2 depends on the nature of the conductor and on the force which urge the free charges through it. The effect of all such forces is most easily calculated in terms of the work performed per unit positive charge. This quantity is called the *potential difference.* The charge flows through the conductor in the direction of decreasing potential. Hence the potential drop

$$V = V_1 - V_2 = -\int_2^1 \mathbf{E} \, . \, d\mathbf{l} = \int_1^2 \mathbf{E} \, . \, d\mathbf{l}$$

$$= \int_1^2 E dl = \int_1^2 \rho \, j \, dl, \qquad \ldots(18)$$

where the path of integration is along any line of flow from S_1 to S_2. If these points are brought together, the above relation reduces to zero. It is therefore clear that *no static distribution of charge can produce a potential difference around a circuit.* If we increase the field and potential everywhere in the same ratio, the flow lines will not change their pattern, but $\mathbf{j}$ will increase everywhere in the same proportion as will i. Thus we can write above relation as

$$V = i \int_1^2 \rho (j/i) dl. \qquad \ldots(19)$$

or $$V=Ri, \qquad \ldots(20)$$

where R is a quantity which remains constant as V and i change but depends on the location of S_1 and S_2 on the flow pattern between these surfaces and on the value of ρ. This constant R is known as *resistance* of the piece of the conductor between surfaces S_1 and S_2 and is measured in ohms (volts/ampere). The relation (20) is also named as Ohm's law. Very large resistances are measured in megohms (1 megohm=1 million ohms).

If the conductor is a straight and long wire of length l and area of cross section A, the lines of current flow and of electric field will be parallel, and thus $\mathbf{E}$ will be constant along any one line and hence, throughout the wire. The $\mathbf{j}$ is uniform in magnitude and direction and we can write $j=i/A$. Thus we have

$$V=\int_1^2 \rho\left(\frac{i}{A}\right)dl=\rho\left(\frac{i}{A}\right)\int_1^2 dl=\frac{\rho il}{A}. \qquad \ldots(21)$$

Combining Eqs. (20) and (21), we get

$$R=\rho l/A. \qquad \ldots(22)$$

The resistivity of the material of a conductor thus can be defined as the resistance offered by a specimen of the material of unit length and of unit area of cross section. It can be found by measuring R, l and A of a sample of the material in the form of wire.

V, i and R are macroscopic quantities as these are properties of a body as a whole. The corresponding microscopic quantities are $\mathbf{E}$, $\mathbf{j}$ and ρ. The former can be found by integrating over the latter quantities, as indicated by the relations (18) and (3). The resistance of a conductor between two equipotential surfaces S_1 and S_2 can be expressed in microscopic terms as

$$R=\frac{V}{i}=\frac{\int \mathbf{E}\cdot d\mathbf{l}}{\int \mathbf{j}\cdot d\mathbf{S}}=\frac{E\int dl}{j\int dS}$$
$$=\frac{E\,l}{j\,A}=\rho\,\frac{l}{A}. \qquad \ldots(23)$$

From Eqs. (10) and (11), we get

$$div\ \mathbf{j}=div\ (\sigma\mathbf{E})=\sigma\ div\ \mathbf{E}=0.$$

Since $\mathbf{E}=-\text{grad}\ V$, hence $\nabla^2 V=0$. $\qquad \ldots(24)$

Thus we see that Laplace' equation holds, as in electrostatics. In the medium, the lines of current flow are orthogonal to the lines of constant V and coincide with lines of $\mathbf{E}$.

Resistance for radial current through a circular disc.

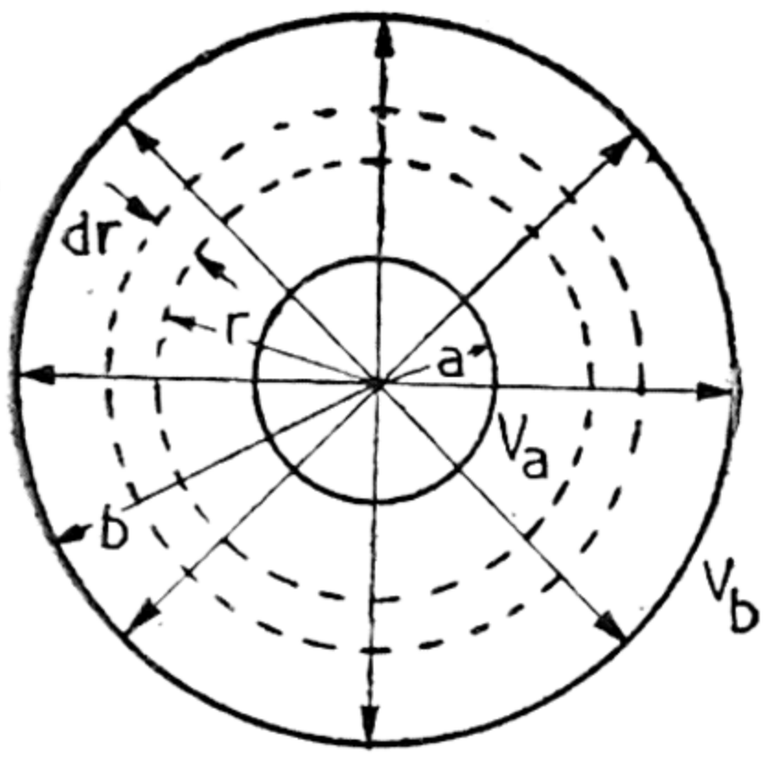


Fig. 6.4.

Let us consider a circular slab of conducting material of thickness t and conductivity σ. Consider the radial flow of current from some radius a at a potential V_a to the outer radius b at a potential V_b. Total current through any cross section will be same as that through any other cross section. Current through a cross section $\perp$ to the current flow at any radius r and thus of area $2\pi rt$ is given by

$$i = jA = \sigma AE = \sigma . 2\pi rt(-dV/dr),$$

or $$dV = -\frac{i}{2\pi t\sigma}\frac{dr}{r}.$$

$$\therefore \text{ Total potential difference } \int_{V_a}^{V_b} dV = \frac{-i}{2\pi t\sigma}\int_a^b \frac{dr}{r}$$

or $$V_a - V_b = \frac{i}{2\pi t\rho}\log_e \frac{b}{a}.$$

$$\therefore \text{ Resistance } R = \frac{V_a - V_b}{i} = \frac{1}{2\pi t\sigma}\log_e \frac{b}{a}. \qquad \ldots(25)$$

If the current is not $\perp$ to the cross section, we must use $i = \mathbf{j}.\mathbf{A}$ and the problem will be complicated.

6.5 STEADY CURRENTS IN MEDIA WITHOUT SOURCE OF EMF

Since charge can neither be piling up or disappearing at a boundary, the current towards the boundary on one side must be equal to that away from it on the other side. Thus

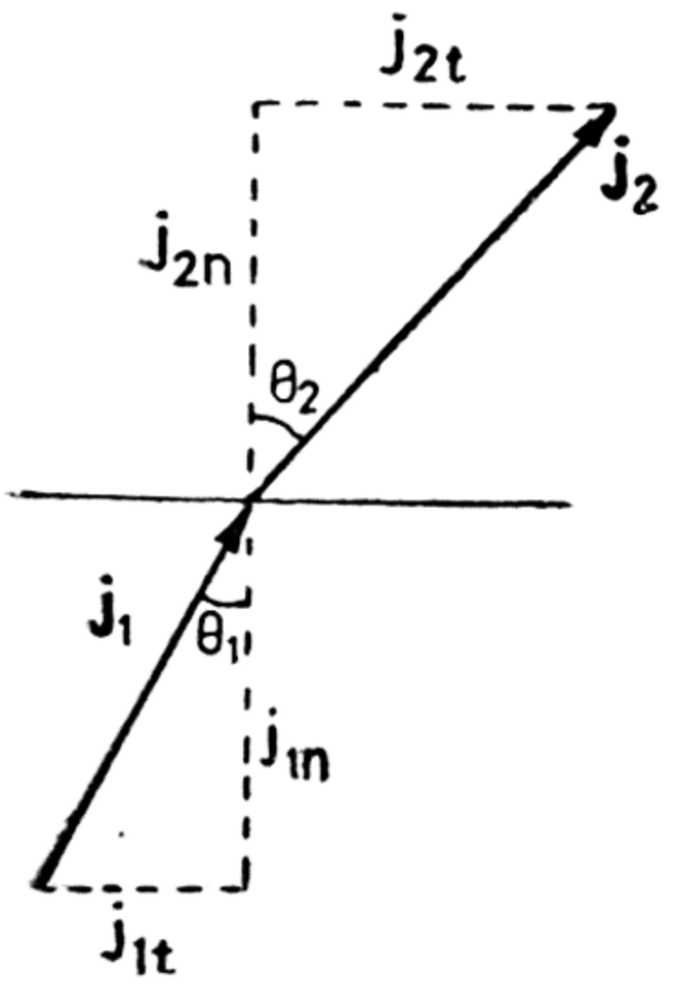


Fig. 6.5.

$$\mathbf{j}_1 . d\mathbf{S} = \mathbf{j}_2 . d\mathbf{S} \text{ or } j_{1n} = j_{2n}, \qquad \ldots(26)$$

or *the normal component of* $\mathbf{j}$ *must be continuous across the interface.* This relation can also be written as

$$\sigma_1 E_{1n} = \sigma_2 E_{2n}. \qquad \ldots(27)$$

This equation replaces the analogous equation for the continuity of D_n across the dielectric interfaces.

So long as there are no sources of emf in either medium, $\oint \mathbf{E} . d\mathbf{l} = 0$ for a closed path which links both media, therefore

$$E_1 t = E_2 t. \qquad \ldots(28)$$

The equation is analogous to the equation for the continuity of E_t in the case of dielectric problems. It can be written as

$$j_{1t}/\sigma_1 = j_{2t}/\sigma_2. \qquad \ldots(29)$$

Combining Eqs. (26) and (29), we get

$$\sigma_1 \cot \theta_1 = \sigma_2 \cot \theta_2.$$

It is the equation for the *law of refraction.* Thus we see that *at the boundary between two media of different conductivities the lines of flow are refracted in a manner analogous to that of the lines of force at the boundary between two dielectrics.*

A second example of the similarity between conductors and the dielectrics is the *resistance-capacitance analogy.* Consider two metallic conductors in a homogeneous ohmic medium of conductivity σ. If a potential difference V is applied between these conductors, then the current is given by

$$i = V/R,$$

where R is the resistance of the medium. Current i can be written in the form of current density $\mathbf{j}$ as

$$i = \int_S \mathbf{j}.\ d\mathbf{S} = \int_S \sigma \mathbf{E}.\ d\mathbf{S} = \sigma \int_S \mathbf{E}.\ d\mathbf{S}.$$

As by Gauss's law integral term is equal to q/ε, hence

$$V/R = \sigma q/\varepsilon \quad \text{or} \quad q = \varepsilon V/\sigma R,$$

where ε is the permittivity of the medium between the conductors.

As $q = CV$, hence

$$C = \varepsilon/\sigma R \quad \text{or} \quad RC = \varepsilon\sigma. \qquad \ldots(30)$$

It is the relation between the resistance of the medium and the capacitance of the equivalent electrostatic problem.

6.6 LORENTZ–DRUDE THEORY OF ELECTRICAL CONDUCTION AND OHM'S LAW

The theory of electrical conductivity in metals was developed by Paul Drude in 1900 and was modified by Lorentz. It is assumed that the valence electrons are not attached to individual atoms but are free to move about within the lattice like the molecules of a perfect gas in a container and are called the *conduction electrons.* As the effective radius of an atom is of the order of 10^{-8} cm, while that of an electron is of the order of 10^{-13} cm, we may neglect the collisions between electrons in comparision with the collisions between electrons and ion cores. The electrons and ion cores collisions are elastic random collisions. As in the case of molecular collisions in the gas, we can describe these collisions by a *mean free*

path λ, the average distance that an electron travels between collisions, and the *mean free time* τ, the average time between collisions. Because of the random nature of this motion, on the average no net charge is transferred in the absence of an external electric field. In the Fig. 6.6 the *dashed lines show the electron path in the presence of the electric field* **E**.

If now an electric field **E** is applied to the metal, the electrons are accelerated in the direction opposite to that of the field and acquire an average drift velocity **v**, anti-parallel to **E**. The ratio v/E, the drift velocity in unit field is known as the *mobility*. The drift velocity is small compared to the thermal velocities so that we can assume these two as independent to each other. The latter is determined by the structure of the conductor and the temperature but not by **E**. As the acceleration experienced by an electron is $-e\mathbf{E}/m$, hence the increase in drift velocity between two collisions will be $-\mathbf{E}\ e\tau/m$. The effect of each collision is to reduce the drift velocity to zero and to restore the random thermal motion. The drift velocity increases between two collisions and reduces to zero after next collision. Thus the average drift velocity over the time between two consecutive collisions will be $-\frac{1}{2}\mathbf{E}e\tau/m$.

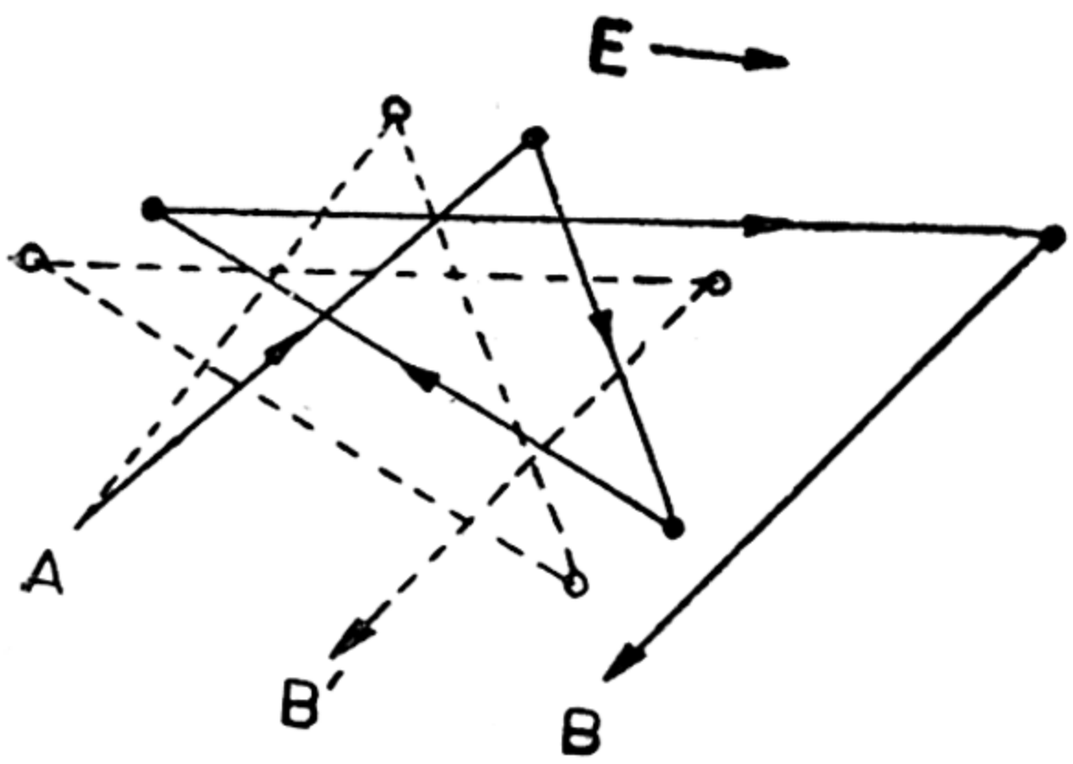


Fig. 6.6.

If n is the conduction electrons per unit volume, then the current density due to the drifting of electrons is given by

$$\mathbf{j} = -ne\mathbf{v} = \tfrac{1}{2}ne^2\tau\mathbf{E}/m. \qquad \ldots(31)$$

If v_{av} is the average thermal velocity of the electrons, then the time between two consecutive collisions will be λ/v_{av}. Thus Eq. (31) can be written as

$$\mathbf{j} = \tfrac{1}{2}ne^2\lambda\mathbf{E}/(mv_{av}), \qquad \ldots(32)$$

the resistivity $\rho = E/j = 2mv_{av}/ne^2\lambda, \qquad \ldots(33)$

and the conductivity $\sigma = 1/\rho = ne^2\lambda/(2mv_{av}). \qquad \ldots(34)$

This is an qualitative agreement with experiment. The quantities v_{av} and λ depend on the velocity distribution of the conduction electrons. Thus the right hand side is independent of **E** and is constant at constant temperature. It shows that metals obey Ohm's law.

Since v_d, the drift velocity ($\simeq 10^{-4}$ *m*/sec) is very small as compared to v($\simeq 10^6$ m/sec), hence we have used $v + v_d \approx v$ and

$$< | v+v_d | > \simeq <v> = v_{av}$$

in the above equations.

Ohm's law can directly be derived from the relation (32). At constant temperature λ and v_{av} are constants, hence

$$j \propto E \quad \text{or} \quad i/A \propto V/l$$

or

$$i \propto V, \qquad \ldots(35)$$

as for a given conductor l and A are fixed. This is the *Ohm's law.*

If the temperature is increased, v_{av} increases and thus the resistivity of a metal increases. In a semiconductor, the number of charge carriers per unit volume, n, increases rapidly with increasing temperature. This increase is greater than that in v_{av} and the resistivity thus decreases. At low temperature, n is very small and the resistivity becomes so large that the material can be considered an insulator.

Ohm's law does not hold good always. The vacuum diode is a good example of nonohmic device. When the supply of electrons is limited by their rate of emission from the cathode, the current is practically independent of the anode voltage, if the anode is positive. Under the condition of space charge limited current, the current in the diode is proportional to the $\frac{3}{2}$ power of the voltage.

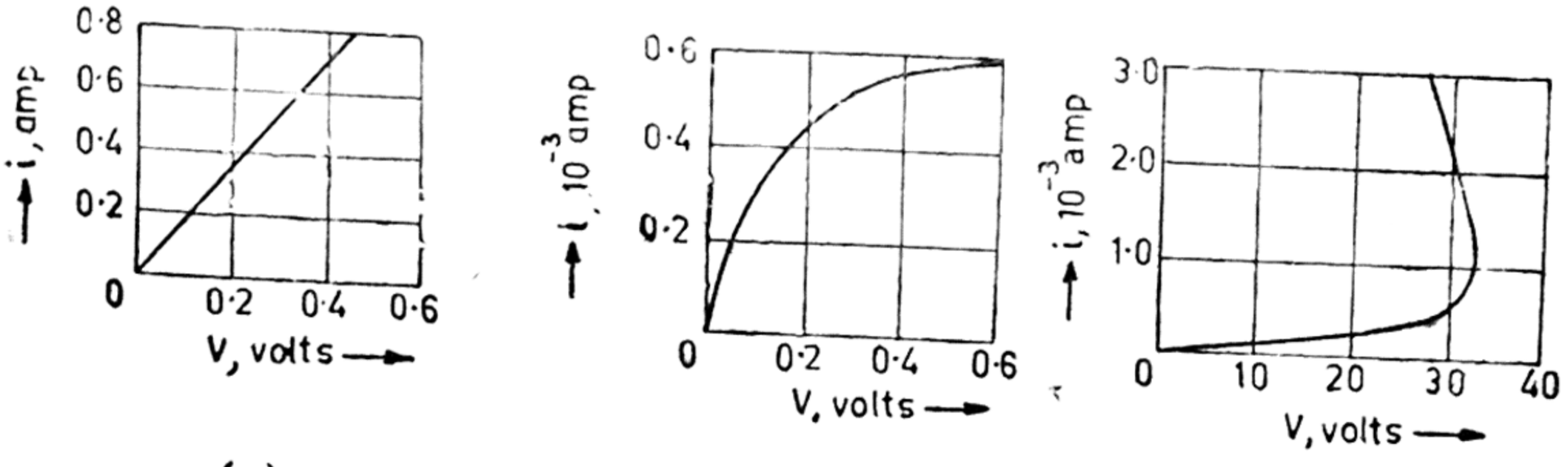


Fig. 6.7.

Fig. 6.7 (*a*) is a curve between i and V for a Cu-conductor. The straight line shows that the resistance of conductor is the same no matter what applied voltage is used to measure it. This result holds for metallic conductors and is known as Ohm's law. Vacuum tubes do not obey Ohm's law. The $V-i$ plot is not straight (Fig. 6.7*b*) and the resistance depends on the voltage applied. Fig. 6.7 (*c*) shows a $V-i$ plot for non-ohmic device, such as *thermistor.* The curve is of such a special type and can be accounted for in terms of the large—ve value of α of the material of which the device is made.

Wiedemann Franz's Law. Wiedemann and Franz gave an empirical law which states that the ratio of the thermal and electric conductivities is proportional to the absolute temperature and is the same at constant temperature for all metals. Later on Lorentz showed that

$$\frac{\text{Thermal conductivity } K}{\text{Electrical conductivity } \sigma} = \frac{\frac{3}{2}k^2T\lambda n/mv_{av}}{ne^2\lambda/2mv_{av}} = 3\left(\frac{k}{e}\right)^2 T$$

or
$$K/\sigma \propto T, \qquad \ldots(36)$$

where constant of proportionality, also known as Lorentz number $L=3\ (k/e)^2$, where k is the Boltzmann constant.

For a number of pure metals this ratio was found experimentally and found to fall within the range 2.3×10^{-8} to 3.2×10^{-8} watt-ohms/deg^2. These results are in fair agreement with the value of $3(k/e)^2$, *i.e.* 2.24×10^{-8} watt ohm/deg^2.

Inspite of the success of the free electron model to explain K and σ, it fails to explain the specific heat of metals. From this metal electron should have a thermal energy $\frac{3}{2}kT$ at temperature T and each electron should contribute to the specific heat of the metal.

6.7. RESIDUAL RESISTIVITY

As discussed in the preceding article, the resistivity ρ of the conductor is due to the scattering of free electrons within the conductor, known as *thermal scattering*. It is true in an ideal lattice only where the ions are arranged with a perfect regularity. In practice, some impurity ions and/or defects are always present. The potential function undergoes a sudden change in the neighbourhood of these impurities and causes an electron to scatter. It gives rise to additional resistivity called a residual resistivity ρ_R. *It is independent of temperature and depends only on the impurities and defects present in the metal hence is constant for a particular specimen.*

Total resistivity ρ = Resistivity due to thermal scattering $\rho_T + \rho_R$.

Hence at absolute zero or very low temperature $\rho = \rho_R$.

6.8. DUALITY BETWEEN j AND D

Since **j** and **D** both are linearly related to the electric field **E** in many materials, hence dual relationship between **j** and **D** exist. In the regions, where nonconservative fields are absent, we have

Conducting media	*Dielectric media*
$\mathbf{j}=\sigma\mathbf{E}$	$\mathbf{D}=\varepsilon\mathbf{E}$
$\nabla.\mathbf{j}=0$	$\nabla.\mathbf{D}=0$
$\nabla\times\mathbf{E}=0$	$\nabla\times\mathbf{D}=0$ ...(37)

In a homogeneous material where ε and σ are constant, we have

$$\nabla \times \mathbf{j}=0 \qquad\qquad \nabla \times \mathbf{D}=0$$

and hence $$\mathbf{j}=-\nabla\,\phi \qquad\qquad \mathbf{D}=-\nabla\phi, \qquad \ldots(38)$$

where ϕ, a potential function, satisfies Laplace's equation.

Above equations show that any solution of **j** can be transformed into a solution for **D** and vice versa, by interchanging as

$$\mathbf{j}\leftrightarrow\mathbf{D} \quad \text{and} \quad \sigma\leftrightarrow\varepsilon.$$

This is also true as we have similar type of relations for the boundary of two media.

Conducting media	*Dielectric media*
$j_{n1}=j_{n2}$	$D_{n1}=D_{n2}$
$j_{t1}/\sigma_1=j_{t2}/\sigma_2$	$D_{t1}/\varepsilon_1=D_{t2}/\varepsilon_2$. ...(39)

Relation (39) shows that a measurement of resistance R will suffice to determine the capacitance C.

6.9. SUPER CONDUCTORS

Ordinary conductors of electricity become better conductors as the temperature is lowered. It was thought that the resistance of a metal would be zero at absolute zero. Kamerlingh Onnes carried out measurements using mercury as his metal at very low temperatures and found that the metal was steadily approaching a very low resistance at zero as the temperature was being lowered with a sudden fall to zero at 4.2°K. The phenomenon was called superconductivity and the metal in that state as *super-conductor.* He succeeded later in showing that lead, tin and indium also become super conductors at 7.2, 3.7 and 3.4°K respectively. Certain alloys become superconductors at rather high temperatures.

The resistance of materials in the super-conducting state is zero and the currents once established in closed super-conducting circuits persist for weeks, even though there is no battery in the circuit. Such currents drop rapidly to zero if the temperature is raised slightly above the super conducting point. Since no heat is generated when a current flows through super conductors, hence are very much used, *e.g.*, in large electromagnets.

6.10. ENERGY DISSIPATION IN A RESISTANCE (*Joule's Law*)

When current passes through a conductor of finite resistance, charge moves from higher potential to lower potential and the work is continuously done by the electric field. For each coulomb of charge that moves through one volt of potential drop, one joule

of work is performed. If dq charge flows between two points differing in potential by V in a time dt, the power loss

$$P=Vdq/dt=Vi=VV/R=V^2/R=i^2R, \qquad ...(40)$$

where R is the resistance between the two points. The unit of power is volt-ampere known as the *watt* or *Joule/sec.* Eq. (40) represents three different forms of Joule's law.

In an extended medium of conductivity σ, the power dissipated in an element of cross section ΔS and length Δl is

$$\Delta P=\Delta V.\, j\Delta S=E\Delta l.\, j\Delta S.$$

or power dissipated/volume $=jE=\mathbf{j}.\mathbf{E}=\sigma\mathbf{E}.\mathbf{E}=\sigma E^2.$ ...(41)

This is the differential form of Joule's law.

If the device is a motor, the dissipated energy appears as mechanical work done by the motor. If the device is a storage battery this energy appears as stored chemical energy. If the device is a resistor energy appears as heat. The flow of electrons through the resistor is like that of the stone through water, where the potential energy mgh of the stone appears as thermal energy in the stone and the surrounding water. On a microscopic scale the transfer of dissipated electric energy in the form of heat can be understood as; *collisions between the electrons and the lattice of ions increase the amplitude of the thermal vibrations of the lattice and this corresponds to a tepmerature increase. This effect is often called the Joule heating and Eq. (40) is known as Joule's law.*

Let us now derive the energy formula for resistance. Consider a conductor with end surfaces S_1, S_2 kept at constant potentials V_1 and V_2. The current density $\mathbf{j}$ flows normal to these end faces and parallel to sides of the conductor. From Eqs. (40) and (41), we have

$$P=i^2R=\int\mathbf{E}.\mathbf{j}d\tau=\int\mathbf{j}.\mathbf{j}\,(1/\sigma)d\tau=\int\sigma(\mathbf{E}.\mathbf{E})d\tau.$$

$$\therefore \qquad R=\frac{1}{i^2}\int\mathbf{E}.\mathbf{j}d\tau=\frac{1}{i^2\sigma}\int\mathbf{j}.\mathbf{j}d\tau, \qquad ...(42)$$

where $d\tau$ is the elementary volume of the conductor.

Let $$\mathbf{E}=-\nabla V$$

$$\mathbf{E}.\mathbf{j}=(-\nabla V).\mathbf{j}=-\nabla.\,(V\mathbf{j})$$

Since $\nabla.\mathbf{j}=0$, hence upon application of the divergence theorem equation becomes

$$-\int\nabla.(V\mathbf{j})d\tau=-\int_s V\,\mathbf{j}.\mathbf{n}\,dS, \qquad ...(43)$$

where S is the total area of the conductor. Since $\mathbf{j}.\mathbf{n}=0$ except on the constant potential surfaces, hence we have

$$i^2R=V_1\int_{S_1}\mathbf{j}.\,(-\mathbf{n})dS-V_2\int_{S_2}\mathbf{j}.\mathbf{n}\,dS=(V_1-V_2)i.$$

It is same as Eq. (40).

6.11. ELECTROMOTIVE FORCE

We know that the closed line integral of electrostatic field $\oint \mathbf{E}.d\mathbf{l}=0$, hence electrostatic field can not cause current flow in the circuit. The source of energy in a circuit may be chemical, mechanical or thermal, or it may result from a charging magnetic field. In an electrolytic cell, the energy is chemical and in a Van de-Graaff generator the energy is mechanical. In a thermocouple, a thermo *emf* is produced when the junctions of a circuit consisting of dissimilar metals are at different temperatures and the source of energy is thermal. In a photovoltaic cell, the photo *emf* is produced when the junction ($p-n$) is irradiated with light, infrared rays or other radiant energy.

In a device such as a dry cell, storage battery, or electrostatic generator, in which nonelectrical energy is converted into electrical energy or *vice versa*, let us consider that the terminal *a* is maintained by the source at a higher potential than terminal *b*. There is therefore an electrostatic field $\mathbf{E}_e$ at all points between and around the terminals. The source is itself a conductor and if only this electrostatic field exerts the positive charges will move from *a* toward *b* and the negative charges from *b* toward *a*. The excess chargeson the terminals wouldddecrease and hence the *pd* between them would decrease and eventually become zero. As it does not happen, we conclude that there must also exit at every point within the source *a* force $\mathbf{F}_n$ of non-electrostatic origin acting on every charged particle and which is equal and opposite to the electrostatic force $\mathbf{F}_e=q\mathbf{E}_e$. We can define the non-electrostatic field $\mathbf{E}_n$ as $\mathbf{F}_n=q\mathbf{E}_n$. When the source is an open circuit, the vector sum of these fields at every point must be zero, *i.e.*

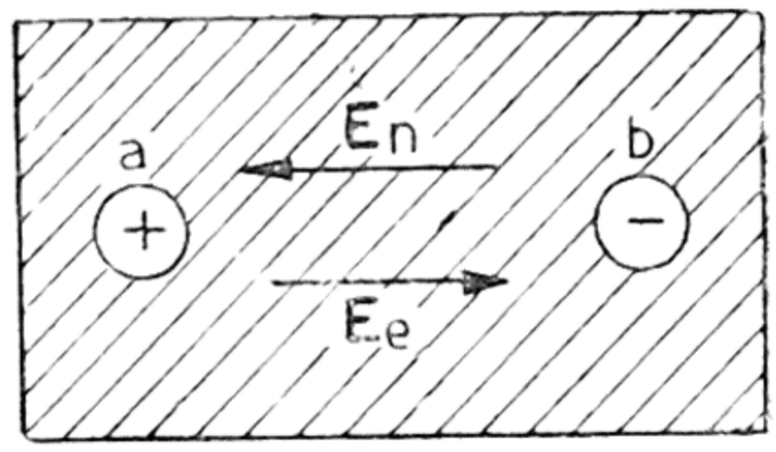


Fig. 6.8.

$$\mathbf{E}=\mathbf{E}_n+\mathbf{E}_e=0 \quad \text{or} \quad \mathbf{E}_e=-\mathbf{E}_n. \qquad \ldots(44)$$

$$\therefore \quad \int_a^b \mathbf{E}_e.d\mathbf{l}=-\int_a^b \mathbf{E}_n.d\mathbf{l}=\int_b^a \mathbf{E}_n.d\mathbf{l}.$$

$$\text{As} \quad \int_a^b \mathbf{E}_e.\, d\mathbf{l}=V_{ab}, \quad \therefore \quad \int_b^a \mathbf{E}_n.d\mathbf{l}=V_{ab}. \qquad \ldots(45)$$

The line integral of the non-electrostatic field from *b* to *a* is called the *electromotive force* of the source and is abbreviated as *emf*. Thus *the emf of the source equals the open circuit potential difference*. We should not confuse between terms *potential difference* and *emf*, as the former is the line integral of an electrostatic field and the latter is the integral of a non-electrostatic field. The

electrostatic field, hence the potential difference depends on the current in the source. The non-electrostatic field, hence the *emf* of the source is a constant independent of the current and this represents the property of the source.

For a closed circuit, *i.e.* when two terminals of a source are connected by a conductor, the resultant field **E** within the source is $\mathbf{E}=\mathbf{E}_e+\mathbf{E}_n$, hence

$$\int_b^a \mathbf{E}.d\mathbf{l}=\int_b^a \mathbf{E}_e.d\mathbf{l}+\int_b^a \mathbf{E}_n.d\mathbf{l}$$

or
$$Ir=-V_{ab}+\mathcal{E}, \qquad \ldots(46)$$

where r is the internal resistance of the source. Here the direction of integration is the direction of current within the source.

If R is resistance of the conductor connecting the terminals a and b, then Eq. (46) can be written as

$$Ir=-IR+\mathcal{E} \quad \text{or} \quad I=\mathcal{E}/(R+r). \qquad \ldots(47)$$

If a circuit consists of any number of sources and conductors connected in series, then the above relation can be written as

$$I=\Sigma\mathcal{E}/\Sigma R, \qquad \ldots(48)$$

where $\Sigma\mathcal{E}$ is the algebraic sum of emf's of the sources and ΣR is the sum of all resistances, external and internal.

6.12. RESISTORS

Resistors may be divided into two groups : resistance boxes and rheostats. The former is used when a definite and known resistance is required and the latter is used when we are concerned with regulation of currents and absorption of power.

In **resistance boxes** coils are connected in various combinations so that any required resistance upto the full capacity of the box may be obtained. The most common types are the *plug box* and the *dial box*. The plug box consists of a number of resistance elements of different magnitudes connected in series. Each junction between elements is connected to a metal block, so that if a metal plug is inserted between adjacent blocks the corresponding resistance is short circuited. Thus the total resistance of the box in circuit is the sum of the resistances whose plugs have been removed. Resistance material used for winding coils is manganin, which is an alloy 84% *Cu*+12% *Mn*+4% *Ni*. It has smaller temperature coefficients and higher resistivity. The dial box also consists of a number of resistance elements in series, the junctions being connected to a set of metal buttons over which a contact arm moves,

Usually in dial boxes the elements are arranged in groups of ten of equal magnitude, successive groups however differing in magnitude by a factor of ten.

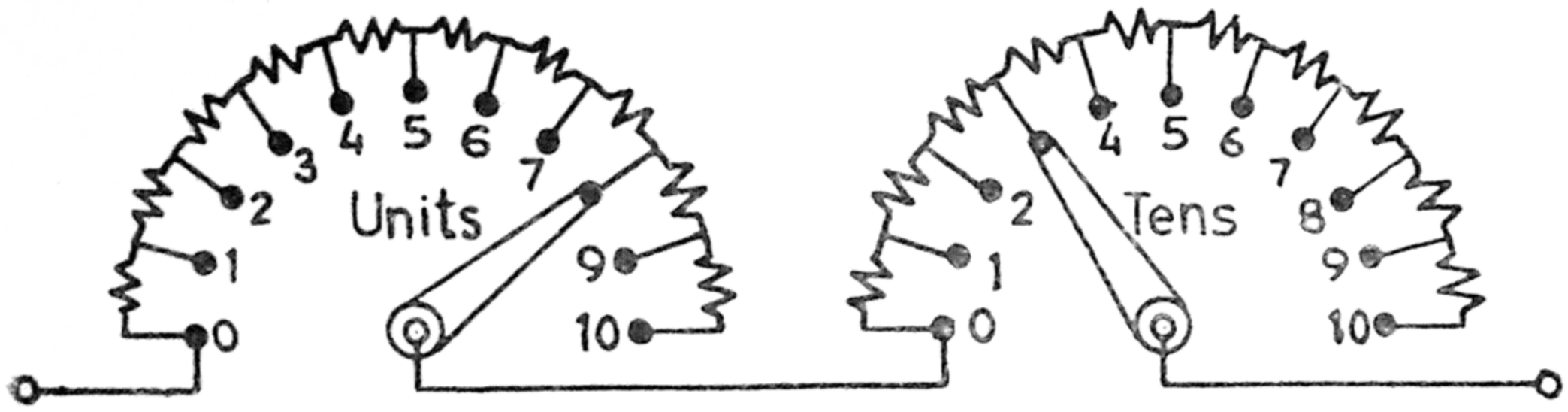


Fig. 6.9.

A **sliding rheostat** usually consists of oxidised nichrome wire (an alloy of *Ni*, *Cr* and *Cu*) wound on an insulating cylinder. A sliding contact is arranged on a metal rod mounted above the winding, parallel to the cylinder. This alloy provides high resistivity. It is taken oxidised to provide insulation between turns. Besides the sliding contact terminal, two terminals are provided at the ends of the wire. These both terminals are used only when the rheostat is used as a potential divider.

In electronic circuits resistors of fixed resistances are required. Most resistors are marked with the normal value of their resistance. The present practice is either to print the value at the body of the resistor in numbers or to put three coloured bands around the body which can be translated into a number by using the standard colour code, given below :

Black	0	Red	2	Yellow	4	Blue	6	Gray	8
Brown	1	Orange	3	Green	5	Violet	7	White	9

The first colour band represents the first figure of the resistance and the second band the second figure and the third band the number of zeros to add the first two figures to get the total resistance. For resistances between 1 and 10 ohms, gold is used for the third band. Thus green, orange and gold bands represent 5.3 ohms. The limit of accuracy is indicated by the colour of the fourth bands, silver being 10% and gold being 5%. Ordinary carbon resistors should not be used when the signal level is 10^{-3} volt or less.

Carbon Resistors. As wire wound resistors are expensive, not suitable to use at very high frequencies and can not be obtained in small size if are of high resistance values. The best alternatives are deposited carbon and carbon-composition rod resistors.

(*a*) *Deposited-carbon Resistors.* The deposited-carbon resistor also known as carbon film resistor consists of a ceramic core on which a thin layer of crystalline metallic carbon is deposited from a hydrocarbon vapour at high temperature. These resistors are much cheaper, nearly stable, having temperature coefficient of

resistivity of the order of 200 to 500 parts per million per °C, very small noise and have superior performance at radio frequencies.

(*b*) *Carbon-composition Resistors.* (*i*) *The carbon-composition-rod type resistor* consists of a solid conducting rod formed of a mixture of fine carbon particles and an insulating binding medium. Although these resistors are widely used in electronic equipments, but are not suitable for precision work because of their poor stability, high value of α, poor performance at high frequencies. (*ii*) *Carbon-composition film type resistor* consists of an insulating support such as a ceramic or glass rod or tube at the surface of which a conducting film of a mixture of carbon particles and insulating binding medium is provided. These are frequently called *metallized resistors.* As the value of α, stability with time and performance at high frequencies are concerned, these are superior to those of carbon rod type resistors. Although these are somewhat inferior to those of the deposited carbon resistors.

Noise. When *dc* is passed through carbon resistor, a noise voltage is developed across its terminals. It is almost proportional to the *dc* voltage applied to the resistor. It varies inversely as the frequency. Different types of resistors differ in the noise that they generate. No noise is observed in wire wound resistances, deposited carbon resistors have the lowest noise ($<0.25\ \mu V$ noise per *dc* volt), carbon rod types have the highest noise and the carbon film resistors have in between two and can thus be assumed as good resistors.

6.13. KIRCHHOFF'S LAWS

Most electrical circuits comprise a number of sources and resistors interconnected in a more or less complicated manner. These are named as networks. All networks can not be reduced to simple series-parallel combinations. There are number of techniques to solve such problems. We shall describe here only one of these, first developed by Gustav Robert Kirchhoff. Let us first define three terms : a *branch point* in a network is a point where three or more conductors are joined, *loop* is any close conducting path and *mesh* is a closed path (loop) which can not be subdivided into other loops. With the restriction that Ohm's law obeys for each part of the network, Kirchhoff gave following two laws.

(1) *The algebraic sum of all the currents meeting at a branch point is zero.* It follows from the equation of continuity, since there can be no accumulation of charge at any point in the circuit. It can be written as

$$\Sigma i = 0. \qquad \ldots(49)$$

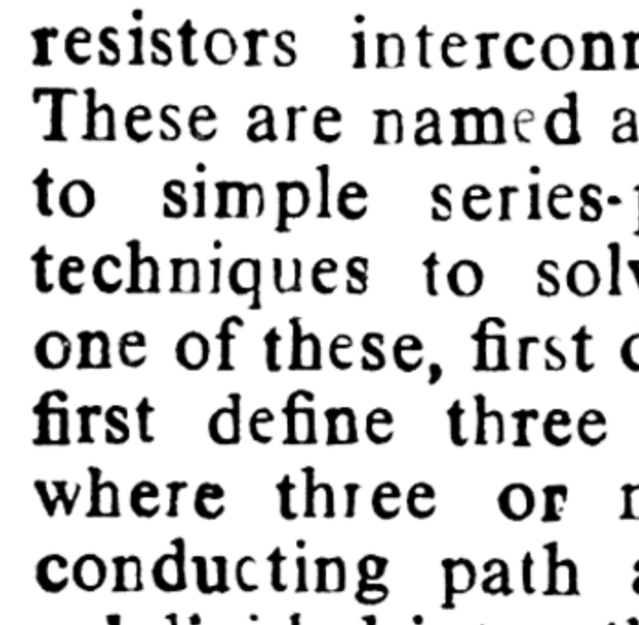


Fig. 6.10.

Hence the net current meeting at any branch point (Fig. 6.10) can be given as

$$i_1 + i_2 - i_3 + i_4 - i_5 = 0. \qquad \ldots(50)$$

Here the convention is that the current flowing towards the branch point or the junction is considered positive, whereas outflowing current is considered negative.

(2) *In any loop in a network of conductors, the algebraic sum of emf's is equal to the algebraic sum of the products of the resistances of each link of the loop and the current flowing through them.* It follows that the total fall of potential round any simple circuit is equal to the total emfs in that circuit. It can be written as

$$\Sigma iR = \Sigma \mathcal{E}. \qquad \ldots(51)$$

Before applying this law, the direction of current in the conductors of the loop is assumed, and the product iR is given a positive sign if the current flowing through the resistance R is along the assumed current direction and the emf $\mathcal{E}$ is given a positive sign if the assumed current direction is from $-$ve to the $+$ve terminal inside the source.

6.14. WHEATSTONE'S BRIDGE

The Wheatstone's bridge is a special type of resistance network, commonly used for comparing resistances. It consists of four resistances R_1, R_2, R_3 and R_4 connected in series. The one pair of opposite junctions of the four resistances, such as A and C being connected by battery of emf $\mathcal{E}$ and internal resistance r and the other pair of opposite junctions B and D through a galvanometer of resistance R_G.

Let i, i_1, i_2, i_3, i_4 and i_g be the currents in the battery circuit, through resistances R_1, R_2, R_3, R_4 and the galvonometer respectively. Then by Kirchhoff's first law, we have *for*

the branch point A $\quad i - i_1 - i_3 = 0$,

,, ,, ,, B $\quad i_1 - i_2 - i_g = 0$,

,, ,, ,, D $\quad i_3 + i_g - i_4 = 0$.

Thus the currents in AB, BC, AD and DC will be i_1, $i_1 - i_g$, $i - i_1$ and $i - i_1 + i_g$ respectively. Assuming the clockwise direction of the current as positive in the meshes, and now applying Kirchhoff's second law to the meshes $ABDA$ and $BCDB$, we get

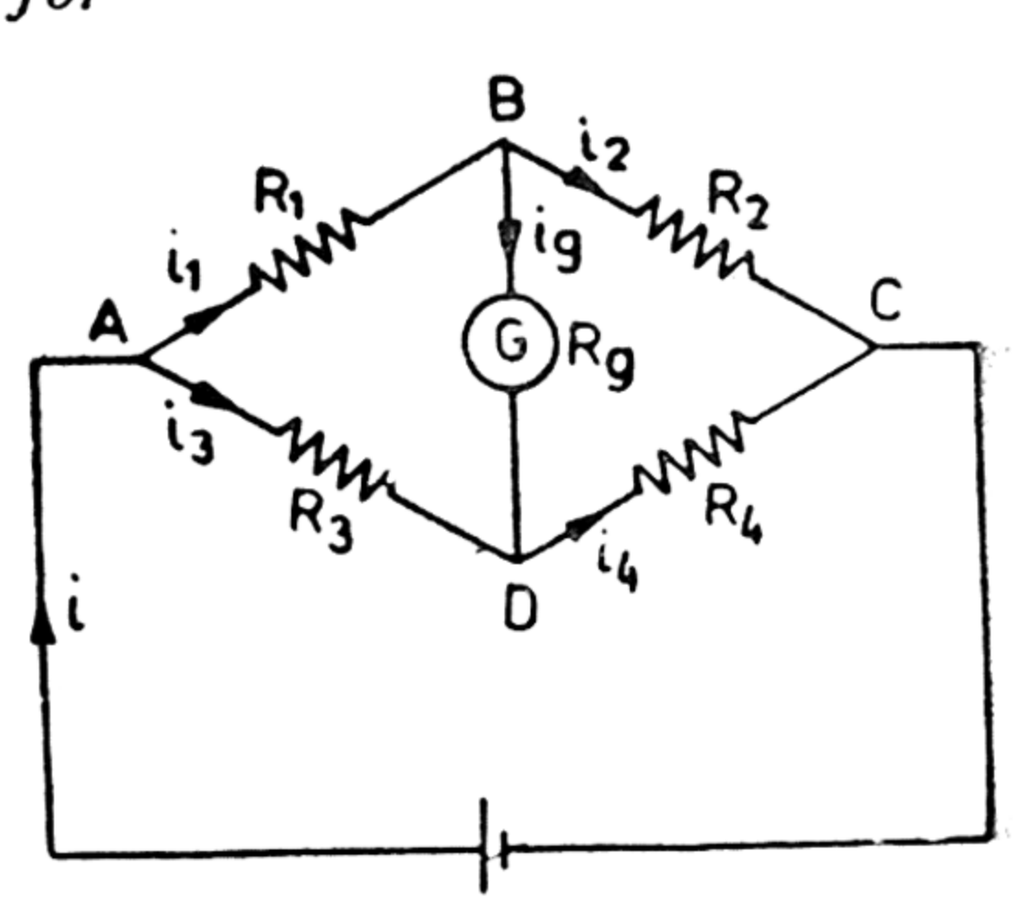


Fig. 6.11.

$$i_1 R_1 + i_g R_G - (i - i_1) R_3 = 0$$

and $$(i_1 - i_g) R_2 - (i - i_1 + i_g) R_4 - i_g R_G = 0.$$

If the bridge is balanced, *i.e.*, no current is flowing through the galvanometer ($i_g = 0$), then we get

$$i_1 (R_1 + R_3) = i R_3$$

and $$i_1(R_2+R_4)=iR_4.$$

$$\therefore \quad \frac{R_1+R_3}{R_2+R_4}=\frac{R_3}{R_4} \quad \text{or} \quad \frac{R_1}{R_2}=\frac{R_3}{R_4}, \qquad ...(52)$$

which is the well known condition for the bridge to be balanced.

Thus two resistances can be compared or an unknown resistance R_4 can be measured in terms of three known resistances R_1, R_2 and R_3. Although this relation is independent of the current in the battery circuit, but the currents must not be large enough to heat the resistances and so alter their values. It is null method, thus does not require any calibration of galvanometer. It can be shown that when the positions of the galvanometer and battery are interchanged the above condition for no deflection of the galvanometer will still hold. These arms are known as 'battery arm' and the 'galvanometer arm', and are known as conjugate to each other when the bridge is balanced. Key's in these arms are known as battery key and galvanometer key respectively.

Sensitivity of a Wheatstone's bridge. The accuracy in the measurement of resistances depends upon the sensitiveness of the bridge. For a better sensitivity of the bridge the resistances in the bridge arms should be so adjusted that the deflection in galvanometer is large even for small deviation from the balance position.

We shall follow a treatment given by Callendar for the conditions of the bridge sensitiveness. The resistances in the four arms of the unbalanced bridge are written in the form $R_4=r$ (unknown), $R_2=mr_1$, $R_3=nr_1$ and $R_1=nmr_1$, where m and n are simple numbers. When $r=r_1$, the bridge is balanced and $i_g=0$.

If i be the current through the resistance r to be measured when the bridge is not balanced and i_g be the current through the galvanometer of resistance R_G, then from Kirchhoff's first law, the current through resistance nr_1 will be $i+i_g$. If i_1 is the current through the resistance nmr_1, then the current through the resistance mr_1 will be i_1+i_g.

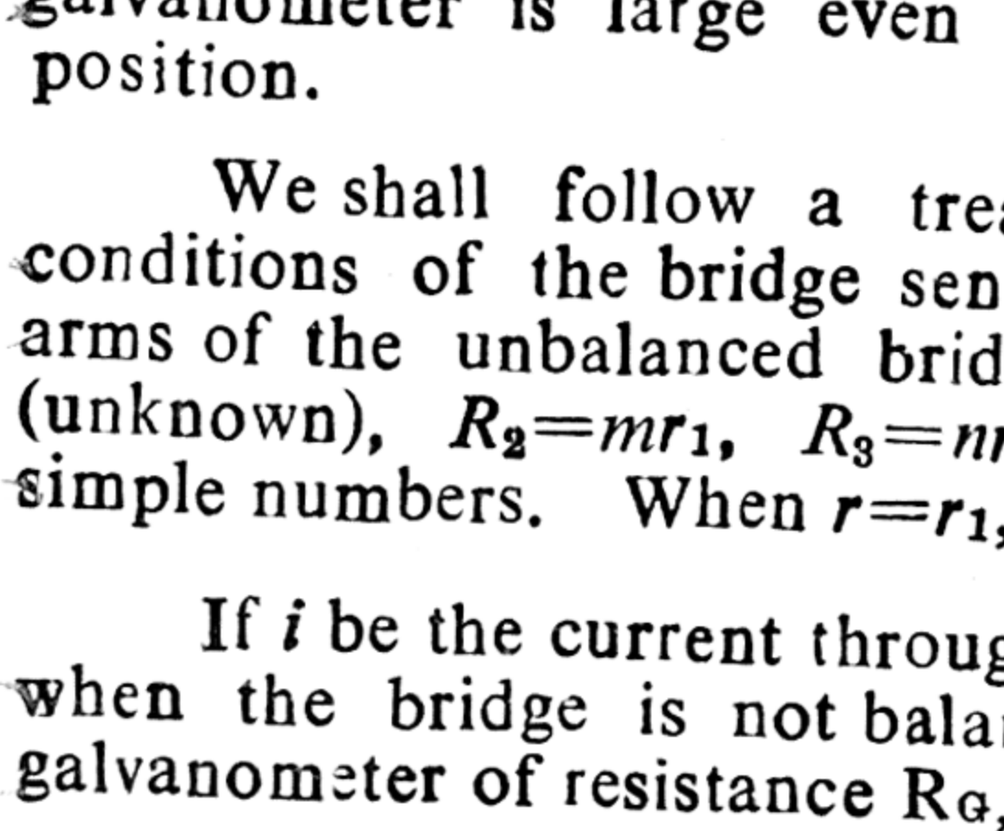


Fig. 6.12.

To solve the problem, let us assume the current in the clockwise direction in the meshes $ABDA$ and $BCDB$ as positive. Now applying Kirchhoff's second law to these meshes, we have

$$nmr_1i_1-R_Gi_g-nr_1(i+i_g)=0$$
$$\text{and} \quad mr_1(i_1+i_g)-ri+R_Gi_g=0.$$

Rearranging these relations and eliminating i_1 from these, we get

$$\frac{i_g}{i} = \frac{n(r-r_1)}{(n+1)\,R_G+n(1+m)r_1}$$

$$= \frac{r-r_1}{(1+m)r_1+(1+1/n)R_G}. \qquad \ldots(53)$$

The ratio i_g/i is a measure of the sensitiveness of the bridge. It is independent of the emf and resistance in the battery circuit. For a given want of balance $r-r_1$, the sensitivity can be increased by (*i*) increasing current i, (*ii*) decreasing galvanometer resistance R_G, (*iii*) decreasing m and (*iv*) increasing n.

The current i through the unknown resistance is limited by the heating effect of this current. Galvanometer resistance can not be decreased below a certain limit as it varies with the number of turns in the coil. Accuracy of the galvanometer is proportional to the number of turns in coil. Thus the decrease in R_G is possible at the cost of accuracy of the galvanometer.

Thus we see that the sensitivity of the bridge can only be increased by decreasing m and increasing n to any extent. The limiting values are $m=0$ and $n=\infty$, in which case

$$\frac{i_g}{i} = \frac{r-r_1}{r_1+R_g}. \qquad \ldots(54)$$

This condition is impracticable because if $n=\infty$, the current in the bridge would be zero, as resistances in arms AB and AD are both infinite ; and if $m=0$, the branch ADC and the galvanometer arm are short circuited. If $m=n=1$, we have

$$\frac{i_g}{i} = \frac{r-r_1}{2(r_1+R_G)}. \qquad \ldots(55)$$

Thus we see that when $m=n=1$, the sensitiveness it still half as great as in the ideal limit ($m=0$ and $n=\infty$). The sensitiveness can be increased by increasing n and decreasing m. It will be greatly reduced if n is made small and m large. Prof. Callendar gave as a rule *that the battery and galvanometer should be connected in such a way that the resistance in series with the unknown resistance be greater than the resistance in parrallel with it.*

As we know that the positions of the galvanometer and the battery can be interchanged but these two arrangements are not equally sensitive. Maxwell gave a rule regarding the best arrangements of these—*of the two resistances that of the battery and that of the galvanometer, the greater connects the junction of the two greatest with that of the two least of the four arms of the bridge.*

6.15. COMPARISON OF TWO RESISTANCES

The Wheatstone's bridge method is the most accurate and frequently used method of comparing resistances. The great advantage in this method is the null detection. Most simple practical application of this bridge is the *meter bridge*, which consists of a straight and uniform wire of manganin or constantan stretched along a meter scale. The balance is obtained on the wire, if lengths of the two parts are l_1 and l_2, then the ratio of two resistances

$$R_1/R_2 = l_1/l_2. \qquad \ldots(56)$$

A compact form of Wheatstone's bridge is a *Post Office box*, originally designed to measure the resistances of electric cables and telegraph wires in Post Offices. It consists of three separate sets of coils of known resistances to form three arms *AB*, *BC* and *AD* of the wheatstone bridge. Two exactly alike and similary placed sets *P* and *Q* are known as *ratio arms* of the bridge. In *P* and *Q*, usually each consists of three resistances of 10, 100, 1000 ohms respectively. The third set of resistance *R* forms the *rheostat arm*, consists of a series of coils of values from 1 to 5000 ohms. The unknown resistance *S* is connected in the fourth arm *CD*. Two spring keys K_1, K_2 are also provided, which when pressed make internal connections with terminals *A* and *B* respectively. Key K_1 is known as *battery key* and key K_2 as the *galvanometer key*.

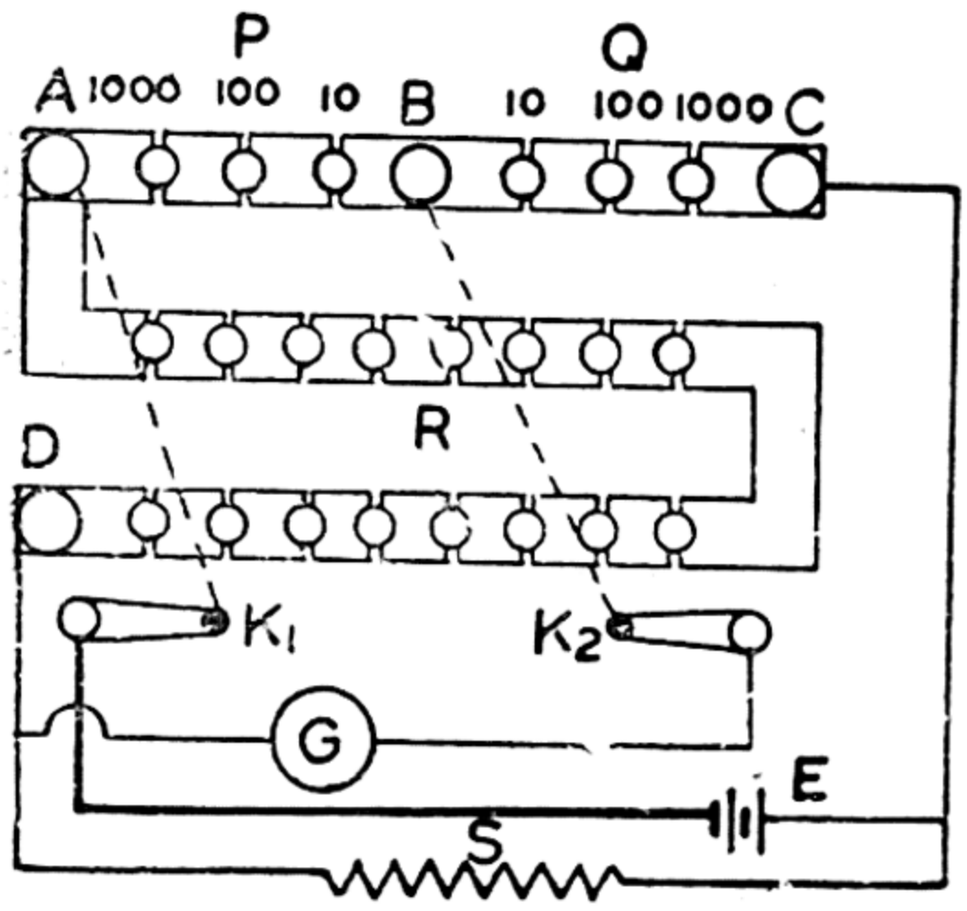


Fig. 6.13.

To determine the unknown resistances, the connections are made as shown in Fig. 6.13. The ratio arms are adjusted to 10 ohms each and rheostat arm is set to 1 ohm. The battery key is first pressed then the galvanometer key is pressed and the direction of the deflection in the galvanometer is noted. Direction of the deflection is also noted for the resistance 2000 from the rheostat arm. If the deflection is in the direction opposite to the previous one the unknown resistance must have value some between 1 and 2000. If two deflections are in same directions, there is either some faulty connection or the value of *S* exceeds 2000. By varying the resistance in the rheostat arm systematically, two resistances, differing by one ohm, with which the deflections are in opposite directions, are found by trial. Now $P=Q$, thus the unknown resistance must lie between these two values, say n and $n+1$.

Now *P* is increased to 100 ohms keeping $Q=10$ ohms, by varying resistance *R*, two values are similarly obtained, which give deflection in opposite directions. Using Eq. (52), the unknown

resistance will be between $n_1/10$ and $(n_1+1)/10$ ohms. Similarly the accuracy can be more increased by taking $P=1000$ ohms.

For the measurements of very high resistance greater than the sum of all resistances in the rheostat arm, Q is increased by keeping $P=10$ ohms.

6.16. KELVIN'S (THOMSON) METHOD OF DETERMINING GALVANOMETER RESISTANCE

This is the Wheatstone bridge method in which galvanometer is removed from galvanometer arm and the balance position is obtained by the galvanometer, whose resistance is to be measured and connected in the fourth unknown arm. The connections are made as shown in Fig. 6.14. Using Kirchhoff's first law, current distribution among the branches is obtained, also shown in figure. Now applying Kirchhoff's second law to the meshes $ABDA$ and $BCDB$ and assuming clockwise direction of the current as positive in these meshes, we get

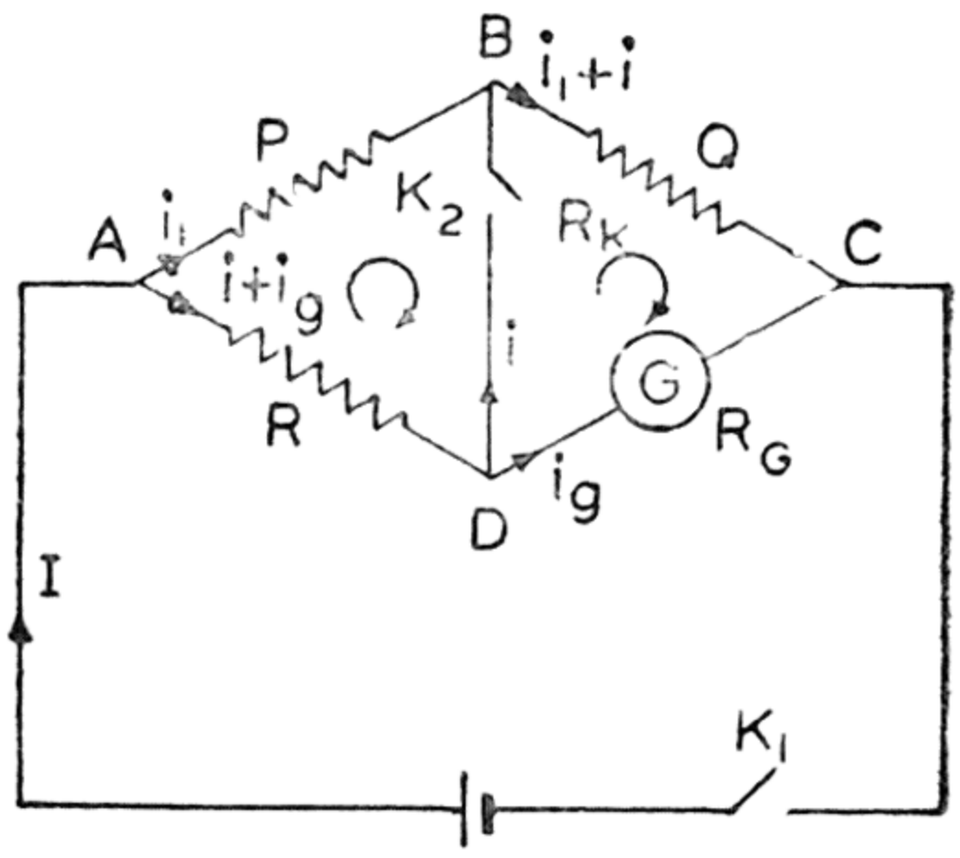


Fig. 6.14.

$$i_1P-iR_k-(i+i_g)R=0$$

and

$$(i_1+i)Q-i_gR_G+iR_k=0.$$

If $i=0$, *i.e.*, the current i_g through the galvanometer is independent of the resistance R_k or remains constant whether the key K_2 is closed or open. Then above relations reduce to

$$i_1P=i_gR \text{ and } i_1Q=i_gR_G,$$

or

$$P/Q=R/R_G. \qquad ...(57)$$

Post Office box is used generally for the measurement of R_G. Galvanometer is connected across the unknown arm CD and the galvanometer arm BD is short circuited by connecting a thick Cu—wire. As soon as we press battery key K_1 the current will flow through the galvanometer and will give large deflection, particularly when the galvanometer is sensitive. Hence to check this large deflection a variable high resistance, say a resistance box adjustable up to 10,000 ohms or a U-tube containing water should be inserted in series with the cell. As clear from the theory given above, relation (57) is valid when i_g, the galvanometer current, is independent of i. Thus by taking any ratio from ratio arms the resistance should be adjusted from the rheosat arm for which deflection of galvanometer does not change when the galvanometer key K_2 is pressed. Knowing P, Q and R the resistance of galvanometer R_G can be calculated.

6.17. MANCE's METHOD OF DETERMINING CELL RESISTANCE

In this method, the cell, resistance of which is to be measured, is connected in the fourth arm of the Wheatstone's bridge and the battery arm is short circuited by connecting a thick Cu-wire. The galvanometer is connected as usual. Applying first Kirchhoff's law the current distribution so obtained is shown in Fig. 6.15. Now applying Kirchhoff's second law to the meshes *ABDA* and *BCDB* and assuming the clockwise direction of current in these meshes as positive we get

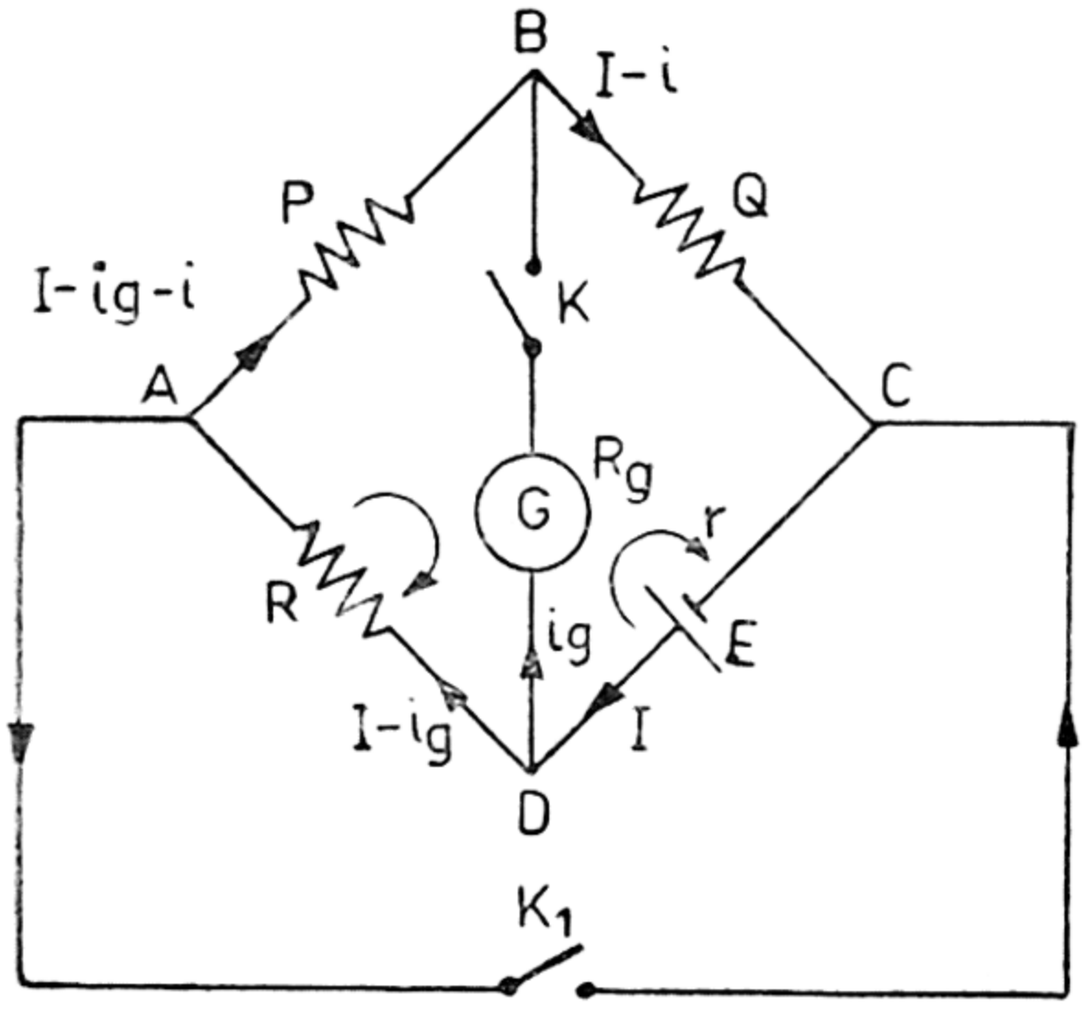


Fig. 6.15.

$$(I-i_g-i)P-i_g R_G+(I-i_g) R=0 \qquad ...(i)$$

and

$$(I-i) Q+Ir+i_g R_G=E. \qquad ...(ii)$$

Rearranging, we get

$$I(P+R)-i_g (R_G+R+P)-iP=0 \qquad ...(iii)$$

and

$$I(Q+r)+i_g R_G-iQ=E. \qquad ...(iv)$$

Multiplying Eq. (*iii*) by $(Q+r)$ and Eq. (*iv*) by $(P+R)$ and then subtracting, we get

$$i_g[R_G(P+R)+(R_G+P+R)(Q+r)]-i[Q(P+R)-P(Q+r)] = E(P+R).$$

Since i is not zero, hence the galvanometer current i_g will be independent of the resistance in the battery arm AC, or remains constant whether the battery key K_1 is closed or open when the coefficient of i is zero, *i.e.*

$$Q(P+R)-P(Q+r)=0.$$

$$\therefore \qquad QR=Pr \quad \text{or} \quad P/Q=R/r. \qquad ...(58)$$

To check high current through the galvanometer, variable high resistance, say a resistance box of 10, 100 Ω or a U-tube containing water, should be placed in series with the galvanometer. In this experiment ratio P and Q is first taken as 10 : 10 and the resistances from rheostat arm R is so adjusted that the the deflection obtained by pressing galvanometer key K_2 does not change while pressing battery key K_1. Now P, Q and R should be noted and r can be calculated.

If the cell resistance is less than 1 ohm as is often the case with a freshly prepared cell, keep the ratio $P/Q=10$ and adjust R and take observations as given before.

6.18. CAREY FOSTER'S BRIDGE FOR MEASURING RESISTANCE

It is a modified form of meter bridge (one meter wire), in which effective length of the wire is considerably increased by including resistances R_1 and R_2 in series with the bridge wire LM. Thus increases the accuracy of the result due to any error in reading the position of the balance, as well as the sensitiveness of the bridge. It is very useful for the comparison of two nearly equal resistances, whose difference is less than the resistance of the bridge wire LM. Two resistances X and Y to be compared are connected in the outer gaps of the bridge in series with the wire LM. Thus we have two arms of Wheatstone's bridge, one AD and other CD, where D is the position of null point on the wire. The remaining

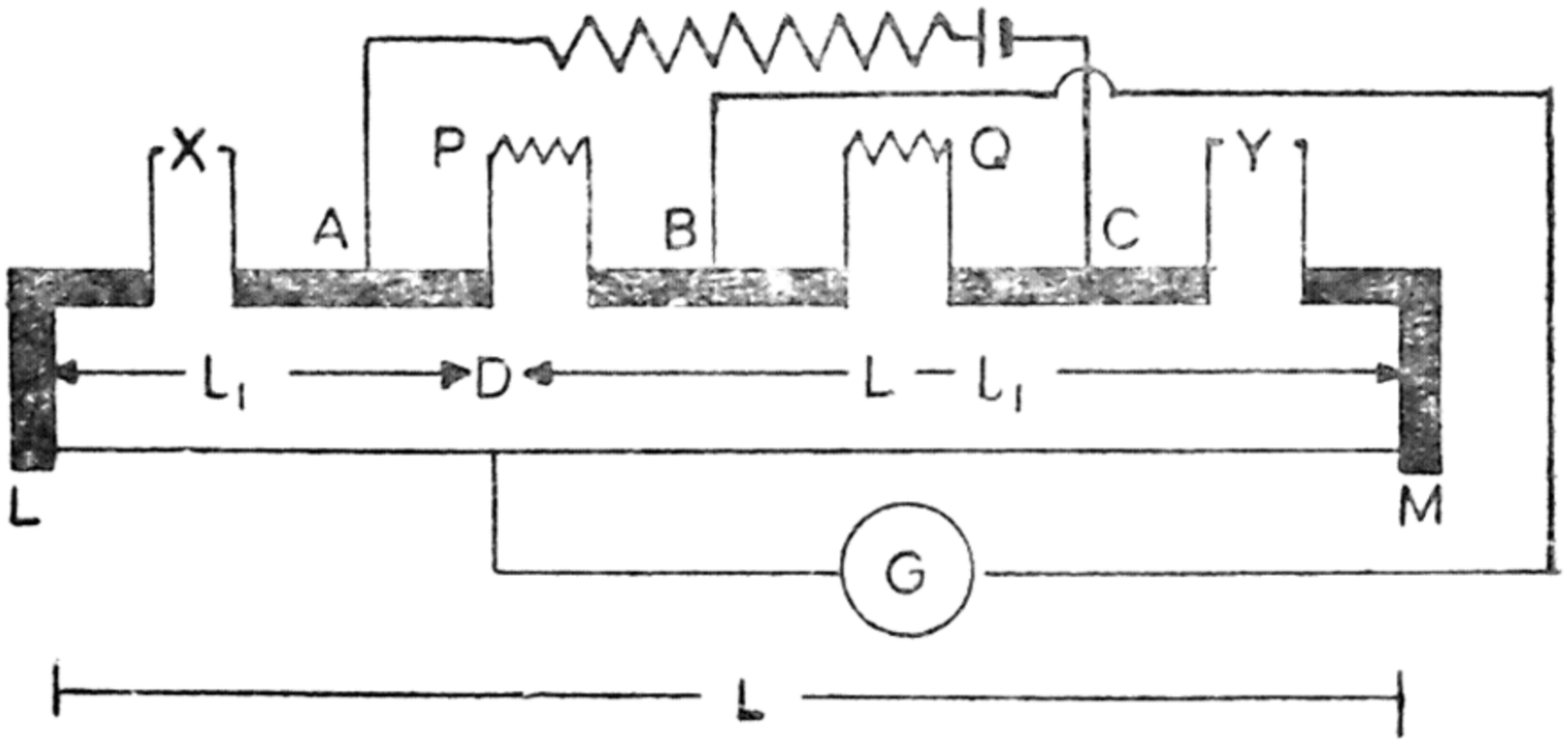


Fig. 6.16.

two arms are formed by two nearly equal resistances P and Q, connected in the inner gaps of the bridge. The cell and galvanometer are connected as indicated in Fig. 6.16. If the wire is of length L and the position of null point is at a distance l_1 from the left hand side, then at the time of balance we have

$$\frac{P}{Q}=\frac{R_1+\rho l_1}{R_2+\rho(L-l_1)}=\frac{X+\alpha+\rho l_1}{Y+\beta+\rho(L-l_1)},$$

where ρ is the resistance per unit length of the bridge wire and α and β are the end corrections at the left and right ends of the bridge wire respectively.

If X and Y are interchanged and the balance point is now obtained at a distance l_2 from the left end on the wire, we have

$$\frac{P}{Q}=\frac{Y+\alpha+\rho l_2}{X+\beta+\rho(L-l_2)}.$$

Comparing these equations, we get

$$\frac{X+\rho l_1+\alpha}{Y+\rho(L-l_1)+\beta}=\frac{Y+\rho l_2+\alpha}{X+\rho(L-l_2)+\beta}.$$

Adding 1 to both sides, we see that the numerators are identical, hence their denominators must also be equal. Thus we can write

$$Y+\rho(L-l_1)+\beta=X+\rho(L-l_2)+\beta,$$

or
$$Y-X=\rho[l_1-l_2]. \qquad ...(59)$$

This equation shows that the end corrections are eliminated without measuring and the difference of two resistances can be obtained by determining the resistance of the bridge wire between two positions of the null points.

ρ, the resistance per unit length can be measured by short circuiting one outer gap by putting a thick Cu-strip and including a standard fractional resistance in the other gap. Two positions of the null point l_1 and l_2 are thus obtained by putting resistance in left, Cu-strip in the right gap and by interchanging their positions. Thus from Eq. (59), we have

$$0-R=\rho[l_1-l_2] \text{ or } \rho=R/(l_2-l_1). \qquad ...(60)$$

Keeping X and Y constant, balance points may be obtained at different parts of the wire by small alterations of the ratio P/Q, thus testing of the uniformity of the wire can be done by seeing that l_1-l_2 remains constant. This method is very useful for the measurement of low resistances. However this method fails if the difference of two resistances $Y-X$ exceeds the total resistance of the bridge wire LM. The unknown resistance X can be measured by connecting it in one gap and the known resistance Y in the other gap. By obtaining two positions of null point corresponding to the X in left and then in right, X can be calculated.

6.19. KELVIN'S DOUBLE BRIDGE FOR THE MEASUREMENT OF LOW RESISTANCES

Wheatstone's bridge is not suitable for the comparison of low resistances because it has low sensitivlty in such circumstances and the resistances of the contact and connecting leads are not negligible in comparison to the resistance to be measured. In the Kelvin's double bridge these errors are eliminated. It is very useful for the measurement of resistances below 0.1 ohm and is also an excellent method for finding the specific resistance of a conducting material taken in the form of a road of uniform cross-section. The principal diagram of the bridge is shown in Fig. 6.17.

R is the unknown low resistance and *S* a standard resistance of the same order of magnitude preferable a continuously variable resistance such as a slide wire. Both the resistances are joined together by a metal bar or a thick wire *BC* of low resistance *r* and are connected in series with a battery, rheostat and key. Sometimes *BC* is replaced by a cup filled with mercury. *M*, *N*, *m* and *n* are four known non-inductive resistances. These are arranged together with a low resistance galvanometer *G* to form a bridge. The ratio *M*/*N* is kept the same as *m*/*n*, the resistance *S* is varied until zero deflection of the galvanometer is obtained.

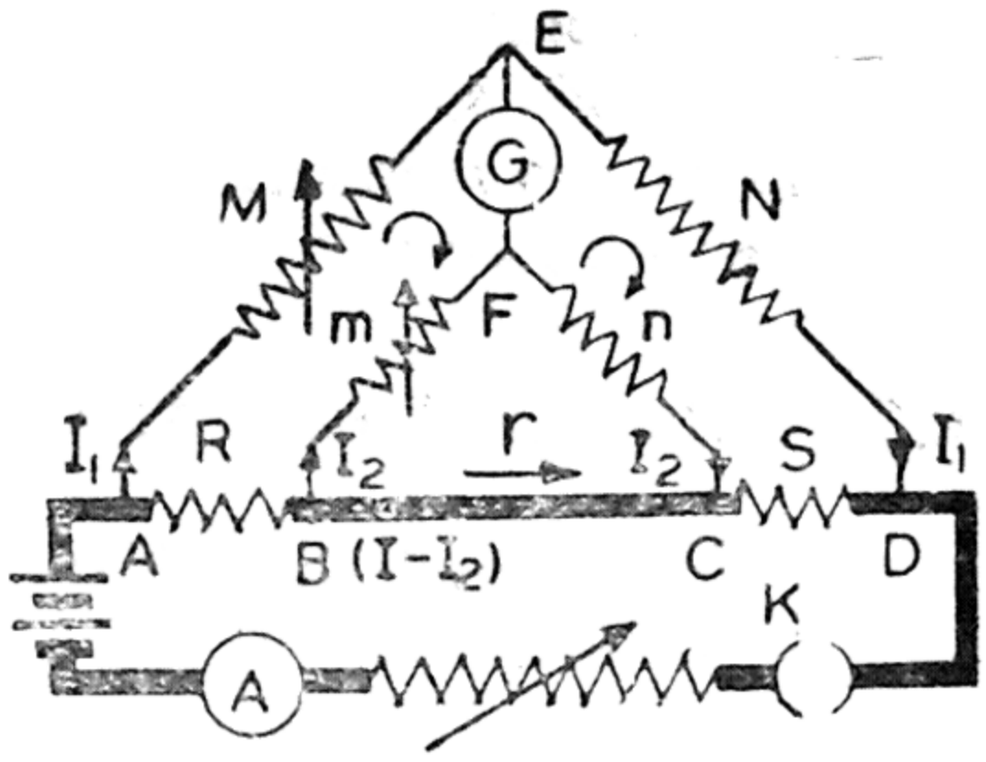


Fig. 6.17.

Let the distribution of currents in the branches is as shown in Fig. 6.17. At the time of balance of the bridge the current I_1 is in the arms *M* and *N*, I_2 in *m* and *n* and *I* in *R* and *S*. Let us now apply Kirchhoff's second law for the meshes *AEFBA* and *EFCDE*, we get

$$MI_1 - mI_2 - RI = 0$$

and

$$NI_1 - nI_2 - SI = 0.$$

As *m* and *n* are in parallel with the resistance *r*, the current *I* in *R* divides so that $[r/(r+m+n)]I$ passes through *m* and *n*. Substituting this value of I_2 in the above equations we have

$$MI_1 = Irm/(r+m+n) + RI$$

and

$$NI_1 = Irn/(r+m+n) + SI.$$

By solving, we get

$$R = \frac{M}{N}S + \frac{nr}{m+n+r}\left(\frac{M}{N} - \frac{m}{n}\right) \qquad \ldots(61)$$

If $M/N = m/n$, then

$$R = MS/N. \qquad \ldots(62)$$

It gives the unknown resistance *R* in terms of the standard resistance *S*.

The *resistivity* ρ of the material can be calculated, by taking the material in the form of a wire. If *A* is the area of cross-section and *l* the length, then

$$\rho = AR/l. \qquad \ldots(63)$$

Different forms of this bridge are manufactured by different manufactureres. The bridge, generally employed in laboratories, consists a compact form of potentiometer and a double ratio box. The value of M/N is made equal to that of m/n by this box and resistance S is measured by measuring the length of the potentiometer wire in circuit knowing that 1 coil=100 cm of wire=0.1 ohm.

6.20. CALLENDER AND GRIFITH's BRIDGE

The resistance variation with temperature is well defined in metals by the relation

$$R_t = R_0(1+\alpha t+\beta t^2), \qquad \ldots(64)$$

where R_t and R_o are the resistances at $t°$ and 0° respectively and α and β are the constant characteristics of the metal. Platinum is the most suitable metal for measurement of temperature, because it does not combine with oxygen even at very high temperature and its resistance varies as given by Eq. (64).

For low temperature measurements the β may be neglected and one may have a relation $R_t=R_0\ 1+\alpha t)$.

If R_{t1} and R_{t2} are the resistances of the specimen at temperatures t_1 and t_2 respectively, then the *temperature coefficient of resistance* α can be calculated by the relation

$$\alpha = \frac{R_{t2}-R_{t1}}{t_2R_{t1}-t_1R_{t2}}. \qquad \ldots(65)$$

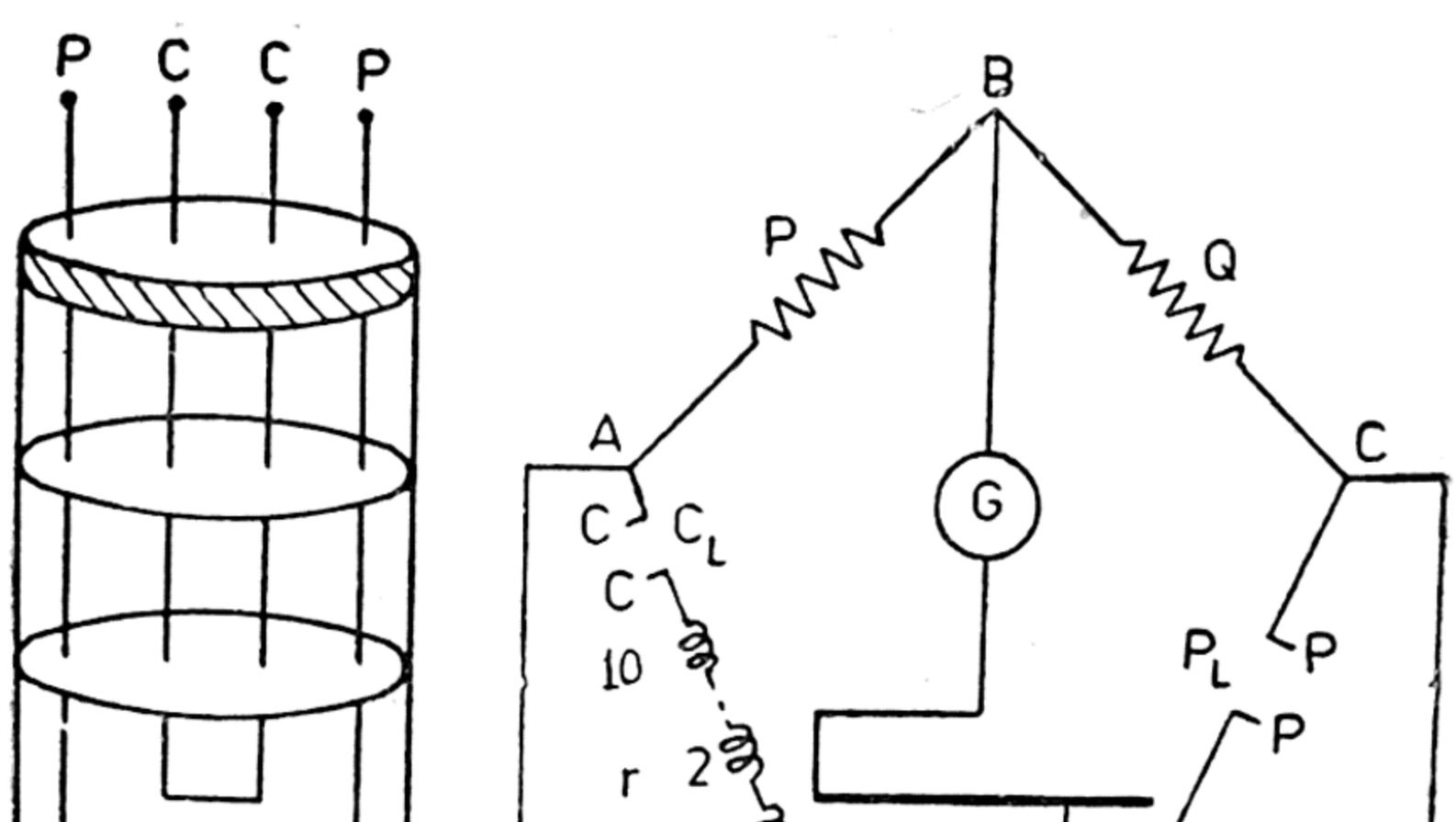


Fig. 6.18.

A platinum resistance thermometer consists of a fine platinum wire usually about 0.2 mm in diameter and of resistance about one ohm, wound non-inductively on a mica frame and placed in a glass tube for temperature upto 300° and in glazed porcelain or fused silica tube for higher temperatures. The ends of the wire are soldered to two copper leads which are connected to the two terminals *PP* at the other end on the top of the tube. The compensating leads or dummy leads, exactly similar in size and material to the leads of the thermometer, are placed parallel to the tube, joined together at the lower end and connected to another pair of terminals *CC* at the top. [Fig. 6.18 (*a*)]

Callender and Grifith's bridge is the modified Wheatstone's bridge for the accurate measurement of resistance of the thermometer. The ratio arms *P* and *Q* have equal resistance coils and C_L and P_L are the gaps for connecting the compensating and resistance leads of the thermometer. In series with a dial type adjustable resistance *r*, a stretched uniform wire *EF* of resistance 1 ohm is connected. A graduated scale with zero at the middle is associated with this wire, as shown in Fig. 6.18 (*b*)

The thermometer is placed in the bath whose temperature is to be measured, the *CC* and *PP* leads are connected across the gaps C_L and P_L respectively, a suitable resistance *r* is introduced in the dial box and the balance is thus observed at point *D* on the wire *EF*. At this stage, we have

$$P=Q \text{ and } C+r+\rho ED=R_t+C+\rho DF$$

or
$$R_t=r+\rho\,(ED-DF)=r+2\rho OD, \qquad \ldots(66)$$

where R_t is the resistance of the platinum wire at $t°C$, *C* of the compensating leads, and *O* is the mid-point of the wire *EF*, having resistance per unit length ρ.

The resistance of the platinum wire is measured at 0°, 100° and the unknown temperature $t°C$. Neglecting β in comparison to α in Eq. (65), we thus have

$$t=\frac{R_t-R_0}{R_{100}-R_0}\times 100. \qquad \ldots(67)$$

This relation is not accurate for high temperature measurements. The coefficient β is also taken into account at these temperatures. For most accurate measurements of temperatures the temperature coefficients α and β are first obtained by measuring resistance of platinum wire at three known temperatures (ice, steam and boiling sulphur) and the use is made of Eq. (64).

6.21 POTENTIOMETER

The potentiometer is an instrument which can measure accurately the emf of source of supply without drawing any current from the source or the potential difference across a circuit element. It

balances an unknown potential difference against an adjustable, measurable potential difference.

The **principle** of the potentiometer is shown in Fig. 6.19. A uniform wire AB of length 100 cm, which in more accurate instruments is 1000 cm, is permanently connected to the terminals of a source of emf $\mathcal{E}_1$. The current through the wire can be varied by means of an auxiliary adjustable resistance R_1 in series with the wire. If terminal A is connected with the +ve terminal of battery there will be a continuous fall in potential from A to B along the wire. Consider a source of emf $\mathcal{E}_2$ in such a way that its +ve pole is connected to A and its negative pole through a galvanometer G to a contact sliding on the wire. If the emf of this source is less than the potential drop along the wire AB, it is always possible to find X on the wire at which galvanometer shows no deflection. At this time the potential drop along AX is equal to the emf $\mathcal{E}_2$ of the second source, *we call the potentiometer as balanced and the point X as the balance position.* If the wire is uniform and its properties are not affected by altering its electrical potential, the potential difference between any two points in the potentiometer wire is directly proportional to the length of wire between them.

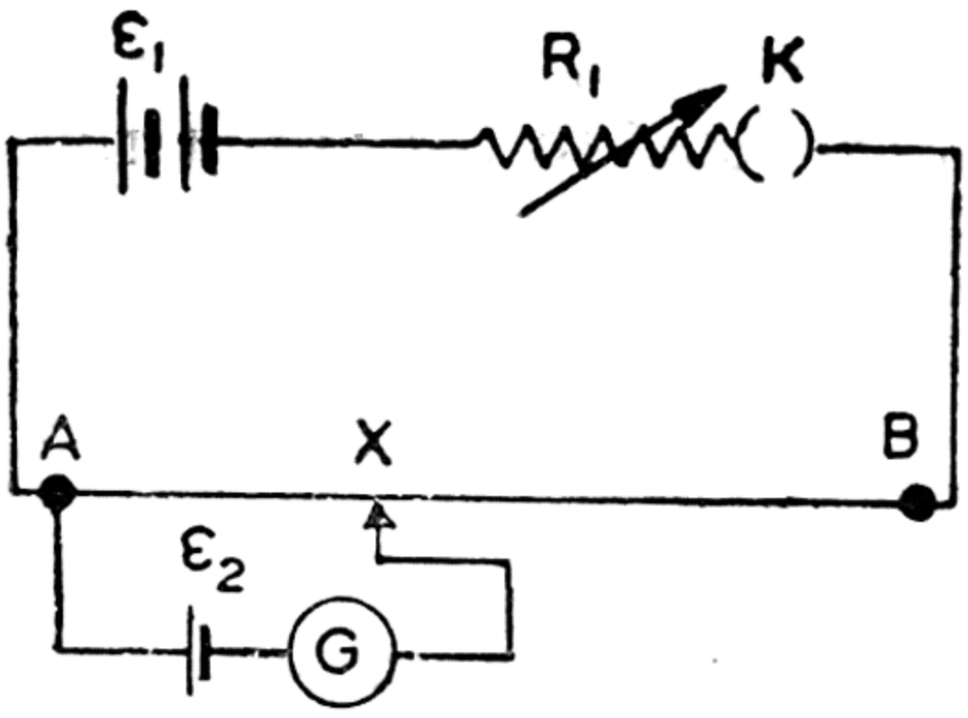


Fig. 6.19.

To compare emf's $\mathcal{E}_1$ and $\mathcal{E}_2$ of two cells, these cells are connected in turn to the galvanometer by means of the two-way key and the position of balance point on the potentiometer wire is obtained for each cell. If the current through the potentiometer wire remains same in each case, then

$$\mathcal{E}_1/\mathcal{E}_2 = l_1/l_2. \qquad \ldots(68)$$

To measure the emf or a potential difference accurately a standard cadmium cell (*emf = 1.01864 volts at 20°C, roughly taken as 1.018 volts*) is used to standarise the potentiometer wire. The potential drop per unit length is thus measured with the help of a standard cell.

To measure the internal resistance r of a cell the emf $\mathcal{E}$ is balanced against a length l_1 of the potentiometer wire. Now known resistance R from a resistance box is placed across the cell. The potential difference across the cell will thus fall to V. It is now balanced against a length l_2 of the potentiometer wire.

$$\mathcal{E} = \rho l_1 \quad \text{and} \quad V = \rho l_2.$$

Since the current in the circuit i is given by

$$i = \frac{\mathcal{E}}{r+R} = \frac{V}{R}$$

$$\therefore \quad r = \frac{\mathcal{E}-V}{V} R = \frac{l_1 - l_2}{l_2} R. \qquad \ldots(69)$$

The calibration of an ammeter is done by using it to measure the current through an accurately known resistance R. The pd V produced across this resistance due to the current I passing through it is measured by the potentiometer. The current must be V/R against the readings in the ammeter.

For this purpose the connections are arranged as shown in Fig. 6.20. With the plug in the gap 1 of two way key K the emf $\mathcal{E}$ of the standard cell is balanced against the pd across a length l_1 (1018 cm for easy calculations) of the potentiometer by adjusting rheostat R_1. Now the potential gradient, of the potentiometer wire will be $\rho = \mathcal{E}/l_1$. The key K is then switched to position 2, the potential V across the resistance R is balanced against the length l_2. Thus the current through the resistance R is

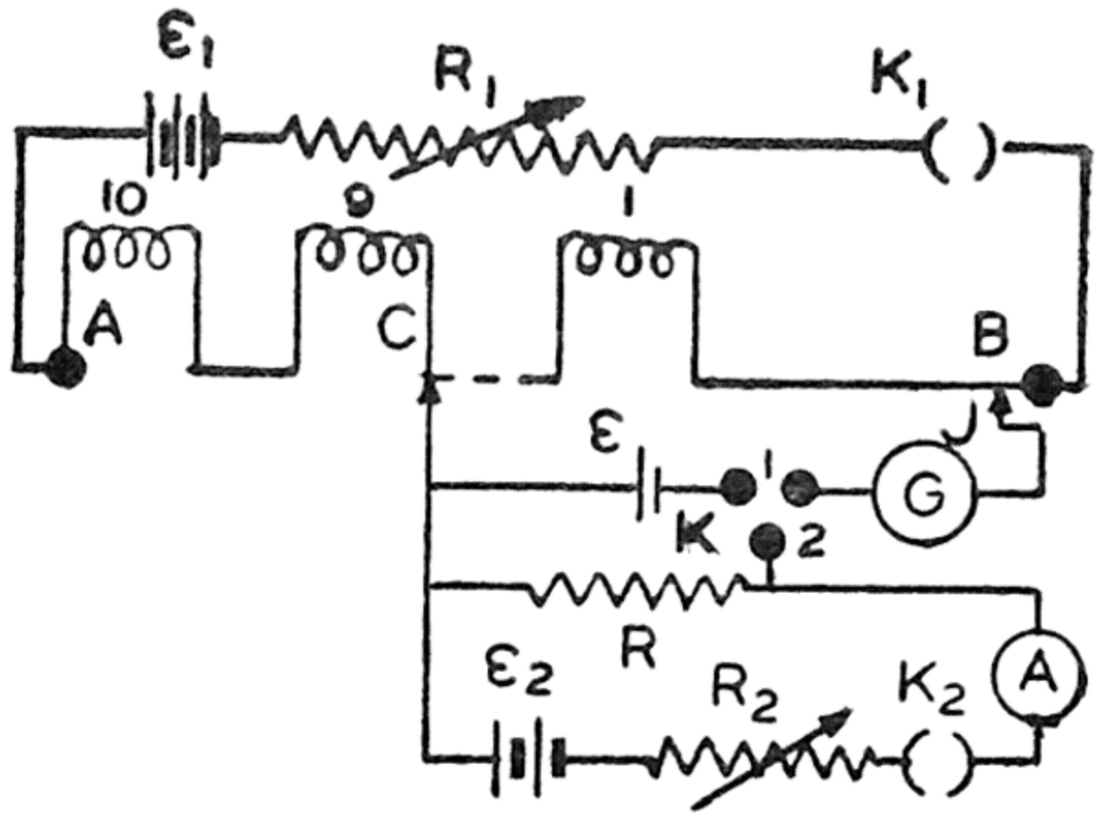


Fig. 6.20.

$$I = V/R = \rho l_2/R$$

$$= \mathcal{E} l_2/l_1 R \qquad \ldots(70)$$

In practice R is generally taken as 1 ohm, $l_2 = 1018$ cm for cadmium cell hence current $I = l_2 \times 10^{-3}$ volt/ohm. By changing current through R by means of the rheostat R_2, several sets of observations for different values of pd across the resistance R are taken. The ammeter readings are also noted at the same time. In this way calibration of ammeter can be done very easily.

To determine the value of an unknown resistance, a steady current of known strength I is allowed to pass through the resistance R to be measured and the pd V across it is determined by means of a potentiometer, accurately calibrated by using a standard cadmium cell using the ciruit diagram as shown in Fig. 6.20, R may be calculated with the help of the Eq. (70).

To compare two resistances, the standard source is not required as we do not want the potential gradient of the potentiometer. Since the potential difference across a resistance is proportional to the product of its resistance and the current passing through it, the ratio of two resistances will be equal to the ratio of the potential differences across them when they are each carrying the same current, hence the ammeter is also not required. Connections are made as shown in Fig. 6.21. A steady current I is allowed

to pass through the resistances R_1 and R_2, to be compared by connecting them in series. Now let the higher potential terminal of the resistance, one of say R_1, be connected to the variable terminal C of the potentiometer in dial type and to the terminal A in 10 wire or simple potentiometer and the lower potential terminal through a galvanometer to the jockey J of the potentiometer. Then if l_1 be the length of the potentiometer wire between C and J for no deflection in galvanometer, the potential difference.

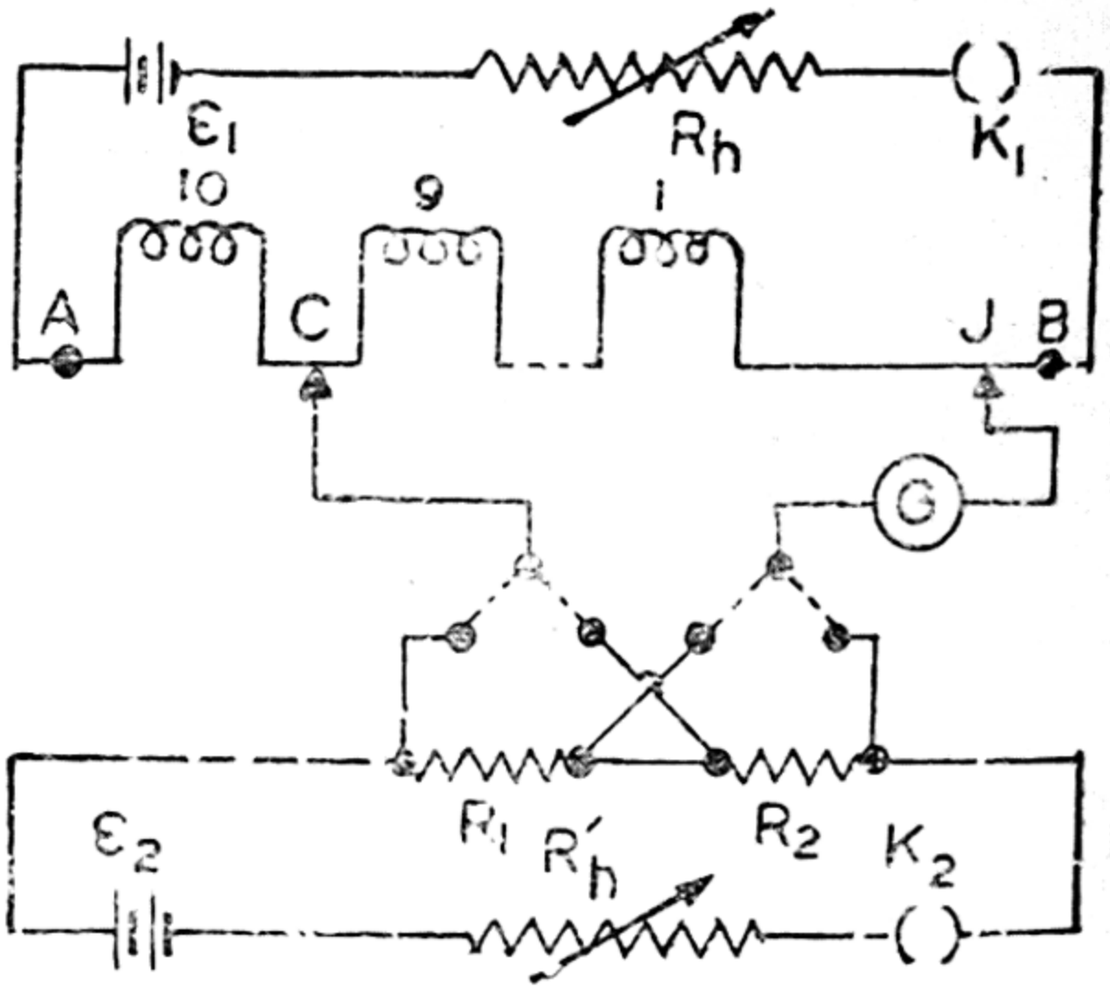


Fig. 6.21.

$$V_1 = \rho l_1 = IR_1. \qquad \ldots(i)$$

If similar observations are taken for the another resistance R_2, then

$$V_2 = \rho l_1 = IR_2 \qquad \ldots(ii)$$

$$\therefore \qquad R_1/R_2 = l_1/l_2. \qquad \ldots(71)$$

At least six sets of observations are taken in the same manner by sliding rheostat R_h'.

To determine a low resistance by means of a potentiometer, this unknown low resistance r is connected in series with a known resistance R and this combination is connected in series with a battery and a rheostat. The higher potential terminal of the resistance R is connected to the variable terminal C, the lower potential terminals of resistances R and r respectively through a two-way key and a galvanometer to the jockey of the potentiometer. First of all the potential difference across the resistance R is balanced by the length l_1 of the potentiometer wire and then the pd across the combination $R+r$ is balanced by the length l_2. Thus we have

$$V_1 = IR = \rho l_1 \text{ and } V_2 = I(R+r) = \rho l_2,$$

$$\therefore \qquad \frac{V_1}{V_2} = \frac{R}{R+r} = \frac{l_1}{l_2} \quad \text{or} \quad r = \left(\frac{l_1}{l_2} - 1\right) R. \qquad \ldots(72)$$

6.22 ELECTROLYSIS

The phenomenon associated with the passage of electricity through liquid conductors was first studied by Faraday. The molecules of a salt in a solution are supposed to break up into two parts of opposite charges, called the *ions*. When an electric field is applied between two rods (or plates) immersed in the solution, these ions move towards the rods (*electrodes*) of opposite

potentials. This phenomenon of transfer of charged ions by an electric current is called *electrolysis*. The solution is known as *electrolyte* and the vessel containing electrodes and electrolyte is called *voltameter*.

Faraday has formulated two laws :

(*i*) The mass of an element or ion deposited or liberated at an electrode during electrolysis is proportional to the total charge passing through the electrolyte, *i.e.*, $m \propto Q$ or $m = Z\,(It)$. Here the constant Z is known as electrochemical equivalent of the element.

(2) This mass is also proportional to the equivalent weight E of the element. This law also tells that the same quantity of electricity is needed to liberate gram-equivalent of any substance. This quantity of electricity is generally referred to as one Faraday. Thus

$$1 \text{ Faraday} = 96500 \text{ coulombs} = E/Z.$$

Two theories have been given to explain electrolysis phenomenon.

(*a*) **Grotthus Theory.** Grotthus suggested that each molecule is consisted of two parts of opposite signs. When a potential difference is applied between the electrodes, the molecules arrange themselves along rows with all +ve ends toward cathode and −ve ends toward the anode. Thus a certain potential difference is required to bring the electrolysis phenomenon. But experimentally it is observed that even a very small pd can produce electrolysis.

(*b*) **Dissociation Theory.** According to Arrhenius (1887) the molecules of the dissolved substance break up into atoms and also there is a constant interchange of atoms between the molecules. These atoms are electrically charged, even before the application of an electric field and are known as ions. The metallic ions and hydrogen (usually) being positive and the nonmetallic ones negative. When the pd is applied across electrodes dipped into the electrolyte, the positive ions move towards the cathode, hence called *cations* and the negative ions move towards the anode, hence called *anions*. The charges carried out by the ions are delivered up at the electrodes, leaving the free atoms.

The state of dynamic equilibrium in a potassium chloride solution can be expressed as

$$KCl \rightleftharpoons K^+ + Cl^- . \qquad \ldots(73)$$

The equilibrium will shift towards the right when the dilution increases. It is observed that the more dilute the solution the more complete are the dissociation. Thus practically all the molecules of the dissolved substance dissociate into free ions in a very dilute electrolyte. The degree of dissociation

$$\alpha = \frac{\text{Number of dissociated molecules in a solution}}{\text{Total number of molecules in a solution}} \quad \ldots(74)$$

The above process can be explained with the help of electrostatic forces. Atoms in the molecules are held together by electrostatic attraction. When the molecules are immersed in a solvent, this force of attraction decreases, as the dielectric medium is changed from air to liquid, The solvents of high permittivities have thus more degrees of dissociation.

Electrical Conductivity of an Electrolyte. It is observed that in the cells in which the electrodes are composed of some other metal not chemically acted upon either by the salt or acid, back emf $\mathcal{E}_B$ is produced and the current is thus proportional to $(V-\mathcal{E}_B)$. This result is true if the current density is low and the applied potential is of very low frequency.

On the application of pd between the electrodes dipping into a solution, the ions will be accelerated and will reach a terminal velocity. While reaching to the electrodes, they collide with ions and molecules. This terminal velocity depends upon the size of the ion, the potential gradient and the viscosity of the solution.

Let n be the number of molecules per unit volume of solute in a solution and nx the number of molecules dissociated. Let u be the velocity of the cations (+ve ions) and v the velocity of the anions (−ve ions). If the electrodes are assumed as parallel planes, then on applying a pd between the electrodes, $n\alpha u$ cations and $n\alpha v$ anions will pass per unit time per unit area of an imaginary plane PQ. If q be the charge on each ion, then the current passing per unit area will be $n\alpha\,(u+v)q$. Hence in unit time $n\alpha\,(u+v)$ cations must be deposited on the cathode and the same number of anions on the anode. As the number of cations reaching the cathode per unit time per unit area is $n\alpha u$ and that of deposited at the cathode is $n\alpha(u+v)$, hence the excess number $n\alpha v$ must come from the electrolyte in the immediate neighbourhood of the cathode. Therefore the electrolyte concentration must decrease. This number of cations must leave a similar number of anions. These anions move towards the anode and decrease electrolyte concentration in the neighbourhood of the anode in the similar way. Hence ratio of loss in concentration in the cathode region to that in anode region is v/u = mobility of the negative ion k_n/mobility of the positive ion k_p.

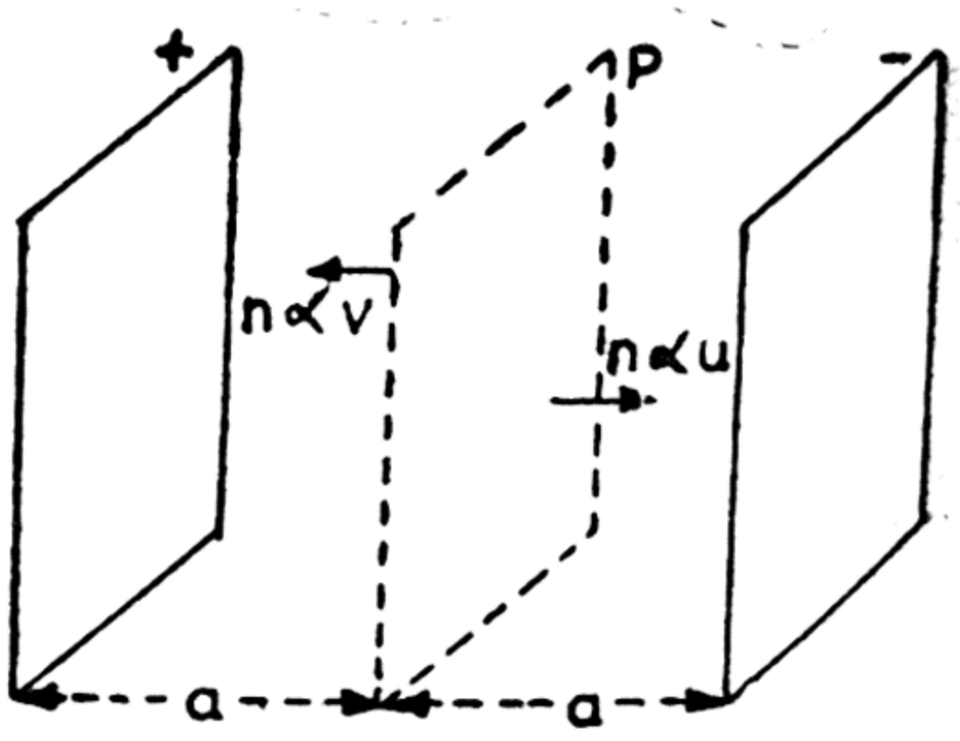


Fig. 6.22.

Instead of n, if the concentration C of molecules is used, then current density

$$j = C\alpha(u+v)q. \quad \ldots(75)$$

As ionic mobility is defined as the velocity of the ion acquired under unit potential gradient, hence for the potential gradient E, we have

$$u = Ek_p \text{ and } v = Ek_n.$$

$$\therefore \quad j = C\alpha E(k_p + k_n)q,$$

or the specific conductivity $\sigma = j/E = C\alpha(k_p + k_n)q$. ...(76)

The ratio $\sigma/C = \alpha(k_p + k_n)q = \Lambda =$ *Equivalent conductivity.*

For dilute solution $\alpha = 1$ and $\Lambda = \Lambda_\infty$.

$$\therefore \quad \Lambda/\Lambda_\infty = \alpha. \qquad ...(77)$$

Hence the degree of dissociation α for a solution of any concentration is found from a graph of Λ against concentration C.

The specific conductivity can be determined by using *Kohlraush's bridge*, will be discussed in chapter 12.

6.23. VOLTAIC CELL

Voltaic cells are simply electrolytic cells in which the electrodes react with the electrolyte in such a way as to produce an internal emf. For theoretical purposes the cells are classified as *reversible* and *irreversible*. The former is one which may be restored to its original condition after discharge by passing a reverse current through it. It requires same energy for charging as obtained during the discharge. In irreversible cell, some chemical action takes place in the course of working which can not be reversed by sending the current in reverse direction. For practical purposes the cells are classified as : *primary* and *secondary*. The former are used for discharge only and are not necessary reversible. The latter, often called a storage cell, are necessarily reversible.

Kelvin suggested that the chemical reactions within the cells were the sole source of energy. By equating this with thermal energy the emf of a cell can be calculated. Let H_1 and H_2 are the heats of reaction at the two electrodes due to the unit charge passing through the cell. If the emf of the cell is $\mathcal{E}$ and q charge is passing through the cell at very slow rate so that joule heating can be avoided then,

$$\text{Energy liberated } J(H_1 \sim H_2)q = \text{Energy consumed } \mathcal{E}q. \qquad ...(78)$$

From this equation emf of the cell $\mathcal{E}$ can be calculated.

Let us apply this equation to a Daniell cell, bearing in mind that the passage of 1 faraday, *i.e.* 96,500 coulombs, corresponds to the formation of 1 gm equivalent of the electrolyte. In this cell, zinc sulphate is formed at the zinc electrode. It can be determined that 1 gm of Zn developes 1670 calories in forming $ZnSO_4$. As atomic weight of Zn is 65.37 and it is divalent, hence,

$$H_1 = 1670 \times 65.37/2 = 54584 \text{ calories per gm equivalent.}$$

At the copper electrode, Cu is deposited from $CuSO_4$ solution. It can be determined that 1 gm of Cu developes 909.5 calories in forming $CuSO_4$. As atomic weight of Cu is 63.57 and it is divalent, hence,

$$H_2 = 909.5 \times 63.57/2 = 28908.5 \text{ calories per gm equivalent.}$$

$$\therefore\ 4.18[54584 - 28908.5] = 96500\ \mathcal{E} \text{ or } \mathcal{E} = 1.112 \text{ volts} \quad ...(79)$$

Gibbs Helmholtz Equation. In the above theory we have used the first law of thermodynamics. However, it has been experimentally determined that a cell has a temperature coefficient of emf. If the charge through the cell flows very slowly, second law of thermodynamics can be applied and the cell can be assumed as reversible engine.

Let a reversible cell of emf $\mathcal{E}$ be connected with external circuit and is maintained at a constant temperature T. The emf of the cell is balanced against an equal and opposite *pd*, derived from the external circuit. The balancing *pd* is reduced slightly so that the cell will derive a charge q at a slow rate round the circuit. This is represented by AB in an indicator diagram The cell will absorb an amount of heat h to keep its temperature constant.

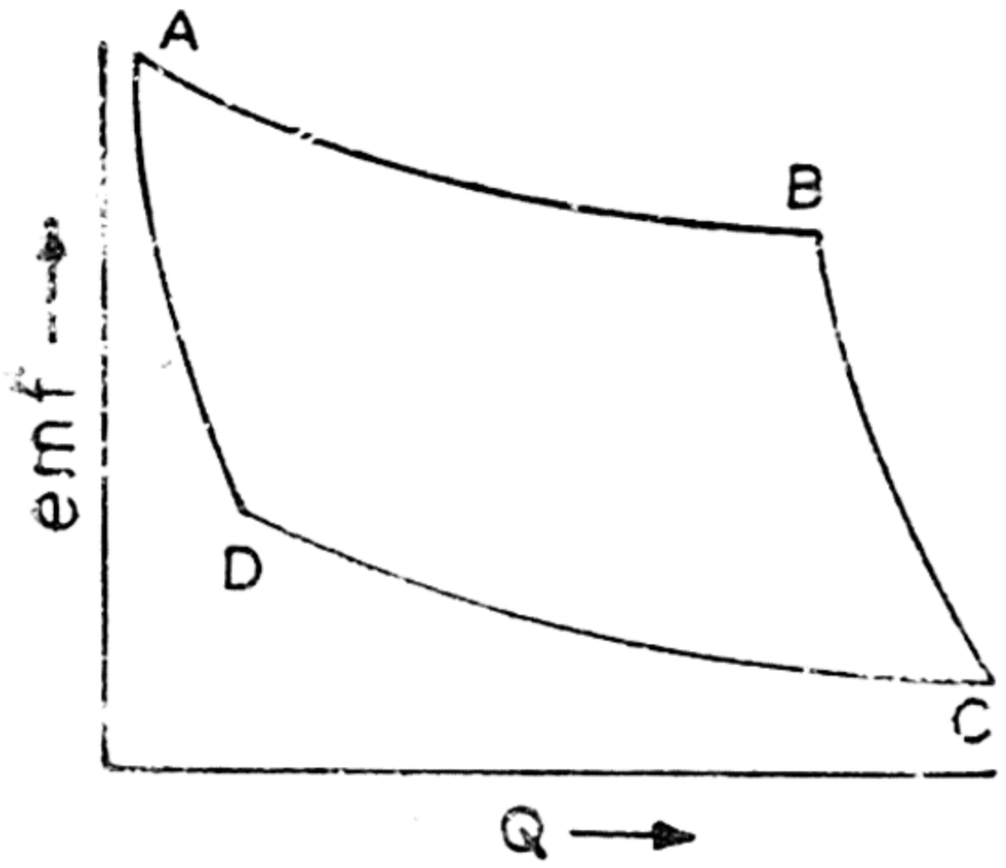


Fig. 6.23.

If the cell is thermally insulated from the surroundings and allowed to send a small charge round the circuit. In this process the cell has to use its own energy and hence temperature of the cell falls and therefore emf will also fall from $\mathcal{E}$ to $\mathcal{E} - (d\mathcal{E}/dT)\,\delta T$. The change is represented by BC.

Now let the cell is maintained at this new temperature $T - \delta T$ in an enclosure. If q be the charge passed into the cell by means of the external circuit of infinitesimally greater emf so that the rate of flow of charge is very slow. It is represented by CD. If the cell is restored to its initial position A, then the work done in a complete cycle.

$$= W_1 - W_2 = \text{Area } ABCD = q.\ (d\mathcal{E}/dT)\delta T.$$

For a reversible cycle

$$\frac{W_1 - W_2}{W_1} = \frac{T_1 - T_2}{T_1} \quad \text{or} \quad \frac{q(d\mathcal{E}/dT)\delta T}{h} = \frac{\delta T}{T}$$

or $$h = qT(d\mathcal{E}/dT). \quad ...(80)$$

Thus the Gibbs-Helmholtz theory modifies the Eq. (78) as

$$J(H_1 \sim H_2)q + h = J(H_1 \sim H_2)q + qT(d\mathcal{E}/dT) = \mathcal{E}q$$

or
$$\mathcal{E} = JH + T(d\mathcal{E}/dT). \qquad \ldots(81)$$

This is known as *Helmholtz-Gibbs equation.* Here $H = H_1 \sim H_2$. It is clear from this equation that for $d\mathcal{E}/dT = 0$ it is same as Eq. (78).

If $d\mathcal{E}/dT > 0$, $\mathcal{E} > JH$ and the heat must therefore be taken in by the cell to keep the temperature constant. Thus the cell will cool while giving a current.

If $d\mathcal{E}/dT < 0$, $\mathcal{E} < JH$ and the heat must therefore be taken out by the cell to keep the temperature constant. Thus the temperature of the cell will increase while giving a current.

Exercises

Example 1. *Calculate the drift velocity for the electrons in a silver wire which has a radius of 0.1 cm and carries a current of 2 amperes. Also find the current density in the wire.*

We know that $i = nq\mathbf{v}$. $\mathbf{A} = nqvA$.

Since silver is univalent, each atom can be assumed to contribute one electron. We know that one gm atom of silver (*i.e.* 108 gm. of silver) has 6.023×10^{23} atoms. As the density of *Ag* is 10.5 gm per cm^3, hence the number of electrons per unit volume

$$n = \frac{6.023 \times 10^{23}}{108 \times 10^{-3}} \times 10.5 \times 10^{-3} \times 10^6 = 5.856 \times 10^{28} \text{ per } m^3.$$

Cross-sectional area of the wire $A = \pi r^2 = 3.14 \times 10^{-6}\ m^2$.

$$\therefore \text{ Drift velocity } v = \frac{i}{nqA} = \frac{2}{5.856 \times 10^{28} \times 1.6 \times 10^{-19} \times 3.14 \times 10^{-6}}$$

$$= 6.79 \times 10^{-5} \text{ m/sec}$$

and current density $j = i/A = 2/3.14 \times 10^{-6}$

$$= 6.37 \times 10^5 \text{ amp/m}^2.$$

Example 2. *In a copper wire of cross-sectional area 2 mm^2, a current of 30 amperes is flowing. Calculate the rms velocity of free electrons at 27°C. Also prove that the drift velocity is very small in comparison to it. Given that Boltzmann constant $k = 1.38 \times 10^{-23}$ joule/°K and density of Cu 8.9 gm/cc.*

We know that the root mean square velocity

$$v_{rms} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{9 \times 10^{-31}}}$$

$$= 1.17 \times 10^5 \text{ m/sec.}$$

Drift velocity is given by the relation $\mathbf{v}_d = -\mathbf{j}/ne$,

Current density $j = i/A = 30/2\times10^{-6} = 1.5\times10^{7}$ amp/m², and the number of free electrons per unit volume.

$$n = 6.023\times10^{23}\times8.9\times10^{6}/63 = 8.48\times10^{28}.$$

$$\therefore \quad \text{Drift velocity} = \frac{1.5\times10^{7}}{8.48\times10^{28}\times1.6\times10^{-19}} = 1.1\times10^{-3} \text{ m/sec}$$

Thus we see that the root mean square velocity is about 10^8 times greater than the drift velocity of free electrons in *Cu*.

Example 3. *An aluminium wire whose diameter is 0.245 cm is welded end to end to a copper wire with diameter 0.1626 cm. The composite wire carries a steady current of 10 ampere. (a) What is the current density in each wire ? (b) Find the drift speed of the electrons in copper wire.*

Assume one free electron per atom in copper. Given Avogadro number $=6\times10^{23}$ *atoms per gm mole, Molecular weight of copper* $=64$, *density of copper* $=9$ *gm/cc.*

(*a*) Since the same current is flowing through *Al* and *Cu*-wires the current density will vary inversely with their area of cross-sections.

$$(A_1)_{Al} = \pi r_1^2 = 3.14\times(0.245\times10^{-2}/2)^2 = 4.711\times10^{-6} \text{ m}^2$$

and $$(A_2)_{Cu} = \pi r_2^2 = 3.14\times(0.1626\times10^{-2}/2)^2 = 2.075\times10^{-6} \text{ m}^2$$

Therefore the current densities in these wires

$$j_{Al} = \frac{i}{A_1} = \frac{10}{4.711\times10^{-6}} = 2.122\times10^{6} \text{ amp/m}^2$$

and $$j_{Cu} = \frac{I}{A_2} = \frac{10}{2.075\times10^{-6}} = 4.819\times10^{6} \text{ amp/m}^2$$

(*b*) The number of electrons per unit volume n in copper wire is related with the Avogadro number N, M the molecular weight of Cu and d its density as

$$n = \frac{dN}{M} = \frac{9\times6\times10^{23}}{64} = 8.437\times10^{22} \text{ electrons/cm}^3$$

$$= 8.437\times10^{28} \text{ electrons/m}^3.$$

Since the current density j is related with the drift speed as

$$j = nqv$$

or $$v_{Cu} = \frac{j_{Cu}}{nq} = \frac{4.819\times10^{6}}{8.437\times10^{28}\times1.6\times10^{-19}}$$

$$= 3.57\times10^{-4} \text{ m/sec}$$

Example 4. *In a copper there are 10^{28} free electrons per cc all of which contribute to a current of 1 amp in a wire of Cu of 0.01 cm^2 cross sectional area. Calculate (1) the average drift speed of the electrons in Cu and (2) the electric field in the wire. Given ρ of Cu as 1.7×10^{-8} ohm-m.*

Drift velocity $v=i/nqA$.

In our problem $i=1$ amp., $n=10^{28}\times10^{6}$ electrons/m^3, $A=0.01\times10^{-4}$ m^2 and charge on the electron $q=1.6\times10^{-19}$ coul.

$$\therefore \quad v=1/10^{28}\times1.6\times10^{-19}\times0.01\times10^{-4}=6.25\times10^{-4} \text{ m/sec.}$$

We know that $j=\sigma E$ or $E=\rho j=\rho i/A$.

$$\therefore \quad E=1.7\times10^{-8}\times1/(0.01\times10^{-4})=1.7\times10^{-2} \text{ volt/m.}$$

Example 5. *Calculate (a) the mean time τ between collisions and (b) The mean free path for free electrons th copper.*

We know that the mean free time

$$\tau=\frac{\lambda}{v_{av}}=\frac{m}{ne^2\rho}=\frac{9.1\times10^{-31}}{(8.48\times10^{28})\,(1.6\times10^{-19})^2\,(1.7\times10^{-8})}$$
$$=2.46\times10^{-14} \text{ sec.}$$

Here the use has been made of example 2.

$$\therefore \quad \text{Mean free path } \lambda=\tau v_{av}=2.46\times10^{-14}\times(1.6\times10_{6})$$
$$=3.94\times10^{-8} \text{ m.}$$

Example 6. *A rod of a certain metal is 100 cm long and 0.55 cm in diameter. The resistance between its ends at 20°C is 2.87×10^{-3} ohm. A round disc of thickness 1 mm and of diameter 2 cm is formed of this material. What is the resistance between the opposing round faces? What is the material?*

Resistivity of the material $\rho=RA/l$

$$=2.87\times10^{-3}\times3.14\times(0.275\times10^{-2})^2$$
$$=6.8\times10^{-8} \text{ ohm-meter.}$$

It is the value for nickel, hence the rod is of the nickel,

Resistance of the disc, $R=\rho l/A=6.8\times10^{-8}\times10^{-3}/(3.14\times10^{-4})$

$$=2.16\times10^{-7} \text{ ohm.}$$

Example 7. *A vacuum diode consists of plane parallel electrodes separated by a distance d and each having area A. When a potential V is applied to the anode with respect to the cathode, a current i flows through the diode. Assuming that the electrons are emitted with zero velocity and do not change the field between electrodes, calculate electron velocity v and charge density ρ at any point between the electrodes.*

We are neglecting the effect of space charge. Thus for steady current

$$\text{div } \mathbf{j}=0 \quad \text{or} \quad \partial j_x/\partial x+\partial j_y/\partial y+\partial j_z/\partial z=0.$$

Flow of electrons is from cathode to anode only in one direction. If it is taken in x-direction only, then $j_y=0$, $j_z=0$ and $\partial j_x/\partial x=0$ or $j_x=$const. or independent of x. We know that $j_x=en_xv_x$, where v_x is the velocity of electrons at a distance x from the cathode and n_x is the number of electrons per unit volume at this distance.

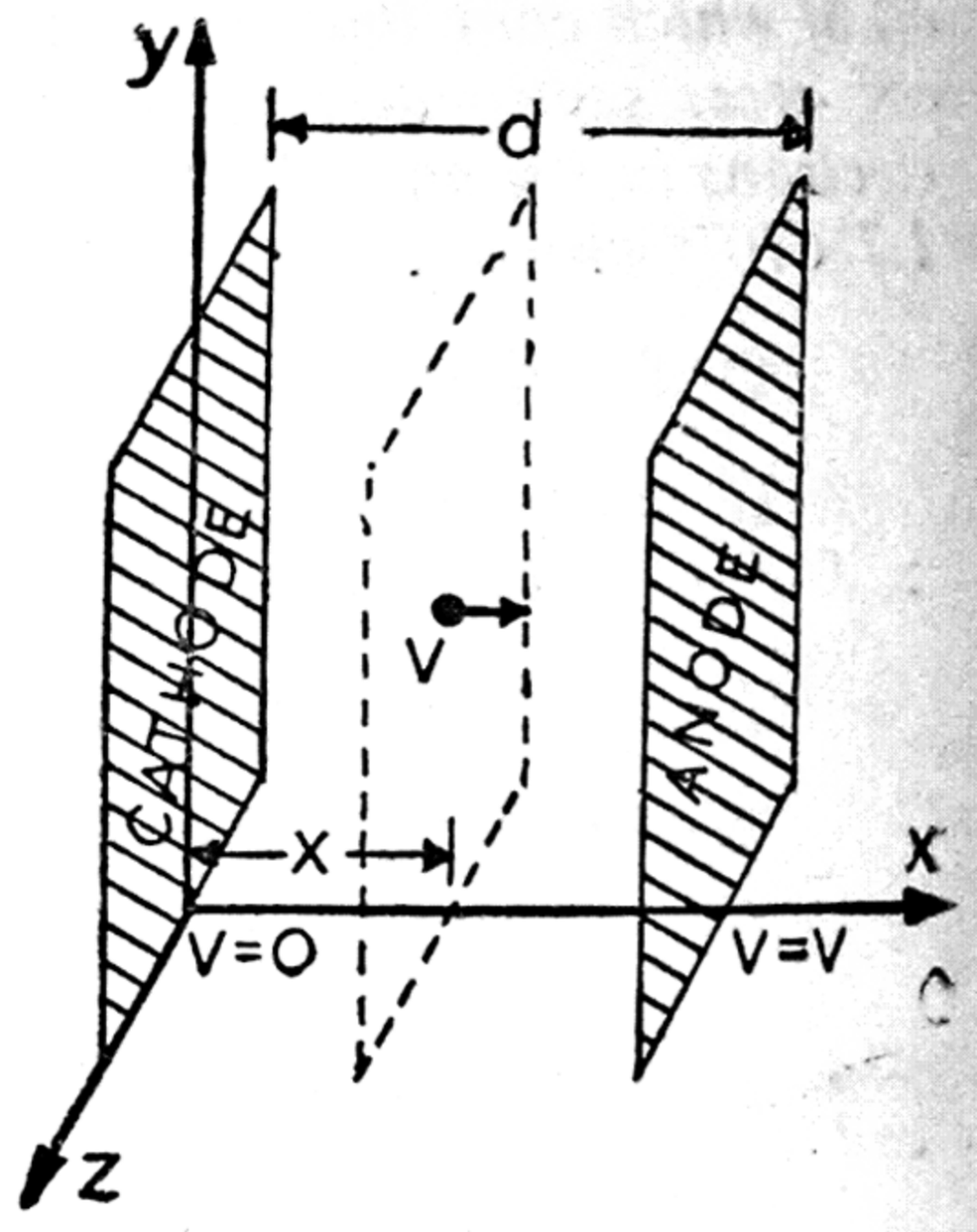


Fig. 6.24.

Force acting on each electrons $=eE=eV/d.$

$\therefore$ Acceleration on the electrons $f=F/m=eV/md.$

Thus the velocity of electrons at a distance x from the cathode is given by

$$v_x^2=0+2fx=2eVx/md$$

or $$v_x=\sqrt{2eVx/md}.$$

$\therefore$ Current $i=j_x\,A=en_xv_x\,A=en_x\,A\sqrt{2eVx/md}$

and space charge density $\rho_x=en_x=i\sqrt{md/2eVxA^2}.$

Example 8. *The region between two concentric spheres of radii a and b is filled with a conducting material. The inner sphere is maintained at a potential V_a and the outer at a lower potential V_b, so there is a current in the radially outward direction. Calculate the electric intensity at any point with in the material.*

Let us consider a cross-sectional area A at a distance r between the spheres. This resistance $\triangle R$ due to the material of thickness $\triangle r$ is

$$\triangle R=\rho\triangle r/A=\rho\triangle r/4\pi r^2,$$

$\therefore$ $$R=\frac{\rho}{4\pi}\int_a^b\frac{dr}{r^2}=\frac{\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right].$$

As the current between the spheres $i=(V_a-V_b)/R,$

hence current density at any point on the sphere of radius r is

$$j=i/A=(V_a-V_b)/4\pi r^2R.$$

Thus the corresponding electric field $E=\rho j=\dfrac{V_a-V_b}{r^2(1/a-1/b)}$

It can be proved very easily that it is same as electrostatic field between two concentric spheres placed in vacuum.

Example 9. *Calculate the resistance of the strip ABCD (shown in Fig. 6.25) between electrodes CD and AB.*

In the medium where the conductivity is constant, the potential equation is $\nabla^2 V=0$. In polar coordinates, $V=c\theta$ is a solution of $\nabla^2 V=0$, if c is a constant. Equipotential surfaces are given by $\theta=$const. The field is given by $\mathbf{E}=-\text{grad } V$ and hence $E=-c/r$ in the direction $\perp$ to the radius vector. Hence the solution for the flow of current in a thin strip bounded by any two concentric circles and having the lines AB and CD as electrodes will be

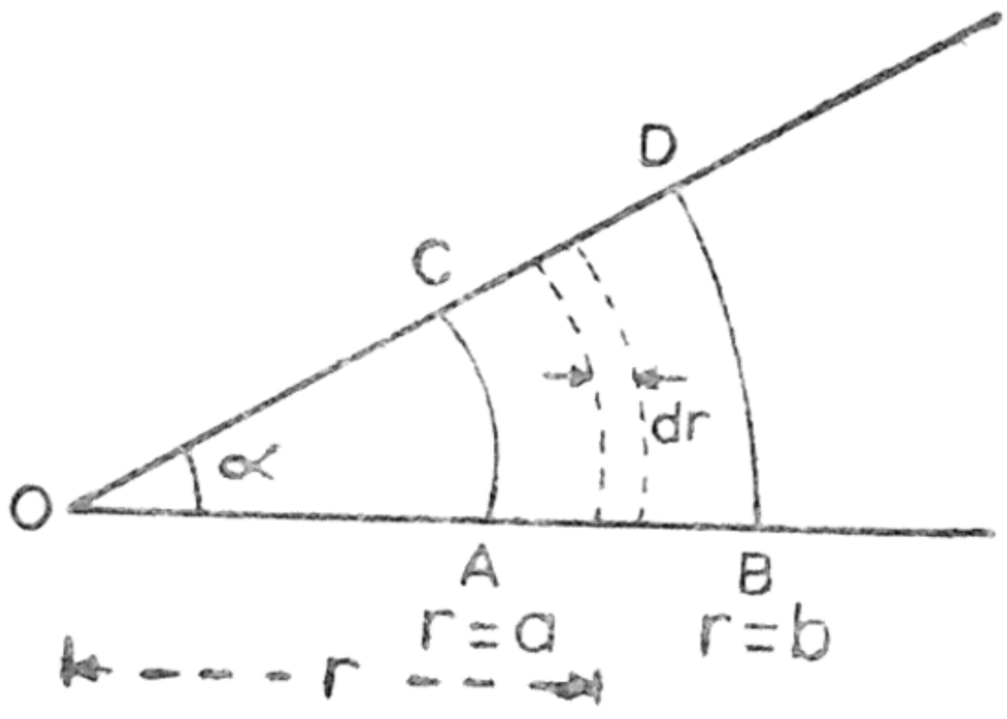


Fig. 6.25.

$$\mathbf{j}=\sigma\mathbf{E}$$

$$\therefore \qquad j=\sigma E=-\sigma c/r.$$

Hence the total current leaving AB is

$$i=\int \mathbf{j}\,.\,d\mathbf{S}=-\int_a^b \frac{\sigma c}{r}\,.\,t\,dr=-c\sigma t\log_e\frac{b}{a},$$

where t be the thickness of the strip.

As the pd between the electrodes AB and CD is $c\alpha$, where $\alpha=\angle AOC$, hence the resistance of the strip

$$R=\frac{-c\alpha}{-c\sigma t\log_e(b/a)}=\frac{\alpha}{\sigma t\log_e(b/a)}.$$

Example 10. *N lamps, each of resistance r, are fed by a machine of resistance R (including leads). If the light emitted by any lamp is proportional to the square of the heat produced, prove that the most efficient way of arranging the lamps is to place them in parallel arcs, each containing n lamps, where n is the integer nearest to $(NR/r)^{1/2}$.*

As each arc is containing n lamps, hence the resistance of each arc is nr and the number of arcs is N/n. Total resistance S is given by

$$\frac{1}{S}=\sum\frac{1}{nr}=\frac{N}{n}\left(\frac{1}{nr}\right) \text{ or } S=\frac{n^2r}{N}.$$

$\therefore$ Total resistance $=R+S=R+n^2r/N$.

If $\mathcal{E}$ is the emf of the machine, the current entering the arcs is $\mathcal{E}/(R+S)$ and in the each arc is $n\mathcal{E}/(R+S)N$. Hence the current passing through each lamp

$$i=\frac{n\mathcal{E}}{N(R+n^2r/N)}=\frac{\mathcal{E}}{N}\left[\frac{R}{n}+\frac{nr}{N}\right]^{-1}.$$

The heat produced per sec in the lamps is $H=Nri^2$ and the light emitted is proportional to H^2. Therefore the light produced is maximum when H^2 and hence H is maximum or factor $R/n+nr/N$ is minimum. We can write

$$\frac{R}{n}+\frac{nr}{N}=\left(\left[\frac{R}{n}\right]^{1/2}-\left[\frac{nr}{N}\right]^{1/2}\right)^2+2\left(\frac{Rr}{N}\right)^{1/2}.$$

It is minimum when $\sqrt{(R/n)}-\sqrt{(nr/N)}=0$ or very small or n is closely equal to $(NR/r)^{1/2}$.

Example 11. *A cable AB of length l of uniform wire develops a leak at a certain point. To locate the fault two observations are made. The resistance between A and the earth through the cable when B is earthed is found to be that of a length a of the wire and that between B and the earth when A is earthed that of a length b. Show that the point at which the fault exists divides AB in the ratio* $[a(l-b)/b(l-a)]^{1/2}$ *and find the resistance of the leak.*

Let ρ be the resistance per unit length of the wire. If the fault exists at a distance x from the end A and r is the resistance of the leak.

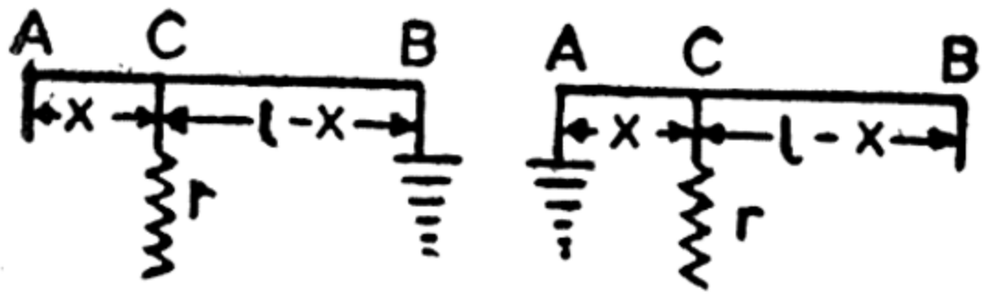


Fig. 6.26.

(*a*) *When B is earthed.* The resistance between A and the earth through the cable

$$R=x\rho+\frac{1}{1/r+1/(l-x)\rho}=a\rho \text{ (given)}$$

or
$$\frac{1}{\rho(a-x)}=\frac{1}{r}+\frac{1}{(l-x)\rho}. \qquad ...(i)$$

(*b*) *When A is earthed.* The resistance between B and the earth through the cable

$$R'=(l-x)\rho+\frac{1}{1/r+1/x\rho}=b\rho \text{ (given)}$$

or
$$\frac{1}{\rho(b-l+x)}=\frac{1}{r}+\frac{1}{x\rho}. \qquad ...(ii)$$

Eliminating r from Eqs. (*i*) and (*ii*), we get

$$\frac{1}{\rho(a-x)}-\frac{1}{(l-x)\rho}=\frac{1}{\rho(b-l+x)}-\frac{1}{x\rho}$$

or
$$\frac{x}{l-x}=\left[\frac{a(l-b)}{b(l-a)}\right]^{1/2}.$$

For the second part of the question, let us assume that the ratio $x/(l-x)=\mu$, or $x=\mu l/(1+\mu)$.

Substituting this value of x in Eq. (i), we get

$$\frac{1}{\rho[a-\mu l/(1+\mu)]}=\frac{1}{r}+\frac{1}{\rho[l-\mu l/(1+\mu)]}$$

$$\frac{1}{r}=\frac{l-\mu l/(1+\mu)-a+\mu l/(1+\mu)}{\rho[a-\mu l/(1+\mu)][l-\mu l/(1+\mu)]}$$

or

$$r=\rho l[a(1+\mu)-\mu l]/(1+\mu)^2(l-a).$$

Example 12. *In a uniform submarine cable there is a leak resistance r per unit length and the resistance of the cable is R per unit length. At distance x along the cable the potential is V and the current is i. Show that* $dV/dx=-Ri$ *and* $di/dx=-V/r$. *Show also that if the cable is of length l and the two ends being at potentials* V_0 *and 0, the potential V is given by the relation*

$$V=V_0 \cosh (R/r)^{1/2} l \sinh (R/r)^{1/2} (l-x).$$

Assuming the submarine cable as the line with continuous leak. As V is the potential at a distance x from end A along the cable and i the current in the cable at this point. The potential at a distance $x+\delta x$ from A will be $V+(dV/dx)\delta x$. The potential current is thus $-(di/dx)\delta x$. Since the resistance of the insulation from x to $x+\delta x$ varies inversely as δx, hence at a distance x,

$$V=\left(-\frac{di}{dx}\delta x\right)\frac{r}{\delta x}=-r\frac{di}{dx}. \qquad \ldots(i)$$

For the element of the cable from x to $x+\delta x$, we get

$$V-\left(V+\frac{dV}{dx}\delta x\right)=i\,R\delta x,$$

or

$$-dV/dx=iR. \qquad \ldots(ii)$$

Eliminating i between Eqs. (i) and (ii), we get

$$V=-r\frac{d}{dx}\left[-\frac{1}{R}\frac{dV}{dx}\right]=\frac{r}{R}\frac{d^2V}{dx^2}.$$

or

$$d^2V/dx^2=(R/r)V. \qquad \ldots(iii)$$

The solution of this equation is

$$V=A \cosh (R/r)^{1/2} x+B \sinh (R/r)^{1/2}x.$$

If $V=V_0$, when $x=0$ and $V=0$ when $x=l$, we find

$$V=V_0 \cosh (R/r)^{1/2} l \sinh (R/r)^{1/2} (l-x).$$

Example 13. *In Fig. 6.27 a second source of emf=18 V and resistance 2 Ω has been connected to the first, like terminals of the sources being connected together, of emf 12 V and resistance 2 Ω. Calculate the total power out put. Find the effect of changing the polarity of the second source.*

Consider the clockwise direction as the postive direction of current.

$$E_1 = -12V,\ E_2 = 18V.$$

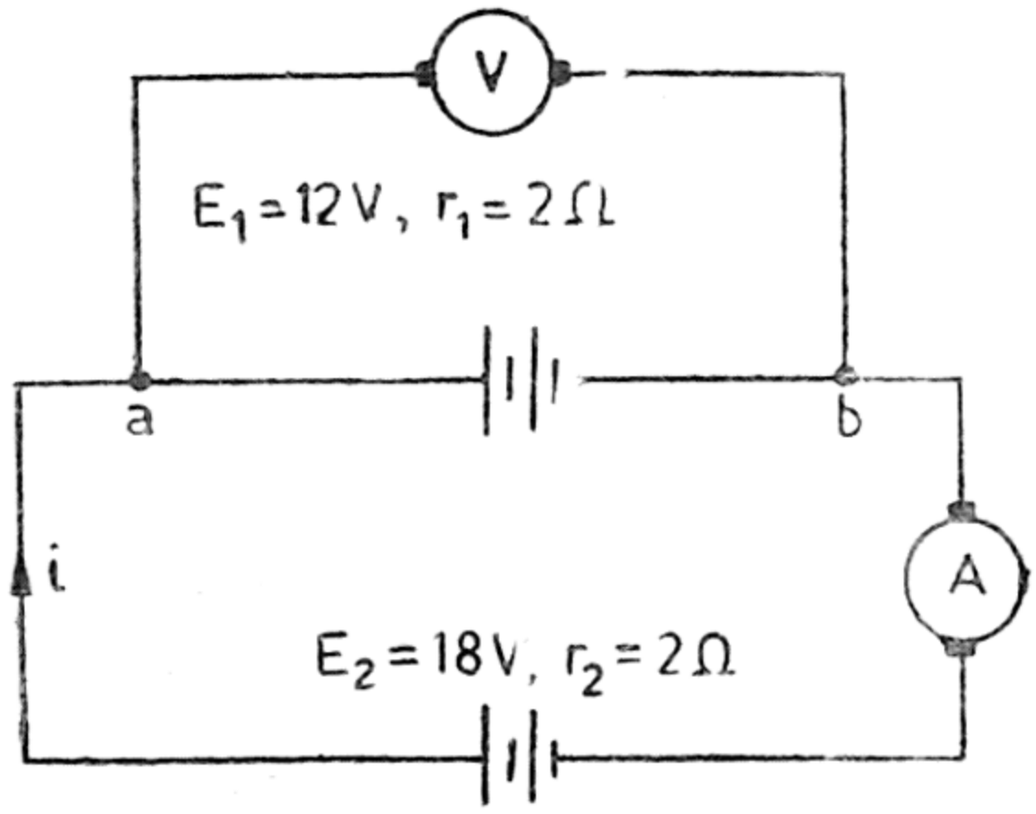


Fig. 6.27.

$\therefore\ \Sigma E = 18V - 12V = 6V.$

Hence the current in the circuit

$I = \Sigma E / \Sigma R = 6/4 = 1.5$ amp.

The rate of energy conversion in source 1st is $E_1 I = 12 \times 1.5 = 18$ watts. Within this source, electrical energy is converted to non-electrical energy. The rate of dissipation of energy in source 1st is $I^2 r_1 = (1.5)^2 \times 2 = 4.5$ watts.

$\therefore$ Total power input to source 1st $= 18 + 4.5 = 22.5$ watts.

The rate of energy conversion in source 2nd is $E_2 I = 18 \times 1.5 = 27$ watts. Within this source, non-electrical energy is converted to electrical energy. The rate of dissipation of energy in source 2nd is $I^2 r_2 = 4.5$ watts.

$\therefore$ Power output $= 27 - 4.5 = 22.5$ watts.

In the second part of question, when the sources are connected with unlike terminals together, the current

$$I = \Sigma E / \Sigma R = 30/4 = 7.5 \text{ amp.}$$

Non-electrical energy is converted to electrical energy in both sources. The rates of conversion are respectively.

$E_1 I = 12 \times 7.5 = 90$ watts and $E_2 I = 18 \times 7.5 = 135$ watts.

The rate of energy dissipation within the resources are respectively $I^2 r_1 = 112.5$ watts and $I^2 r_2 = 112.5$ watts.

$\therefore$ Total rate of energy conversion $= 225$ watts.

Hence the power output of source 2nd is $135 - 112.5 = 22.5$ *W*.

This equals the power input to source 1st. *Hence we see that the power output or input remains same and does not change with the change in polarity of a source.*

Example 14. *Using Kirchhoff's laws, find the current in each branch of the circuit shown in Fig. 6.28.* $V_1 = 5$ *volts,* $V_2 = 2$ *volts,* $R_1 = 3$ *ohms,* $R_2 = 2$ *ohms and* $R_3 = 4$ *ohms.*

Applying Kirchhoff's first law to the point represented by the dot, we get

$i_1+i_2-i_3=0$ or $i_3=i_1+i_2$.

Assuming the sources of negligible resistance and applying second law of Kirchhoff's to both the loops, we get

$$i_1R_1+(i_1+i_2)R_3=V_1$$
and $$i_2R_2+(i_1+i_2)R_3=V_2.$$

Solving these equations for i_1 and i_2, we get

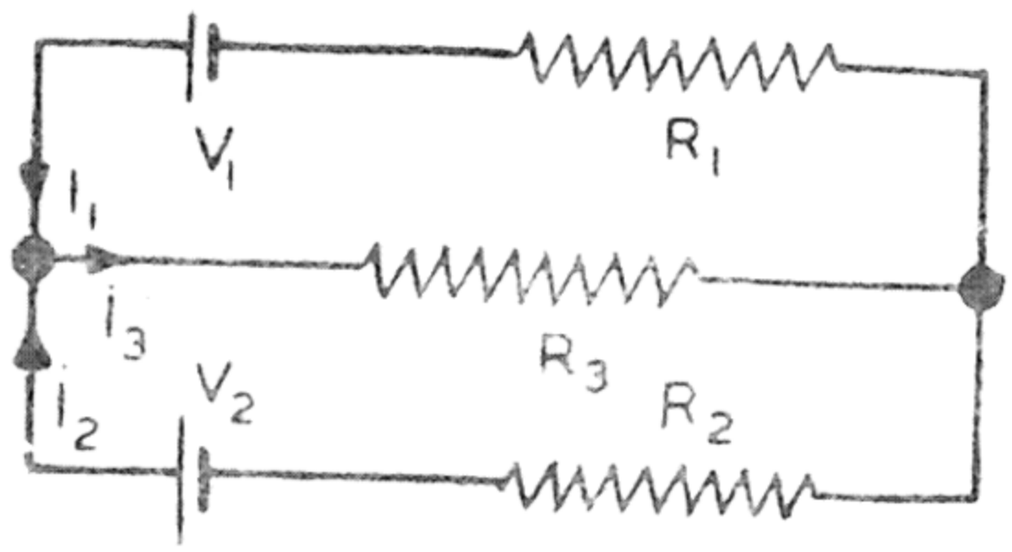


Fig. 6.28.

$$i_1=\frac{V_1(R_2+R_3)-V_2R_3}{R_1R_2+R_1R_3+R_2R_3} \quad \text{and} \quad i_2=\frac{V_2(R_1+R_3)-V_1R_3}{R_1R_2+R_1R_3+R_2R_3}$$

Substituting the numerical values, we get

$i_1=11/13$ amp. and $i_2=-3/13$ amp.

Example 15. *A system of conducting wires, all of equal resistance, from the edges of a cube. Find the currents carried by the wires if a battery is connected without extra resistance to adjacent vertices of the cube.*

By symmetry and Kirchhoff's first law the distribution of currents in various parts of the circuit is as shown in Fig. 6.29. If $\mathcal{E}$ is the emf, r the internal resistance of the battery and R the resistance of each edge of the cube.

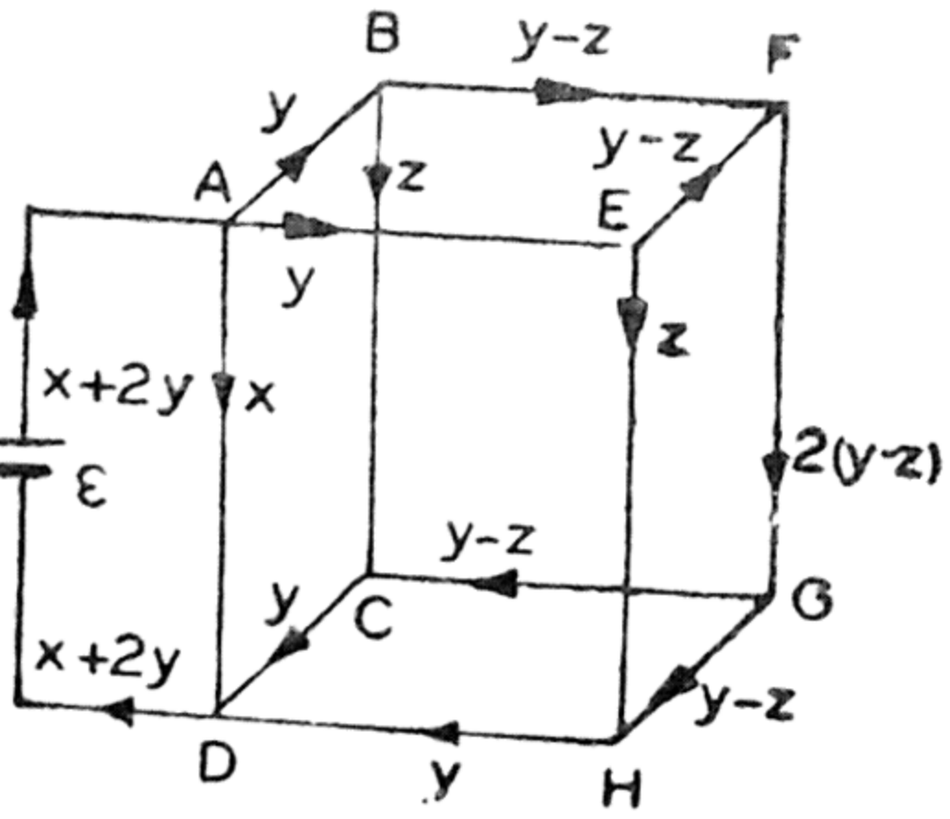


Fig. 6.29.

On applying Kirchhoff's second law to the mesh $ASDA$, we get

$$(x+2y)r+xR=\mathcal{E}. \qquad \ldots(i)$$

For the closed loop $ABCDA$, we have

$$yR+zR+yR-xR=0$$
$$(2y+z)-x=0 \text{ or } x=(2y+z) \qquad \ldots(ii)$$

For the closed loop $BFGCB$,

$$(y-z)R+2(y-z)R+(y-z)R-zR=0$$
$$\therefore \quad 4(y-z)-z=0 \quad \text{or} \quad y=5z/4. \qquad \ldots(iii)$$

Substituting this value in Eq. (ii), we get

$$x=5z/2+z=7z/2. \qquad \ldots(iv)$$

Substituting the values of x and y in relation (1), we get,

$$(\tfrac{7}{2}z+2\,.\,\tfrac{5}{4}z)r+\tfrac{7}{2}zR=\mathcal{E}$$

or

$$z=2\mathcal{E}/(7R+12r).$$

$$\therefore \qquad x=\frac{7\mathcal{E}}{7R+12r} \text{ and } \quad y=\frac{5\mathcal{E}}{14R+24r}.$$

Hence current along AB, AE, CD and HD

$$=y=5\mathcal{E}/(14R+24r).$$

Current along BC and $EH=z=2\mathcal{E}/(7R+12r)$.

Current along BF, EF, GC and $GH=y-z=\mathcal{E}/(14R+24r)$.

Current along $FG=2(y-z)=\mathcal{E}/(7R+12r)$.

Current through the battery$=x+2y=12\mathcal{E}/(7R+12r)$.

Example 16. *Twelve equal wires, each of resistance R ohm are connected so as to form a skelton cube. An electric current enters this cube from one corner and leaves out at the diagonally opposite corner. Calculate the total resistance of this assembly.*

Let $ABCDEFGH$ be the skelton cube formed by Twelve equal wires, each of resistance R ohm. Let the current entering at the corner A add and leaving at the diagonally opposite corner G be I. Since the resistance of each wire is the same, the total current I is divided at a point A into three equal parts, *i.e.*, $I/3$ along AB, AE and AD arms. At points B, E and D, this current is further divided into two equal parts, so that the current in each of the arms BC, BF, EF, EH, DH and DC is $I/6$. At points C, F and H, the currents combine, so that the current in each of the arms CG, FH, and HG is again $I/3$. These currents combine again at point G, such that the current leaving at point G is the current entering at point A, *i.e.*, I.

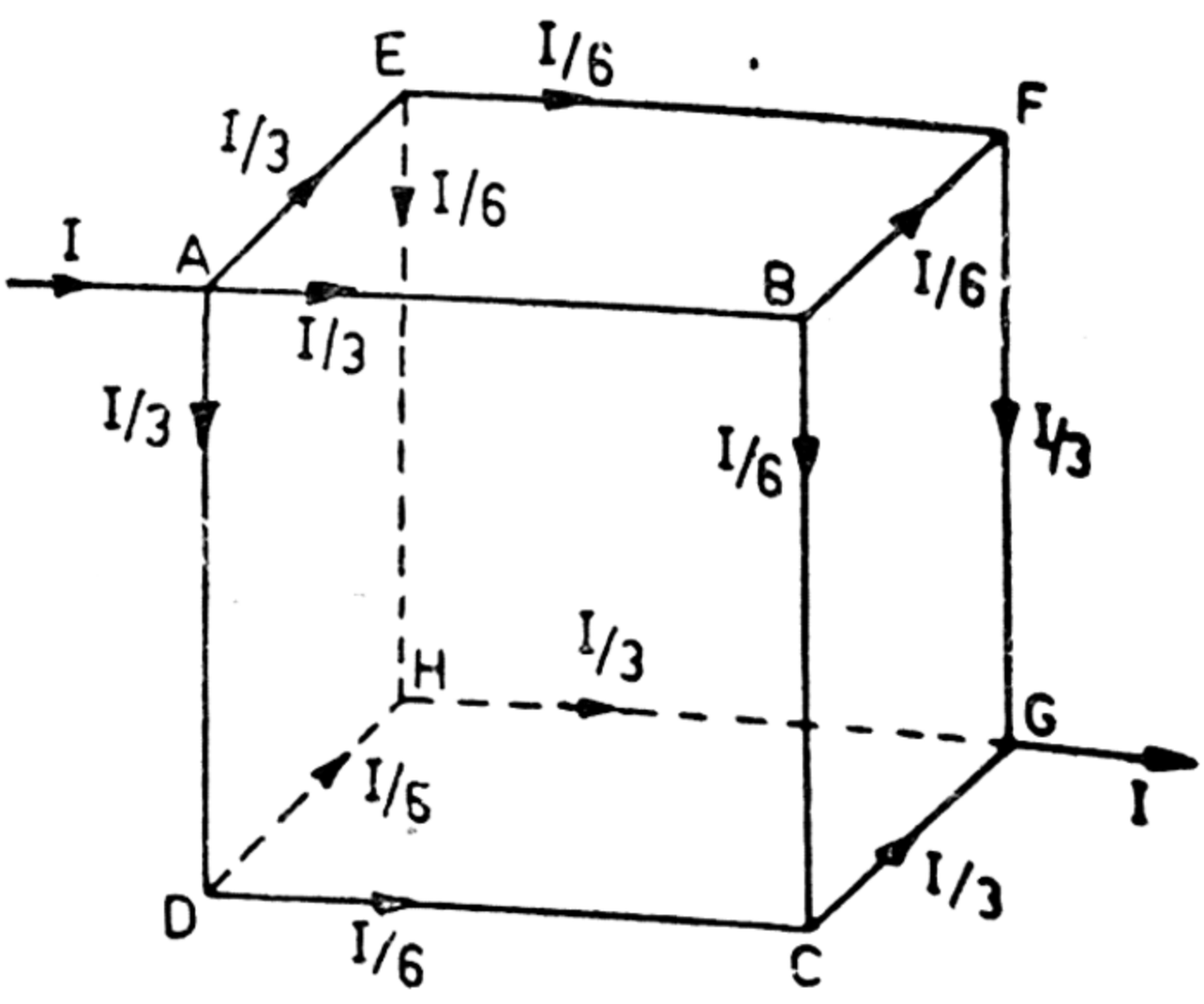


Fig. 6.30.

Let $\mathcal{E}$ be the potential difference between the points A and G. Applying Kirchhoff's second law to any one of the paths between these points, *i.e.*, $ABCG$, we get

$$\tfrac{1}{3}I\,R+\tfrac{1}{6}I\,R+\tfrac{1}{3}I\,R=\mathcal{E}=I\,R_{eq},$$

where R_{eq} is the equivalent resistance between the corners A and G.

$$\therefore \qquad R_{eq}=R\tfrac{1}{3}R+\tfrac{1}{6}R+\tfrac{1}{3}R=\tfrac{5}{6}R.$$

Example 17. *In a Wheatstone bridge, which is nearly balanced, the resistances of the arms are 1000, 10, 1000, 100.01 ohms. The cell has an emf of 2 volts and the negligible resistance. The galvanometer has a resistance of the 10 ohms and sensitivity 0.5 cm per 0.1 μA. Calculate values for the galvanometer deflections in the two possible arrangements of cell and galvanometer.*

Let the bridge be arranged as shown in Fig. 6.31 (*a*). The effective resistance of the bridge is 100 ohms and hence the current from the cell

$$I=\mathcal{E}/R=2/100=0.02 \text{ amp.}$$

By Kirchhoff's first law, we get

$$I=I_1+I_2 \qquad \text{or} \qquad I_2=0.02-I_1.$$

By applying Kirchhoff's second law to the loops *ABDA* and *BCDB* we get

$$100I_1+10I_g-1000(0.02-I_1)=0$$

$$10(I_1-I_g)-100.01(0.02-I_1+I_g)-10I_g=0$$

Solving these equations, we get

$$I_g=-1.49-10^{-7} \text{ amp}=-0.149 \ \mu A.$$

The −ve sign indicates that the current is from *D* to *B* through galvanometer.

∴ Deflection in galvanometer coil

$$=(0.5/0.1)\times 0.149=0.745 \text{ cm}$$

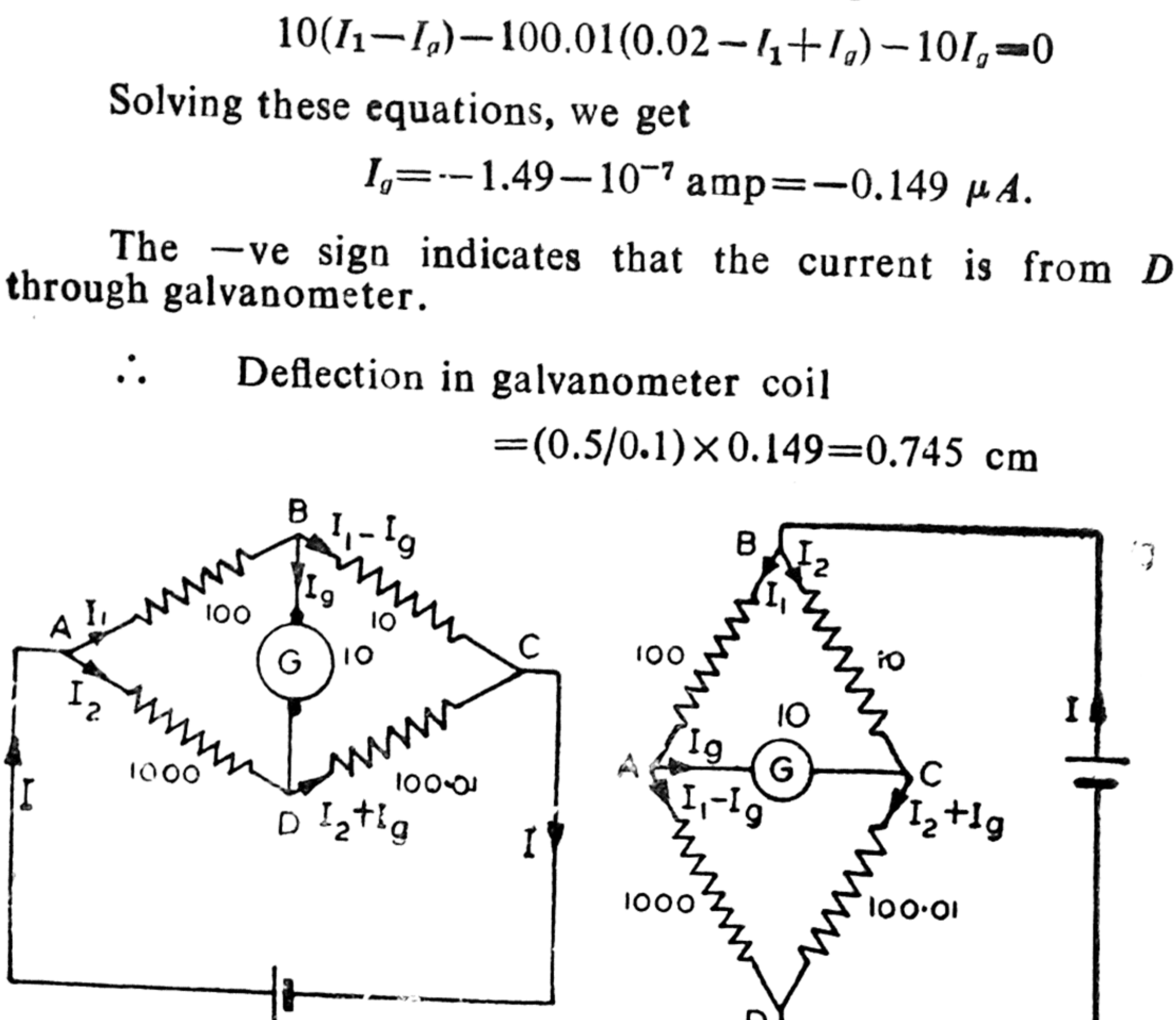


Fig. 6.31.

When the cell and galvanometer are interchanged, the current from cell is again 0.02 amp. and the currents are as shown in Fig. 6.31. (*b*). By Kirchhoff's first law, we have

$$0.02 = I_1 + I_2 \quad \text{or} \quad I_2 = 0.02 - I_1.$$

Applying Kirchhoff's second law to the meshes *ABCA* and *ACDA*.

$$100I_1 + 10I_g - 10(0.02 - I_1) = 0.$$

and
$$1000(I_1 - I_g) - 100.01(0.02 - I_1 + I_g) - 10I_g = 0.$$

Solving these equations, we get

$$I_g = -1.50 \times 10^{-7} \text{ amp} = -0.15\ \mu A.$$

Hence the current through the galvanometer flows from *C* to *A*.

∴ Deflection in galvanometer coil = 0.75 cm.

Thus we see that the current through the galvanometer is practically independent of the arrangement of the cell and the galvanometer. After calculating I_g in both the arrangements, it can also be noted that in the first instance the current through 100.01 Ω (unknown resistance) is much less than in the second arrangement thus a correction for the effect of joule heating will have to be made, if this resistance has a large resistance temperature coefficient, in the second arrangement.

Example 18. *A tetrahedron frame ABCD is formed by six wires, the resistances of opposite edges being equal. Prove that the whole resistance of the frame for a current entering at A and leaving at D is* $(r_1r_3 + 2r_1r_2 + r_2r_3)\ r_3/2(r_1 + r_3)\ (r_2 + r_3)$, *where* r_1 *is the resistance of AB or CD,* r_3 *that of AC or BD and* r_2 *that of AD or BC.*

By the help of Kirchhoff's first law the distribution of current is as shown in the figure. By applying Kirchhoff's second law to the mesh *ABDA*, we have

$$r_1I_1 + r_2(I_1 + I_3) - r_3\ (I - I_1 - I_3) = 0,$$

For the mesh *ACBA*

$$r_2I_2 + r_3I_3 - r_1I_1 = 0,$$

For the mesh *CDBC*

$$r_1(I_2 - I_3) - r_2(I_1 + I_3) - r_3I_3 = 0.$$

On rearranging, we get

$$(r_1 + r_2 + r_3)\ I_1 + r_3I_2 + r_2I_3 = r_2I,$$

$$-r_1I_1 + r_3I_2 + r_3I_3 = 0,$$

and $$-r_2I_1 + r_1I_2 - (r_1 + r_2 + r_3)\ I_3 = 0.$$

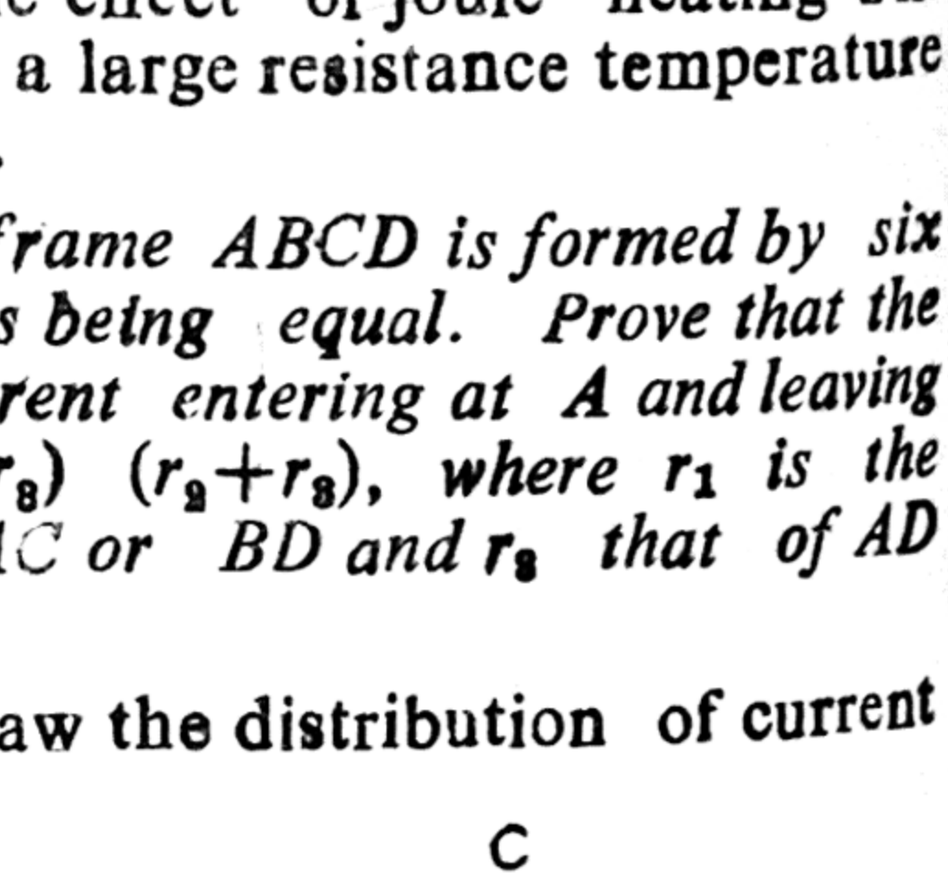


Fig. 6.32.

Solving these equations, we get

$$\text{Current through } AD = I - I_1 - I_3 = \frac{r_3\ (r_1 + r_2) + r_1(r_2 + r_3)}{2(r_1 + r_2)(r_2 + r_3)}\ I.$$

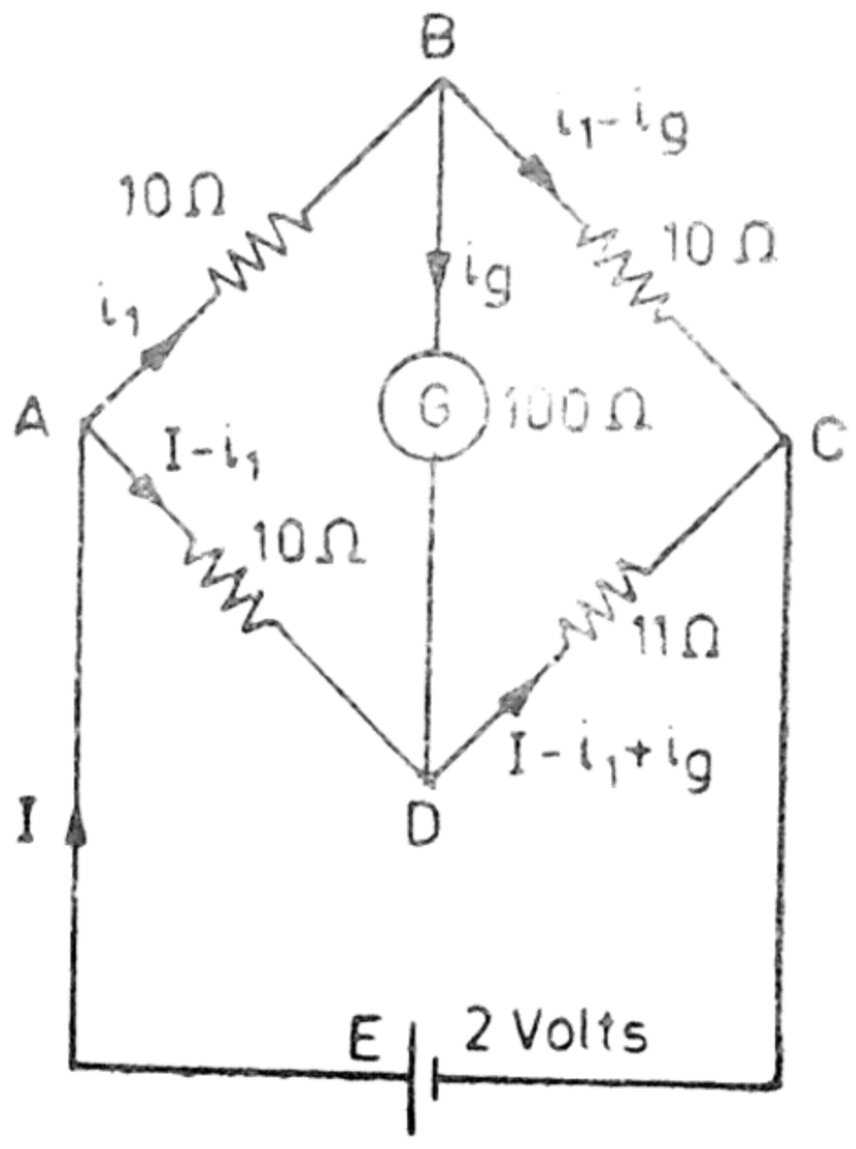
If R is the whole resistance of the frame for a current entering at A and leaving at D, then

$$IR=(I-I_1-I_2)r_3=\frac{r_2(r_1+r_3)+r_1(r_2+r_3)}{2(r_1+r_3)(r_2+r_3)}I\,r_3$$

or
$$R=(r_1r_3+r_2r_3+2r_1r_2)r_3/2(r_1+r_3)(r_2+r_3).$$

Example 19. *A Wheatstone bridge has three arms of 10 Ω each and one of 11 Ω. Find the current in the various arms of the bridge, taking the applied potential as 2 volts, the galvanometer resistance as 100 Ω and the battery resistance as negligible.*

The given resistances, 10, 10, 10 and 11 Ω form the four arms of a wheatstone bridge. There are two possible arrangements of galvanometer and battery. Let us solve the problem using an arrangement shown in Fig. 6.33.



Flg. 6.33.

If I be the current in the bettery arm, i_1 along AB, and i_g along BD, then Kirchhoff's first law gives the current in various arms as shown in Fig. 6.33. Now applying Kirchhoff's second law for the mesh $ABDA$.

$$i_1\times 10+i_g\times 100-(I-i_1)\times 10=0$$
or $20\,i_1+100\,i_g-10\,I=0$ ...(*i*)
for the mesh $BCDB$

$$(i_1-i_g)\times 10-(I-i_1+i_g)\times 11-i_g\times 100=0$$
or $21\,i_1-121\,i_g-11\,I=0$ ...(*ii*)
for the mesh $AECDA$

$$(I-i_1)\times 10+(I-i_1+i_g)\times 11=2$$
$$-21\,i_1+11\,i_g+21\,I=2 \quad \text{...(iii)}$$

Solving these equations, we get

$$i_1=\frac{231}{2315},\quad i_g=-\frac{1}{2315}\quad\text{and}\quad I=\frac{452}{2315}\ \text{amp.}$$

Therefore the currents through the arms AB, BC, AD, DC, DB and EA are $\frac{231}{2315}$, $\frac{232}{2315}$, $\frac{221}{2315}$, $\frac{220}{2315}$, $\frac{1}{2315}$ and $\frac{452}{2315}$ amp respectivity.

Example 20. *An accumulator having a steady emf of 2 volts is conneted across the potentiometer wire of length 10 meters. The*

potentiometer wire is of manganin having resistance of 11.5 ohm/meter. A cell gives a null point at 690 cm. If a resistance of 5 Ω is put in series with the potentiometer wire, find the new position of the null point.

Length of the potentiometer wire=10 meters.

$\therefore$ Fall in potential per meter=ρ=2/10=0.2 V/m.

As the cell gives a null point at 690 cm, hence its emf

$$=\rho l=0.2\times 6.9=1.38 \text{ volts.}$$

Resistance of the potentiometer wire=$11.5\times 10=115$ ohms.

When a resistance of 5 Ω is connected, total resistance becomes $115+5=120$ ohms and the current $i=V/R=2/120$ amp.

$$\therefore \text{ Potential fall per meter } = \frac{115}{120}\times 2\times \frac{1}{10}=\frac{23}{120}V/m.$$

If l' be the position of null point on the potentiometer wire, then

$$\frac{23}{120}l'=1.38 \quad \text{or} \quad l'=\frac{1.38\times 120}{23}=7.2 \text{ meters.}$$

Oral Questions

1. Name the physical quantities which are scalars but represented by an arrow in diagram.
2. What is the distinction between drift and rms velocities ?
3. Find the effect on the electron drift velocity of (*a*) doubling V, (*b*) doubling l and (*c*) doubling d in a wire of length l, diameter d across which a potential difference V is applied.
4. What is the physical meaning of nondivergence of **j** ?
5. When current passes through a conductor, which of the two i or j remains constant across every sectional area ?
6. What do you mean by non ohmic resistor ?
7. A current i enters the top of a coppor sphere and leaves at its diametrically opposite point. Whether the energy is dissipated equally from each point of the sphere ?
8. The Joule heating in a resistor known to very according to the relations $P=i^2$ $R=V^2/R$. How do you explain this different effect of resistance, so appreared ?
9. Giving some other examples, how can you explain that current is not used up as it flows through a resistor ? What is lost in this case ?
10. If two capacitors are charged to the same potential when one grasp the two wires leading from the two plates of a charged capacitor. Why the cause of sensation of shock given to the body is greater for a capacitor of higher capacitance ?

11. Why do you not use wire wound resistors in high frequency assembly ?

12. What is the most important point which must remain in your mind while using Post Office box for Mance's and Thomson's methods ?

Problems

1. Calculate the drift velocity for the copper wire of dia, 0.064 inch when a current of 10 amp passes through it. $(3.6\times10^{-4}\ m/sec)$

2. The electron of a hydrogen atom is considered to be revolving round a proton in a circular orbit of radius $\hbar^2/me^2$ with a velocity $e^2/\hbar$. Find an expression for current, the circulating charge is equivalent to.
$(me^5/2\pi\hbar^3)$

3. A copper wire having a cross section of 0.01 cm^2 carries a current of 1.0 ampere. If the number of free electrons in copper is $8.0\times10^{22}/cm^3$, calculate the average drift velocity of free electrons. $(0.78\times10^{-4}\ m/sec)$

4. In a neon gas discharge tube 2.9×10^{16} Ne^+ ions move to the right through a cross section of the tube each second, while 1.2×10^{18} electrons move to the left in the same time. Find the magnitude and the direction of the current.
(2.384 amp, left to right)

5. A silver wire 1 mm in a diameter carries a charge of 90 coulombs in 75 min. Silver contains 5.8×10^{22} free electrons per cc. Calculate (*a*) the current in the wire and (*b*) the drift velocity of the electrons in wire in meters/sec.
$(0.02\ amp.,\ 2.74\times10^{-6}\ m/sec)$

6. In vacuum diode, the anode is parallel to the cathode at a distance 0.2 cm. If a potential of 100 volts is applied to the anode with respect to cathode, calculate the velocity of electrons at the points at the middle of the electrodes.
$(4.2\times10^{-6}\ m/sec)$

7. A current of 5 amp exists in a 10 ohm resistance for 4 min. Calculate the number of electrons passing in this time through any cross section of the resistor.
(7.5×10^{31})

8. A steady beam of α-particles travelling with constant *K.E.* 20 *MeV* carries a current 2.5×10^{-7} amp. (*a*) How many α-particles strike the surface $\perp$ to the direction of beam in 3 sec. ? How many α-particles are there in a given 20 cm length of the beam at any instant ? $(2.3.\times10^{12},\ 5.0\times10^{3})$

9. A wire with a resistance of 20 ohms is drawn out so that its length becomes thrice the original length. Find the resistance of the streched wire.
(18 ohms)

10. The space between two coaxial cylinders of radii a and b and of length l is filled with a medium of conductivity σ. Show that the resistance between the two cylinders is $(1/2\ \pi\sigma t)\ \log_e (b/a)$.

11. Find the leakage conductance per meter of a cylindrical coaxial cable whose inner conductor has a radius of 0.125 inch and whose outer conductor has an inner radius of 0.375 in., if the space between is filled with material whose conductivity is 6.0×10^{-8} mho/m.
$(3.3\times10^{-7}\ mho)$

12. Show that if the arcs AC and BD in example 7 are taken to be two electrodes, the resistance of the conductor is changed to $(1/\sigma\alpha t)\ \log_e(b/a)$.

13. A fault of given resistance develops in a telegraph wire. Show that with a given battery transmitting signals, the current received is least when the fault is at the middle of the wire.

14. A telegraph wire joining two places A, B drops from one of its supports at a place C and rests on another wire which is earthed at both ends. If λ is the ratio of the current strength at A to that at B when the current in AB is sent from A and μ is the ratio when the current is sent from B, show that C divides AB in the ratio $(1/\mu - 1) : (\lambda - 1)$.

15. Express the rate of dissipation of energy in a resistor in terms of (1) potential difference and current, (2) resistance and current, (3) potential difference and resistance. Energy is dissipated in a resistor at the rate of 40 watts when the *pd* between its terminal is 60 V. What is its resistance ?
(90 Ω)

16. Five points are connected by ten wires, each pair being joined by a wire of the same resistance R. Show that the resistance to a current entering at one of the points and leaving at any other point is $\frac{2}{5}R$.

17. A battery of 6 volts and internal resistance 0.5 ohm is joined in parallel with another battery of 10 volts and internal resistance 1 ohm. The combination sends a current through an external resistance of 12 ohms. Find the current through each battery.
(—2.27 amp, 2.865 amp)

18. A power of 36 watts is to be expended in a resistor connected across the terminals of a battery having an emf of 20 volts and an internal resistance of 1 ohms. (*i*) What values of resistance will satisfy this condition. (*ii*) What will be the total power expenditure in each case.
(9 and 1/9 ohm ; 40 and 360 volts)

19. A source of emf $\mathcal{E}$ volts and internal resistance r ohms, supplies current to a heating coil. Calculate the resistance R of the heating coil in terms of r, so that (*i*) heating takes place most rapidly, (*ii*) three fourth of the total energy delivered by the source goes into the heating coil.
(r, 3r)

20. Eleven equal wires each of resistance r ohm form the edges of an incomplete cube. Find the total resistance from one end of the vacant edge of the cube to the other.
(1.4 r ohm)

21. The sides of a hexagon $ABCDEF$ are formed of wires each of resistance R and wires each of the same resistance R join BF and BE. Show that the equivalent resistance of the conductors between A and D is $40R/29$.

22. The resistances of three wires BC, CA and AB of the same uniform cross section and material are a, b and c respectively. Another wire from A of constant resistance d can make a sliding contact with BC. If a current enters at A and leaves at the point of contact with BC, show that the maximum resistance of the network is $(a+b+c)d/(a+b+c+4d)$.

23. A wire forms a regular hexagon and the angular points are joined to the centre by wires each of which has a resistance $1/n$ of the resistance of a side of the hexagon. Show that the resistance to current entering at one angular point of the hexagon and leaving it by the opposite point is $2(n+3)/(n+1)\ (n+4)$ times the resistance of a side of the hexagon.

24. Suppose a voltmeter reads the voltage of a very old cell to be 1.40*V* while a potentiometer reads its voltage to be 1.55*V*. What is (*a*) the emf of cell, (*b*) its internal resistance and (*c*) the current it would supply to a 5 Ω resistor ? Assume the voltmeter resistance to be 280 Ω.
(1.55 V, 30 Ω, 0.044 A)

25. A capacitor of capacitance 2μF, a resistor of resistance 10$M\Omega$ and a battery of emf 100 *V* are connected in series. The capacitor is initially uncharged. The switch is thrown to the up position for 20 sec and then quickly reversed to the down position. How much energy is eventually dissipated in the resistor ? *(10^{-2} joule)*

26. A nichrome heater dissipates 500 watts when the applied potential difference is 110 volts and the wire temperature is 800°C. If the wire is immersed in a bath of cooling oil of temp. 200°C, calculate the power dissipated. The applied *pd* remains the same and α for nichrome is about 4×10^{-4}/°C.
(620 watts)

7

Thermo-Electricity

7.1 SEEBECK EFFECT

In 1822 Thomson Johann Seebeck discovered that an emf could be produced by thermal means alone, in a circuit composed of two different metals with their junctions at different temperatures. This effect is called *Seebeck effect*, after his name. The two metals constitute a thermocouple, the emf produced in a circuit is called *thermo emf* (*thermoelectric motive force*) or a *Seebeck emf*, and the current in the circuit due to this emf is known as *thermo-electric current.*

In a *copper-iron couple*, the current flows from copper to iron through the hot junction and from the iron to copper through the cold junction. Similarly current flows from bismuth to antimony at the hot junction. A series of metals was given by Seebeck in the order that, when a circuit is made up of any two metals in the series, the current flows across the hot junction from the earlier occurring metal in the series to the one occurring latter. It is called *Seebeck's series*, and is *Bi, Ni, Co, Pd, Pt, U, Cu, Mn Ti, Hg, Pb, Sn, Cr, Mo, Rh, Ir, Au, Ag, Zn, W, Cd, Fe, As, Sb, Te.* The separation of the metals in this series is the representative of the magnitude of the emf to be expected with a given temperature difference between the junctions. Thus we see that the emf produced between the junction of a *Bi—Sb* couple is greater than that of *Cu—Fe* couple. The metals to the left of *Pb* are called thermo-electrically negative and those to its right are thermoelectrically positive. The amount of emf produced for different thermocouples, taking lead as one of the two metals and cold junction at 0°C, is given below :

Other metal	*Sb*	*Fe*	*Rhodium*	*Cu*	*Pt*	*Ni*	*Constantan*	*Bi*
$\times 10^{-9} V/°C$	*31*	*14*	*2*	*−3*	*−4*	*−18*	*−39*	*−69*

Thus emf for Sb−Bi couple=31−(−69)=100 micro volts/°C and for Cu −Fe couple=−3−(+14)=−17 micro volts/°C.

For Cu—Fe couple, it is useful to remember Hot-coffee indicating that current flows from Cu to Fe at the hot junction. For Bi—Sb couple the word ABC is useful to remember. It indicates that current flows from antimony **(A)** to bismuth **(B)** at cold junction.

Let us consider a thermocouple of two metals, *Fe* and *Cu* (say). Their junctions are kept cold and hot. If a galvanometer G is connected in the circuit the galvanometer deflection is observed which corresponds to thermoelectric current. If the temperature of the hot junction be continuously increased, thermoelectric current increases until maximum value is reached. The temperature of the hot junction at which maximum current flows is constant for a given thermocouple. It is independent of the temperature of cold junction and is called the *neutral temperature* for the couple. On further raising the temperature of hot junction the current decreases to zero and is then reversed. The temperature at which the current is zero and its reversal begins is called the *temperature af inversion or inversion temperature*. It depends on the temperature of cold junction and the chosen couple. It is always as much above the neutral temperature as the cold junction is below it, *i.e.*, $T_{inversion} - T_{neutral} = T_{neutral} - T_{cold}$. The graph between the emf and the temperature difference of the two junctions is of the parabolic form for majority of couples. Thus emf can be given as

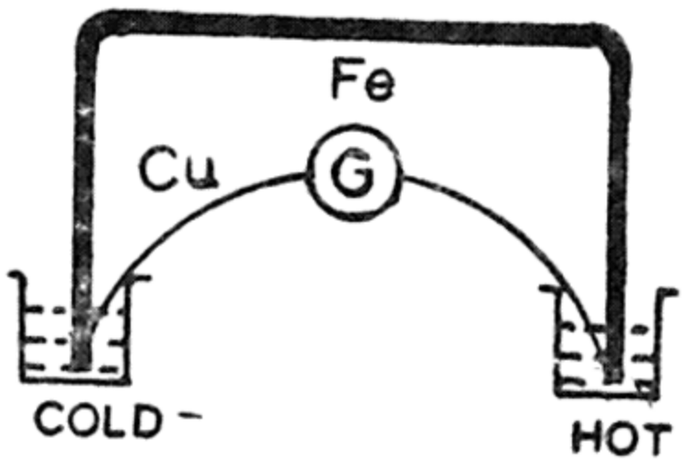


Fig. 7.1.

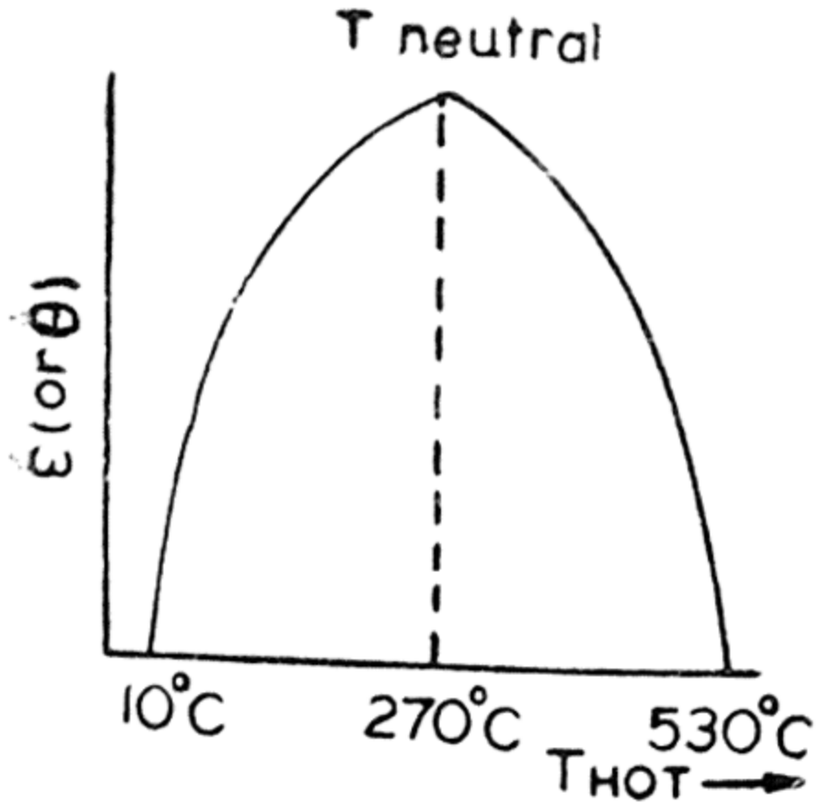


Fig. 7.2.

$$\mathcal{E} = a + bt + ct^2. \qquad \ldots(1)$$

The production of the emf at junction of two metals can be explained by *Modern theory*, which gives us knowledge of the structure of conducting substances. It suggests that all metals contain large number of free electrons. These electrons behave like the molecules of a gas and are referred often as *electronic gas*. The density of free electrons is different in different metals and depends upon the temperature. When two different metals are connected to form two junctions and these junctions are kept at different temperatures, free electrons will diffuse into one another metal at the junctions. Net flow is from higher electron pressure to the metal of lower electron pressure. If the junctions are at the same temperature the emf at one junction will be equal and opposite to that at the other junction. If these junctions are at different temperatures the electron diffusion takes place at different rates at the junctions and there is a net motion of the electrons. This sets emf between the junctions. The Seebeck emf is the line integral of the field, which drives the electrons, around the thermocouple.

$$\mathcal{E}_{BA\ (T_1, T_2)} = \frac{1}{e}\int_{T_1}^{T_2} (S_A^* - - S_B^*)dT, \qquad \ldots(2)$$

where S_A^* is the property of the metal, a function of temperature, a positive quantity and is known as *entropy transport per electron.*

If S_A^* is greater than S_B^*, electrons are transferred more rapidly along metal *A* toward lower temperatures than along metal *B* and the ^+ve current flows from *A* to *B* at the hot junction. The symbol $\mathcal{E}_{BA}(T_1,T_2)$ can be read as *emf from B to A at temperature* T_1 *less than* T_2.

7.2. PELTIER EFFECT

In 1834, Jean C.A. Peltier discovered that if a current be allowed to pass through the junction of two different metals, the heat is either evolved or absorbed at the junctions, *i.e.*, the junction gets either heated or cooled. This effect is known as *Peltier effect* and the heat evolved or absorbed as *Peltier heat*, after the discoverer. Experiment has shown that the Peltier heat transferred at any junction is proportional to the quantity of electricity crossing the junction and changes its sign (cold junction becomes hot and hot becomes cold) when reversing the direction of current. Thus we see that the Peltier effect is *reversible*. The Peltier effect must not be confused with the Joule's effect. The main differences are :

Peltier's Effect	*Joule's Effect*
1. It is a reversible effect. 2. It takes place only at the junctions. 3. It may be heating or a cooling effect. 4. It is proportional to current only. 5. It depends upon the current direction.	1. It is an irreversible effect. 2. It takes place at all points of the circuit and arises from its resistance. 3. It is always a heating effect. 4. It is proportional to the square of current and to the resistance. 5. It is independent of the sign and direction of current.

The Peltier effect may be explained as : metals contain free electrons and can be considered as moving like the gas molecules. Since the number of electrons per unit volume differs in different metals, there will be a flow of electrons from one metal to the other when they are in contact. This flow continues until sufficient charge has been carried at the junction to build up a potential difference of a value high enough to prevent any further movement at the junction. Work must be done for the passage of current. The energy absorbed (or evolved) at the junction when one coulomb of electricity flows is called the *Peltier Coefficient*. It is denoted by π. It is ^-ve, when heat is evolved. It varies as the absolute temperature of the junction and also depends on the metals used. Its unit is joules/coulomb, *i.e.*, volts and is given by

$$\pi_{AB} = (T/e)\,[S_B^*(T) - S_A^*(T)]. \qquad \ldots(4)$$

If the temperatures of the two junctions are the same, the Peltier coefficients have the same value. Let the junctions are at

temperatures T_2 and T_1, where $T_2 > T_1$. Then the Peltier coefficients under these conditions are $[\pi_{AB}]_{T_2}$ and $[\pi_{AB}]_{T_1}$. Let dq amount of charge moves round the circuit in a clockwise direction. Hence in one complete round, energy gained by the charge dq

dW = Heat absorbed at the hot junction − Heat liberated at the cold junction

$$= \{[\pi_{AB}]_{T_2} - [\pi_{AB}]_{T_1}\}\, dq.$$

If the Peltier effect is the only thermoelectric effect in the circuit, then the electromotive force, known as Peltier *emf* $\mathcal{E}$, is given as

$$\mathcal{E} = dW/dq = [\pi_{AB}]_{T_2} - [\pi_{AB}]_{T_1}.$$

This Peltier *emf* must not be confused with the constant potential difference between metals. The latter depends upon the work function ϕ, the work required to remove free electrons from the metal. The distinction is clear on considering an open circuit formed by two metals A and B. The contact potential difference V_{AB} between the metals A and B is defined as the pd between a point P just outside metal A and a point Q just outside metal B. If ϕ_A and ϕ_B be the work functions of metals A and B respectively and let π_{AB} be the Peltier coefficient at the junction R. The potiential difference

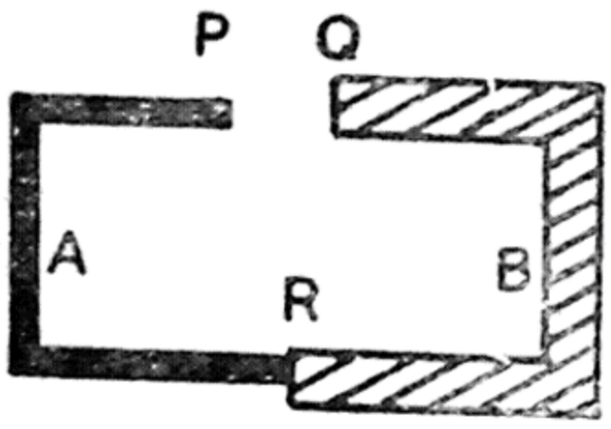


Fig. 7.3.

$$V_{AB} = -\phi_A/e - \pi_{AB} + \phi_B/e = (\phi_B - \phi_A)/e - \pi_{AB}$$

$$= (\phi_B - \phi_A)/e \text{ (Approx)}. \qquad \ldots 5)$$

Here the second term is neglected as the thermo *emf* is of the order of a millivolt, while the contact potential is of the order of 1 volt.

In experiments, the Joule heating effect is dominant, hence it is very difficult to measure the Peltier coefficient by direct determining of heating and cooling at the junction. An attempt for direct measurement was made by Le Roux in 1867. A $Cu - Fe$ junction was immersed in a water bath and a current I was allowed to pass for time t in the direction iron to copper. The heat produced H_1 was given as

$$JH_1 = I^2Rt + \pi It,$$

where R was the total resistance of the immersed metals. The direction of current was reversed. The heat produced H_2 in this case was given as

$$JH_2 = I^2Rt - \pi It. \qquad \ldots(7)$$

$$\therefore \quad \pi = J\,(H_1 - H_2)\,/2It. \qquad \ldots(8)$$

The existence of the Peltier effect is confirmed if H_1 is not equal to H_2. Peltier's original arrangement was as is shown in Fig. 7.4. Thick bars of antimony and bismuth were soldered at junctions

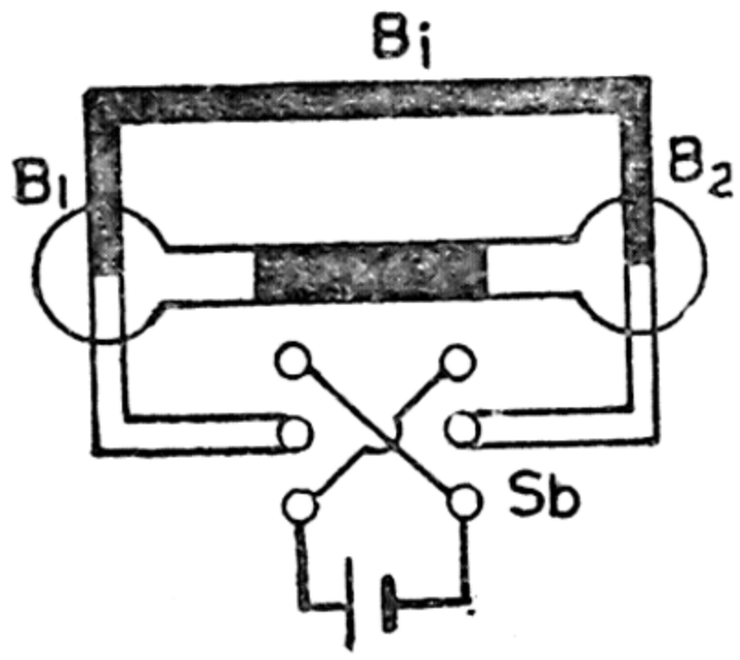


Fig. 7.4.

and the junctions were enclosed in evacuated glass bulbs B_1 and B_2 joined by a thin capillary tube. When the current was allowed to pass from a battery in one direction, mercury thread in the capillary shifted in one direction and when the direction of current was reversed, the thread shifted in the opposite direction. As *Bi* and *Sb* were taken in the form of thick rods, the Joule heating effect was negligible and the shift in thread of mercurry was due to Peltier effect only.

Thermodynamics of Peltier effect. Since the Peltier effect is a reversible effect and the heat is absorbed at the hot junction and evolved at the cold junction in a thermocouple. Lord Kelvin suggested that like in a reversible heat engine in thermocouple arrangement the ratio of the heat absorbed at temperature T_1 to that given up at T_2 ($T_2 < T_1$) should be the same as that of T_1 to T_2, where T_1 and T_2 are absolute temperatures. In other words *we can write that for all the sources and sinks in a reversible cycle the sum of the quantities dH/T is zero, i.e.,*

$$\oint dH/T = 0. \qquad \ldots(9)$$

This relation is only correct when the irreversible processes occurring in the circuit produce negligible effects. The Joule effect is reduced by using metals in the form of thick bars and by passing very small current in the circuit.

If π_1 and π_2 are the Peltier coefficients at temperatures T_1 and T_2 respectively. From the definition of Peltier coefficient, heat absorbed at the hot junction is $\pi_1 q$ and the heat given out to the cold junction is $\pi_2 q$, when the charge q moves round the circuit at infinitely small speed. Hence,

$$\frac{\pi_1 q}{\pi_2 q} = \frac{T_1}{T_2} \text{ or } \pi_1 - \pi_2 = \frac{\pi_2}{T_2}(T_1 - T_2).$$

The left hand side represents the value of Peltier effect and can be written as $\mathcal{E}$, the Peltier emf. Thus

$$\mathcal{E} = \pi_1 - \pi_2 = \frac{\pi_2}{T_2}(T_1 - T_2). \qquad \ldots(10)$$

If one junction is maintained at constant temperature T_2 then π_2 is constant and $\mathcal{E} \propto (T_1 - T_2)$. But this is contrary to experience, experiments show that this emf is a parabolic function of the temperature difference. Lord Kelvin (Prof. W Thomson) concluded in 1851 that the Peltier emf's at the junctions are not the only emfs acting in circuit. He predicted, from theoretical considerations, the existence of reversible heat effects in metals due to the differe-

nce of temperature across their ends. This effect is known as *Thomson effect*, after the discoverer.

7.3. THOMSON EFFECT

If the temperature at any two points in a conductor is different, the density of the electron gas must differ, as a result potential gradient is developed along the conductor between these points. This effect was first predicted by Thomson and is known as *Thomson effect*. Thus *Thomson effect is the absorption or evolution of heat due to flow of current in an unequally heated conductor.* In the thermocouple heat is absorbed or evolved at the junctions due to Peltier effect and along one or both the conductors forming this thermocouple due to Thomson effect. Thomson effect is also a reversible effect and is of similar origin to the Peltier effect, as both are due to non-uniform distribution of electrons in metal. Although the former is due to a change of temperature while the latter is due to a change of metal. The *emf* produced across the conductor is called the *Thomson emf*. It is positive from cold to hot, as the electron density is greater at lower temperatures. The Thomson *emf* between two points on a conductor which differ in temperature by dT may be expressed in the form $\sigma\, dT$, where σ is known as *Thomson coefficient* or the *specific heat of electricity*. σ is a function of temperature (changes from point to point) and also depends upon the material of the conductor. If the ends of a conductor are at temperatures T_1 and T_2 then the total energy absorbed or evolved when unit charge flows through it is the Thomson *emf* and is given by

$$\mathcal{E} = \int_{T_2}^{T_1} d\mathcal{E} = \int_{T_2}^{T_1} \sigma\, dT. \qquad \ldots(11)$$

If a current is passed along a copper bar AC, heated at the middle B, heat will be absorbed in the part AB and evolved in the

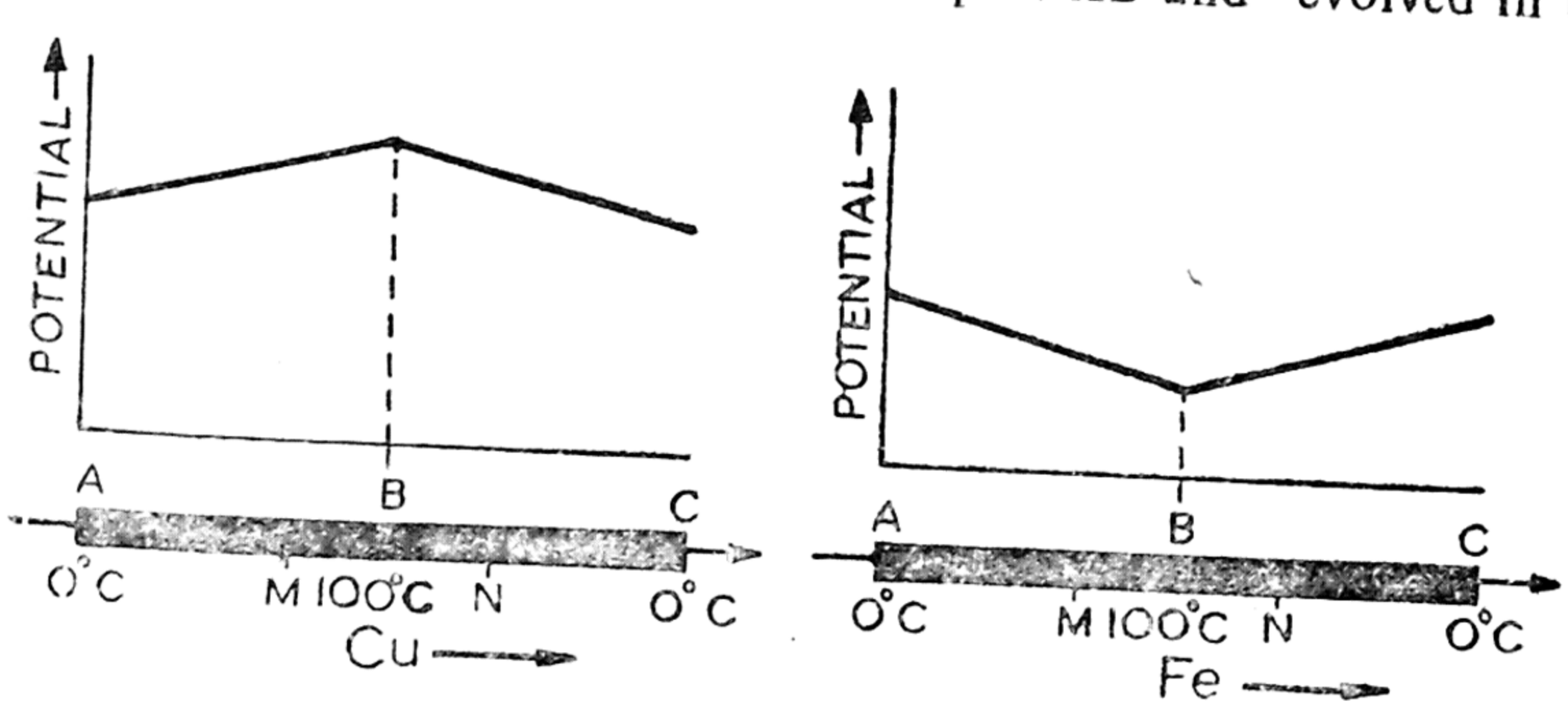


Fig. 7.5.

part BC. It can be shown by selecting two equidistant points M and N from point B, the N will be at a higher temperature than M because of Thomson effect. Thus we see that in copper *the heat is transferred, due to current flow in the direction of current. Hence σ for Cu is*

positive. Thomson effect and thus Thomson coefficient σ is positive also for *Cd*, *Zn*, *Ag* and *Sb*. In the case of iron, the colder parts are at higher potentials. If the current passes along *ABC* in the wire, heat will be evolved in *AB* and absorbed in *BC*. Among two equidistant points *M* and *N*, *N* will be at lower temperature than *M* because of the Thomson effect. Thus in iron, heat is transferred due to current flow in the direction opposite to the direction of current flow and hence is said to have *negative Thomson effect*. Iron like other metals are *Bi*, *Co*, *Pt* and *Ni*. The Thomson effect is zero in lead. *Experiments show that the metals coming before Pd in Seebeck series show positive Thomson effect and the metals after Pb in that series show negative Thomson effect.*

The idea of the specific heat of electricity was given by Lord Kelvin as the Thomson phenomenon in *Cu* is analogous to the one which occurs when heat is absorbed by a liquid flowing in a tube, heated at one end. The amount of heat absorbed or evolved by unit mass of liquid in passing from one point to another, having temperature difference *dT*, is *sdT*. The amount of heat absorbed or evolved when a unit quantity of electricity passes through the conductor having temperature difference *dT* between its ends is σdT. Thus we see that σ is analogous to *s*, the *specific heat*. Metals for which σ is positive have a positive specific heat of electricity and are analogous with a real liquid as regards absorption or evolution of heat. Metals having negative specific heat of electricity behave like a hypothetical liquid which absorbs heat on cooling and evolves on heating. Comparing with a liquid, it can be shown that the *Peltier coefficient π is analogous with a latent heat of liquid L.*

The existence of Thomson effect can also be explained on the electron theory, similar to that of Peltier effect. However the electron theory can explain the existence of the positive Thomson effect but can not explain the possibilities of zero and — ve values of σ.

Demonstrations of the Thomson effect. For this purpose, Lord Kelvin used a thick strip of iron bent in zig-zaz manner. The free ends of this strip were connected to a battery through a reversing key. The corners *B* and *F* were kept cold and the corner *D* was kept hot by heating it to a higher temperature. When no current was passing the corners *C* and *E* were found at equal temperatures on account of thermal conduction. When a current was allowed to pass through this strip, the thermometer at *C* indicated a higher temperature than the point *E*. The thermometer readings were reversed when the direction of current was reversed.

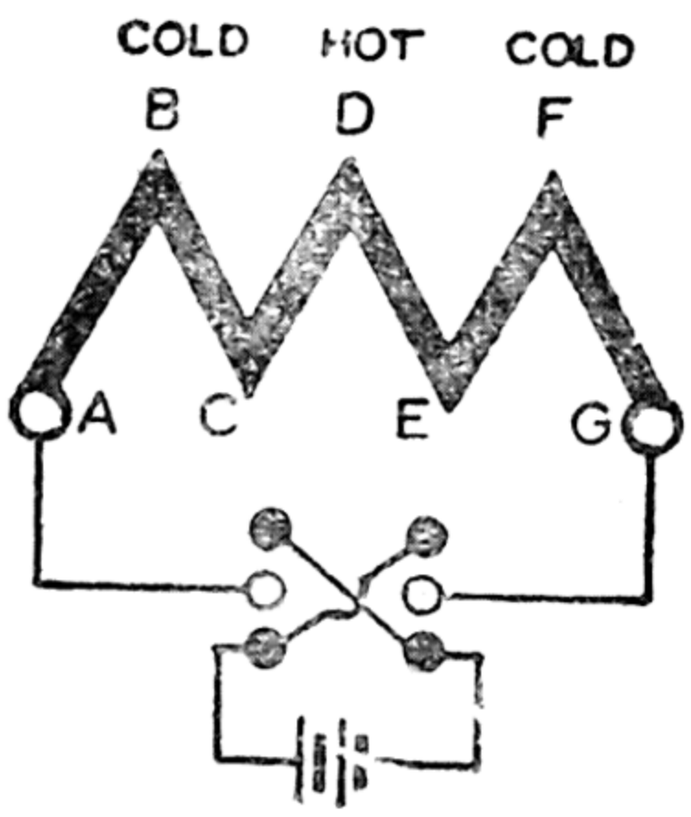


Fig. 7.6.

Another demonstration of the Thomson effect is due to *S. G.* Starling. In this experiment a large thick iron rod is bent into a U-shape, the bent is made red hot and the ends, kept cool in mercury baths *A* and *B*, are connected to a source of heavy current. The resistance coils R_1 and R_2 are wound very close together on the limbs and packed round with asbestos wool. These are placed in gaps of a meter bridge. The balance is first obtained without passing any current through the rod. Now a heavy current is passed through this rod, it causes a disturbance in the balance. This shift in the balancing point indicates the change in resistance of the coils due to change of temperature on account of absorption or evolution of heat along the limbs of the U-shaped iron rod. It thus confirms the existence of Thomson effect.

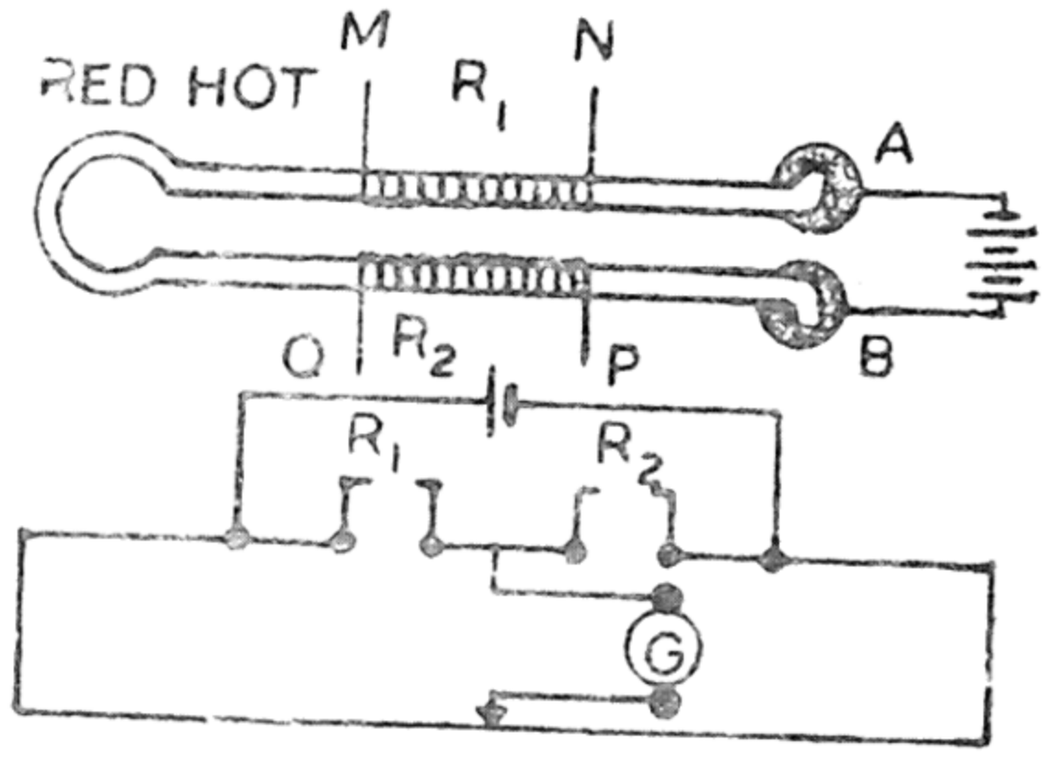


Fig. 7. 7.

7.4. TOTAL EMF IN A THERMOCOUPLE

Let us consider a thermocouple, composed of conductors *X* and *Y*. Let the current be positive from *X* to *Y* at the hot junctions and the temperatures of the hot and cold junctions be T_1 and T_2 respectively. The total emf responsible for the thermocurrent must be the sum of potentials due to Peltier and Thomson effects. Let π_1 and π_2 be the Peltier coefficients at the junctions at T_1 and T_2 respectively and σ_x and σ_y be the Thomson coefficients for the metals *X* and *Y* respectively. When the electric charge flows in the thermocouple, the heat is absorbed and evolved at the junction due to Peltier effect and all along the conductors due to Thomson effect. The total emf round the circuit $\mathcal{E}$ is the energy gained by unit charge when passing round the circuit.

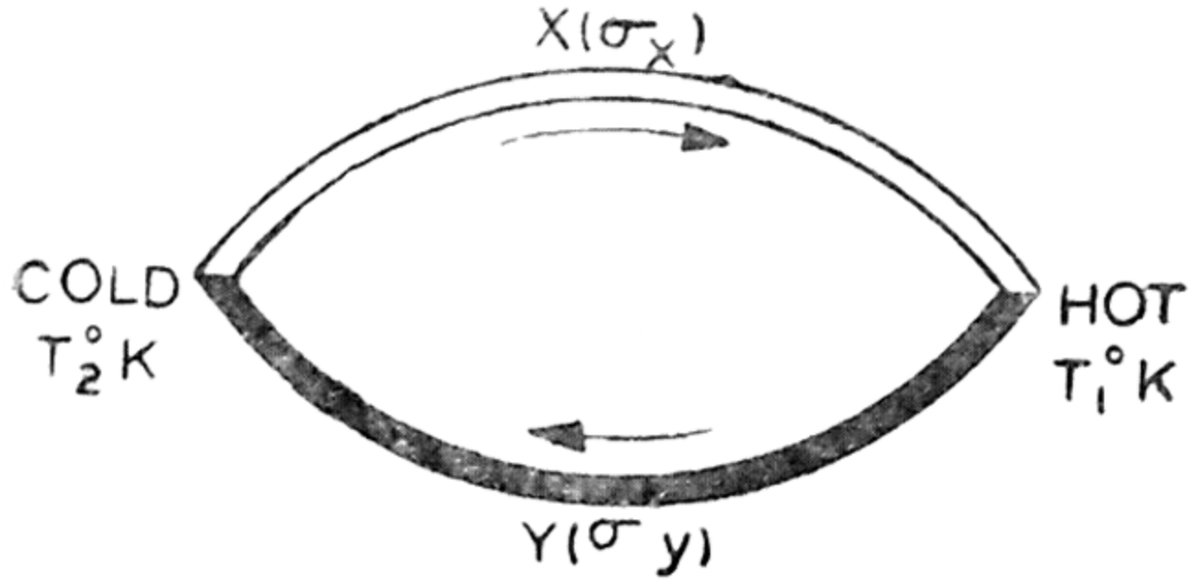


Fig. 7.8.

$$\therefore \quad \mathcal{E} = \text{Sum of pds. across various parts of the circuit}$$

$$= \pi_1 - \pi_2 + \int_{T_2}^{T_1} \sigma_x dT - \int_{T_2}^{T_1} \sigma_y dT$$

$$= \pi_1 - \pi_2 + \int_{T_2}^{T_1} (\sigma_x - \sigma_y) dT. \qquad \ldots(12)$$

It is equal to the total Seebeck emf available in the circuit.

Thermodynamics of Thermo emf. Let us now compare thermoelectric circuit to a *reversible heat engine*, as the Peltier and Thomson effects are reversible and Joule heating effect is negligible. Let the two junctions of a thermocouple be at temperatures $T+dT$ and T and the corresponding values of Peltier coefficients are $\pi+d\pi$ and π. From second law of thermodynamics we know that the total change of entropy in traversing one complete cycle of a perfectly reversible engine is zero. Hence we have

$$\frac{\pi+d\pi}{T+dT}-\frac{\pi}{T}+\frac{(\sigma_x-\rho_y)dT}{T}=0.$$

or
$$\frac{Td\pi-\pi dT}{T(T+dT)}+\frac{(\sigma_x-\sigma_y)dT}{T}=0.$$

As dT is the small change in T, hence $T(T+dT)$ in the denominator of the first term can be written as T^2. Thus we have

$$\frac{\pi}{T}=\frac{d\pi}{dT}+(\sigma_x-\sigma_y). \qquad ...(13)$$

Considering the temperature of cold junction T as constant, the temperature of hot junction is varied. The variation of emf with the temperature of hot junction may be obtained by differentiating relation (12) with respect to T, as

$$\frac{d\mathcal{E}}{dT}=\frac{d\pi}{dT}+(\sigma_x-\sigma_y). \qquad ...(14)$$

This quantity is called the *Thermoelectric power* or *relative Seebeck coefficient* for the two metals and is denoted as P_{xy}.

From relations (13) and (14), we get

$$\pi/T=d\mathcal{E}/dT \quad \text{or} \quad \pi=T(d\mathcal{E}/dT). \qquad ...(15)$$

Thus for any two metals the Peltier coefficient at a given temperature is given by the product of the thermoelectric power $(d\mathcal{E}/dT)$ at that temperature and the absolute temperature of the junction.

Differentiating Eq. (15) w.r. to T, we get

$$\frac{d\pi}{dT}=T\frac{d^2\mathcal{E}}{dT}+\frac{d\mathcal{E}}{dT}. \qquad ...(16)$$

Combining this with Eq. (14), we get

$$\sigma_x-\sigma_y=-T(d^2\mathcal{E}/dT^2). \qquad ...(17)$$

It gives a relationship for the difference between the two Thomson coefficients and second derivative of the graph of $\mathcal{E}$ against T. Hence by measuring Seebeck *emf* against the temperature, $(\sigma_x-\sigma_y)$ can be calculated.

σ is zero for lead, is+ve for *Cu*-like metals and is—ve for *Fe*-like metals. Hence σ can be obtained by using one metal as lead and other the unknown metal. Thus

$$\sigma_{Cu} = -T(d^2\mathcal{E}/dT^2) \text{ and } \sigma_{Fe} = +T(d^2\mathcal{E}/dT^2).$$

7.5. EXPERIMENTAL LAWS OF THERMOELECTRIC CIRCUITS

There are two simple and experimentally established simple laws.

(1) **Law of successive temperatures** (law of intermediate temperatures). This states that for a given thermocouple, the emf for any specified temperature difference is equal to the sum of the emf's corresponding to any smaller intervals into which the given range of temperature may be subdivided. Thus

$$\mathcal{E}_{T_1}^{T_n} = \mathcal{E}_{T_1}^{T_2} + \mathcal{E}_{T_2}^{T_3} + \mathcal{E}_{T_3}^{T_4} + \cdots \mathcal{E}_{T_{n-1}}^{T_n} . \qquad \ldots(18)$$

It can be proved as under : Let us chose any temperature T intermediate between T_1 and T_2. If $\mathcal{E}_1$ and $\mathcal{E}_2$ be the emf's for the temperature intervals $T_1 - T$ and $T - T_2$ respectively, we have

$$\mathcal{E}_1 = \pi_{T_1} - \pi_T + \int_T^{T_1} (\sigma_x - \sigma_y)dT \qquad \ldots(19)$$

and $$\mathcal{E}_2 = \pi_T - \pi_{T_2} + \int_{T_2}^{T} (\sigma_x - \sigma_y)dT. \qquad \ldots(20)$$

Adding these

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \pi_{T_1} - \pi_{T_2} + \int_{T_2}^{T_1} (\sigma_x - \sigma_y)dT. \qquad \ldots(21)$$

Which is the emf for the temperature difference $T_1 - T_2$, we may therefore divide $T_1 - T_2$ into any number of intervals with the same result.

(2) **Law of intermediate metals.** This states that the insertion of an additional metal or metals into any thermoelectric circuit does not alter the effective emf of the circuit, provided that the both ends of each such conductor are at the same temperature. For this let us include a third metal Z in the thermocouple consisting of two metals X and Y. The total emf is thus given by

$$\mathcal{E} = [\pi_{xy}]_{T_1} + \int_{T_1}^{T_2} \sigma_y dT + [\pi_{yz}]_{T_2} + \int_{T_2}^{T_2} \sigma_z dT + [\pi_{zx}]_{T_2} + \int_{T_2}^{T_1} \sigma_x dT.$$

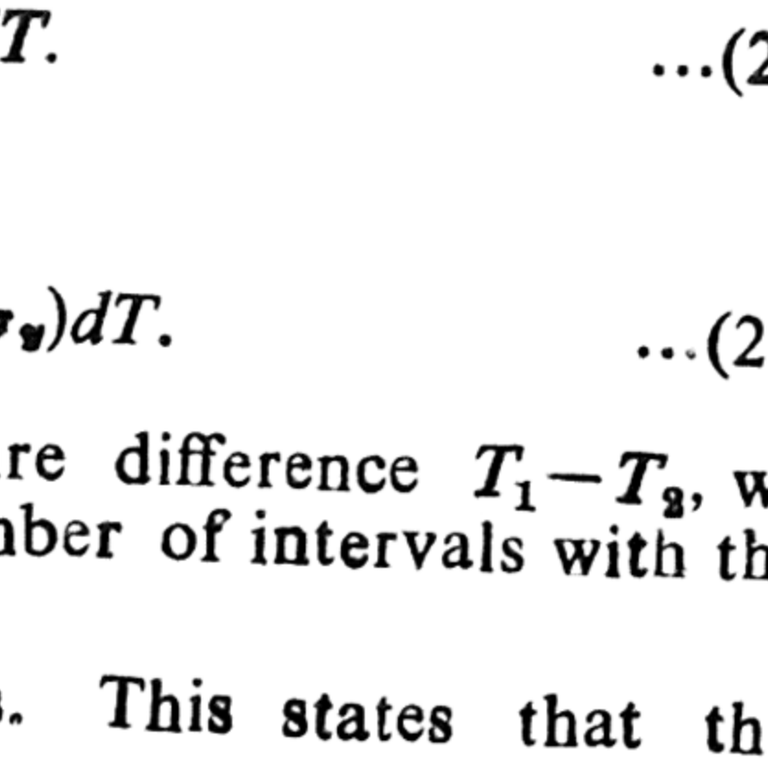


Fig. 7.9.

$$\text{or} \quad \mathcal{E}=[\pi_{xy}]_{T_1}+[\pi_{yz}+\pi_{zx}]_{T_2}+\int_{T_2}^{T_1}(\sigma_x-\sigma_y)dT. \qquad \ldots(22)$$

For a closed circuit composed of three metals which are at the same temperature

$$\pi_{xy}+\pi_{yz}+\pi_{zx}=0 \quad \text{or} \quad \pi_{yz}+\pi_{zx}=-\pi_{xy}.$$

$$\therefore \quad \mathcal{E}=[\pi_{xy}]_{T_1}-[\pi_{xy}]_{T_2}+\int_{T_2}^{T_1}(\sigma_x-\sigma_y)dT. \qquad \ldots(23)$$

$=$emf of the circuit composed of two metals X and Y.

This may be generalized to any number of intermediate metals, whose ends are at the same temperature. Thus we see that the thermocouple is unaffected by the introduction of other metals under same conditions. This is the most important property of thermocouple circuits, as *the introduction of measuring instruments, Cu-connecting wires and other accessories required for the measurement-purposes will not effect the emf.*

7.6. THERMOELECTRIC DIAGRAMS

Experiment shows that when the cold junction of a thermocouple is at constant temperature and the temperature of the hot junction is varied, the Seebeck emf is almost a parabolic function of the temperature, *i.e.*,

$$\mathcal{E}=\alpha t+\tfrac{1}{2}\beta t^2, \qquad \ldots(24)$$

where t is the temperature difference between the junctions of a thermocouple in degrees Celsius (°C) and α and β are constants, known as *thermoelectric constants.* These constants are characteristic of the metals forming thermocouple. Such parabolic curves, showing variation of thermo emf with temperature difference, are called *thermo-electric curves.*

The rate of change of $\mathcal{E}$ with the temperature T, $d\mathcal{E}/dT$ is numerically equal to $d\mathcal{E}/dt$ and is given by

$$d\mathcal{E}/dt=\alpha+\beta t. \qquad \ldots(25)$$

This quantity is called the *thermoelectric power* or the relative Seebeck coefficient and is denoted by the symbol ρ. Thermoelectric power of metal X with respect to metal Y at any temperature T_1 is obtained by the measurement of slope of the thermoelectric curve at temperature T_1. From relation (25) it is clear that the graph between $d\mathcal{E}/dt$ and t will be a straight line, the slope of which gives the constant β and the intercept at $t=0$ gives the constant α. The values of these constants for same metals and alloys relative to lead are given in the following Table 7.1.

These values should be regarded as indicative values only, as the effect of the state of purity, method of preparation and previous history of the metals count much. Here $\mathcal{E}_{metal,\ lead}$ is assumed positive when current flows at the hot junction from lead to the metal listed.

7.1. Thermoelectric Constants

Metal	$\alpha[\mu V/^\circ C]$	$\beta[\mu V/^\circ C^2]$
Aluminium (Al)	—0.76	0.0039
Antimony (Sb)	35.58	0.146
Bismuth (commercial) Bi	—43.7	—0.465
Cadmium (Cd)	3.06	0.029
Constantan (60% Cu+40% Ni)	—38.1	—0.089
Copper (Cu)	2.76	0.012
Gold (Au)	2.90	0.009
Iron (soft) Fe	16.65	—0.030
Manganin (84% Cu, 12% Mn, 4% Ni)	1.366	0.0008
Mercury (Hg)	—8.81	—0.033
Molybdenum (Mo)	5.89	0.043
Nickel (Ni)	—19.1	—0.030
Platinum (Pt)	—3.04	—0.033
Platinum-iridium		
85% Pt, 15% Ir	14.08	0.0106
90% Pt, 10% Ir	13.21	0.0075
Platinum rhodium		
85% Pt, 15% Rh	6.69	0.0107
90% Pt, 10% Rh	7.01	0.0064
Silver (Ag)	2.50	0.012
Steel	10.8	—0.016
Tungston (W)	1.594	0.0341
Zinc (Zn)	3.05	—0.01

To find the values of α and β for the thermocouple consisting two metals X and Y, we can make use of relations

$$\alpha_{xy}=\alpha_{x\ ,\ lead}-\alpha_{y,\ lead} \text{ and } \beta_{xy}=\beta_{x,\ lead}-\beta_{y,\ lead}.$$

As the thermoelectric power is independent for the temperature of the cold junction, hence it is not necessary to specify this temperature. The procedure of finding total emf of a thermocouple from thermoelectric curves is more tedious. Tait has shown that it is possible to compute the emf produced in a thermocouple by

constructing diagrams, known as *Tait diagrams or thermoelectric diagrams.* These are the curves between the rate of change of emf with temperature $d\mathcal{E}/dT$ and the temperature of the hot junction T for the thermocouple, consisting of lead and another metal. One metal is taken lead since for this Thomson effect is either zero, or negligibly small. The Tait diagram is a straight or a slightly curved line, as shown in Fig. 7.10.

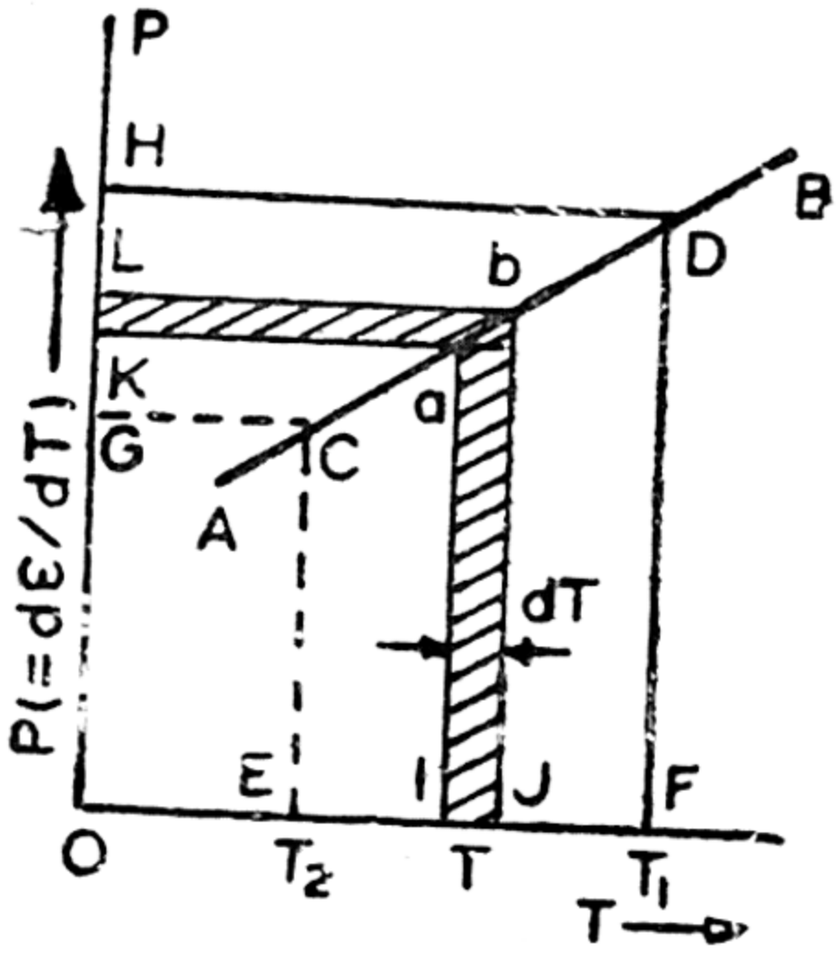


Fig. 7.10.

Let us consider a small strip *abIJ* of thickness dT at temperature T. If the thickness is assumed very small, the thermoelectric power P can be represented by aI for all the points over ab. Thus the emf $d\mathcal{E}$ between T and $T+dT$ is given by

$$d\mathcal{E}=(d\mathcal{E}/dT)dT=aI\times IJ=\text{Area of the strip } abJI.$$

$\therefore$ Total emf between temperatures T_2 and T_1 is

$$\mathcal{E}=\int_{T_2}^{T_1} d\mathcal{E}=\int_{T_2}^{T_1}\frac{d\mathcal{E}}{dT}\,dT=\textit{Area } CDFE. \qquad \ldots(26)$$

$$=FE\times\tfrac{1}{2}(CE+DF)=(T_1-T_2)\times\tfrac{1}{2}(P_2+P_1)$$

$$=\textit{Difference of temperatures}\times\textit{Average value of thermoelectric powers at these temperatures.}$$

The Peltier coefficient or the Peltier emf at temperature T_1 is given by

$$\pi_1=T_1(d\mathcal{E}/dT)_{T1}=T_1P_1=\textit{Area } OFDH. \qquad \ldots(27)$$

Similarly $\pi_2=T_2(d\mathcal{E}/dT)_{T_2}=T_2P_2=\textit{Area } OECG. \qquad \ldots(28)$

$\therefore \quad \pi_1-\pi_2=\textit{Area } OFDH-\textit{Area } OECG.$

We know that $\sigma_{lead}-\sigma_{metal}=-T(d^2\mathcal{E}/dT^2)$.

$$\therefore \quad \sigma_{metal}=T\frac{d^2\mathcal{E}}{dT^2}=T\frac{d}{dT}\left(\frac{d\mathcal{E}}{dT}\right)=T\frac{dP}{dT}$$

or $\sigma_{metal}dT=TdP=\text{Area of the strip } abLK \quad (\because \ \sigma_{lead}=0)$

$$\therefore \quad \int_{T_2}^{T_1}\sigma_{metal}\,dT=\text{Area } CDHG \qquad \ldots(29)$$

Hence the total emf is given by

$$\left[\mathcal{E}_{lead,\ metal}\right]_{T_2}^{T_1}=\pi_1-\pi_2-\int_{T_2}^{T_1}(\sigma_{lead}-\sigma_{metal})$$

$$=\textit{Area } OFDH-\textit{Area } OECG-\textit{Area } CDHG$$

$$=\textit{Area } CDFE. \qquad \ldots(30)$$

This is the same result as given by Eq. (26). This emf is such that the current flows from lead to metal across the hot junction.

If we require total thermo emf generated in a thermocouple constructed of two metals a and b (or a special case Cu and Fe). Two thermoelectric lines are constructed on the Tait diagram, one for Cu and other for Fe.

$$\left[\mathcal{E}_{Pb,\ Cu}\right]_{T_2}^{T_1} = Area\ BEFC \quad ...(31)$$

$$\left[\mathcal{E}_{Pb,\ Fe}\right]_{T_2}^{T_1} = Area\ AEFD \quad ...(32)$$

$$\therefore\ \mathcal{E}_{Cu,\ Fe} = \mathcal{E}_{Cu,\ Pb} + \mathcal{E}_{Pb,\ Fe}$$

$$= -\mathcal{E}_{Pb,\ Cu} + \mathcal{E}_{Pb,\ Fe}$$

$$= Area\ ABCD \quad ...(33)$$

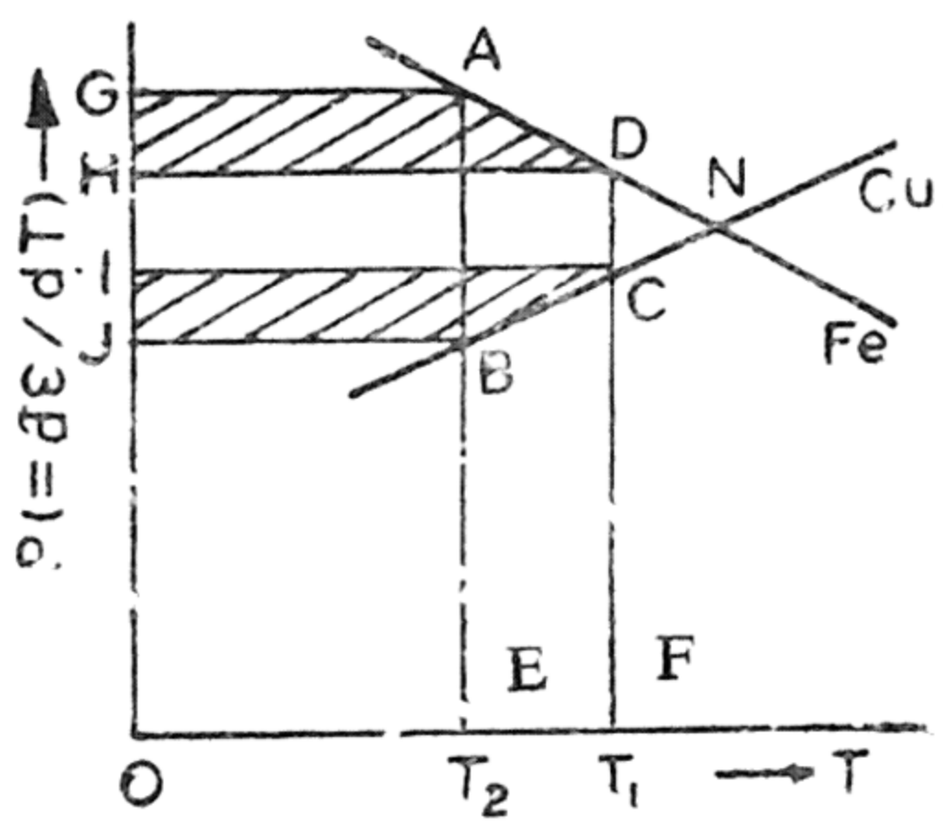


Fig. 7.11.

The direction of emf is indicated by the anticlockwise direction. *i.e.*, at hot from Cu to Fe and at cold from Fe to Cu.

Peltier coefficient at the hot junction for $Cu-Fe$ is given by

$$[\pi_{Cu-Fe}]_{T_1} = [\pi_{Cu,\ Pb}]_{T_1} + [\pi_{Pb,\ Fe}]_{T_1} = [\pi_{Pb,\ Fe}]_{T_1} - [\pi_{Pb,\ Cu}]_{T1}$$

$$= T_1\left[\left(\frac{d\mathcal{E}}{dT}\right)_{T_1}\right]_{Fe} - T_1\left[\left(\frac{d\mathcal{E}}{dT}\right)_{T_1}\right]_{Cu} = T_1FD - T_1FC$$

$$= T_1CD = Area\ HICD.\ (\text{Energy absorbed}) \quad ...(34)$$

Similarly $[\pi_{Cu\ Fe}]_{T_2} = T_2AB$

$$= Area\ GJBA\ (\text{Energy given out}) \quad ...(35)$$

Since $\mathcal{E}_{Cu,Fe} = \mathcal{E}_{Pb,\ Fe} - \mathcal{E}_{Pb,\ Cu}$

$$P_{Cu,\ Fe} = d\mathcal{E}/dT = P_{Pl,\ Fe} - P_{Pb,\ Cu}. \quad ...(36)$$

Thus the thermoelectric power of two metals with respect to each other at a given temperature is simply given by the difference of the respective ordinates for the thermoelectric lines of the two metals, each with respect to the standard metal (lead taken generally). Since $\sigma_{lead} = 0$, hence

$$\int_{T_2}^{T_1} \sigma_{Cu}\, dT = \int_{T_2}^{T_1} T\, dP = Area\ IJBC\ (\text{Energy absorbed})$$

Similarly $\int_{T_2}^{T_1} \sigma_{Fe}\, dT = \int_{T_2}^{T_1} T\, dP$

$$= Area\ GHDA\ (\text{Energy absorbed})$$

$$\therefore\ Total\ emf = Area\ (HICD + IJBC + GHDA - GJBA)$$

$$= Area\ ABCD. \quad ...(37)$$

The *neutral temperature* of a thermocouple is the temperature corresponding to the point at which two thermoelectric lines intersect. At this temperature the thermoelectric power ($d\mathcal{E}/dT$) of the two metals is zero, or the emf $\mathcal{E}$ is maximum. Hence Peltier coefficient $\pi=0$ and the emf is wholly contributed by the Thomson effect.

For elements, having —ve values of Thomson coefficient, thermoelectric lines slope downwards and the area representing Thomson effect is negative and reverse is the case for elements having +ve values of σ. For the thermocouple consisted of two metals having opposite signs of σ, the Thomson emf due to one increases and due to other decreases with the temperature. At low temperature emf increases with temperature. At the neutral temperature, two emf's are just equal and the total emf in the circuit is thus zero. Beyond this temperature the decreasing factor dominates and the net emf decreases with temperature. If the temperature is increased, such that the temperature of the hot and cold junctions are equally removed from the neutral temperature and on both sides, then the total emf becomes zero. This temperature is known as *temperature of inversion.*

7.7 MEASUREMENT OF THERMO-EMF

Thermocouples are of great value for the measurement of temperatures upto 1600°C. *Cu* and constantan thermocouples are satisfactory upto 300°C. Iron-constantan serves upto 800°C. For higher temperatures platinum and some platinum alloy must be used. Since thermal emf to be measured is small, a potentiometer wire having a small potential drop per unit length along it is required. An ordinary potentiometer is modified for this purpose by connecting it in series with a resistance box. Experimental procedure is given below.

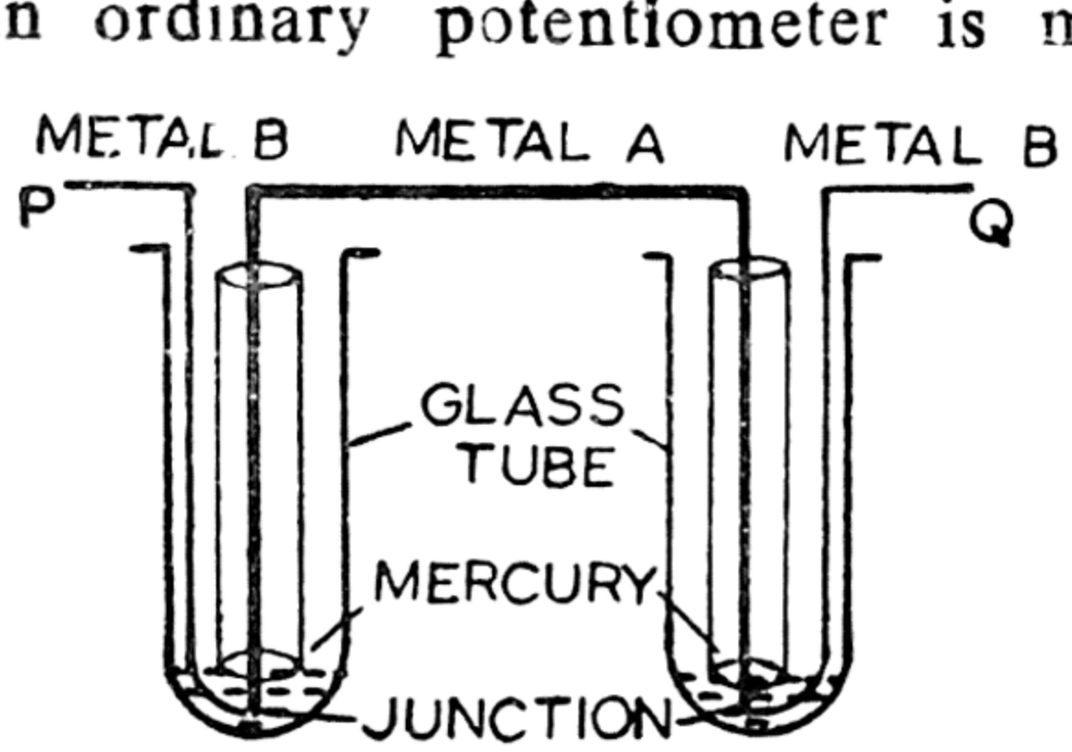


Fig. 7.12.

The thermocouple is first constructed by passing one end of the metal (*A*) wire down a thin glass tube and joining to a metal (*B*) wire. To ensure good contact at the junction the combination is placed in a test tube containing a small amount of mercury and the junction is dipped in the mercury. The other end of metal (*A*) wire is allowed to pass through glass tube and is connected with the one end of metal (*B*) wire, as shown in Fig. 7.12. Free ends (*P* and *Q*) of the metal (*B*) wire are connected to the measuring device. One junction is placedin air and other in an oil bath or in a sand bath.

A high resistance *R* is connected in series with the potentiometer wire *AB* and the combination is connected to a battery E_1 in series with a rheostat R_1 and key K_1. The current enters in the combination at *A'* and leaves at *B*. The +ve pole of the standard

cell E_2 is connected through a protective resistance R_2 ($10K\Omega$) to the higher potential terminal A', the —ve pole of this cell is connected through a two way key to the one terminal of the galvanometer, whose second end is connected to the variable terminal C of the potentiometer. The higher potential of the thermocouple is connected to the one terminal of the two way key and the low potential terminal of which is connected to the sliding, terminal J.

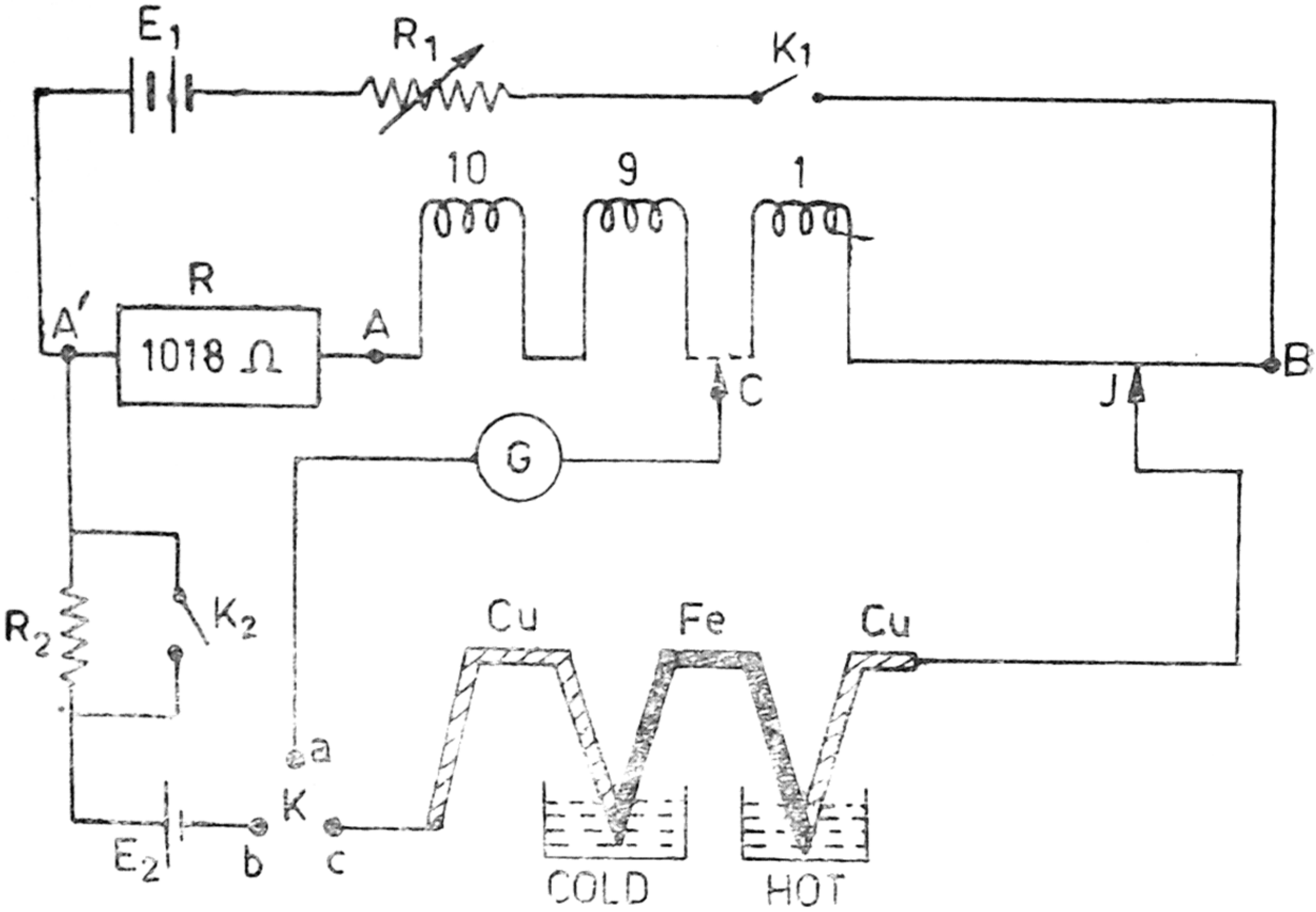


Fig. 7.13.

The resistance box R is adjusted to about 1000 ohms and the standard cell is connected to the galvanometer by putting plug in ab gap of two way key K. The variable contact C is adjusted to end A (or dial reading ten) and the rheostat R_1 is so adjusted that the galvanometer shows no deflection. The final adjustment is made by removing shunt from the galvanometer and by closing key K_2. At this time the pd across R is balanced by the emf $\mathcal{E}$ of the standard cell and the current in the main circuit is $\mathcal{E}/R$.

The emf across the thermocouple is now balanced by keeping R_1 unchanged by putting plug in the gap ac of two way key by adjusting C and J in such a way that there is no deflection in galvanometer. At this time the thermo emf is equal to the potential drop across the potentiometer wire between C and J. If ρ is the resistance per unit length of the wire and l is the length of the wire between C and J, then

$$\text{Thermo emf} = \rho l I = \rho l \mathcal{E}/R. \qquad \ldots(38)$$

In practice standard cadmium cell is used, for this R is taken 1018 ohms as $\mathcal{E}=1.018$ volts for this cell. The potentiometer wires are generally having value of ρ as 0.01 ohm/cm. If l is in cms, the

$$\text{emf} = 10^{-5} \times l \text{ volt.}$$

To investigate how the thermoemf varies with the temperature, keeping the cold junction at constant temperature, the hot

junction is heated to different temperatures. A graph is drawn between emf and the temperature of the hot junction. From this curve the neutral temperature, temperature of inversion and the constants α and β of the equation of this curve are obtained.

7.8 APPLICATIONS OF THERMO EMF

Thermo—electric effect is very useful in measuring several physical quantities in addition to temperature.

(*i*) **Thermopile.** One of the best known applications is the detection of small amounts of radiant heat by means of the device known as *thermopile*. To increase the sensitivity the effect produced by one junction is multiplied by arranging a number in series. To reduce the resistance the metals are used in the form of thick bars. Antimony and bismuth bars are used, as these metals are widely separated in the thermoelectric series and a comparatively large emf is produced for a small difference of temperature between the junctions. Antimony and bismuth bars are alternately connected in series. The free ends of the thermopile are connected to a galvanometer of which the resistance is preferably equal to the resistance of the thermopile. One set of alternate junctions is exposed to the radiation and the other set of the junctions is covered with a brass cap to maintain the temperature of these junctions constant. The galvanometer deflection will be due to thermo—emf produced which is approximately proportional to the radiant energy falling on the thermopile. It can easily measure difference of 10^{-5}°C.

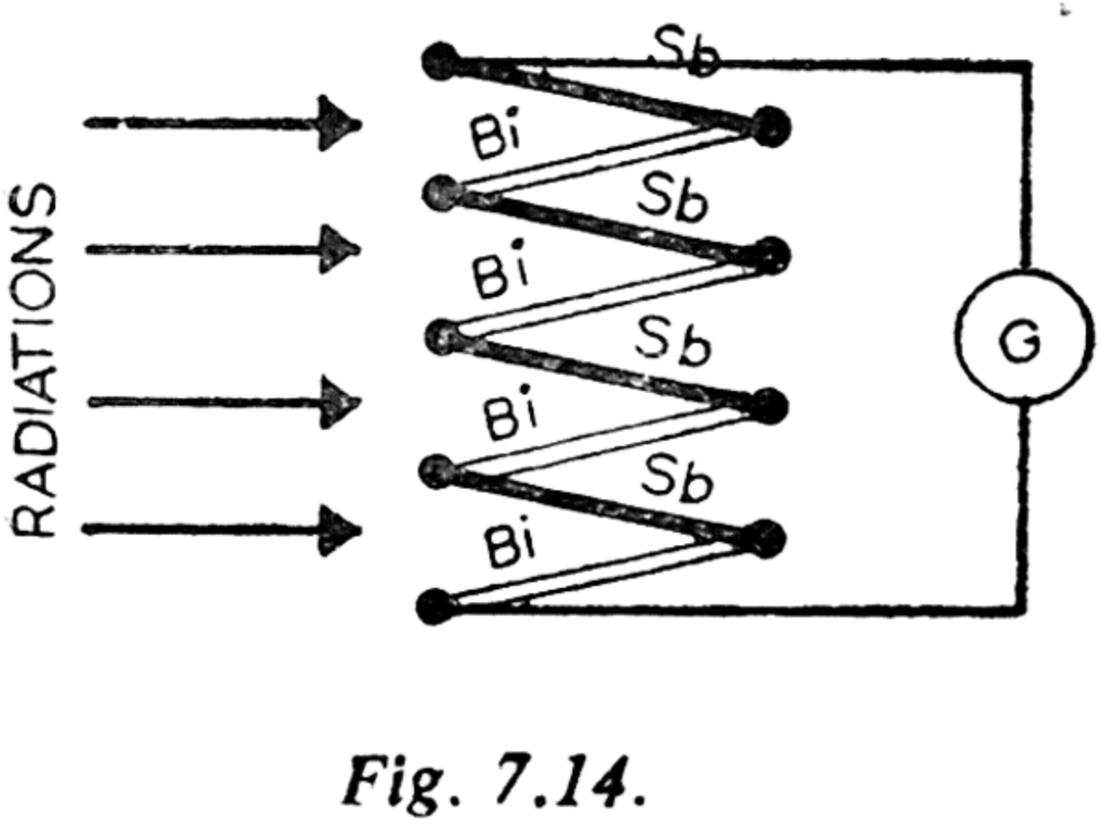


Fig. 7.14.

A relation between the rate at which radiant energy is being received and the thermoelectric current produced, is obtained by calibration. From the rate of reception of energy the rate of emission of energy can be culculated, if the distance and area of the source are known.

(*ii*) **Bolometer.** The principle of the Bolometer is similar, but it consists of a piece of resistance wire forming one arm of a bridge. When it receives energy its temperature rises, consequently its resistance rises and the balance of the bridge is destroyed. The current therefore flows through the galvanometer and a relation between this current and the rate of reception of energy can be found by calibration. The Bolometer is calibrated by passing current of different values in the main circuit and observing change of corresponding resistance of the Bolometer wire. Thus a relation between absorption of energy and change of resistance of the Bolometer wire can be obtained.

(*iii*) **Boy's radio micrometer.** It is a more sensitive arrangement, and is the combination of the couple and the galvanometer.

A moving coil consists of a single loop of *Cu* suspended between the poles of a strong magnet by a quartz fibre. The ends of this loop are soldered to short rods of antimony and bismuth which are soldered at the other ends to a very thin blackened copper disc. When the radiation to be measured is directed on to the disc, the junction of antimony and bismuth is heated and the thermo current is produced in the loop. As the loop is hanging in magnetic field hence experiences a couple. This couple deflects the coil and the deflection is measured by the mirror and scale arrangement.

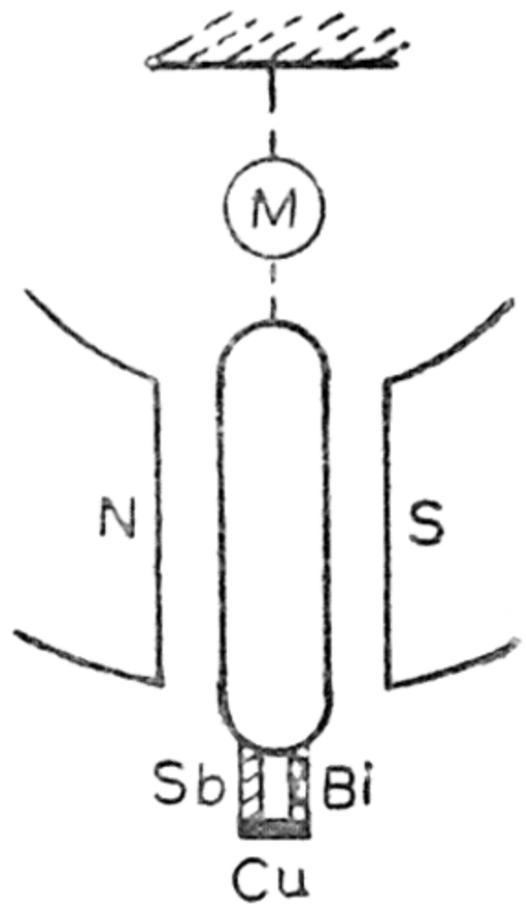


Fig. 7.15.

(*iv*) **Duddell thermo-galvanometer.** It is a modification of Boy's radio-micrometer for the measurement of small alternating or varying currents. A loop of silver wire hangs by a quartz fibre in between the poles *N*, *S* of a permanent magnet. The lower ends of the loop are attached to little pieces of bismuth and antimony respectively. These pieces are in contact at the bottom. Below this junction is situated the heater consisting of a straight filament about 3 mm. long with two straight legs leading to the terminals through which the current to be measured is allowed to pass in the heater. The heater element is taken straight so that its inductance and capacitance can be assumed as negligible and the instrument can be used for high frequency current. When the current passes through the heater, part of the heat developed is radiated to the *Bi Sb* junction, the result is that another current (thermo-current) is set up in the loop and the loop is deflected as it is suspended between the poles *N*, *S*. The deflection is read by lamp and scale arrangement and is proportional to the square of the current passing through the heater.

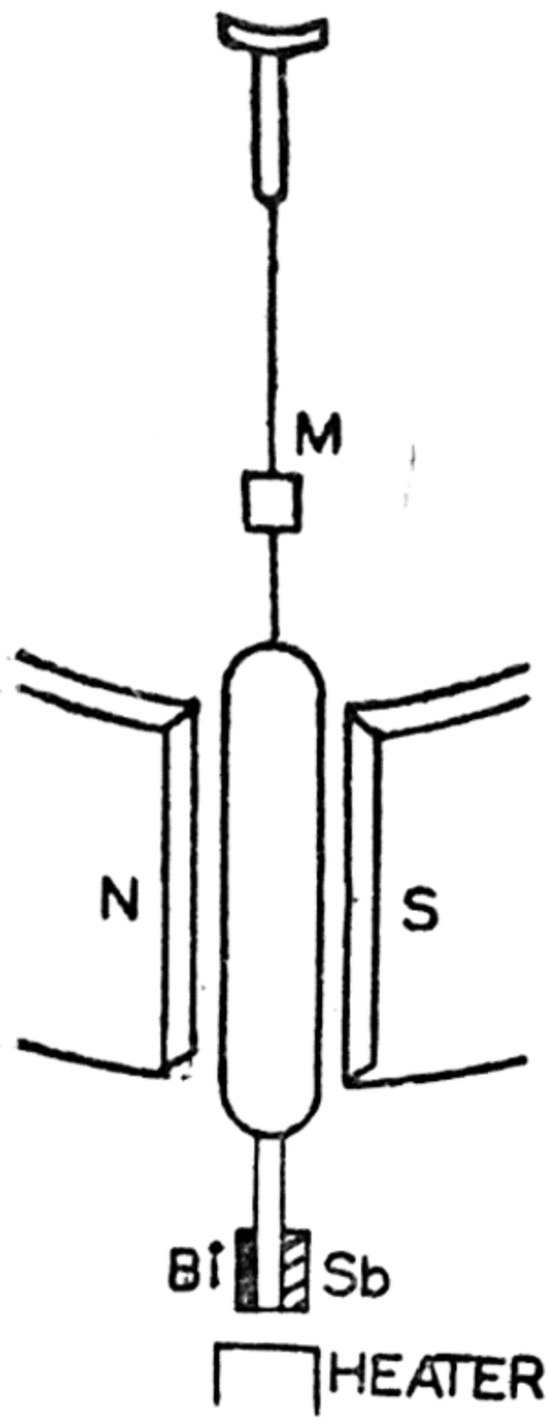


Fig. 7.16.

(*v*) **Thermoelectric pyrometer.** We know that some of the thermoelectric lines, for example those of *Cu* and *Ag*, are nearly parallel. If they are actually parallel, the emf round one of these couples would be proportional to the difference of the temperature of the junctions. This fact is made use of in the case of pyrometers constructed for measurement of high temperatures. Couple is usually of pure platinum and an alloy of platinum and iridium, or platinum and rhodium. In the Fery radiation pyrometer, the radiations are received on a concave mirror, which brings them to focus at the point where a sensitive couple is situated. The

thermo-emf generated is read by a millivoltmeter connected to the free ends of the couple.

(*vi*) **Thermo-milliammeter.** Sir J.A. Fleming devised a sensitive form of ammeter for the measurement of small alternating or continuous currents in which the heating produced by the current passing through a fine constantan wire *AB* warms the junction *C* of a tellurium-bismuth couple. The wire *AB* and the couple are in vacuum. A sensitive galvanometer *G* is connected in series with the thermocouple. The deflection in the galvanometer is proportional to the current passing in wire *AB*.

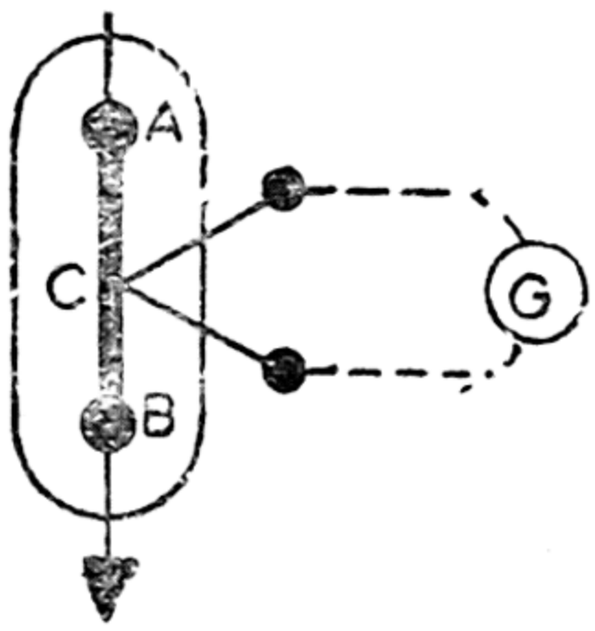


Fig. 7.17.

(*vii*) **Ratio-balance.** The idea that the absorption of heat at a thermal junction, when the direction of the current is the same as that of the Peltier emf has been employed by Prof. Callendar in 1910 for the measurement of the radiant heat. Two similar discs *A* and *B*, each supported by four stout iron and four constantan wires, are the junctions of an iron-constantan couple. The discs also form the junction of the single wire iron and constantan circuit in which the galvanometer *G* is included. Another circuit contains a battery, an adjustable resistance and a milliammeter. When radiation falls on the disc (say *A*), a thermo current flows and the galvanometer is deflected. Resistance *R* is adjusted until the two discs remain at the same temperature as indicated by the zero deflection in galvanometer. At this time the radiant energy absorbed will be equal to Peltier cooling. Knowing the current *i* and Peltier coefficient π, the rate of absorption of radiant energy is calculated. The radiation may also be allowed to fall on the disc *B* instead of *A*, for the purpose of making a control measurement. In the later design the discs are replaced by small cups.

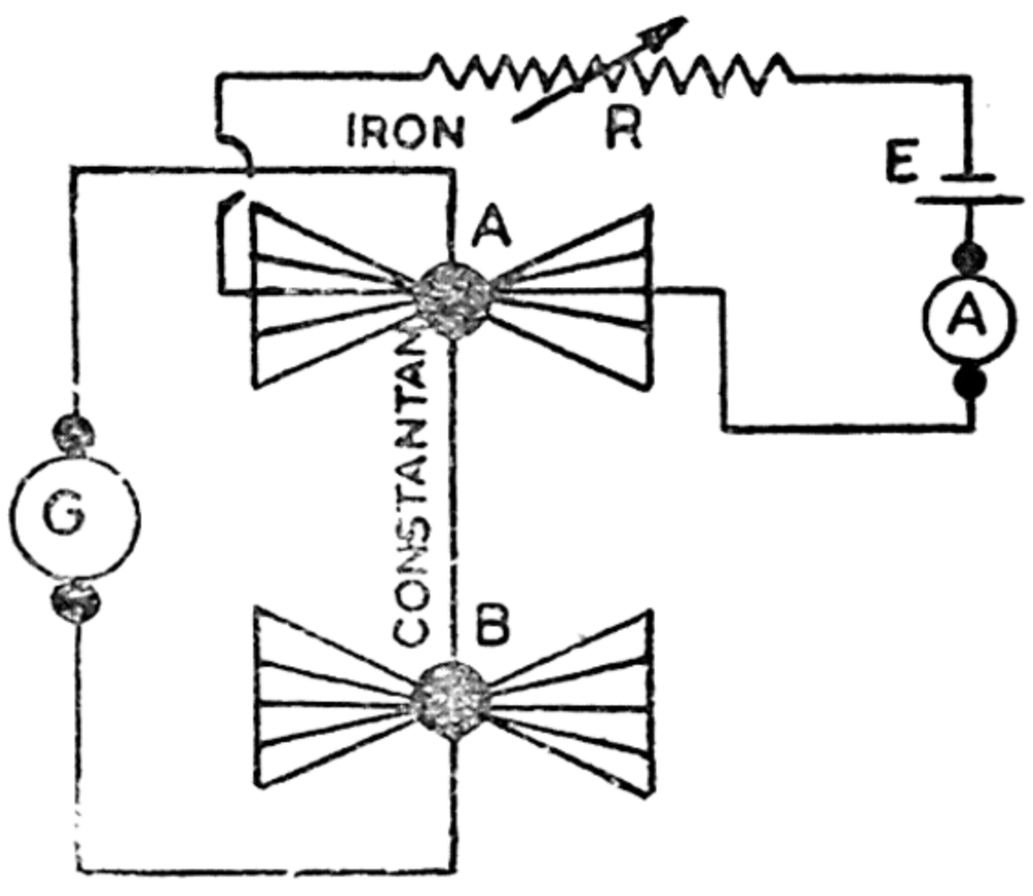


Fig. 7.18.

Exercises

Example 1. *Find the emf of a Cu-Fe thermocouple when temperature of hot is 100°C and that of cold is 0°C.*

We know that

$$\mathcal{E}_{Cu,\ Fe} = \alpha t + \tfrac{1}{2}\beta t^2.$$

By the law of intermediate metals, we have

$$\mathcal{E}_{Cu,\ Fe} = \mathcal{E}_{Cu,\ Pb} + \mathcal{E}_{Pb,\ Fe} = \mathcal{E}_{Cu,\ Pb} - \mathcal{E}_{Fe,\ Pb}$$

$$\mathcal{E}_{Cu,\ Fe}=(\alpha_{Cu}\,t+\tfrac{1}{2}\beta_{Cu}\,t^2)-(\alpha_{Fe}\,t+\tfrac{1}{2}\beta_{Fe}\,t^2)$$
$$=(\alpha_{Cu}-\alpha_{Fe})\,t+\tfrac{1}{2}(\beta_{Cu}-\beta_{Fe})t^2$$
$$=(2.76-16.65)\ 100\mu V+\tfrac{1}{2}(0.012+0.030)\ 100^2\mu V$$
$$=-1.179mV,$$

where the minus sign indicates that the direction of the current at the junction is from iron to copper at 100°C.

Example 2. *The emf in a thermocouple, one junction of which is kept at 0°C is given by* $\mathcal{E}=\alpha t+\frac{1}{2}\beta t^2$. *Find the neutral temperature and the Peltier and Thomson coefficients.*

In absolute scale of temperature, the relation can be written as

$$\mathcal{E}=\alpha(T-273)+\tfrac{1}{2}\beta(T-273)^2$$

$\therefore$ $\quad d\mathcal{E}/dT=\alpha+\beta(T-273)$ and $d^2\mathcal{E}/dT^2=\beta$.

At the neutral temperature $d\mathcal{E}/dT=0$.

or

$$(T_n-273)=t_n=-\alpha/\beta.$$

Peltier coefficient $\pi=T(d\mathcal{E}/dT)=T[\alpha+\beta(T-273)]=T(\alpha+\beta t)$

Thomson coefficient $\sigma=T(d^2\mathcal{E}/dT^2)=\beta T$.

Example 3. *A certain thermocouple, with one junction at 0°C, has an emf given by* $\mathcal{E}=[80t-15000\ \{1-exp(-0.0025t)\}]\mu V$. *Determine the Peltier coefficient at 0°C and show that the difference between the Thomson coefficients is practically stationary in a small temperature range in the neighbourhood of 127°C.*

Peltier coefficient at 0°C$=(273°K)=T(d\mathcal{E}/dT)_{273}$
$$=273[80-15000\times0.0025\ \exp\ (0)]\ \mu V$$
$$=11.6mV.$$

Difference between the Thomson coefficients

$$=\sigma_x-\sigma_y=-T(d^2\mathcal{E}/dT^2)=-(273+t)(d^2\mathcal{E}/dt^2)$$

But $d^2\mathcal{E}/dt^2=15000\times(0.0025)^2\ \exp\ (-0.0025\ t)$

$\therefore$ $\quad \sigma_x-\sigma_y=-(273+t)\ [93.75\times10^{-3}\ \exp\ (-0.0025\ t)]$.

For the stationary value of $\sigma_x-\sigma_y$, its first derivative must be zero.

$$\frac{d}{dt}[\sigma_x-\sigma_y]=0=-[93.75\times10^{-3}\ \exp\ (-0.0025\ t)].$$
$$+(273+t)[93.75\times10^{-3}(0.0025)\ \exp\ (-0.0025t)]$$

$\therefore$ $\quad 1-(273+t)\ (0.0025)=0$ or $t=127°C$.

Example 4. *Calculate the Peltier coefficients, the neutral temperature, the temperature of inversion and the total emf for a Cu—Fe thermocouple whose junctions are at 0°C and 100°C.*

By the law of intermediate metals, we have

$$\alpha_{Cu,\,Fe} = \alpha_{Cu,\,Pb} - \alpha_{Fe,\,Pb} = 2.76 - 16.65 = -13.89 \mu V/°C$$

$$\beta_{Cu,\,Fe} = \beta_{Cu,\,Pb} - \beta_{Fe,\,Pb} = 0.012 + 0.030 = 0.042 \ \mu V/°C^2.$$

Neutral temperature $t_n = -\alpha/\beta = 330.7°C.$

$\therefore$ Temperature of inversion $= 2(t_n - t_{cold}) = 661.4°C.$

Total emf $\mathcal{E} = \alpha t + \frac{1}{2}\beta t^2$

$$= -13.89 \times 100 + \tfrac{1}{2} \times 0.042 \times 100^2 = -1179 \mu V.$$

Thermoelectric power $d\mathcal{E}/dT = d\mathcal{E}/dt = \alpha + \beta t.$

$\therefore$ Peltier coefficient $\pi = T\,(d\mathcal{E}/dT) = T(\alpha + \beta t)$

at $100°C$ $\pi_{373} = 373(-13.89 + 0.042 \times 100) = -3614.37 \mu V$

at $0°C$ $\pi_{273} = 273(-13.89) = -3791.97 \mu V.$

Difference in Thomson coefficients $\sigma_{Cu} - \sigma_{Fe} = -T d^2\mathcal{E}/dT^2.$

$\therefore$ Total Thomson emf $= \int_{T_1}^{T_2} (\sigma_{Cu} - \sigma_{Fe}) dT$

$$= \int_{T_1}^{T_2} \left(-T \frac{d^2\mathcal{E}}{dT^2} \right) dT$$

$$= \int_{T_1}^{T_2} -T\beta dT = -\frac{\beta}{2}\left[T_2^2 - T_1^2\right]$$

$$= -\tfrac{1}{2} \times 0.042[373^2 - 273^2] = -1356.6 \mu V$$

Hence the total emf $= -3614.37 - (-3791.97) - 1356.6$

$$= -1179 \ \mu V,$$

which is the same as calculated by direct method.

Example 5. *Find the relation for Peltier coefficient in terms of electron density, using classical electron theory.*

Consider two metals X and Y in contact. The junction is maintained at temperature T. The electron densities are n_1 and n_2 in the metals X and Y respectively. The higher electron pressure in X will tend to drive electrons from X to Y. These electrons will set up an electric field and an equilibrium will be achieved when the electric field pressure is equal to the diffusion pressure of electrons, which corresponds to a pd between X and Y or the Peltier emf

The Peltier coefficients π, defined as the work required to transport unit quantity of charge across the junction, will be given by

$$\pi = \int_{V_1}^{V_2} P dV.$$

where V_1 is the volume occupied by the electrons in unit-charge in metal X and V_2 in metal Y.

Assuming the electrons as the molecules of a perfect gas,

$$P = \tfrac{1}{3} mnC^2 = \tfrac{2}{3} nk\ T.$$

If volume V occupied unit charge the number of electrons in a unit volume

$$n = 1/eV$$

and hence

$$P = 2kT/3eV.$$

$$\therefore \quad \pi = \int_{V_1}^{V_2} \frac{2}{3} \frac{kT}{eV} dV = \frac{2}{3} \frac{kT}{e} \int_{V_1}^{V_2} \frac{dV}{V}$$

$$= -\frac{2}{3} \frac{kT}{e} \log_e \frac{V_2}{V_1} = \frac{2}{3} \frac{kT}{e} \log_e \frac{n_1}{n_2}$$

Oral Questions

1. What will be the direction of current at the hot junction of *Ni—Fe* thermocouple ? Whether the emf produced between the junction is greater or smaller than that in *Cu—Fe* thermocouple ?
2. The neutral temperature of a thermocouple depends on materials forming the couple only. The inversion temperature depends on which factors ?
3. Distinguish the effects—(*a*) Peltier's effect, (*ii*) Joule's effect, and (*iii*) Thomson's effect.
4. What do you mean by +ve and —ve Thomson's effects ?
5. How can you use *Cu*—leads for connecting galvanometer in the thermocouple circuit of materials other than *Cu* ?
6. What do you conclude from two lines constructed on the Tait diagram ?
7. Why do you use a resistance of 1018 ohms in series of a potentiometer ?
8. What do you expect if no balance is obtained while balancing the thermo-emf across the potentiometer ?

Problems

1. Compute for a *Cu—Vi* thermocouple at temperature 100°C (*a*) thermal emf, (*b*) $d\mathcal{E}/dt$, (*c*) the Peltier emf, (*d*) Peltier heat transferred at the hot junction if a current of 1 *mA* is maintained for 1 hr ?
   $-1.42\ mV, -12.1\ \mu V/°C, -4.51\ mV, -3.88 \times 10^{-3}\ cal.$
2. Calculate π_{ab} and $(\sigma_a - \sigma_b)$ for a bismuth silver thermocouple at 0°C.
   $2.12 \times 10^{-2} V$, $6.6\ \mu V/°C$
3. If θ is the temperature of the hot junction of a thermocouple and the cold junction is at 0°C, then its emf $\mathcal{E} = \alpha\ \theta + \frac{1}{2}\beta\theta^2$. Discuss whether or not the emf of such a thermocouple may be represented by $\mathcal{E} = p + q\theta + r\theta^2$, where p, q and r are constants and the cold junction is not at 0°C.

4. Two semicircular rings, one of *Cu* and one of constantan are joined together to form a ring whose mean diameter may be taken as 14cm. Each metal has a cross-sectional area of 2 cm^2. Calculate the current through this ring if the junctions are at 0°C and 200°C. (17.4 *amp*)

5. Calculate the emf of a *Ag*–*Fe* thermocouple with junctions at 20°*C* and 80°*C*. The thermo-electric powers of *Ag* and *Fe* are $3.34+0.008\,t$ and $16.65-0.03\,t$ respectively. (685 μV)

6. For a chromel-alumel thermocouple, when cold junction is at 0°*C*, the emf is 18.37 μV and 44.24 μV when the hot junction is at 444.6°C and 1083°C respectively. Calculate (*a*) Thermo emf, (*b*) the Peltier coefficient, (*c*) the thermo electric power and (*d*) the Thomson coefficient at 600°C. 24.71 μV, 0.04076 $\mu V/°C$, 35.58μV, $\sigma_x-\sigma_y=6.43\times10^{-4}\ \mu V/°C$

7. A thermoelectric circuit consists of two bars of *Cu* and *Fe* with the ends joined and the junctions are maintained at 19° and 20°C respectively. Compare the heat dissipated due to Joule effect with that conducted along the bars. Given the thermo-electric power at 20°*C* as 14 $\mu\ V/°C$, the resistivities of *Cu* and *Fe* as 1.6 and 9.8 $\mu\ V/cm$ and the thermal conductivities as 0.9 and 0.2 cal $cm^{-1}\ sec^{-1}$ respectively. (7.4×10^{-6})

8. In a thermocouple constructed from lead and a semiconductor, the thermoelectric power is given by $d\mathcal{E}/dt=a+b/(t+273)$, where t is the temperature of one juuction, the other being kept at 0°*C*. Calculate rate of absorption of energy in the semiconductor when a current I flows through it and the temperature difference between its ends is 1°*C*. [$bI/(273+t$]

8

Magnetostatics

8.1. MAGNETISM

Few substances have the power of attracting iron, but not materials such as wood, amber etc. Such forces are called *magnetic forces* and the bodies giving rise to them are called *magnets* or the *magnetized* substances. The first magnetic phenomena to be observed were those associated with so called, natural magnets, the fragments of an ore of iron known as lodestone. The word magnetism comes from the district of Magnesia in Asia Minor, which is one of the places at which the lodestones were found. If a lodestone is put in iron filings, the filings are found to cluster more thickly about two regions on the stone than anywhere else. These regions are named as "*poles*". When it is suspended freely, the one end pointing north is called the north pole of the lodestone and the other the south pole. The earth itself is a great magnet having south polarity somewhere near its geometrical north pole and north polarity somewhere near its geometrical south.

Now-a-days electricity and magnetism is related by *electromagnetism*. The laws of attraction or repulsion are quite similar. There are certain foundamental differences and certain resemblances between electric charges and magnetic poles (or charged bodies and magnets).

1. A magnet attracts pieces of magnetic material only ; *a charged body attracts any light object.*

2. A magnet orients itself in a particular direction (geometrical north-south) suspended freely ; *a charged body does not.*

3. A magenet has polarity, northward force on the north pole is just balanced by the southward force on the south pole ; *a body charged inductively has polarity +ve and −ve, while a body with a free charge has not.*

4. If a magnet is cut in two, we find two complete magnets eachof which has equal north and south poles. Thus it is impossible to isolate north and south poles on different poles, *it can be done in the case of electric charges of opposite sign.*

5. A magnet magnetizes a piece of magnetic material when placed in its neighbourhood, the induced magnetism disappears (for soft iron) when the magnet is removed ; *a charged body has exactly corresponding properties, but will induce a charge on any insulated conductor instead of an particular ones only.*

6. There is no effect on a magnet when touched with the hand ; *the charged body is discharged.*

7. If we bring two magnets together, nothing happens except that they exert a force on each other ; *if two oppositely charged bodies are brought together a spark will pass between them.*

8. No substance is known in which magnetic poles move under the influence of magnetic forces with the *freedom that electric charges move in a conductor when subject to electric forces.*

9. No force is exerted by an electric charge on a magnetic pole when both are at rest relative to the observer.

8.2. MAGNETIC FIELD

Upto 1820, no connection was known between electrical and magnetic phenomena. In 1820, Danish Scientist Hans Christian Oersted observed the magnetic effects of current and thus showed a connection between electricity and magnetism. Twelve years later, Faraday found that a momentary current existed in a circuit while the current in a nearby circuit was being started or stopped. The same effect was also observed by the motion of a magnet toward or away from the circuit. Thus we see that *the work of Oersted demonstrated the magnetic effects by moving electric charges and that of Faraday the electric effects (production of current) by moving magnets. Now-a-days it is believed that all so-called magnetic phenomena result from forces between electrostatic charges in motion and the magnetism can not be considered as a separate entity.* The possibility that the magnetic properties of matter were the result of tiny atomic currents (due to continuous rotation of electrons in atoms about the axis passing through the atomic nuclei) was first suggested by Ampere in 1820.

We define the space around a current carrying conductor (or a magnet) as the site of a magnetic field, as we define the space near a charged rod as the site of an electric field. The magnetic field around a moving charge exists in addition to the electrostatic field which surrounds the charge whether the charge is in motion or not. If a second charged particle is placed near the moving charge it will experience a *force due to the electric field whether it is in motion or at rest and a force due to the magnetic field only if it is in motion.* The magnetic field, like the electric field, is a vector field and its

magnitude and direction at any point are specified by a vector **B**, called the *magnetic induction* or *magnetic flux density.*

Magnetic field can be represented by lines, called *lines of induction*, just as the electric field is represented by lines of force. **B** is related to its line of induction in the following ways.

(*i*) *The tangent drawn at any point to a line of induction gives the direction of* **B** *at that point.*

(*ii*) *The number of lines per unit area gives an idea about the magnitude of the magnetic field vector* **B**. *This field is large if lines are closer and is small if they are far apart.*

In a uniform magnetic field, where the **B**-vector has the same magnitude and direction at all points, the lines of induction are parallel and straight. The lines of induction simply give a graphical representation of the variation of **B** throughout a certain region of space.

The magnetic flux Φ_B across a surface is defined in the same way as the electric flux Φ_E for the electric field. Thus the flux $d\Phi_B$ across an area $d\mathbf{A}$ is,

$$d\Phi_B = B_n dA = \mathbf{B}.d\mathbf{A}.$$

$\therefore$ Total flux across an finite area

$$= \Phi_B = \int B_n dA = \int \mathbf{B}.d\mathbf{A}. \qquad \ldots(1)$$

If **B** is uniform and normal to a finite area **A**, then the flux across the area is $\Phi_B = \mathbf{B}.\mathbf{A}$.

8.3. THE DEFINITION OF B

An unknown electric field can be determined by magnitude and direction of the force on a test charge q_0 at rest. To explore an unknown magnetic field, we must measure the magnitude and direction of the force on a moving test charge. Let us assume that there is no electric field present.

Let us consider a positive test charge q_0 moving with velocity **v** through any point P in space. If a side ways deflecting force **F** acts on it, one can say that a magnetic field is present at P and the vector **B** can be defined in terms of **F** and other measured quantities. If the direction of **v** is varied through point P, keeping its magnitude constant, we find that although **F** remains $\perp$ to **v** its magnitude changes and there is one direction of **v** in which a magnetic field exerts no force on the moving charge. *This direction is defined as the direction of the* **B**-*vector.*

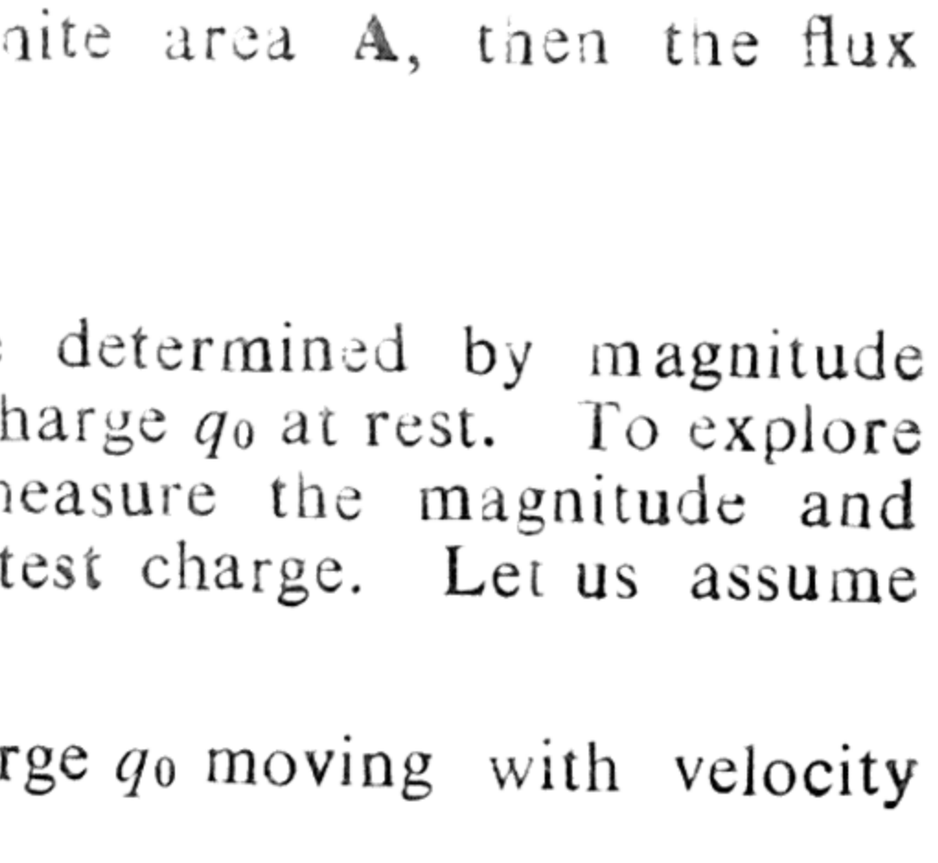


Fig.8.1.

Having determined the direction of **B**, let us now keep the velocity of the moving charge perpendicular to the magnetic field. The force is found perpendicular to both the **B** and **v**. If the velocity is not perpendicular to the direction of the magnetic field, but makes an angle ϕ with the field, the force acting on the moving charge is perpendicular to both the magnetic field and to $v \sin \phi$ (the component of **v** perpendicular to **B**) and has a magnitude proportional to $v \sin \phi$. The magnitude of the magnetic field vector **B** at any point P is given as

$$B = F/q_0 v \sin \phi \text{ or } F = q_0 v B \sin \phi. \quad \ldots(2)$$

In vector form, it can be written as

$$\mathbf{F} = q_0(\mathbf{v} \times \mathbf{B}). \quad \ldots(3)$$

From these relations we see that (*a*) **F** *is perpendicular to the plane formed by* **v** *and* **B** *and thus always is a sideways force*, (*b*) **F** *vanishes as* (*i*) $\mathbf{v} \to 0$, (*ii*) $\mathbf{B} = 0$, *or* (*iii*) **v** *and* **B** *are either parallel or antiparallel*, (*c*) $\mathbf{v} \perp \mathbf{B}$, *the magnetic force* F *has its maximum value, given as* $F = q_0 v B$ *and* (*d*) *the work done by the force in a static magnetic field is zero, as the direction of* F *and the direction of motion of charge are always at right angles i.e.*, *K.E. of a moving charge remains unaltered in a static magnetic field.*

The definition of **B** is similar in spirit to the definition of the electric field **E**, which is defined as $\mathbf{F} = q_0\mathbf{E}$. In this case the only characteristic direction is that of **F**, but in defining **B**, two characteristic directions those of **v** and of **F**, appear. The unit of magnetic field **B** can be obtained from relation (3) and is the (newton/coulomb)/(meter/sec). This is given the special name weber/meter^2 or *tesla*. The unit of the magnetic fiux is *weber*.

If a charged particle moves through a region in which both an electric field and a magnetic field are present, the resultant force

$$\mathbf{F} = q_0\mathbf{E} + q_0(\mathbf{v} \times \mathbf{B}). \quad \ldots(4)$$

This relation is known as *Lorentz relation* and **F** as Lorentz force.

The direction of **F** is $\perp$ to the plane containing **v** and **B** and is given by Fleming's left hand rule. According to this rule, the forefinger, the central finger and thumb of the left hand are stretched in such a way that they are mutually perpendicular to each other. If the central finger shows the direction of current (or **v**) and the forefinger shows the direction of magnetic field or pointing magnetic field vector **B**, then the thumb will give the direction of deflecting force **F**.

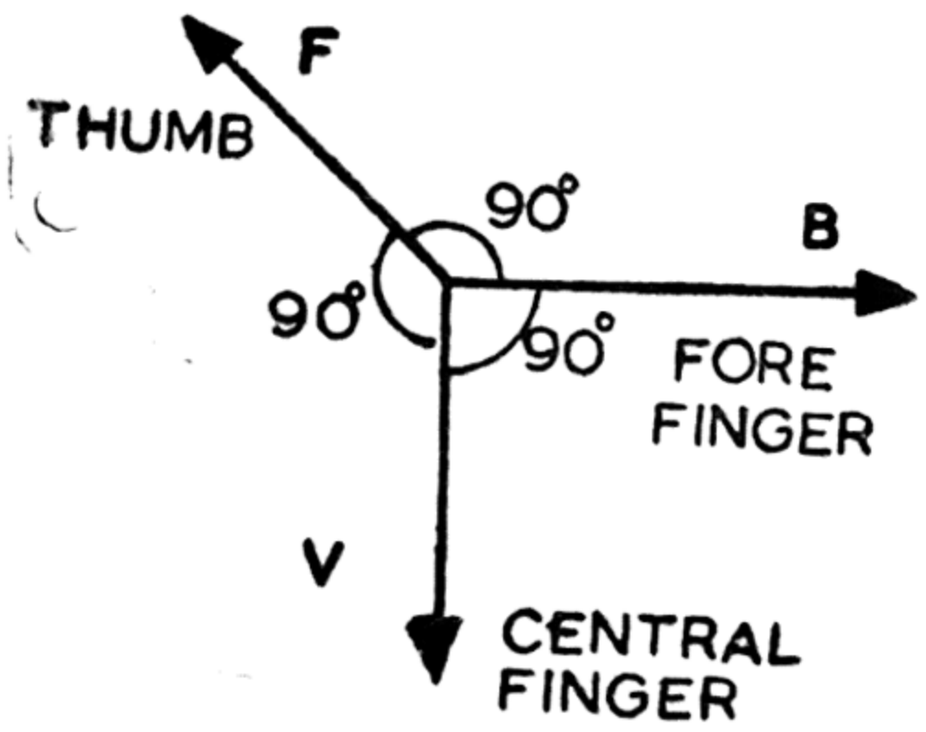


Fig. 8 2.

8.2. MOVING CHARGES IN ELECTRIC FIELD

We consider following cases.

(*a*) **Motion in Uniform Electric Field.** The force **F** acting on an ion or charge q moving in a uniform electric field **E** is $q\mathbf{E}$. It experiences a constant acceleration f in the direction of **F**.

$$\mathbf{F}=m\mathbf{f}=q\mathbf{E} \qquad \text{or } \mathbf{f}=(q/m)\ \mathbf{E}. \qquad \ldots(5)$$

On integration, we get

$$\mathbf{v}=d\mathbf{r}/dt=(q/m)\ \mathbf{E}t+C.$$

As at $t=0$, $\mathbf{v}=\mathbf{u}$, hence $\mathbf{v}=\mathbf{u}+(q/m)\ \mathbf{E}t$.

Again integrating, we get $\mathbf{r}=\frac{1}{2}(q/m)\ \mathbf{E}t^2+\mathbf{u}t+C'$.

As at $t=0$, $\mathbf{r}=\mathbf{r}_0$ (initial position), hence

$$\mathbf{r}=\tfrac{1}{2}(q/m)\ \mathbf{E}t^2+\mathbf{u}t+\mathbf{r}_0. \qquad \ldots(6)$$

There are two particular cases.

(*i*) *Longitudinal Field.* When the electric field acts along the driection of motion of the particle *i.e.*, the case of longitudinal field, the both **u** and $\mathbf{r}_0$ are directed along the electric field **E**. In this case the resultant position vector **v** is also along the direction of **E** and the problem acts as if it is one dimensional one.

If at the initial position $\mathbf{r}_0=0$ and $\mathbf{u}=0$, then we have

$$\mathbf{v}=(q/m)\ \mathbf{E}t \text{ and } \mathbf{r}=\tfrac{1}{2}\ (q/m)\ \mathbf{E}t^2. \qquad \ldots(7)$$

Thus we again have one dimensional problem.

(*ii*) *Transverse Electric Field.* Consider a charged particle of mass m and charge q, which enters with the initial velocity vx along x-axis between two parallel plates A and B. If the plates are of length l, separated by a distance d and kept at a potential difference of V volts, the electric field $E=V/d$

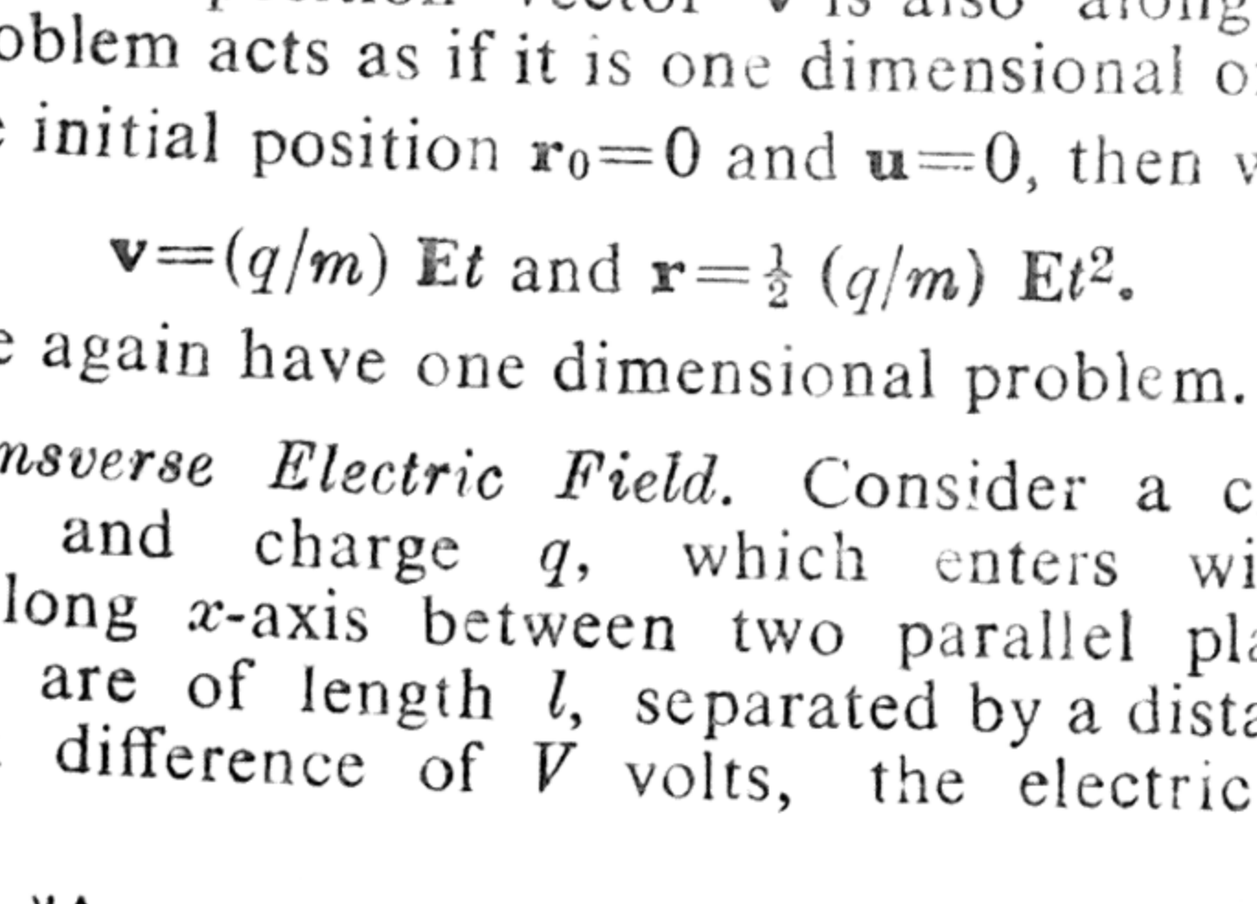


Fig. 8.3.

will act along $+y$-axis. As there is no force along the x-axis, hence the time taken by the particle to pass through the plates $T=l/vx$. As a force $q\mathbf{E}$ acts along y-axis, hence the acceleration produced in the particle will be along y-axis and given *by* $\mathbf{f}=q\mathbf{E}/m$. Since there was no component of $\mathbf{v}$ along y-axis at the time of entrance hence the velocity vy along y-axis at any instant t is

$$v_y=f_y t=(qE/m)\ (x/vx).$$

and the displacement $y=\frac{1}{2}\ fyt^2=\frac{1}{2}\ \ (qE/m)(x/vx)^2$. ...(8)

Here x is the distance travelled along x-axis in time t (position O').

Thus the path of the charged particle in between the plates is a parabola in the x-y plane. When the particle leaves the plates, it will travel in a straight line alohg the tangent to the parabola at the point where the particle leaves the plates. The angle is given by

$$\tan\theta=(dy/dx)x_{=l}=qEl/mv_x^2.$$

The total y-displacement at a distance L from the ends of the plate is given by

$Y=$displacement between plates$+$outside the plates

$$=\frac{1}{2}\frac{qE}{m}\ \frac{l^2}{v_x^2}+L\tan\theta=\frac{qEl}{mv_x^2}\left(L+\frac{l}{2}\right). \qquad ...(9)$$

This principle is used in *cathode ray tubes*, where electrons are allowed to pass through the parallel plates and the beam strikes the fluorescent screen.

(*b*) **Motion in Alternating Electric Field.** If the electric field is an alternating, given by $\mathbf{E}=\mathbf{E}_0\sin\omega t$, then above relations reduce to

$$m\mathbf{f}=m\ d\mathbf{v}/dt=m\ d^2\mathbf{r}/dt^2=q\mathbf{E}_0\sin\omega t. \qquad ...(10)$$

On integrating with the condition $\mathbf{v}=0$ at $t=0$, we get

$$\mathbf{v}=\ -\frac{q\mathbf{E}_0}{m\omega}\cos\omega t+\frac{q\mathbf{E}_0}{m\omega}=\frac{q\mathbf{E}_0}{m\omega}\ [1-\cos\ \omega t].$$

On integrating with the condition $\mathbf{r}=0$ at $t=0$, we get

$$\mathbf{r}=-\frac{q\mathbf{E}_0}{m\omega^2}\sin\omega t+\frac{q\mathbf{E}_0}{m\omega}t. \qquad ...(11)$$

(*c*) **Motion in non-uniform Electric Field.** The motion of charged particle (electron) in nonuniform fields is of considerable

interest in the field of *electron optics*. The electron is allowed to pass in a region of a variable field. The component of velocity parallel to the equipotential surface is unperturbed. The velocity changes in the direction of potential gradient and is given by the equation

$$\triangle \mathbf{v}=\mathbf{F}\triangle t/m=\nabla V(e/m)\triangle t. \qquad ...(12)$$

As will be seen in *electrostatic focusing*, the velocities are related as

$$\frac{v_1}{v_2}=\frac{\sin\theta_2}{\sin\theta_1}=\left(\frac{V_1}{V_2}\right)^{\frac{1}{2}} \qquad ...(13)$$

8.5. MOVING CHARGES IN MAGNETIC FIELD

(*a*) If a charged *particle is moving along the line of magnetic field*, it will move as it is, as it will experience no force ($\mathbf{v} \parallel \mathbf{B}$).

(*b*) If a *particle of charge q is introduced with its initial velocity* $\mathbf{v}$ *in the plane normal to* $\mathbf{B}$ then the force $\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$ will act on it in this plane in the direction $\perp$ to $\mathbf{v}$ and $\mathbf{B}$. Since the direction of this force is and remains normal to the velocity, hence,

$$\mathbf{F}.\mathbf{v}=q(\mathbf{v}\times\mathbf{B}).\ \mathbf{v}=0$$

or
$$m\frac{d\mathbf{v}}{dt}.\ \mathbf{v}=\frac{d}{dt}\left(\frac{mv^2}{2}\right)=0.$$

and therefore,
$$\tfrac{1}{2}\,mv^2=\text{constant}. \qquad ...(14)$$

This equation shows that the magnetic field performs no work and produces no change in the kinetic energy of the particle. Since the force $\mathbf{F}$ due to magnetic field is always perpendicular to the direction of motion and $\mathbf{v}$ and $\mathbf{B}$ are constant in magnitude, hence force $\mathbf{F}$ will also be constant and the particle will move along a circular path of radius R given by

centripetal force $mv^2/R=$ *Force* qvB

or
$$R=mv/qB.$$

The corresponding angular velocity

$$\omega=v/R=qB/m, \qquad ...(15)$$

or the frequency $f=\omega/2\pi=qB/2\pi m$.

This relation shows that the charged particle rotates in a plane $\perp$ to the magnetic field with the angular frequency independent of its linear velocity v. If the particle loses or gains energy continuo-

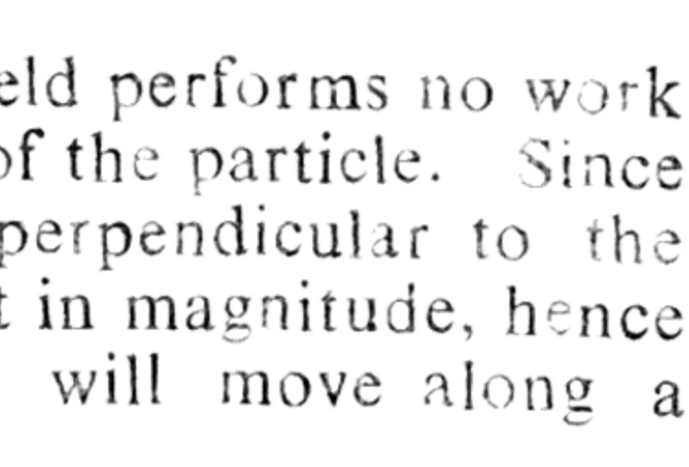


Fig. 8.4.

usly the path will be a spiral. The radius of curvature decreases in the former case while increases in the latter case.

(*d*) *If the charged particle is neither parallel nor perpendicular but making an angle* θ *with the direction of* **B** (as shown in Fig. 8.5), then one can split the velocity vector into two components, v_x along **B** and v_y normal to **B**. Since $\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$, or $Fx=qv_xB \sin 0=0$ hence v_x is unaffected by the field. Thus the particle will move in the direction of **B** with a constant velocity v_x. As v_y is the velocity component in the direction $\perp$ to **B**, the particle will travel in a circle, which is in y-z plane, of radius $R=mv_y/qB$. Thus the

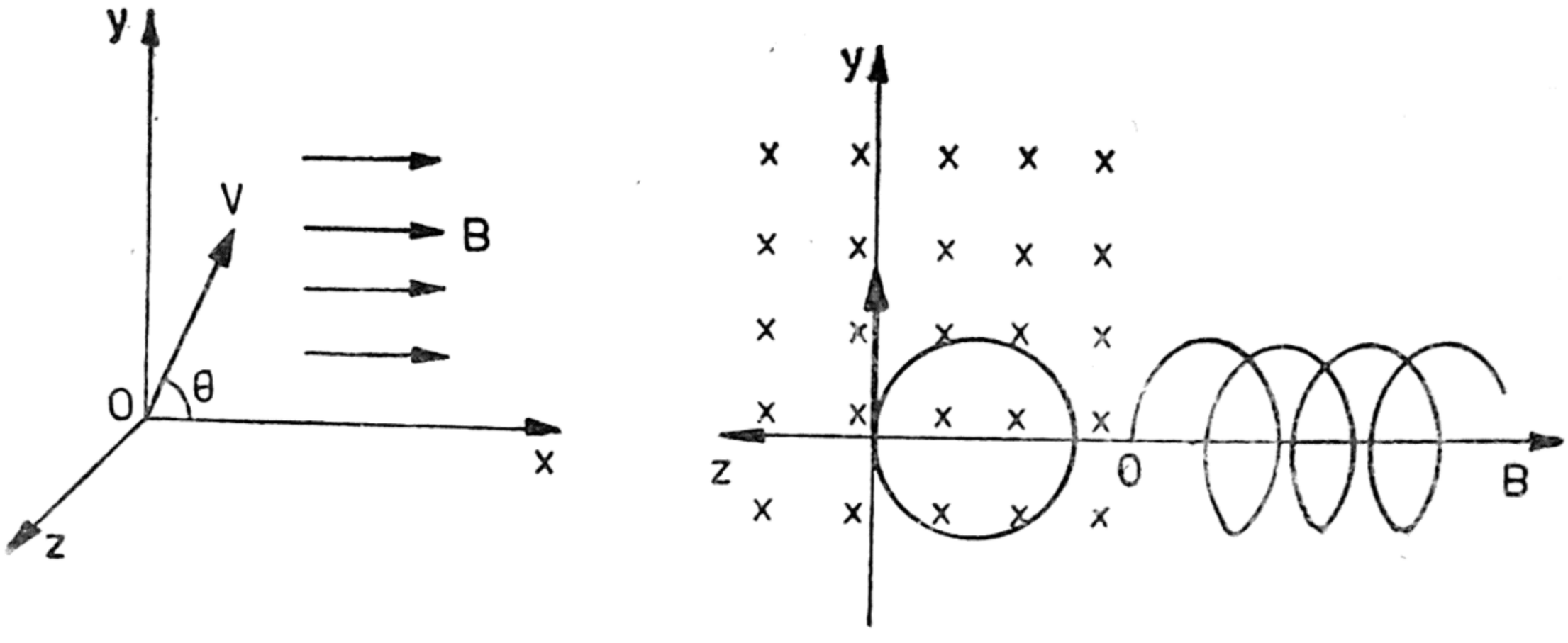


Fig. 8.5.

resultant motion will be the superposition of two motions, one constant velocity v_x along **B** and other circular motion $\perp$ to **B**. Hence the resultant path will be a helix, with its axis along x-axis and radius in y-z plane. As the pitch of a helix is the distance travelled along its axis in one revolution, hence

$$\text{pitch of } q = v_x \times \text{time period } T = v_x \times 2\pi m/qB = 2\pi m v_x/qB \quad \ldots(16)$$

In general the velocity of the particle may be represented as

$$\mathbf{v}=\mathbf{v}_{\perp}+\mathbf{v}_{\parallel}, \quad \ldots(17)$$

where $\mathbf{v}_{\perp}$ is the component of velocity perpendicular to the magnetic field and $\mathbf{v}_{\parallel}$ is the component of velocity in the direction of the magnetic field. Therefore the component of Lorentz force in the direction of magnetic field is zero, *i.e.*,

$$\mathbf{F}_{\parallel}=q(\mathbf{v}\times\mathbf{B})_{\parallel}=0. \quad \ldots(18)$$

$$\therefore \quad m\frac{d\mathbf{v}_{\parallel}}{dt}=0 \text{ and } \mathbf{v}_{\parallel}=\text{constant}. \quad \ldots(19)$$

Substituting value of $\mathbf{v}$ in the equation of motion $m d\mathbf{v}/dt = q(\mathbf{v} \times \mathbf{B})$, we get

$$m \frac{d}{dt}(\mathbf{v}_\perp + \mathbf{v}_\parallel) = q(\mathbf{v}_\perp + \mathbf{v}_\parallel) \times \mathbf{B}.$$

or $$m \frac{d\mathbf{v}_\perp}{dt} = q\mathbf{v}_\perp \times \mathbf{B}. \quad \ldots(20)$$

The angle between $\mathbf{v}_\perp$ and $\mathbf{B}$ remains constant throughout the motion and is equal to $\pi/2$. $\mathbf{v}_\perp$ and $\mathbf{B}$ do not change in magnitude. Hence the force $m d\mathbf{v}_\perp/dt$ is constant in magnitude and is perpendicular to $\mathbf{v}_\perp$. Therefore Eq. (20) describes the circular motion. The radius of the circle is given by the relation

$$mv_\perp^2/R = qv_\perp B. \quad \ldots(21)$$

The superposition of motions due to $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$ gives the helix.

This principle is used in many instruments. Let us discuss below few of them.

Magnetic Focussing. In general the ions emitted from the ion source are of different velocities, the ions of same velocity also emit in different directions.

(*i*) To focus ions of different velocities, they are collimated by an entrace slit *S*. The magnetic field is applied $\perp$ to the plane of a paper and particles are bent in a circular path. After a turn of 180°, they are recorded on a photographic plate *P*. The particles of different velocities are focussed at defferent points on the plate. By measuring the distances one can calculate velocity and energy of different particles.

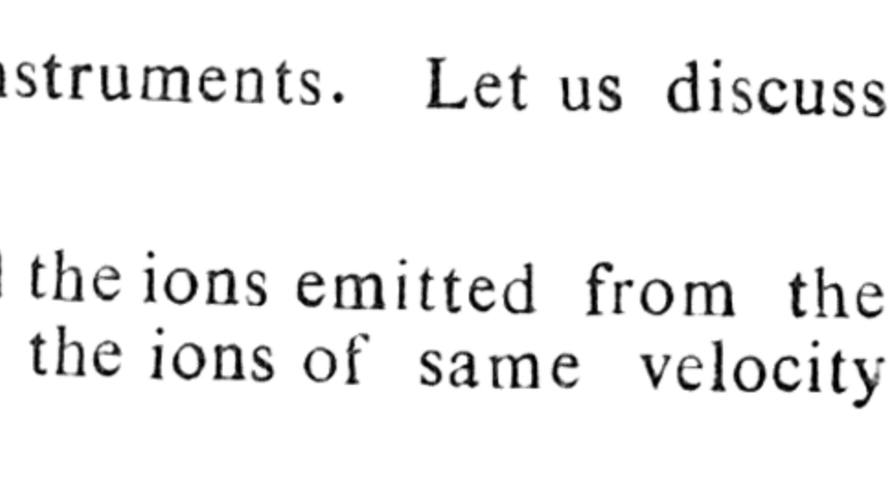


Fig. 8.6.

$$mv^2/R = qvB \text{ or } v = qBR/m$$

and energy $= \frac{1}{2}mv^2 = q^2B^2R^2/2m$.

This device is used in isotope separation.

(*ii*) To focus ions of same velocity travelling in different directions, the ions are allowed to pass through a fine slit *S*. After a turn of 180°, in a perpendicular magnetic field, they are focussed on a plate. All the particles leaving the slit normally will strike the

plate at point A normally. The other particles making initial angle θ with the normal strike the plate at other point B. The width of the focussed image

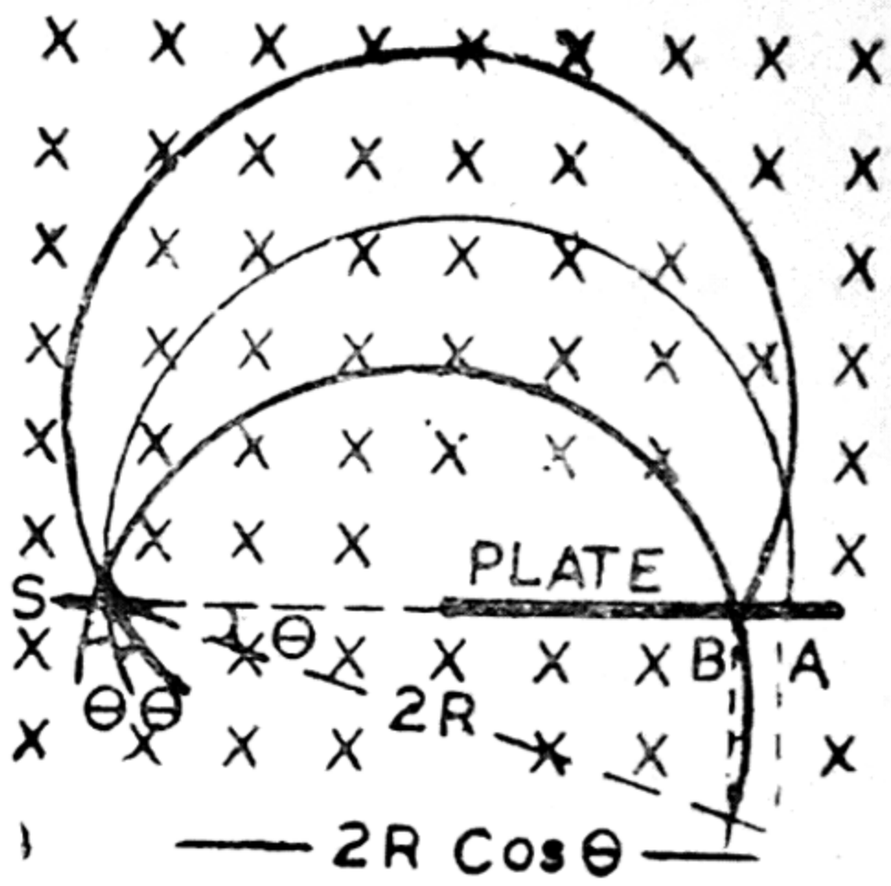


Fig. 8.7.

$$AB=SA-SB=2R-2R\cos\theta$$

$$=2R(1-\cos\theta)=2R[1-1+\theta^2/2\ !]$$

$$=R\theta^2. \qquad ...(22)$$

As θ is very small. The focussing power or the resolution

$$P=AB/SA=R\theta^2/2R=\tfrac{1}{2}\theta^2.$$

If the slit width is s, then

$$P=(s+R\theta^2)/2R=s/2R+\theta^2/2.$$

This device is very useful in β-ray spectrometers.

8.6. CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

Consider a particle of charge q and mass m, emitted at the origin with zero initial velocity into a region of uniform electric and magnetic fields. The field **E** is acting along x-axis and field **B** is along y-axis. The force $q\mathbf{E}$ due to electric field will act along x-axis. As the charged particle is placed in a magnetic field **B** it will experience a force $q(\mathbf{v}\times\mathbf{B})$ acting in the direction perpendicular to **B** and **v** both. Hence the particle will bent. The resultant force also known as Lorentz force is thus given by

$$\mathbf{F}=q[\mathbf{E}+\mathbf{v}\times\mathbf{B}]. \qquad ...(23)$$

In our particular case $\mathbf{E}=\mathbf{E}\mathbf{i}$, $\mathbf{v}=v_x\mathbf{i}+v_y\,\mathbf{j}+v_z\,\mathbf{k}$ and $\mathbf{B}=B\mathbf{j}$.

$$\therefore\quad \mathbf{F}=q[E\mathbf{i}+v_xB\mathbf{k}-v_z\ B\mathbf{i}]=q[E-v_z\ B]\mathbf{i}+qv_xB\mathbf{k}.$$

To find velocity components v_x and v_z, above relation can be written as

$$F_x=m(dv_x/dt)=qE-qv_z\ B,\ F_y=m(dv_y/dt=0$$

and $F_z=m(dv_z/dt)=qv_xB.$

Combining both by differentiating first part and then substituting value of dv_z/dt, we get

$$m\frac{d^2v_x}{dt^2}=-qB\ \frac{dv_z}{dt}=-\ \frac{q^2B^2v_x}{m}\ \text{or}\ \frac{d^2v_x}{dt^2}+\frac{q^2B^2}{m^2}v_x=0.$$

This is the general equation of S.H.M. Its solution is

$$v = A \sin(\omega t + \phi), \text{ where } \omega = qB/m. \qquad \ldots(24)$$

As at $t=0$ $v=0$, hence $\phi=0$.

Again $dv/dt = A\omega \cos \omega t$. Hence at $t=0$, $dv/dt = A\omega$, which should be same as qE/m.

$\therefore \quad A\omega = qE/m$ or $A = qE/m\omega = E/B$.

Hence Eq. (24) becomes $vx = (E/B) \sin \omega t$.

Now substituting value of vx in the equation for Fx, we get

$$m \frac{E}{B} \omega \cos \omega t = qE - qvz\, B \text{ or } vz = \frac{E}{B}\left[1 - \cos \omega t\right]. \qquad \ldots(25)$$

On integrating equations for vx and vz and knowing that at $t=0$, $x=0$ and $z=0$, we get

$$x = \frac{E}{B\omega}\left[1 - \cos \omega t\right] \text{ and } z = \frac{E}{B\omega}\left[\omega t - \sin \omega t\right] \qquad \ldots(26)$$

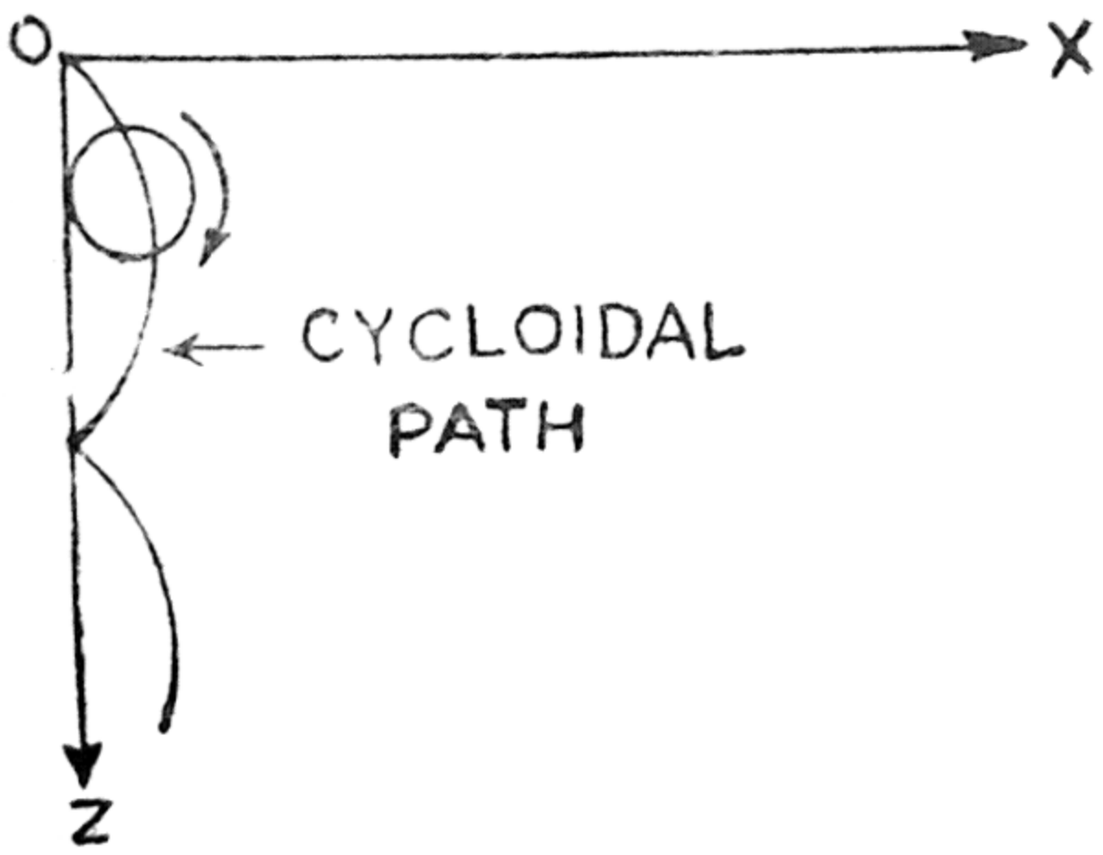


Fig. 8.8.

These equations are the equations for a *cycloi*, which is defined as the path generated by the point on the circumference of a circle, rolling along a circle. In the present case the radius of the rolling circle is $E/B\omega$, the maximum displacement along x-direction is $2E/B\omega$. The x-displacement becomes zero at $t=0$, $2\pi/\omega$, $4\pi/\omega$. The z-displacement in time $t=2\pi/\omega$ is $2\pi\ E/B\omega$.

Let us now discuss few important applications of the combined electric and magnetic fields acting perpendicular to each other.

(*i*) **Velocity Selector.** Since most of the electrons or ions emitted from the source are with a wide range of velocities, a velocity selector is often essential to obtain ions of same velocity. This device requires both electric and magnetic forces. A capacitor like arrangement AB provides a uniform electric field **E** in the plane of the paper. A uniform magnetic field **B** is provided $\perp$ to the plane of the paper. Collimated charged particles move along SC if the force due to **E** is balanced by the

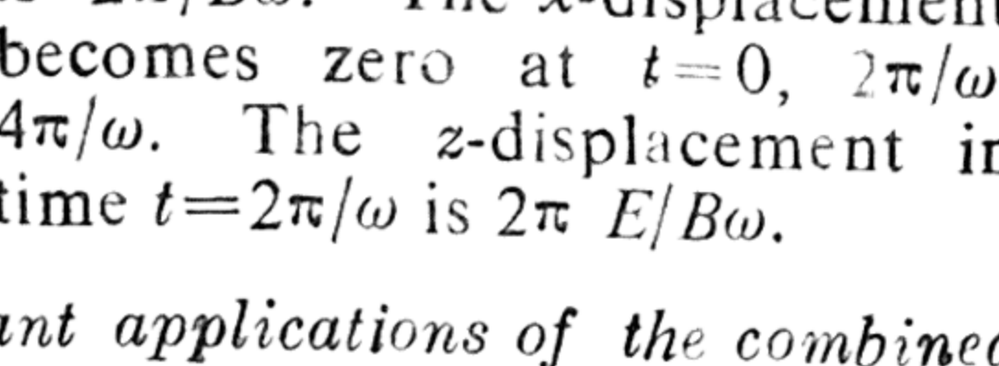


Fig. 8.9.

force due to **B**. Hence the particle will move with velocity v, given by $qE=qv\,B$ or $v=E/B$. Thus by adjusting E and B, particles of different velocities are obtained in the SC direction.

(*ii*) **Hall Effect.** In 1879 E.H. Hall devised an experiment that gives the useful information regarding current carriers in conductors. Suppose we have a steady current i, hence current density **j**, flowing in a uniform conducting strip along the x-axis. If this strip is placed in a uniform magnetic field **B**, acting along the y-axis. The field **B**, will exert a deflecting force **F** on the current carriers, may be electrons or holes (missing electrons), in the direction ⊥ to the plane containing vector **B** and current density vector **j**. If the current is carried by positive charge q moving with velocity **v** along **j**, then the force $\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$. As clear in Fig. 8.10, this force is in upward direction whether the charge carriers are positive or negative. If the chage carriers are electrons, an excess negative charge accumulates at the upper edge of the slab, leaving an excess positive charge at its lower edge. Thus a transverse potential difference, known as *Hall emf* is produced. This name is after its discoverer. The sign of this emf will be opposite if the charge carriers are positive or holes. Thus the sign of the charge carriers is determined by the sign of this Hall emf, which can be measured with a potentiometer. Experiment shows that in metals the charge carriers are electrons. However semiconductors exhibited a Hall emf opposite to that of the metals as if their charge carriers were positively charged. These charge carriers are now known as *holes*.

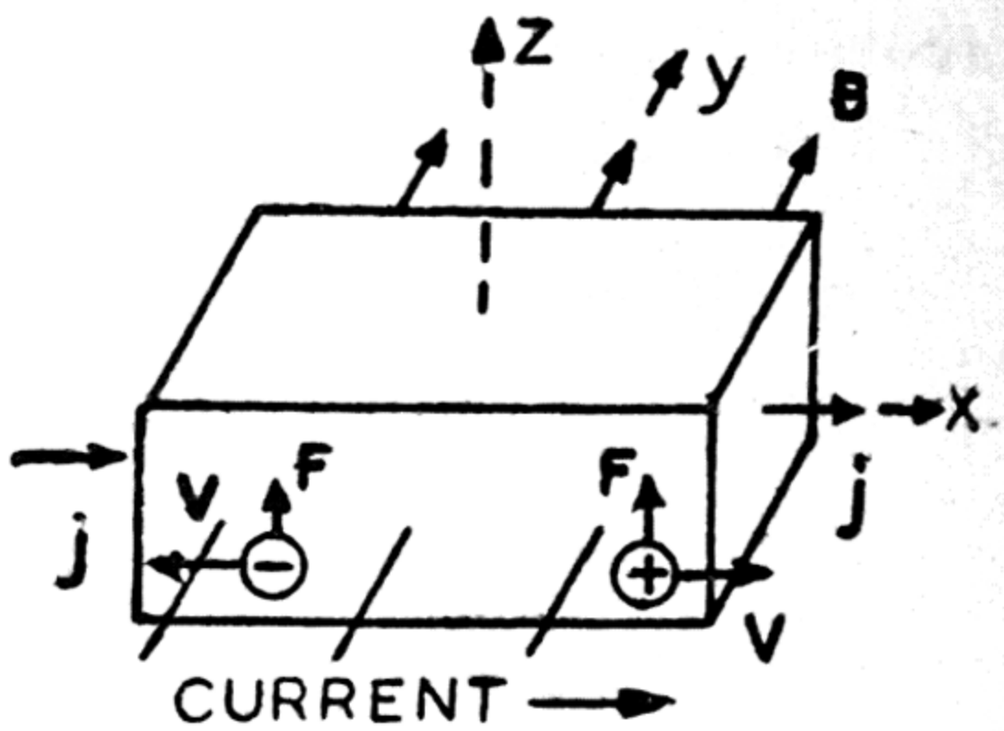


Fig. 8.10.

The displacement of charge carriers gives rise to a transverse field, known as *Hall electric field* $\mathbf{E}_H$, which acts inside the conductor, to oppose the side ways drift of the carriers. An equilibrium is reached in which the magnetic deflecting force on the charge carriers is just balanced by the electric force caused by the Hall electric field.

$$q(\mathbf{v}\times\mathbf{B})+q\mathbf{E}_H=0 \text{ or } \mathbf{E}_H=-\mathbf{v}\times\mathbf{B}. \qquad ...(27)$$

The velocity of the deflecting charge carriers **v** can be measured by measuring $\mathbf{E}_H$ and **B**, where E_H=Hall emf (V_H)/distance(d) between two edges. The Hall emf is thus given by

$$V_H=\mathbf{E}_H.\,\mathbf{d}\ =vBd.$$

We know that the current density $\mathbf{j}=nq\mathbf{v}$, hence $E_H=jB/nq$. or density of charge carriers $n=jB/qE_H$.

The ratio of the Hall electric field to the product of current density j and magnetic induction B is known as the *Hall coefficient* or simply *Hall constant* R. Thus

$$R = E_H/jB = 1/nq. \qquad \ldots(28)$$

The Hall constant R is negative, if charge carriers are electrons ($-$ve q) and is positive if these are holes ($+$ve q). Observed and calculated values of the Hall constant are in good agreement for most of the metals. Its unit is m^3/coul.

The exact result for semiconductors is slightly different from Eq. (28) because the electrons in the semiconductor are not entirely free. The hall coefficient is found to be

$$R = 3\pi/8nq = 1.18/nq. \qquad \ldots(29)$$

Since the current density is defined as

$$j = nq\langle v\rangle,$$

hence we get

$$E_H = jBR = nq\langle v\rangle B.\ (1/nq)$$

$$= \langle v\rangle B = \frac{\langle v\rangle}{\mathcal{E}}\,\mathcal{E}B = \mu\mathcal{E}B, \qquad \ldots(30)$$

where $\mathcal{E}$ is the emf responsible for current density j and μ is called the mobility of charge carriers. The mobility can be determined by measuring E_H when a known $\mathcal{E}$ and B are applied to the crystal. The mobility may also be measured by measuring the *Hall angle*, defined as

$$\tan\theta = E_H/\mathcal{E} = \mu B. \qquad \ldots(31)$$

The mobility of the charge carriers may also be determined by measuring the conductivity σ, defined as

$$\sigma = nq\mu = \mu/R,$$

$$\therefore \quad \mu = \sigma R. \qquad \ldots(32)$$

Measurement of Hall coefficient gives us the number of current carriers per unit volume and the relaxation time (the average time between two collisions of an electron). Its sign tells us whether the current carriers are positive or negative. The mobility of the carriers can be measured by the conductivity of the material and the Hall coefficient. Hall effect may be used to measure magnetic fields by using germanium of high resistance.

Hall Probe—Hall effect may be used to measure the flux density of a magnetic field. A simple instrument used for this purpose is known as *Hall probe*. In it a wafer of semiconductor has two contacts on opposite sides which are connected to a high impedance voltmeter V. An electric current i($<$ 1 amp) is allowed to pass through

the semiconductor in the normal direction and is measured with the ammeter *A*. Araldite encapsulation is done so that the wires may not be detached from the wafer. Combining relations

$$E_H = j\,B/nq,\ V_H = E_H d$$

and $i = j\,A$, we get

$$V_H = \frac{jBd}{nq} = \frac{iBd}{nqA}$$

or

$$B = \frac{V_H\, nqA}{id} = \frac{V_H}{i}\left(\frac{nqA}{d}\right) \quad \ldots (33)$$

Here (nqA/d) is constant for a given semiconductor and can be determined easily. Thus the measurement of V_H and i, *i.e.*, the voltmeter and ammeter readings will give the value of B of magnetic field in which the semiconductor or Hall probe has been placed.

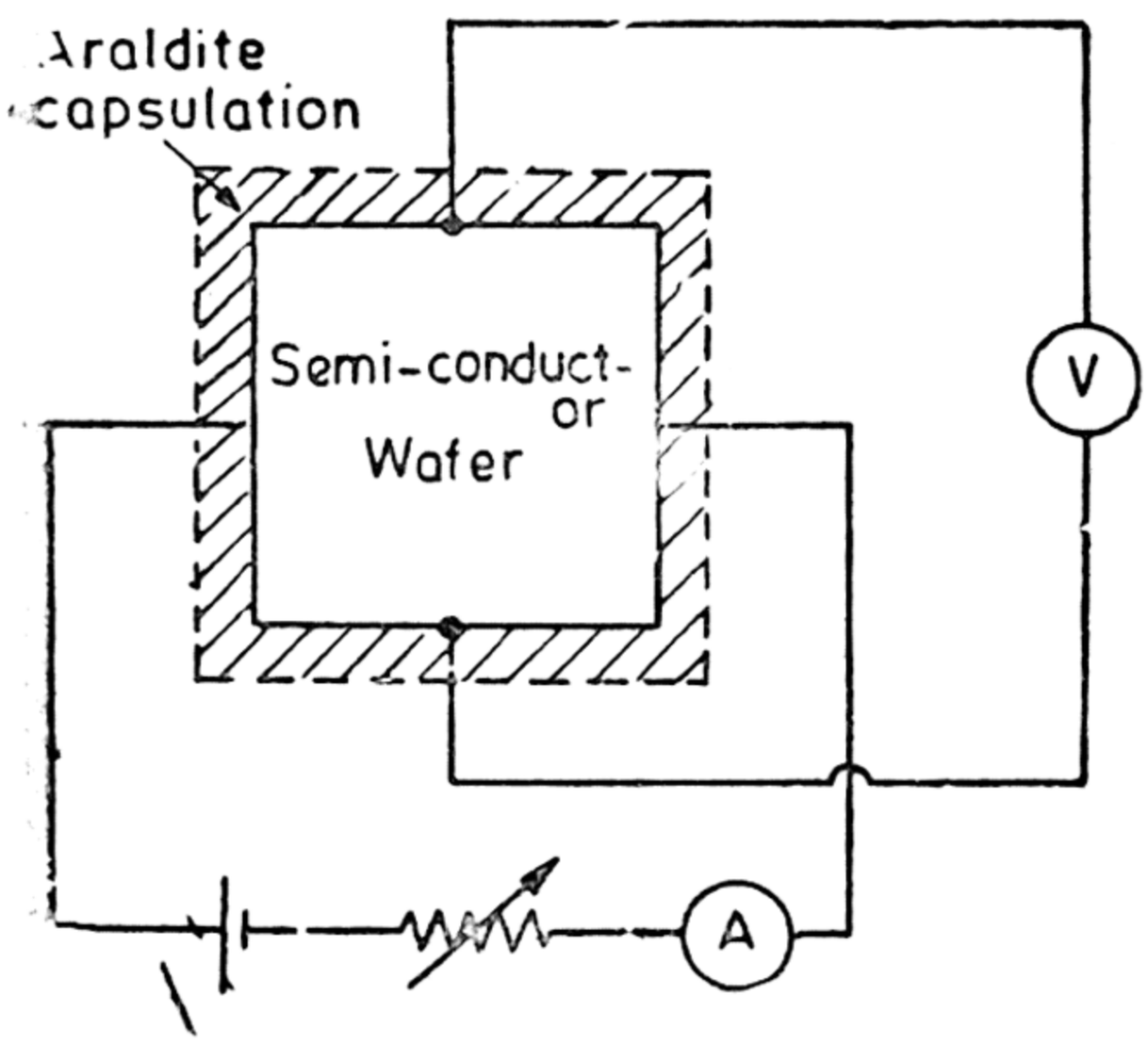


Fig. 8.11.

(*iii*) **Zeeman Effect**. In 1896, Zeeman discovered that spectral lines are split up into components when the source emitting the lines is placed in a very strong magnetic field. The simplest case is known as the *normal Zeeman Effect*. A single line of wavelength λ_0 in the absence of a magnetic field is split into three components with wavelength $\lambda_0 - \Delta\lambda$, λ_0 and $\lambda_0 + \Delta\lambda$ when viewed at right angles to the magnetic field **B**. The shifted lines are plane polarized with the electric vector perpendicular to **B**, while the line of wavelength λ_0 is plane polarized with the electric vector parallel to **B**. Only the two components with wavelengths $\lambda_0 \pm \Delta\lambda$ are seen when viewed along **B**, they are circularly polarized.

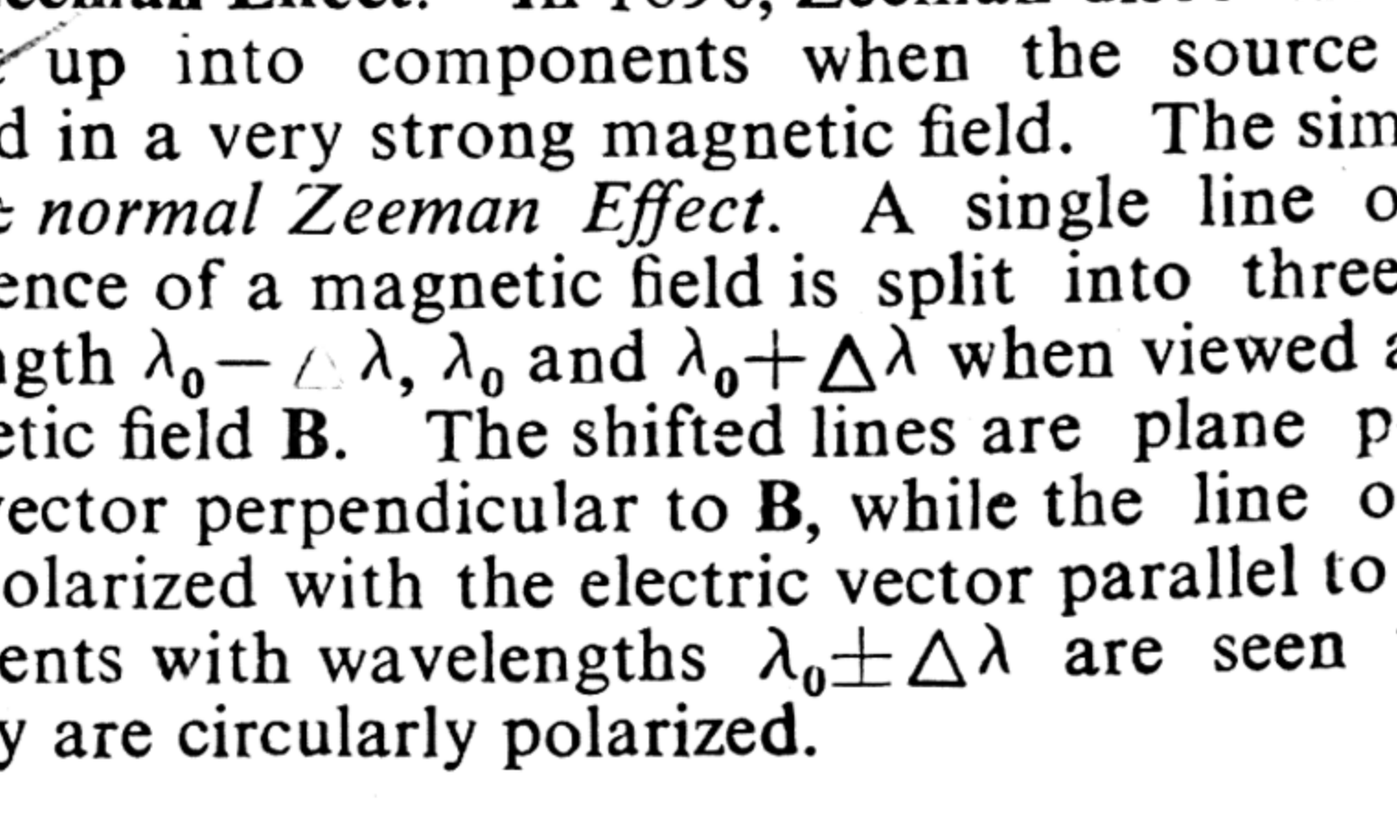


Fig. 8.12.

Classically a charge performing simple har monic motion radiates light. In the absence of a magnetic field, the electron of

mass m and charge e is revolving round the nucleus in an orbit of radius r, such as the centripetal force $mv^2/r = m\omega_0^2 r$ is balanced by an elastic force proportional to the dis placement, *i.e.*

$$m\omega^2 r = kr$$

and the frequency $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$. ...(34)

In the presence of magnetic field **B**, the component of the motion ∥ to **B** is unaffected and the angular velocity changes to ω due to the perpendicular component. In general any simple harmonic vibration can be regarded as the sum of two superposed circular motions with the same frequency but in opposite directions. Due to the additional force $e(\mathbf{v} \times \mathbf{B})$, the frequencies of these two circular motions are different. Viewed perpendicular to **B**, radiation from the circular motion would be plane polarized with the electric vector ⊥ to **B**. Now

$$m\omega^2 r = kr \pm ev B$$

$$= m\omega_0^2 r \pm er\omega B.$$

or $$\omega^2 - \omega_0^2 = (\omega + \omega_0)(\omega - \omega_0) = \pm e\omega B/m.$$

Since $\omega \simeq \omega_0$ and $\omega = 2\pi\nu$, hence the modified frequency

$$\nu = \nu_0 \pm eB/4\pi m. \qquad ...(35)$$

As the electron is negatively charged, the slower rotation (longer λ) occurs in the anticlockwise direction as seen by an observer looking at the source through the hole in the pole piece. From his results Zeeman showed that the radiating particle was negative and e/m was found 1.6×10^{11} C/kg.

(*iv*) **Determination of e/m of an Electron (Thomson Method).** J.J. Thomson in 1897, devised an experiment for the determination of e/m (specific charge) of the electron by using electric and magnetic fields in mutually ⊥ directions. The electrons are allowed to pass in the plane of **E** but in the direction ⊥ to **E** and **B**.

The discharge is maintained by the application of high *pd* between the cathode *C* and anode *A* of a discharge tube containing air at a very low pressure ($\sim 10^{-2}$ mm of *Hg*). The electrons so produced are allowed to pass through slits A_1 and A_2 also kept at the potential of *A*. The beam then passes along the axis of the tube and produces a spot of light at *O* on the fluorescent screen *S*. The electric field **E** is applied between two horizontal plates *P* and *Q*. The magnetic field **B** is applied in the direction ⊥ to the paper plane by passing the current through coils, in the region within the dotted circle. It is clear from Fleming's left-hand rule, that **F** due to **E** is in upward direction, while due to **B** in downward direction.

Hence fields **E** and **B** can be adjusted so that the electrons suffer no deflection and strike at point O on the screen. In this case

$$eE=evB \text{ or } v=E/B. \qquad ...(36)$$

Now electric field is switched off, the electrons thus describe the circular arc and fall at O' on the screen. In this case the force

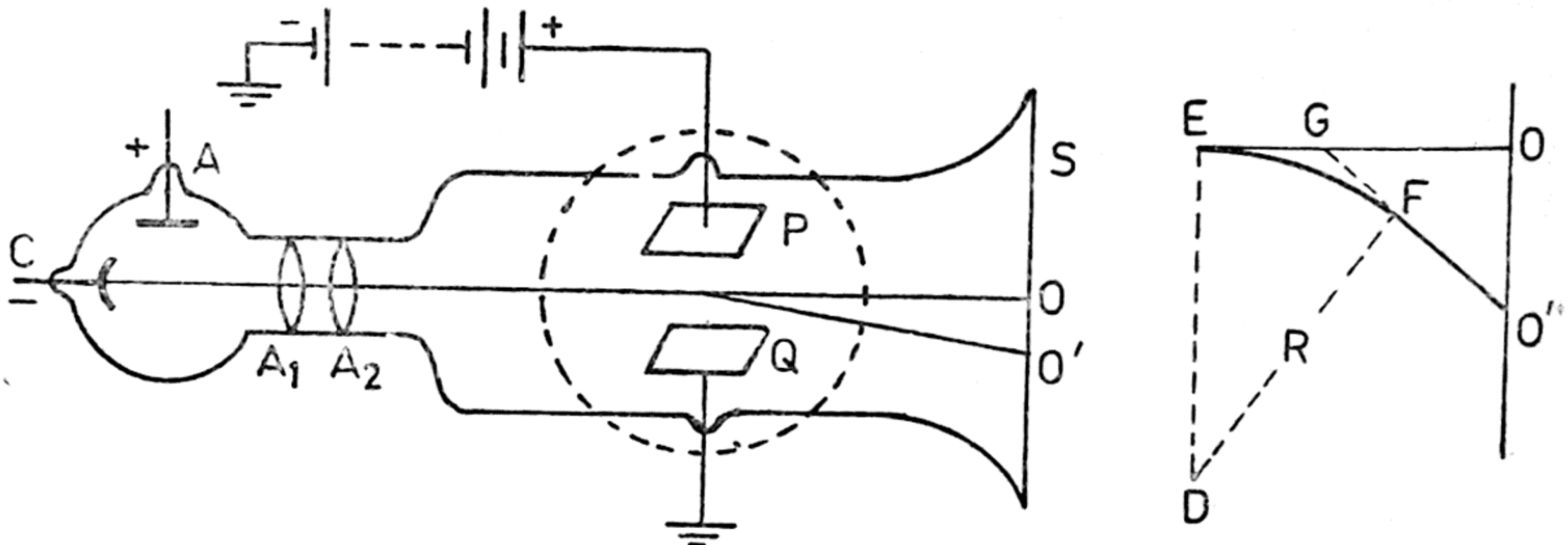


Fig. 8.13.

$F=Bev$ bends the electron beam in a circular arc, such that it is balanced by the centripetal force mv^2/R.

$$\because \quad Bev=mv^2/R \text{ or } v=BRe/m. \qquad ...(37)$$

Combining Eqs. (36) and (37), we get

$$e/m=E/RB^2. \qquad ...(38)$$

As E and B are known. To find R consider arc EF of the circular path in the magnetic field region. From the geometry of the Fig. 8.13 (right), we get

$$OO'/GO=EF/R \text{ or } R=EF\times GO/OO'. \qquad ...(39)$$

Practically EF is replaced by the width of the magnetic flux region and G is taken at the middle of the region. Thomson's value for e/m was 1.7×10^{11} coul/kg, which is in excellent agreement with the modern value of 1.75890×10^{11} coul/kg.

(*iv*) **Cyclotron**. In 1932, Lawrence developed a machine named cyclotron, for the acceleration of charged particles, such as protons or deuterons. These particles (ions), starting from the central source are caused to move in circular orbits by magnetic field and are accelerated by the electric field.

In its simplest form, it consists of two flat semi-circular metal boxes, called dees because of their shape. These hollow chambers have their diametric edges parallel and slightly separated from each other. A radio-frequency alternating potential of the order of megacycles per second is applied between the dees, which act as

electrodes. These dees are surrounded by a closed vessel containing gas like hydrogen, helium, deuterium at very low pressure. The vessel is placed between the poles of a strong electromagnet which provides a magnetic field perpendicular to the plane of the dees. A

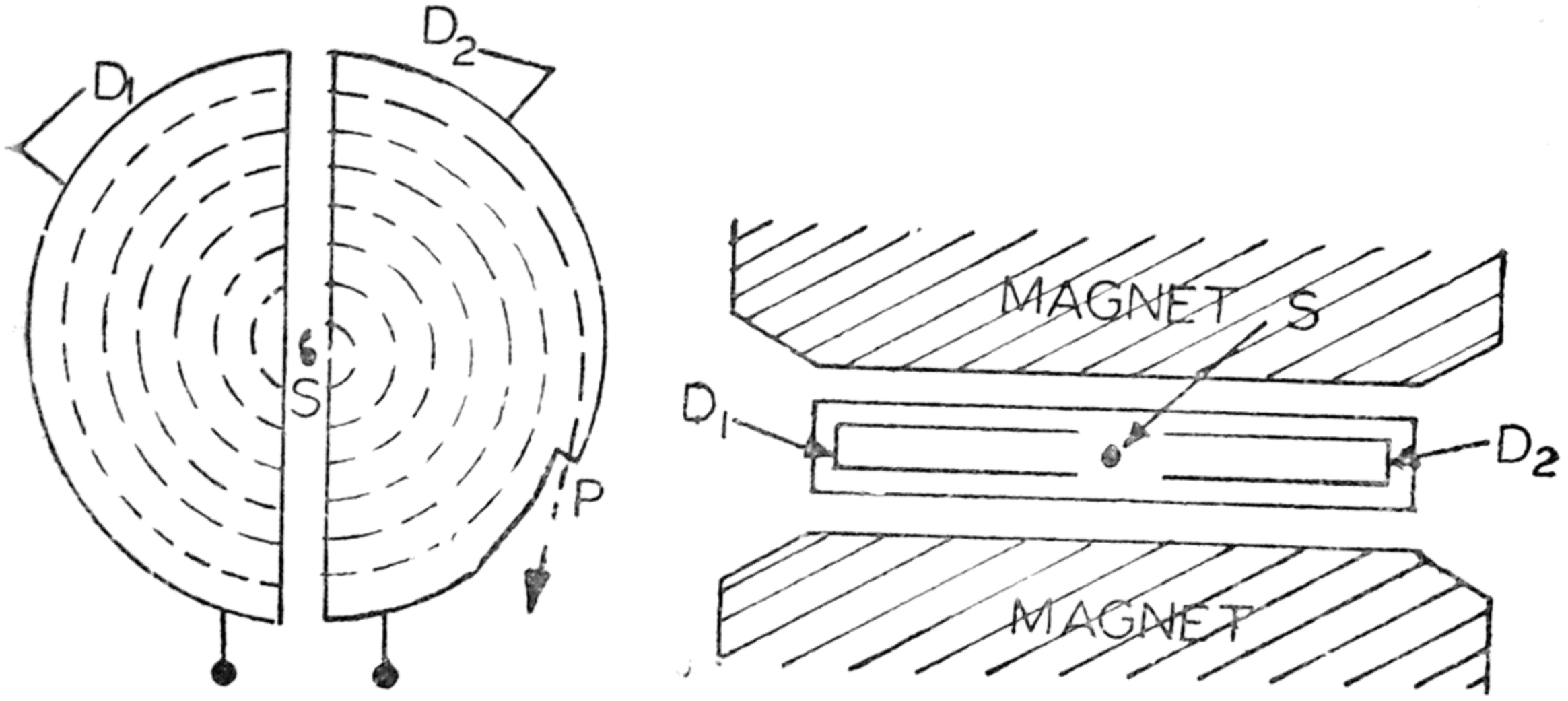


Fig. 8.14.

filament situated at the centre of the chamber is heated and a small potential difference is applied between the filament and the metal box to increase the energy of emitted electrons. These electrons ionize some of gas atoms and thus produce positive ions. Another way of producing ions is the use of ion source.

Suppose that at any particular instant the alternating potential is in the direction which makes D_1 positive and D_2 negative. A positive ion of mass M, charge q and an initial velocity v starting from the source S will be attracted by the dee D_2. Due to magnetic field B it will move in a circular path of radius r inside the dee D_2, where

$$r = Mv/Bq.$$

In the interior of the dee, the speed of the ion remains constant. After it has traversed half a cycle the ion comes to the edge of D_2. If in the meantime, the potential difference between D_1 and D_2, has changed direction so that D_2 is now positive and D_1 negative, the positive ion will receive an additional acceleration, while going across the gap between the dees and then travel in a circular path of larger radius inside D_2 under the influence of magnetic field. After traversing a half cycle in D_2 it will reach the edge of D_1 and receive an additional acceleration between the gaps because in the meantime the direction of potential difference between the dees has changed. The ion will continue travelling in a semi-circle of increasing radii, the direction of pd changes every-time the ion goes from D_1 to D_2 and from D_2 to D_1. The time taken by the charged particle to traverse the semicircular path in the dee is given by

$$t=\pi r/v=\pi M/Bq. \qquad ...(40)$$

This relation indicates that time t is independent of the velocity of the particle and of the radius. For any given value of M/q, it is determined by the magnetic field intensity. By adjusting the magnetic field intensity the time can be made the same as that required to change the potentials, On the other hand the oscillator frequency can also be adjusted to the nature of a given ion and to the strength of the magnetic field. The frequency of the oscillations required to keep the ion in phase is given by the relation

$$f=1/T=1/2t=Bq/2\pi M. \qquad ...(41)$$

If the oscillation frequency is adjusted to keep the charged ion always in phase, each time the ion crosses the gap it receives an additional energy and at the same time it describes a flat spiral of increasing radius. Eventually, the ion reaches the periphery of the dee, where it can be brought out of the chamber by means of a deflecting plate charged to a high negative potential. This attractive force draws the ion out of its spiral path and thus can be used easily. If R is the radius of the dee, kinetic energy of the ion emerging from the cyclotron is thus given by

$$E=\tfrac{1}{2}M(BqR/M)^2=B^2R^2q^2/2M=2\pi^2R^2f^2M. \qquad ...(42)$$

This relation indicates that the maximum energy attained by the ion is limited by the radius R, magnetic field B or the frequency of the alternating potential f. It is independent of the alternating voltage. It can be explained by the fact that when the voltage is low the ion makes a large number of turns before reaching the periphery, but when the voltage is high the number of turns is small. The total energy remains same in both the cases provided B and R are unchanged.

Cyclotrons are usually described in terms of the diameter of the pole faces of the magnet. The first machine constructed by Lawrence and Livingston in 1932 had a magnet with pole faces 11 inches in diameter and produced 1.2 MeV protons. The first cyclotron (26" dia) in operation in India is at Physics Department, P.U. Chandigrah. The first Indian 88" dia cyclotron is at Calcutta.

(*v*) **Mass Spectrograph.** Instruments are used to measure mass numbers, to determine the relative abundance of isotopes or to produce separated isotopes. Instruments in which we use a photographic plate and obtain the series of lines (*called the mass spectrum*) on it are known as *mass spectrographs*. Instruments in which as electric or a magnetic field is adjusted to make each part of the spectrum in turn to fall on a fixed detecting slit and measured electrically are called *mas spectrometers*.

All mass spectroscopes start with an ion source where the ions are produced either by electron bombardment of gases or by heating the suitably coated filaments. The ions are first set in motion with the help of an accelerating potential. In the spectrograph designed by Bainbridge of Harvard University, thus known as *Bainbridge mass spectrograph*, a velocity selection is used to obtain ions of particular velocity v. The velocity selector allows a beam of a positive ions having the same velocity v to pass undeviated through crossed electric and magnetic fields **E** and **B**. In this case forces due to these fields are equal and opposite.

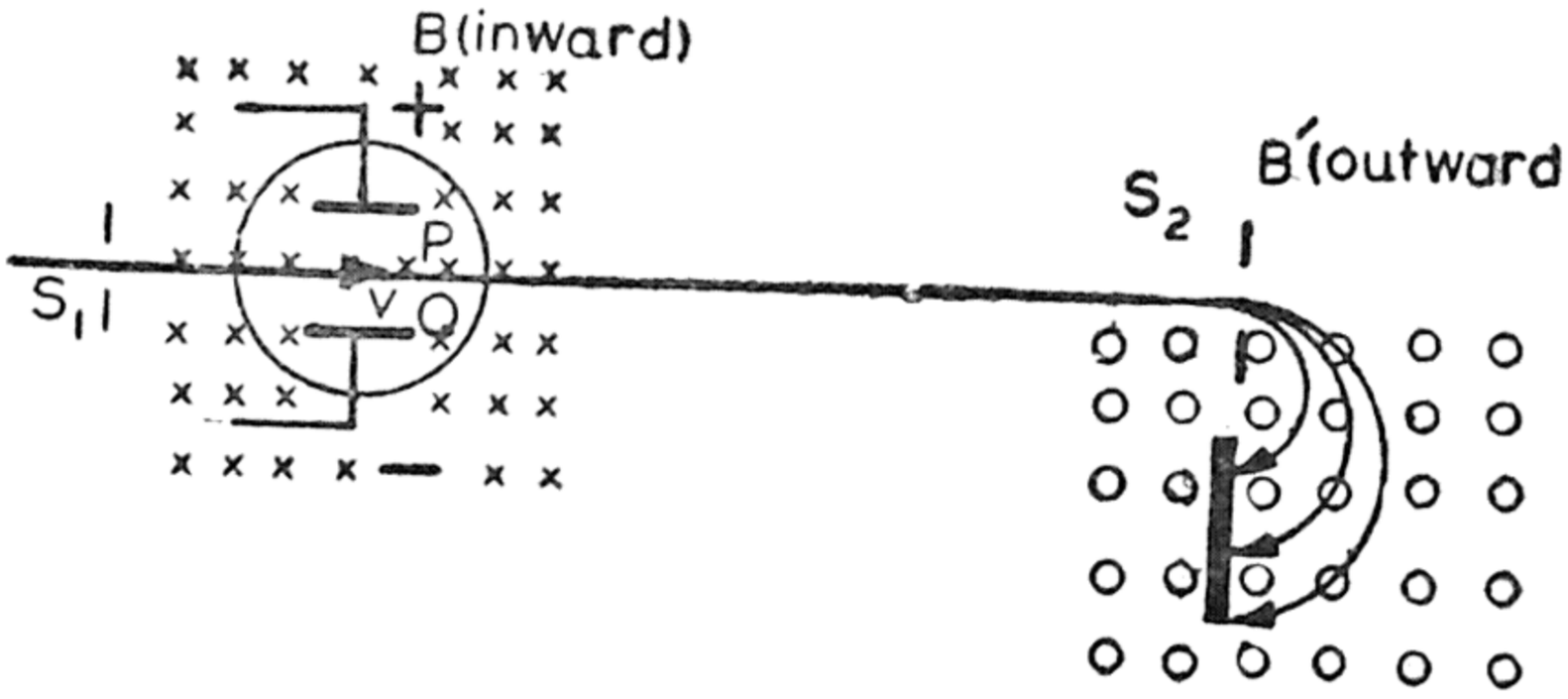


Fig 8.15.

Force due to electric field qE = *Force due to magnetic field* Bqv

or $$v = E/B. \qquad \ldots(43)$$

All the ions having the same velocity ($v = E/B$) enter the analysing chamber through the slit S_2. In this chamber another magnetic field B is applied perpendicular to the plane of paper and in outward direction. Due to this field ions of different masses move in circles of different radii such that

Magnetic forcc qvB' = centrifugal force mv^2/r

or $$r = mv/qB' = (v/qB')m. \qquad \ldots(44)$$

This relation shows that the radius of circular orbit is directly proportional to the mass m. Therefore the ions having same value of q/m are focussed on to the different positions on the photographic plate. The value of r is obtained from the position of the line on the photographic plate and hence q/m is calculated.

Let us now consider the effect of simultaneous electric and magnetic fields acting parallel to each other on a beam of charged particles. Thomson designed a mass spectrograph using such fields for the determination of q/m for positive rays. It is called *Thomson's mass spectrograph.*

Positive rays also called *canal rays* are produced in a discharge tube consisting of an anode (an aluminium disc) and a cathode (rod of *Al* in which is embedded a narrow copper tube of bore 0.1 mm). These rays are allowed to pass through the strong electric and magnetic fields acting parallel to each other and perpendicular to the ray direction. Let us calculate the effect of these fields.

Consider a particle of charge q and mass m moving along the negative z direction with an initial velocity v.

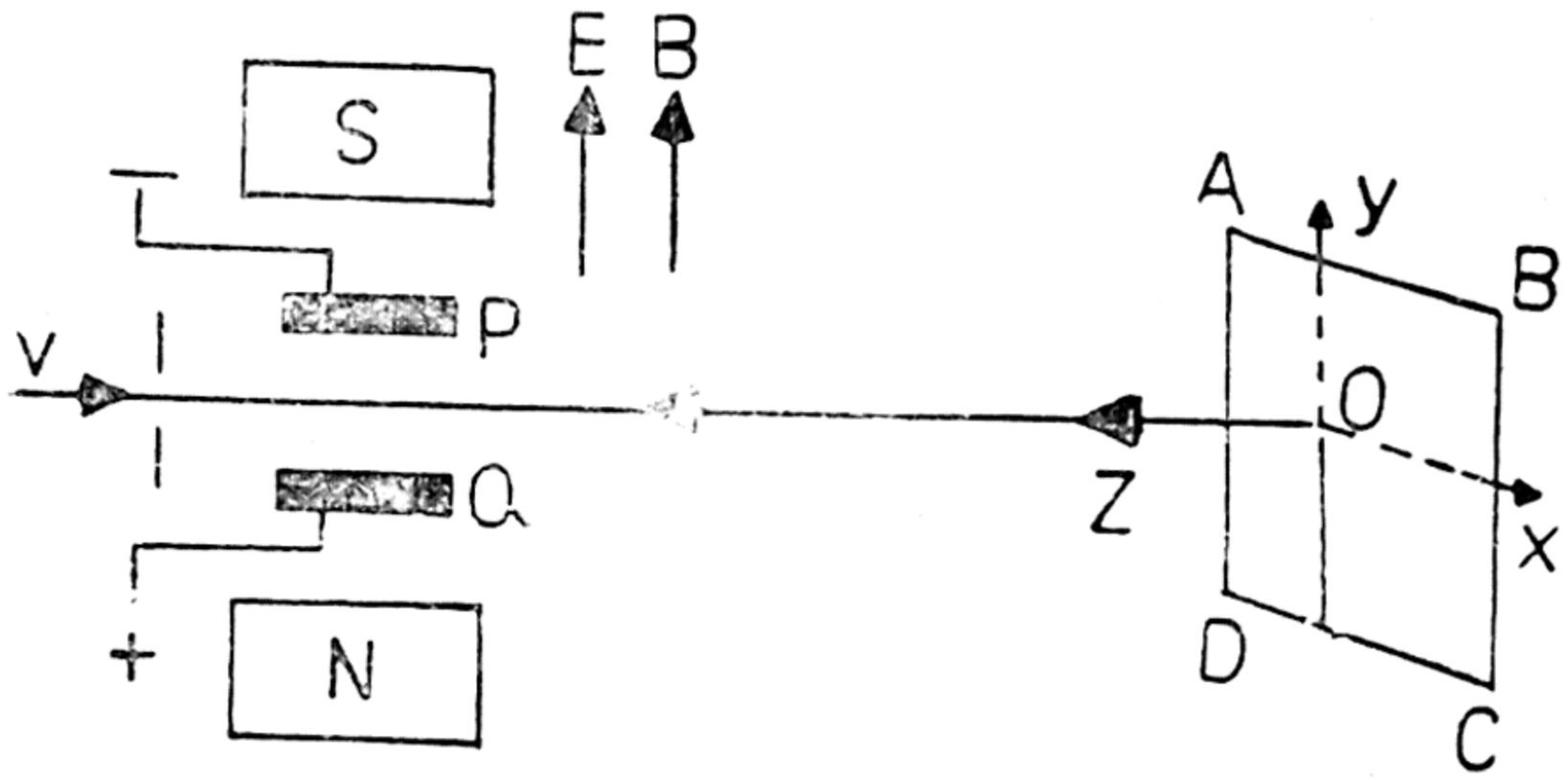


Fig. 8.16.

Let **E** and **B** fields be along the y-direction. Due to the magnetic field **B**, a force $q\mathbf{v}\times\mathbf{B}$ will act on the charge particle along x-axis. Hence the acceleration produced will be given by the relation.

$$a_x = qBv/m.$$

If this field exists in the length l_1, the time taken by the particle to traverse the magnetic field with velocity v is

$$t_1 = l_1/v.$$

Hence the displacement of the particle in the x-direction at the end of the magnetic field

$$x_1 = \tfrac{1}{2}\, a_x t_1^{\,2} = \tfrac{1}{2}(qBv/m)\,(l_1/v)^2.$$

If the screen is placed at a distance L_1 from the end point of magnetic field the further displacement of the charged particle along x-axis is

$$x_2 = L_1 \tan\theta = L_1 \quad \frac{v_x}{v} = \frac{L_1}{v}\; a_x t_1 = \frac{qBL_1\, l_1}{mv}.$$

$$\therefore \text{ Total displacement } x = x_1 + x_2 = \frac{qB}{mv}\left(\frac{l_1^2}{2} + L_1\, l_1\right) \qquad \ldots(45)$$

As discussed in article 8.4, the displacement along y-direction due to the electric field in y-direction on a charged particle moving along z-direction with a velocity v is given as

$$y = \frac{qE}{mv^2}\left(\frac{l_2^2}{2} + L_2 l_2\right). \qquad \ldots(46)$$

where l_2 is the length of electric field region and L_2 is the distance from the end of electric field to the screen.

Eliminating v from Eqs. (45) and (46), we get

$$x^2 = \frac{q}{m}\frac{B^2}{E}\frac{(\frac{1}{2}l_1^2 + l_1L_1)^2}{(\frac{1}{2}l_2^2 + l_2L_2)}\ y = \frac{q}{m}Ky, \qquad \ldots(47)$$

where the constant K can be calculated from the knowledge of the **E** and **B** fields and the geometry of the apparatus.

For the particles having same q/m ratio, this equation represents a parabola in x-y plane. By measuring ordinates of any point on a parabola the value of q/m can be determined. The particles of different q/m will trace the different parabolas on the screen.

8.7. FORCE ON A CURRENT CARRYING CONDUCTOR

Let us consider a conducting wire of length (l) and cross-sectional area A, in which the current i is passing. If this wire is placed in a magnetic field **B**, which is perpendicular to the plane of the diagram pointing in downward direction, shown by sign $\times$ in Fig. 8.17.

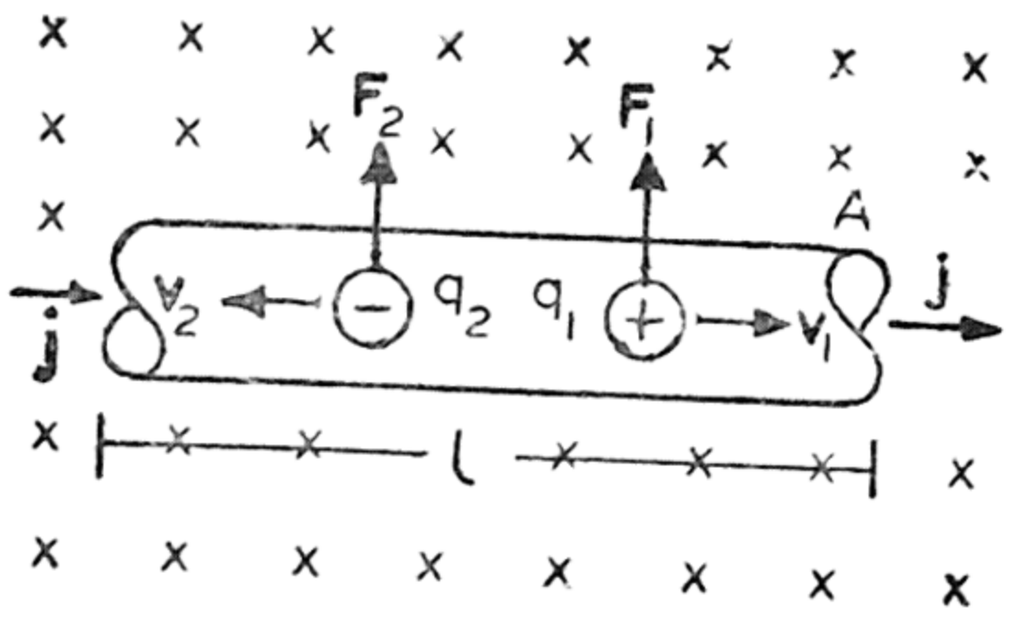


Fig. 8.17.

The current i in a metal wire is carried by the negative charges (free, or conduction electrons) and the positive charges (positive ions). If the positive charge q_1 is moving with a drift velocity **v**, it is acted on by an force $\mathbf{F}_1$, given as

$$\mathbf{F}_1 = q_1(\mathbf{V}_1 \times \mathbf{B}).$$

The direction of $\mathbf{F}_1$ is upward, if the direction of the current density **j** is from left to right.

The negative charge q_2 moving in opposite direction is acted on by a force

$$\mathbf{F}_2 = -q_2(\mathbf{v}_2 \times \mathbf{B}).$$

It is also in upward direction as the velocity v_2 is opposite to v_1. If n_1 is the number of +ve charge carriers per unit volume and n_2 that of – ve charge carriers per unit volume, then the total force **F** on all charge carriers of the given conductor

$$\mathbf{F} = (n_1 Al)q_1(\mathbf{v}_1 \times \mathbf{B}) - (n_2 Al)q_2(\mathbf{v}_2 \times \mathbf{B}) = \Sigma Aln_i q_i(\mathbf{v}_i \times \mathbf{B}).$$

But we know that the vector current density $\mathbf{j} = \Sigma n_i q_i \mathbf{v}$, hence

$$\mathbf{F} = Al(\mathbf{j} \times \mathbf{B}).$$

As $i = Aj$ and the direction of the current is along the length or displacement vector **l**, hence the above relation can the expressed as

$$\mathbf{F} = i(\mathbf{l} \times \mathbf{B}). \qquad \ldots(48)$$

This relation is very useful when the conductor is of any shape and makes any arbitrary angle with the field. Actually it is equivalent to the relation (3) which is for a single carrier.

If we consider a differential element of a conductor of length $d\mathbf{l}$, then the force $d\mathbf{F}$ is given by

$$d\mathbf{F} = i(d\mathbf{l} \times \mathbf{B}). \qquad \ldots(49)$$

The Eqs. (48) and (49) describe both the magnitude and direction of the force. The direction of **F** (or $d\mathbf{F}$) is $\perp$ to the plane containing **j** (or **l**) and **B**.

8.8. FORCE AND TORQUE ON A CURRENT LOOP IN A UNIFORM MAGNETIC FIELD

Let us consider a very simple shape of the loop, *a rectangular loop*, as shown in Fig. 8.18. Its sides are of lengths a and b. It is free to rotate about the axis through its centre of mass *i e.* axis OO' which is $\perp$ to the direction of the uniform magnetic field **B**. Let the normal to the plane of the loop makes an angle θ with the field direction **B**.

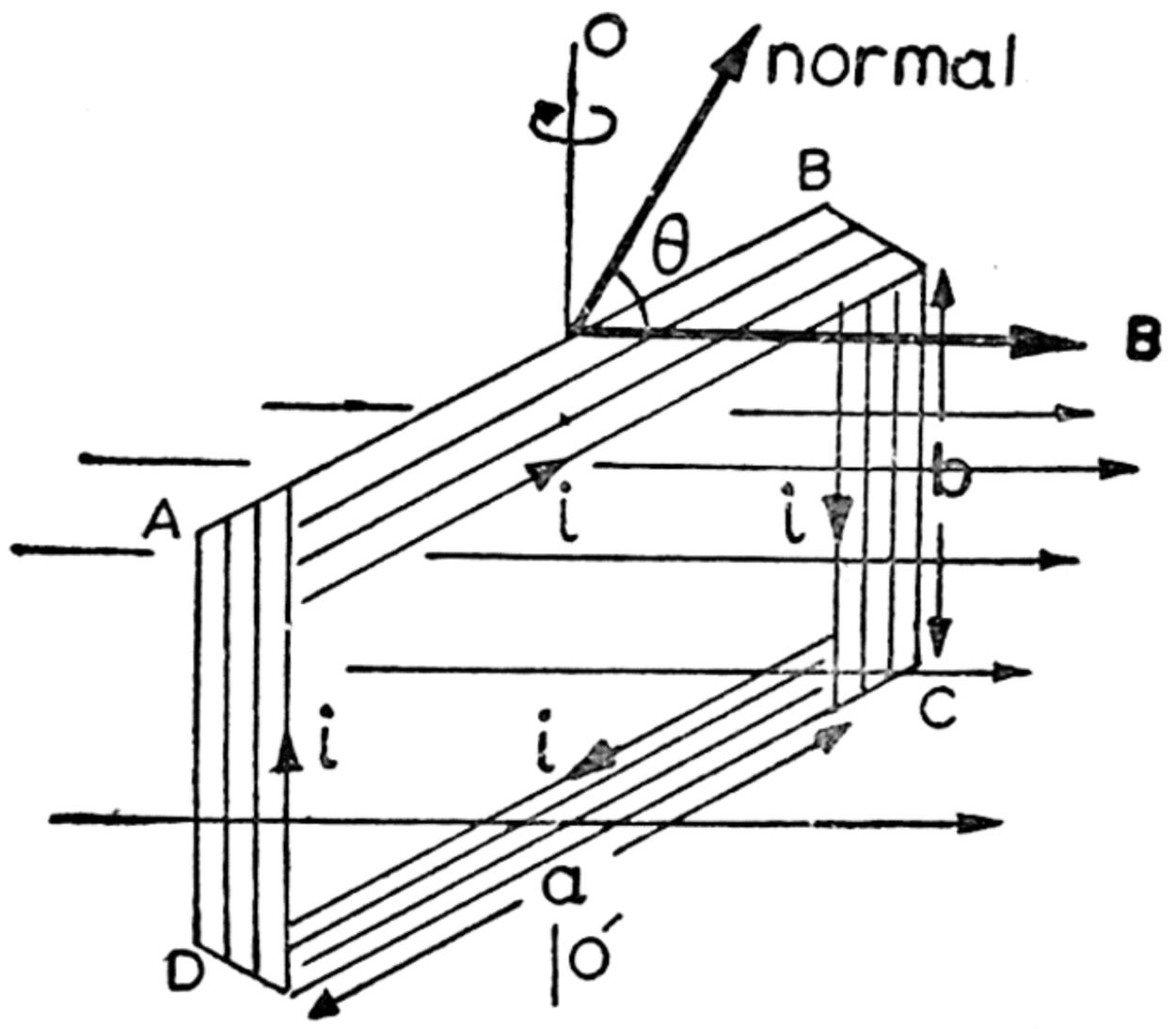


Fig. 8.18.

If the current i is allowed to pass through the loop, say in clockwise direction when viewed from the right of the Fig. The net force on the loop is the resultant of the forces on its four sides. Using the force Eq. (48), applied to the top side AB we find

$$\mathbf{F}_1 = i(\mathbf{l} \times \mathbf{B}) \text{ or } \qquad F_1 = iaB \sin(90° - \theta) = iaB \cos\theta.$$

The vector cross product indicates that this force is in the downward direction. One can show very easily that the force $\mathbf{F}_3$ on the side CD has the same magnitude but points in the upward direction. As these forces $\mathbf{F}_1$ and $\mathbf{F}_3$ have the same line of action, hence the net force on the loop from these elements AB and CD is zero and the net torque due to these forces is zero.

The force $\mathbf{F}_2$ on the side BC is ibB. It is along the outward normal to the plane of paper. The force $\mathbf{F}_4$ on the side AD has the

same magnitude ibB but is along the inward normal to the plane of paper. These forces $\mathbf{F}_2$ and $\mathbf{F}_4$ are thus oppositely directed. These do not have the same line of action and thus constitute a couple. The net torque tends to rotate the loop in the direction of the torque. Using Fig. 8.19 the view looking downward on the loop, we find for the total torque or the moment of the couple $\tau = ibB \times a \cos(90-\theta) = iBA \sin\theta$, where A is the area of the loop.

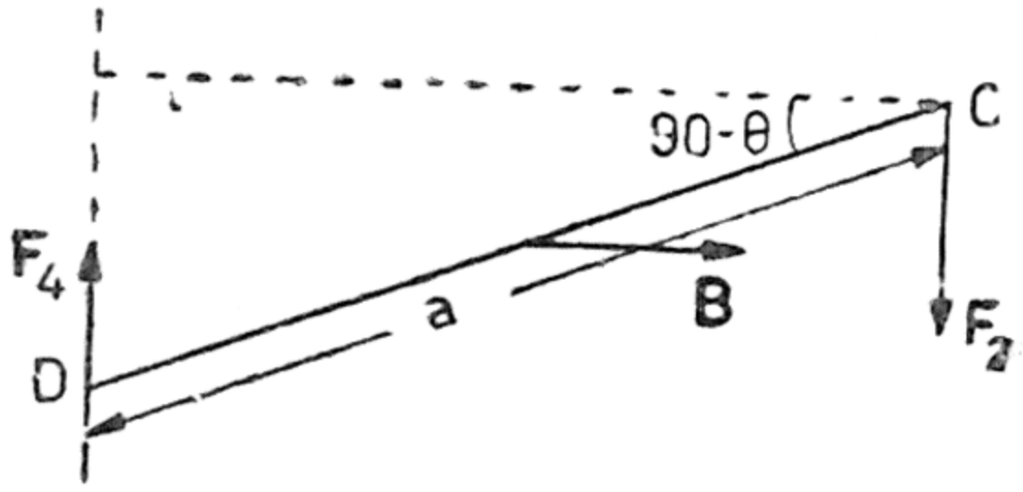


Fig. 8.19.

In the vector form, it can be written as

$$\tau = i(\mathbf{A} \times \mathbf{B}). \qquad \ldots(50)$$

where **A** is the vector area of the loop, having direction given by the right hand screw rule as : *If the current loop is rotated in the direction of the current, then* **A** *points in the direction of travel of a right hand screw subject to such a rotation.*

Eq. (50) also holds good for all plane loops of any shape having an area A (To be proved in example 12).

If a coil consists N turns, the torque acting on the entire coil is

$$\tau' = N\tau = iN\,AB \sin\theta. \qquad \ldots(51)$$

The coil is rotated by this torque until the equilibrium position reaches in which area vector **A** is in the same direction as the field **B**.

Let us now compare the relation (51) with the expression for the torque acting on the electric dipole of dipole moment **p** placed in an external electric field **E**, *i.e.* $\tau = \mathbf{p} \times \mathbf{E}$ or $\tau = pE \sin\theta$. We see that the coil has a magnetic dipole moment $\mathbf{p}_m (= iN\mathbf{A})$ analogous to the electric dipole moment of an electric dipole. Hence relation (51) can be written as

$$\tau = \mathbf{p}_m \times \mathbf{B}. \qquad \ldots(52)$$

The direction of $\mathbf{p}_m$ is the direction of thumb of the right hand if the fingers of this curl around the loop are in the direction of the current.

When the current loop or a magnetic dipole is placed in a magnetic field **B**, work must be done by an external agent to change the orientation of the dipole. This work done is the potential energy of the magnetic dipole. By analogy with the assumption made for electric dipoles, magnetic energy is assumed zero when $\mathbf{p}_m$ is $\perp$ to **B**. If the magnetic dipole is turned through an angle θ from the zero energy position, then

$$U=\int_{90^\circ}^{\theta}\tau d\theta=p_m B\int_{90^\circ}^{\theta}\sin\theta\, d\theta=-p_m B\cos\theta$$

$$=-\mathbf{p}_m.\ \mathbf{B}. \qquad \ldots(53)$$

It is same as for energy of an electric dipole in an electric field **E**, *i.e.* $U=-\mathbf{p}_E.\ \mathbf{E}$.

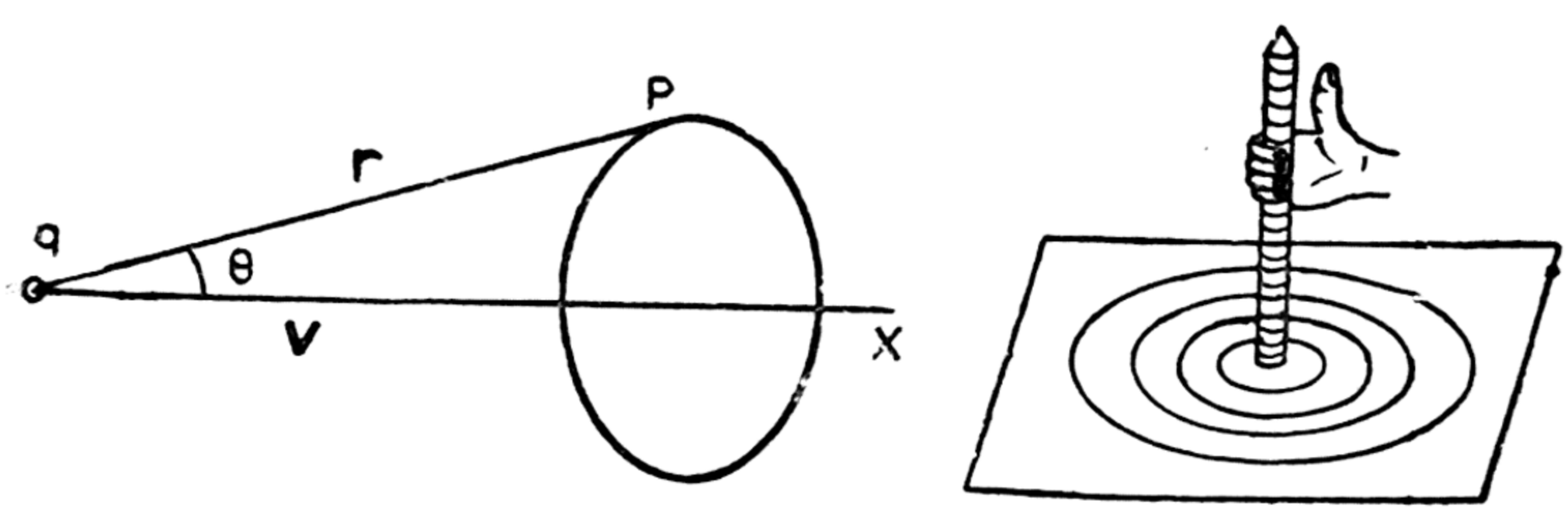
8.9. MAGNETIC FIELD OF A MOVING POINT CHARGE

We know that a point charge q, at rest in the observer's inertial system, produces an electric field along the radius vector and is given as

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r}. \qquad \ldots(54)$$

If this charge is moving relative to the observer's inertial system, it produces a magnetic field as well as an electric field. The magnitude of the magnetic field is proportional to the speed of the charge relative to the observer (provided speed $v<c$, the velocity of light), An observer travelling with a moving charge detects no magnetic field. The magnetic field vector **B** at the point P at a distance **r** from the charge q moving with a velocity **v** is found to be

$$\mathbf{B}=\frac{\mu_0 q}{4\pi r^3}(\mathbf{v}\times\mathbf{r}). \qquad \ldots(55)$$



(*a*) (*b*)

Fig. 8.20.

The direction of **B** is thus perpendicular to the plane of **v** and **r** in the direction of advance of a right handed screw rotated from **v** to **r**. Its magnitude is

$$B=\frac{\mu_0}{4\pi}\frac{q\ v\sin\theta}{r^2}.$$

The **B** is thus zero at all points on a line through the charge in the direction of **v**, as $\theta=0$ or π, $\sin\theta=0$, for all such points. **B** is maximum in the plane perpendicular to the **v** and through the charge q, as $\sin\theta=1$ at all points in this plane. **B** remains unaltered in magnitude at all points on the circumference of circle passing

through P and lying in a plane $\perp$ to $\mathbf{v}$ with its centre on the $\mathbf{v}$ direction. Thus we see that unlike the $\mathbf{E}$ field of the charge which is radial, the lines of induction of $\mathbf{B}$ are circles in plane $\perp$ to the velocity vector $\mathbf{v}$, with centres on a line through this vector. The direction of $\mathbf{B}$ is given by either :—

(*a*) *Maxwell's cork-screw rule.* If the direction of the current through the conductor is represented by the linear motion of the cork-screw motion, then the direction of the magnetic field can be represented by the direction of rotation of the cork-screw.

(*b*) *Right hand claps rule.* If a conductor carrying a current is clapsed in the right hand so that the thumb indicates the current direction, then the direction of the magnetic field is represented by the finger tips round the wire.

The constant of proportionality is $\mu_0/4\pi$ and is having value 10^{-7} weber/amp-m. Thus quantity

$$\mu_0 = 4\pi \times 10^{-7}\ WbA^{-1}\ m^{-1} \text{ (or newton sec}^2\text{/coulomb}^2\text{)}$$

This is called the permeability of free space. The reason for this choice of the numerical value will be explained in the next article. It is very clear that—

$$\frac{\text{Constant of proportionality in electric field relation}}{\text{Constant of proportionality in magnetic field relation}} = \frac{1/4\pi\varepsilon_0}{\mu_0/4\pi}$$

$$= \frac{9.0 \times 10^9 \text{ Newton-meter}^2/\text{coul}^2}{10^{-7} \text{ Newton sec}^2/\text{coul}^2} = 9 \times 10^{16} \text{ metre}^2/\text{sec}^2,$$

which is equal to the square of the speed of light c. This suggests that there may be a *close relation between electricity, magnetism and light, i.e.*, $\varepsilon_0\mu_0 = 1/c^2$.

Using relation (3), the force acting on a charge q', moving with velocity $\mathbf{v}'$, in a magnetic field $\mathbf{B}$ given by

$$\mathbf{F} = q'\mathbf{v}' \times \mathbf{B}.$$

If this field $\mathbf{B}$ is set up by a charge q moving with velocity $\mathbf{v}$, then

$$\mathbf{F} = q'\mathbf{v}' \times \frac{\mu_0 q}{4\pi r^3}(\mathbf{v} \times \mathbf{r}) = -\frac{\mu_0 q q'}{4\pi\ r^3}\ \mathbf{v}' \times (\mathbf{v} \times \mathbf{r}). \qquad \ldots(56)$$

which corresponds to coulomb's electrical force between the charges q and q' moving with velocities v and v' respectively relative to an observer at rest.

8.10. THE BIOT SAVART LAW

It is the relation for the magnetic field by the current element and is based upon the experimental facts. It is the improved form of

the relation (53). Let us consider a small section dl of a wire of a cross section A carrying a current i. If n is the number of charge carriers per unit volume, each of charge q and moving with a drift velocity $\mathbf{v}$, then

$$i = nq\mathbf{v}.\ \mathbf{A} = nqAv \text{ or charge } dQ = Anqdl.$$

Here $vdt = dl$. The magnetic field induction due to the current element dl at any point P at a distance $\mathbf{r}$ from this element is

$$d\mathbf{B} = \frac{\mu_0 dQ}{4\pi r^3}(\mathbf{v}\times\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Anqdl\,(\mathbf{v}\times\mathbf{r})}{r^3}$$

$$= \frac{\mu_0}{4\pi}\frac{Anqv(d\mathbf{l}\times\mathbf{r})}{r^3} = \frac{\mu_0 i}{4\pi r^3}(d\mathbf{l}\times\mathbf{r}), \qquad \ldots(57)$$

This is called *Biot-Savart law* for an element. Here, $d\mathbf{l}$ is a vector of length dl which is along the direction of $\mathbf{v}$ or current. Here we see that the direction of $d\mathbf{B}$ is perpendicular to the plane of $d\mathbf{l}$ and $\mathbf{r}$. The resultant field $\mathbf{B}$ at this point P due to the complete current circuit is thus given by

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{i}{r^3}(d\mathbf{l}\times\mathbf{r}). \qquad \ldots(58)$$

Relation (57) can not be verified experimentally, however $\mathbf{B}$, given by Eq. (58), can be measured experimentally.

To define unit of current, *ampere*, let us consider two elements A and B of lengths $d\mathbf{l}_1$ and $d\mathbf{l}_2$ having currents i_1 an i_2 respectively. In calculating the force between these elements, the first step is to calculate the magnetic induction $d\mathbf{B}$ at the right hand conductor (say) due to the current in the left hand conductor.

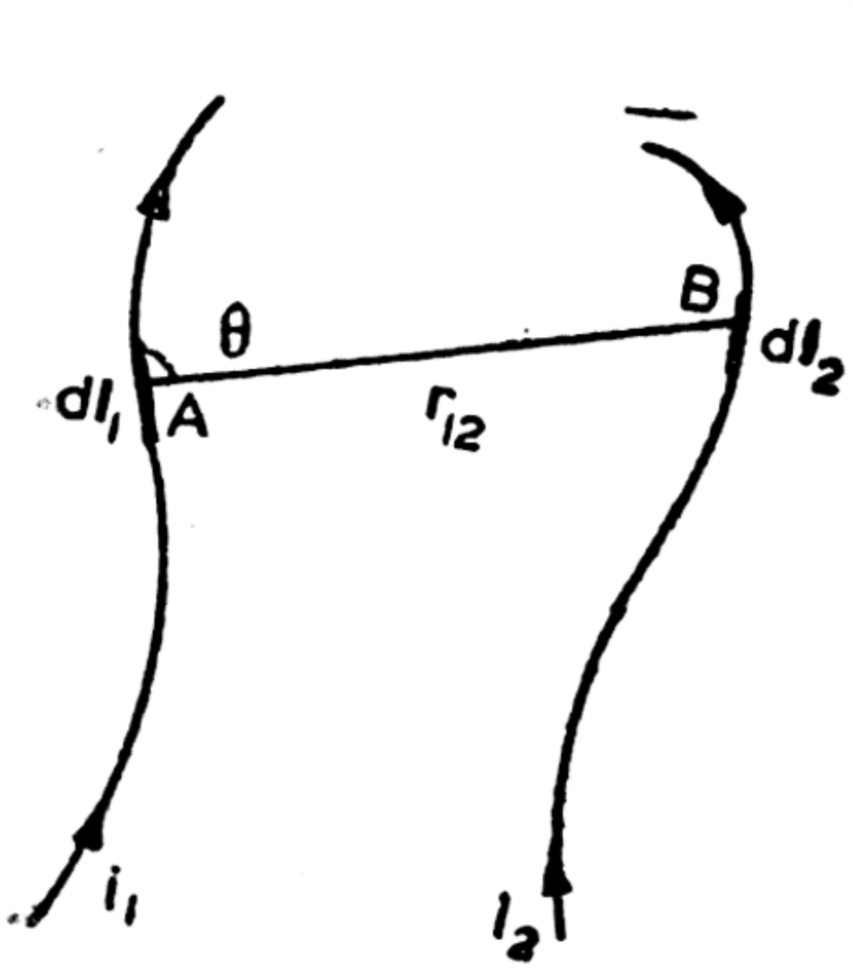


Fig. 8.21.

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\, i_1 \frac{d\mathbf{l}_1\times\mathbf{r}_{12}}{r_{12}^3}.$$

In this case $d\mathbf{B}$ is $\perp$ to $d\mathbf{l}_1$ and $\mathbf{r}_{12}$ and is directed into the paper, as $\mathbf{r}_{12}$ points from A to B. Next the force equation is applied to the right hand element, giving

$$d\mathbf{F} = i_2 d\mathbf{l}_2\times d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i_1 i_2}{r_{12}^3} d\mathbf{l}_2\times(d\mathbf{l}_1\times\mathbf{r}_{12}). \qquad \ldots(59)$$

The total force on the second wire due to the current in the entire length of the first wire is obtained by integrating this equation over the whole lengths l_1 and l_2.

$$\therefore \quad \mathbf{F}=\int_{l_1}\int_{l_2} d\mathbf{F}=\frac{\mu_0}{4\pi}\, i_1 i_2 \int_{l_1}\int_{l_2} \frac{d\mathbf{l}_2\times(d\mathbf{l}_1\times\mathbf{r}_{12})}{r_{12}^{\,3}}\,. \qquad \ldots(60)$$

This equation does not appear to be symmetrical in respect of $d\mathbf{l}_1$ and $d\mathbf{l}_2$, *i.e.* the force on one wire does not appear to be the force on the other wire. A use of vector algebra shows that this equation is symmetrical and thus not violating Newton's third law. This relation is equivalent of Coulomb's law equation for the force between static charges. If these current elements are each of unit length and at a unit distance apart and are so oriented that the factor $d\mathbf{l}_2\times(d\mathbf{l}_1\times\mathbf{r})/r^3$ is unity. Then the current in the elements are defined to be an ampere, if the resulting force is 10^{-7} newton. From this we get $\mu_0=4\pi\times10^{-7}$.

Thus we see that μ_0 is obtained from the definition hence cannot be measured experimentally. On the other hand ε_0 is a measured quantity and can be found from Coulomb's law.

8.11. CALCULATION OF B BY USING BIOT-SAVART LAW

(a) **Long straight wire.** A point P is chosen at a fixed distance R from the long straight wire AB, carrying a current i. Let us consider a small element dx at a distance x from point O, the foot of perpendicular from point P to the wire. Using Biot-Savart law for this element, we get

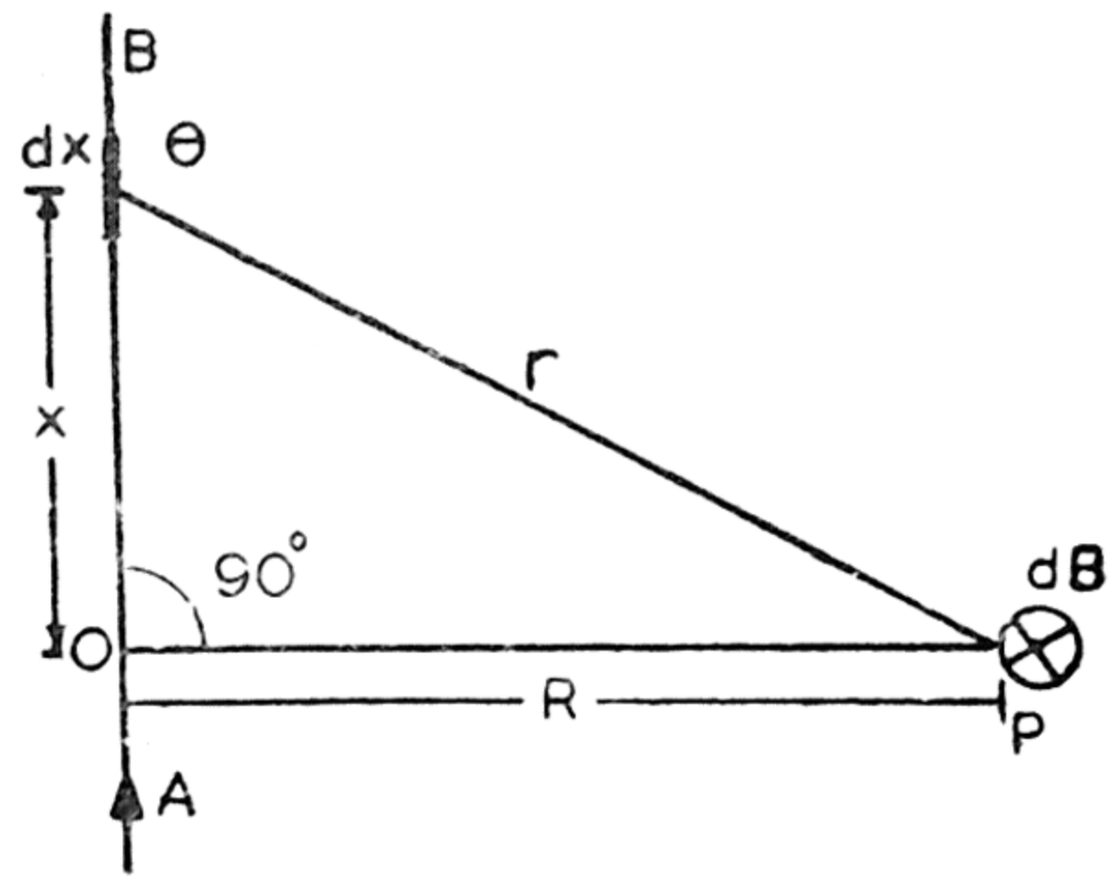


Fig. 8.22.

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{i}{r^3}\, d\mathbf{x}\times\mathbf{r}.$$

The direction of $d\mathbf{B}$ is perpendicular to the plane of $d\mathbf{x}$ and $\mathbf{r}$ and is directed into the plane of paper. This is the direction for all current elements. Hence the total magnetic field

$$\mathbf{B}=\int d\mathbf{B}=\frac{\mu_0}{4\pi}\, i \int \frac{d\mathbf{x}\times\mathbf{r}}{r^3}\,, \text{ or } B=\frac{\mu_0}{4\pi}\, i \int_{r=-\infty}^{r=\infty} \frac{dx\,\sin\theta}{r^2}.$$

From Fig. 8.22, $\sin(180-\theta)=R/r$ or $r=R\,\text{cosec}\,\theta$ arnd $x=-R\cot\theta$ or $dx=R\,\text{cosec}^2\,\theta\, d\theta$.

$$\therefore \quad B=\frac{\mu_0}{4\pi}\, i \int \frac{R\,\text{cosec}^2\,\theta\,\sin\theta\, d\theta}{R^2\,\text{cosec}^2\,\theta}=\frac{\mu_0}{4\pi}\,\frac{i}{R}\int_0^{\pi}\sin\theta\, d\theta$$

$$=\frac{\mu_0}{4\pi}\,\frac{i}{R}\cdot 2=\frac{\mu_0 i}{2\pi R}. \qquad \ldots(61)$$

This relation indicates that the value of B is same for all points at the same distance from the wire, thus the lines of magnetic

induction are circles concentric with wire and lying in planes ⊥ to it. This relation is analogous to the expression for **E** due to a long charged rod $(1/4\pi\varepsilon_0)(2\lambda/r)$ and thus shows its *electrostatic equivalent nature.*

(*b*) **Circular current loop.** Let us consider a circular loop of radius a, having turns n and carrying a current i. The magnetic field $d\mathbf{B}$ at point P on its axis at a distance x from its centre, due to a current element AB of length $d\mathbf{l}$ is

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\, i\, \frac{d\mathbf{l}\times\mathbf{r}}{r^3}.$$

If the plane of the loop or coil is ⊥ to the plane of the paper, the vector $d\mathbf{l}$ for a current element at the top of the loop points perpendicularly out of the page. Since **r** is perpendicular for all elements around the loop, $\mathbf{r} \perp d\mathbf{l}$ hence $d\mathbf{B}$ is always ⊥ to the plane consisting of **r** and $d\mathbf{l}$. Thus $d\mathbf{B}$ must lie in the plane of paper, as shown in Fig. 8.23. It can be resolved into two components, one $d\mathbf{B} \parallel$ (or $dB \sin\alpha$) along the axis of the loop and the other $d\mathbf{B} \perp$ (or $dB \cos\alpha$) at right angles to the axis. By symmetry the component of $d\mathbf{B}$ perpendicular to the axis vanishes for the complete loop as the component due to the top element AB is cancelled by the field component due to bottom element $A'B'$ and so on and the

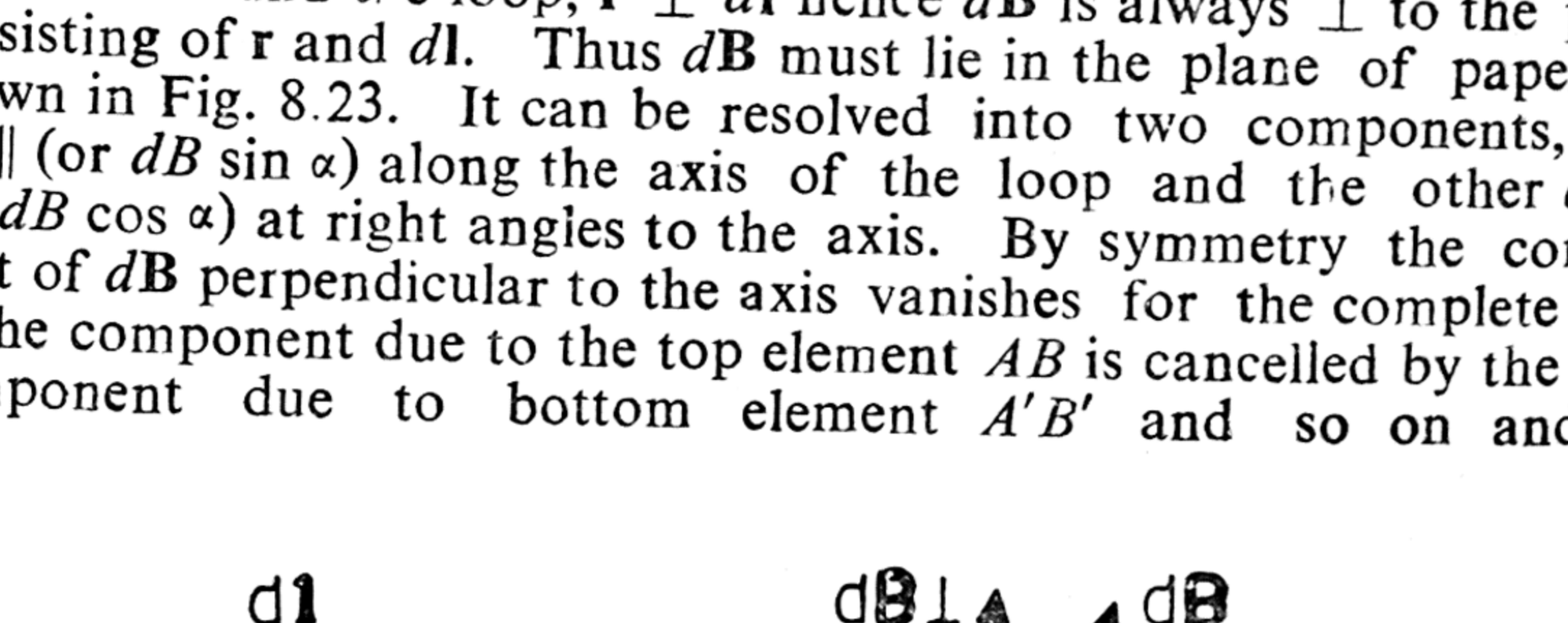


Fig. 8.23.

resultant B is in the direction of the axis and is equal to $\int dB \sin\alpha$, where α is the angle between **r** and the axis of the loop and $\sin\alpha = a/r$. Hence

$$B=\int \frac{\mu_0}{4\pi}\, i\, \frac{dl}{r^2}\cdot \sin 90^\circ \sin\alpha$$

$$=\frac{\mu_0}{4\pi}\frac{i}{r^2}\sin\alpha\int dl,$$

As α and r are constants, so they are taken outside the integral. The integral of dl around the loop is $2\pi a$. If r and α are expressed in terms of the constants a and x, we then get

$$B=\frac{\mu_0}{2}\frac{ia^2}{(a^2+x^2)^{3/2}}\ weber/m^2. \qquad ...(62)$$

At the centre of the loop, $x=0$ and $B=\mu_0 i/2a$, pointing along the axis of the loop in a direction depending on the direction of

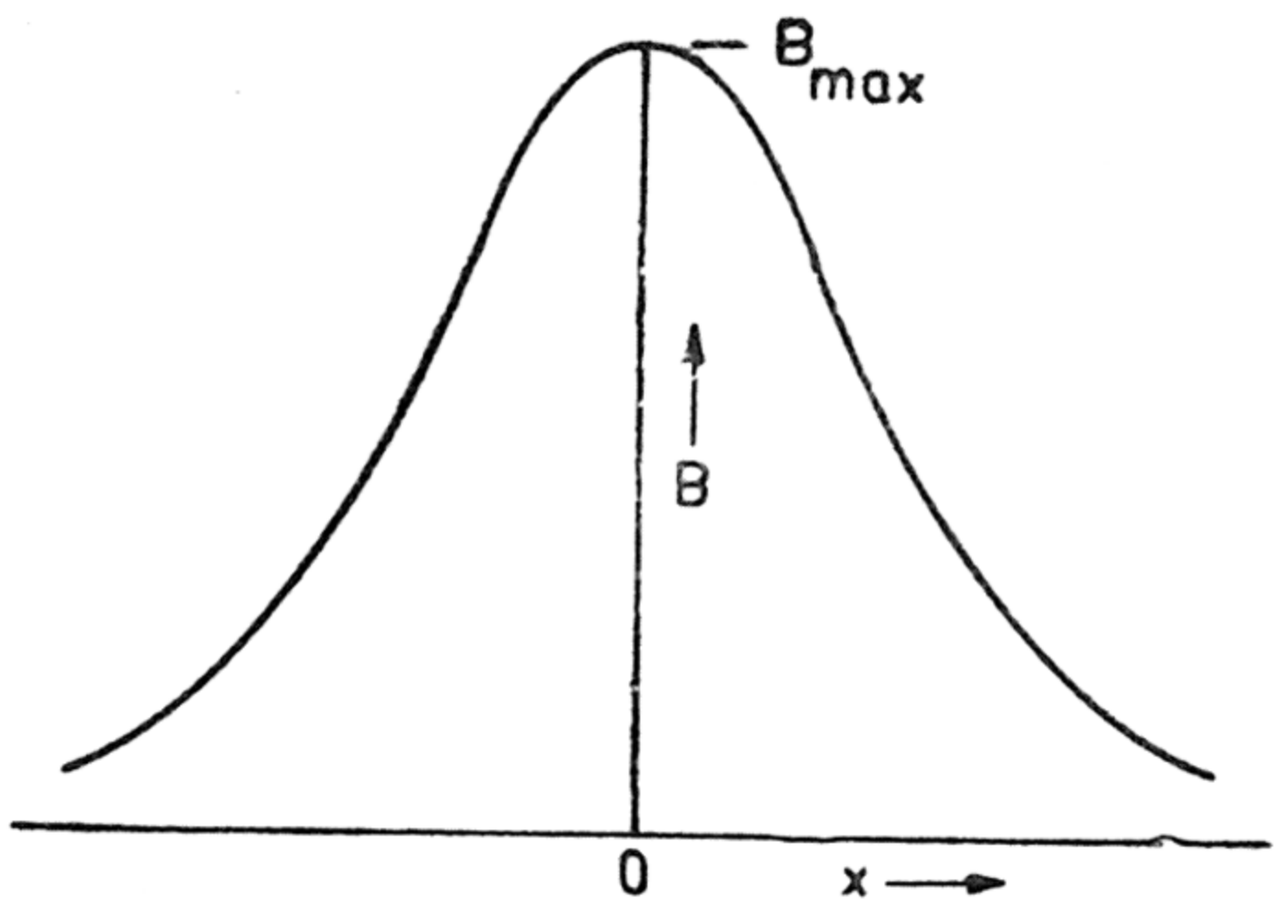


Fig. 8.24.

current flow. **B** is assumed positive when it is directed toward the viewer and the current is assumed positive when it is directed counterclockwise as we view it. If the coil has N closely packed turns, the result is multiplied by N. Thus

$$B=\frac{\mu_0}{2}\frac{iNa^2}{(a^2+x^2)^{3/2}}. \qquad ...(63)$$

We have not considered the effect of current leads to the loop, their contribution to the field can be made vanishingly small by keeping them close together. As the lines of force are concentric circles around the wires in opposite directions for the two wires, they cancel when close together as in the lead wires.

As $\pi a^2=A$, the area of the loop, and $r=x$ approximately for the points for which $x\gg a$, we therefore get

$$B=(\mu_0/4\pi)\ (2iNA/x^3). \qquad ...(64)$$

This relation shows that the current loop generates its own magnetic field and hence can be regarded as magnetic dipole. Thus the axial B – field of the loop at large distances is like the E – field of an electric dipole $[E=(1/4\pi\varepsilon_0)\ (2p_E/x^3)]$ on the axis of an electric dipole and the term (iNA) corresponds to electric dipole moment p_E of the dipole. Therefore the term iNA is called the *magnetic dipole moment of the loop* and is represented by p_m (or m)

$$\therefore \qquad \mathbf{B}=(\mu_0/4\pi)\ 2\mathbf{p}_m/x^3, \qquad ...(65)$$

It was pointed out in article 8.8 that a current loop is acted on by a torque when placed in an external magnetic field. The loop is in stable equilibrium when its plane is at right angles to the external field. But we have just proved that this coil has its own magnetic field along its axis, *i.e.*, $\perp$ to its plane, hence the equilibrium position is such that within the area enclosed by the loop, the loop's own field is in the same direction as that of the external field. The lines of force due to the current through the coil with in space nearer to it is shown in Fig. 8.25. It shows that at the centre the field is nearly constant, *i.e.*, the lines of force are approximately parallel equidistant straight lines.

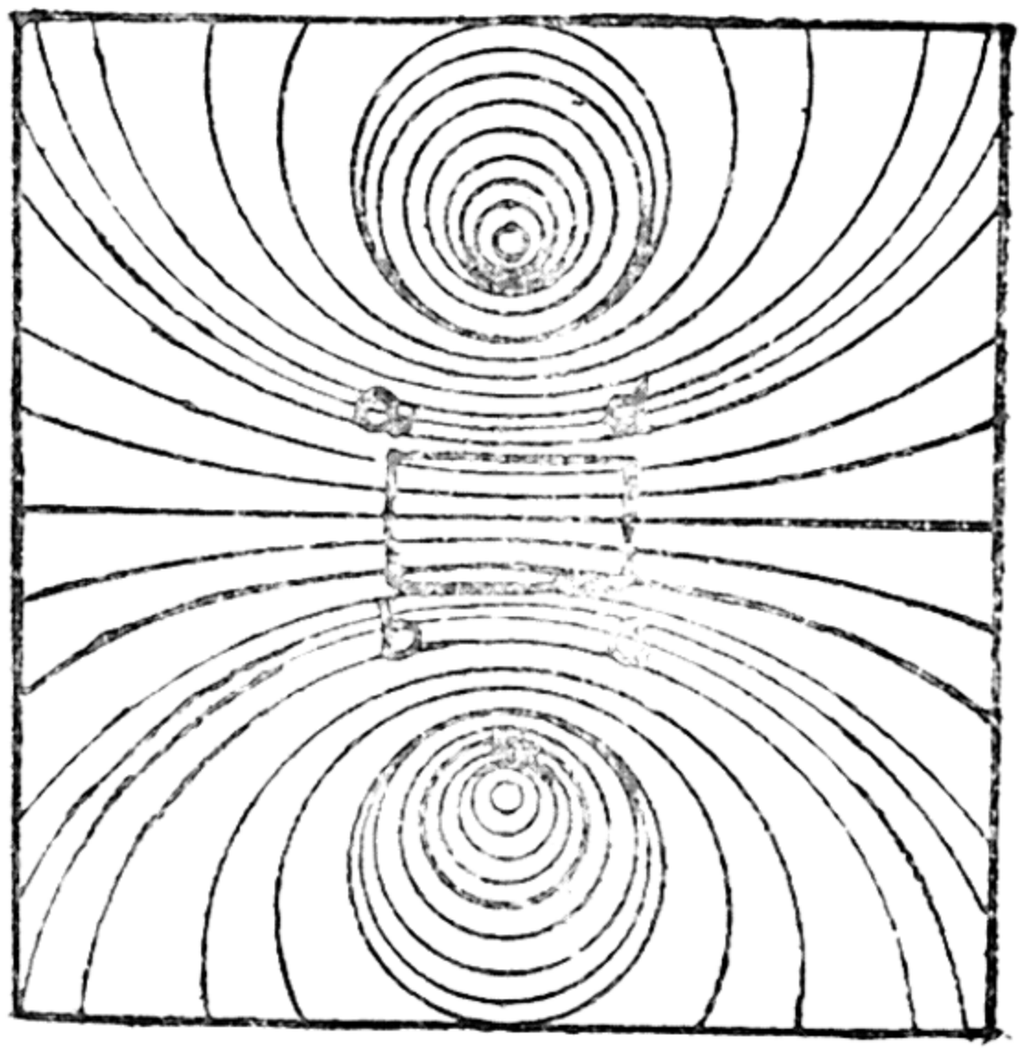


Fig. 8.25.

(*c*) **Field midway between two similar coaxial circular coils.** In the preceding article we have seen that the field B varies with the distance x. It is maximum at the centre of the coil and decreases with x on both sides of the coil. Thus if two similar coils are connected in series, wound in the same sense and are placed exactly parallel to each other, then the magnetic field at any point on the axis (or the line joining centres of the coils will be additive.

Let the coils each having N turns and carrying current i are placed a distance $2d$ apart. Let B be the magnetic field at point P on the common axis, which is at a short distance x from the mid-way point O. Assuming the current as flowing in the same direction in each coil and the radius of each coil as a, we get

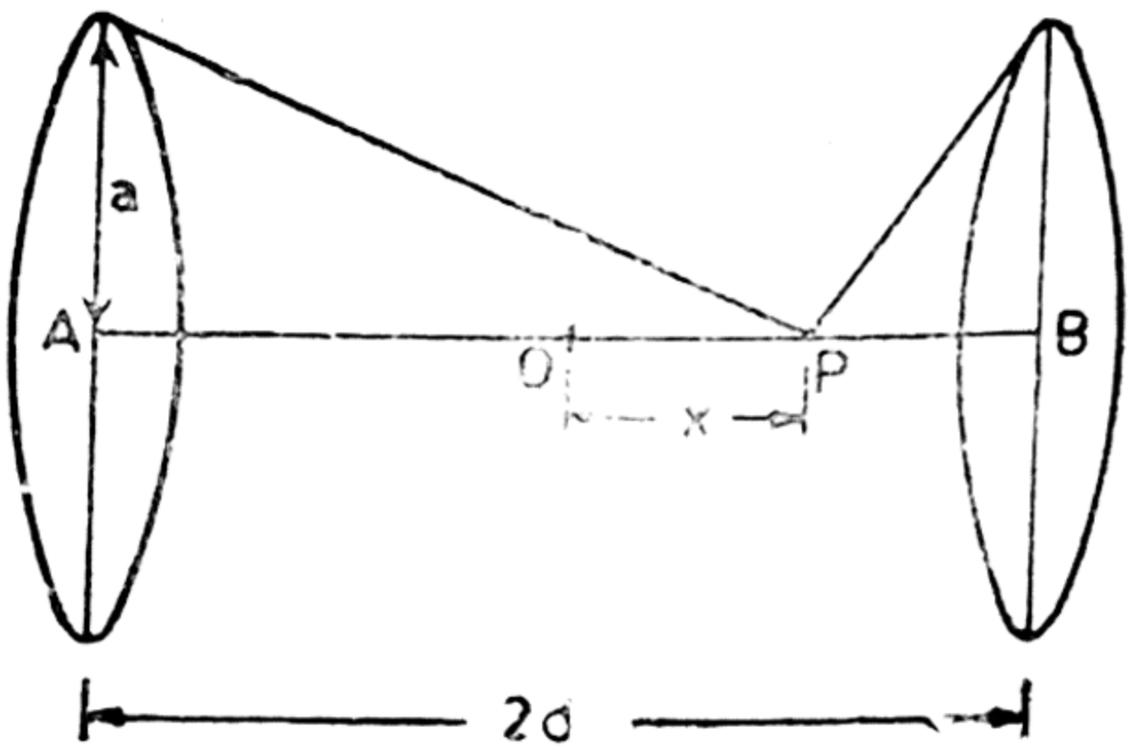


Fig. 8.26.

$$B=\frac{\mu_0}{2} ia^2N\left[\frac{1}{[(d+x)^2+a^2]^{3/2}}+\frac{1}{[(d-x)^2+a^2]^{3/2}}\right]$$

The field will be uniform (or constant) if $dB/dx=0$, *i.e.*

$$\frac{\mu_0}{2}ia^2N\left[\frac{-3(d+x)}{[(d+x)^2+a^2]^{5/2}}+\frac{3(d-x)}{[(d-x)^2+a^2]^{5/2}}\right]=0$$

or
$$(d+x)[(d+x)^2+a^2]^{5/2}=(d-x)[(d+x)^2+a^2]^{5/2}. \quad \ldots(66)$$

Now $[(d+x)^2+a^2]^{5/2}=(d^2+a^2+2xd)^{5/2}=(d^2+a^2)^{5/2}\left[1+\dfrac{2xd}{d^2+a^2}\right]^{5/2}$

$$=(d^2+a^2)^{5/2}\left[1+\frac{5xd}{d^2+a^2}\right] \qquad \frac{2xd}{d^2+a^2}<<1$$

Similarly $[(d-x)^2+a^2]^{5/2}=(d^2+a^2)^{5/2}\left[1-\dfrac{5xd}{d^2+a^2}\right]$.

Substituting these values in Eq. (66), we get

$$(d+x)\left[1-\frac{5xd}{d^2+a^2}\right]=(d-x)\left[1+\frac{5xd}{d^2+a^2}\right]$$

or $5d^4/(d^2+a^2)=1$, therefore $d=a/2$ or $2d=a$.

Thus we see that, to secure a very uniform field in the central region between the coils, the distance apard $2d$ must be equal to a, the radius of each coil. The field B at point O due to current i in the coils

$$B=2\,\frac{\mu_0}{2}\,ia^2N\,\frac{1}{[(a/2)^2+a^2]^{3/2}}=\frac{8\mu_0 iN}{5\sqrt{4a}}. \qquad \ldots(67)$$

The resultant field in the space between the coils is the sum of the fields due to the each coil individually, and may be traced out graphically by erecting ordinates at a number of points equal to the sum of the ordinates at those points. The graph of the resultant is a straight line for a considerable distance on either side of the mid-point O. It is due to the fact that on either side of O, field due to one coil increases while the field due to other coil decreases and the rates of increase and decrease are equal at points close to O. This

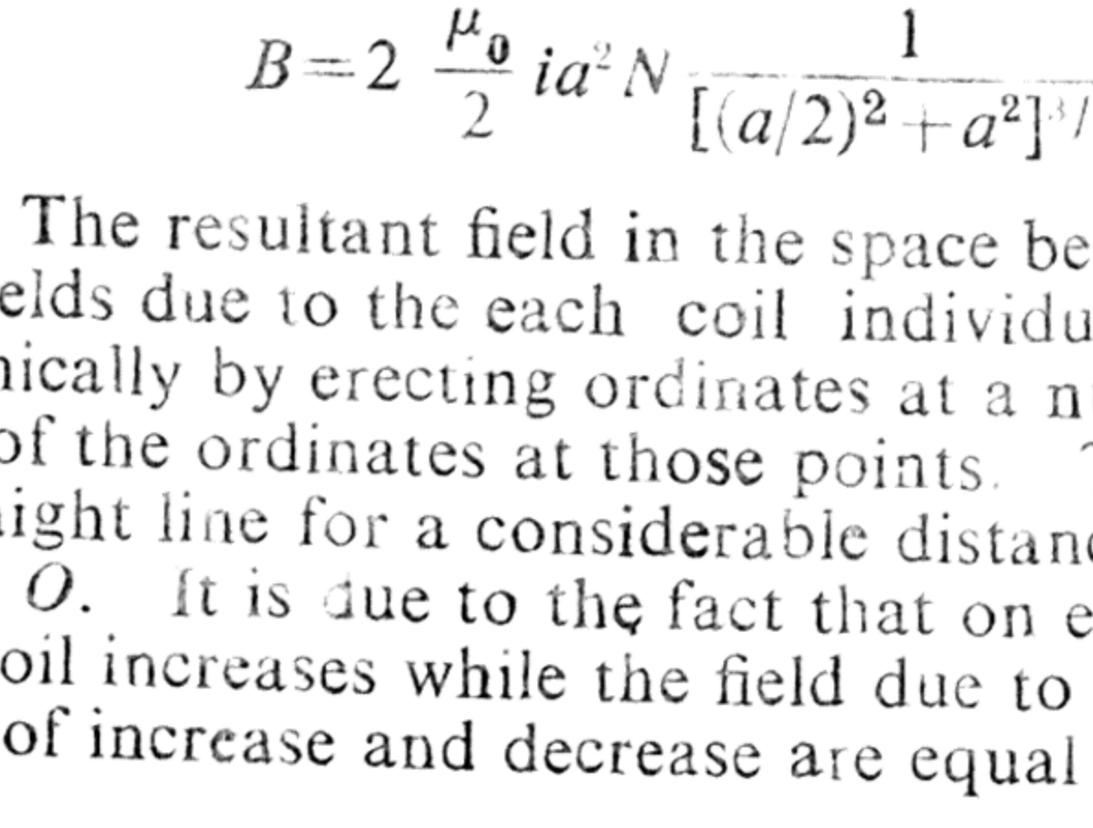


Fig. 8.27.

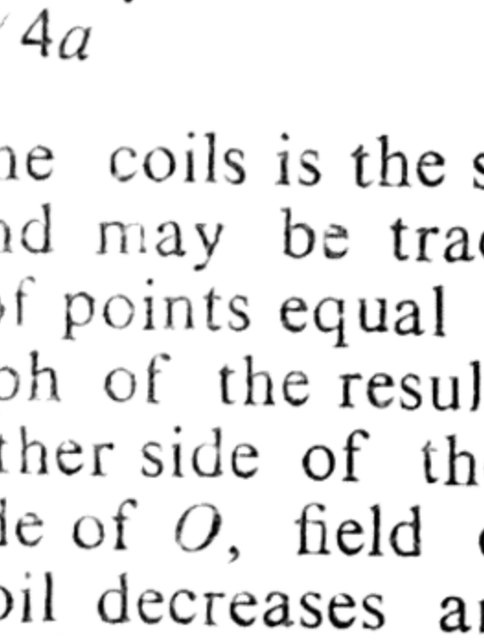


Fig. 8.28.

is shown graphically in Fig. 8.27. The dotted curves show the fields for the separate coils and the full line curve. the resultant field and the horizontal part indicates uniformity of field. Fig. 8.28 shows the lines of force between the coils and indicates that there is a region, in the middle of the field, where the field is practically uniform as the lines of force are nearly horizontal within this region.

An arrangement of two identical circular coils mounted coaxially in series with their centres at a distance apart equal to the radius of each coil is used in *Helmholtz galvanometer*. A suspended or a pivoted magnet, usually a deflection magnetometer, is placed at the mid-point O on the common axis. As the field is uniform around this point O, the error due to the finite size of the needle in tangent galvanometer is avoided in this case. If θ is the deflection in the needle, then tangent law gives

$$\frac{8\mu_0 Ni}{5\sqrt{5a}} = H \tan\theta \quad \text{or} \quad i = \frac{5\sqrt{5aH}}{8\mu_0 N} \tan\theta, \qquad \ldots(68)$$

where H is the horizontal component of earth magnetic field. The coils of Helmholtz galvanometer are to be placed in magnetic meridian, *i.e.*, parallel to the H direction.

(*d*) **Field along the axis of a solenoid.** The name solenoid was first given by Ampere to a wire wound in a closely spaced spiral over a hollow cylindrical non-conducting core, as he thought the cylindrical coil as a canal confining lines of magnetic field in it. The name solenoid is from the Greek word for channel. If n is the number of turns per unit length, each carries a current i, uniformly wound round a cylinder of radius a, the number of turns per length dx of the solenoid is ndx. Thus the magnetic induction field at the axial point O due to this element dx is

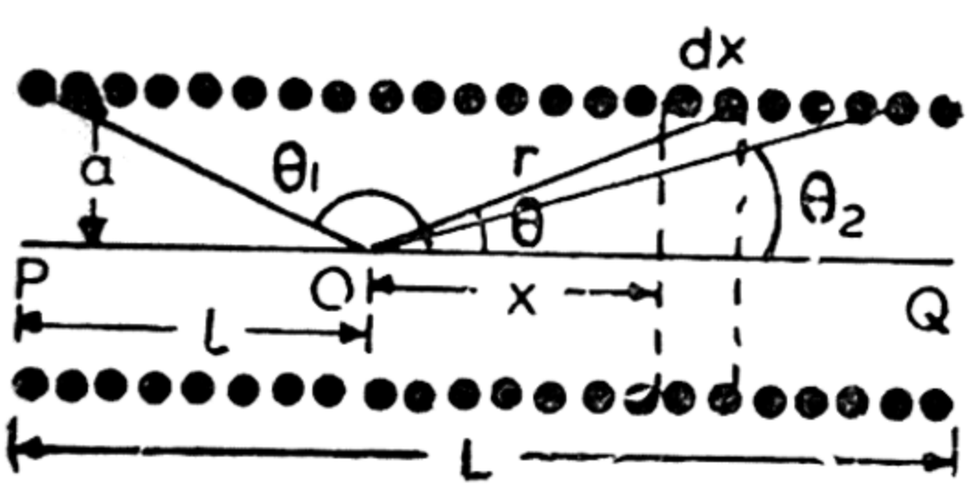


Fig. 8.29.

$$dB = \frac{\mu_0}{2} \frac{i\, n\, dx\, a^2}{(a^2+x^2)^{3/2}}. \qquad \ldots(69)$$

Its direction is along the axis of the solenoid and in the sense of advance of a right handed screw. From the geometry of the Fig. 8.29, we know

$x = a \cot\theta$ and $dx = -a \operatorname{cosec}^2\theta\, d\theta$ and hence

$dB = -\frac{1}{2}\mu_0 ni \sin\theta\, d\theta.$

Total field B due to the entire solenoid is

$$B = \tfrac{1}{2}\mu_0 ni \int_{\theta_1}^{\theta_2} (-\sin\theta)\, d\theta = \tfrac{1}{2}\mu_0 ni\, [\cos\theta_2 - \cos\theta_1]. \qquad \ldots(70)$$

$$= \frac{\mu_0 ni}{2}\left[\frac{L-l}{[a^2+(L-l)^2]^{1/2}} + \frac{l}{[a^2+l^2]^{1/2}}\right], \qquad \ldots(71)$$

where L is the total length of the solenoid. The value of B can be easily determined from this equation at any point inside the solenoid on its axis.

If the solenoid is of infinite length ($L>a$) and the point O is chosen at the middle, *i.e.*, if $\theta_1=180°$, $\theta_2=0$, then we get

$$B=\mu_0 ni. \qquad ...(72)$$

At the end of the solenoid,

$\theta_2=0$, $\theta_1=90°$, we get

$$B=\tfrac{1}{2}\mu_0 ni. \qquad ...(73)$$

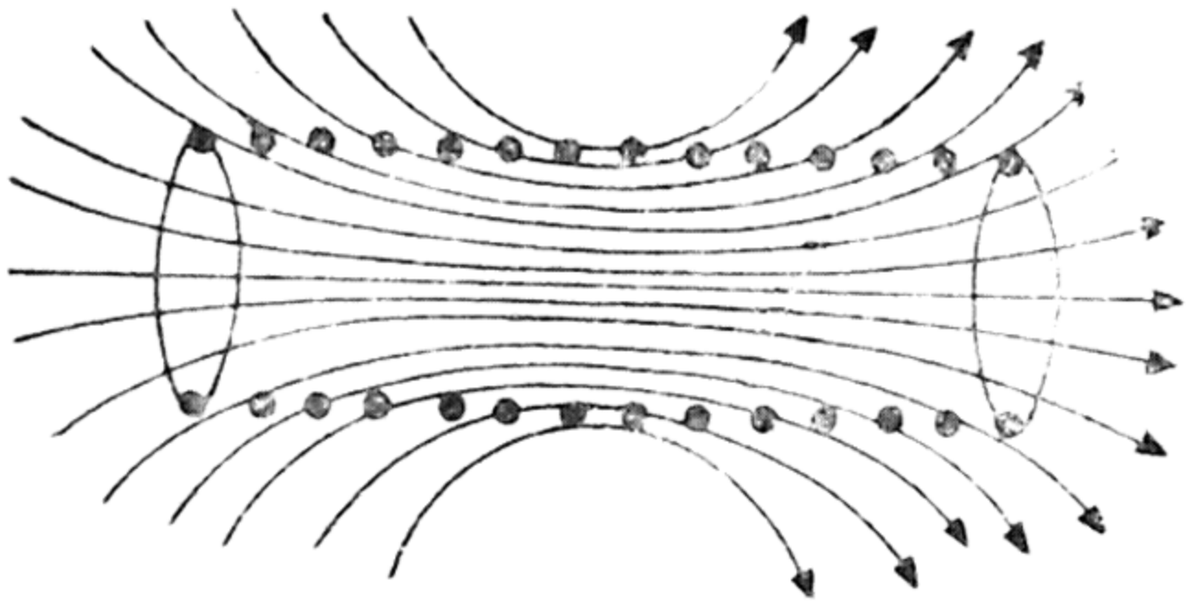


Fig. 8.30.

Thus the field strength at the end of a long solenoid is just one half that at the centre and the half the lines of force passing through the central section of a long solenoid pass out through the sides before reaching the end. (Fig. 8.30).

The quantity ni, the number of amperes circulating around the coil per metre of length of coil, is known as *solenoidal current density* and is denoted as $\mathbf{j}^*$ (or λ) which is to be distinguished from the conventional current density $\mathbf{j}$.

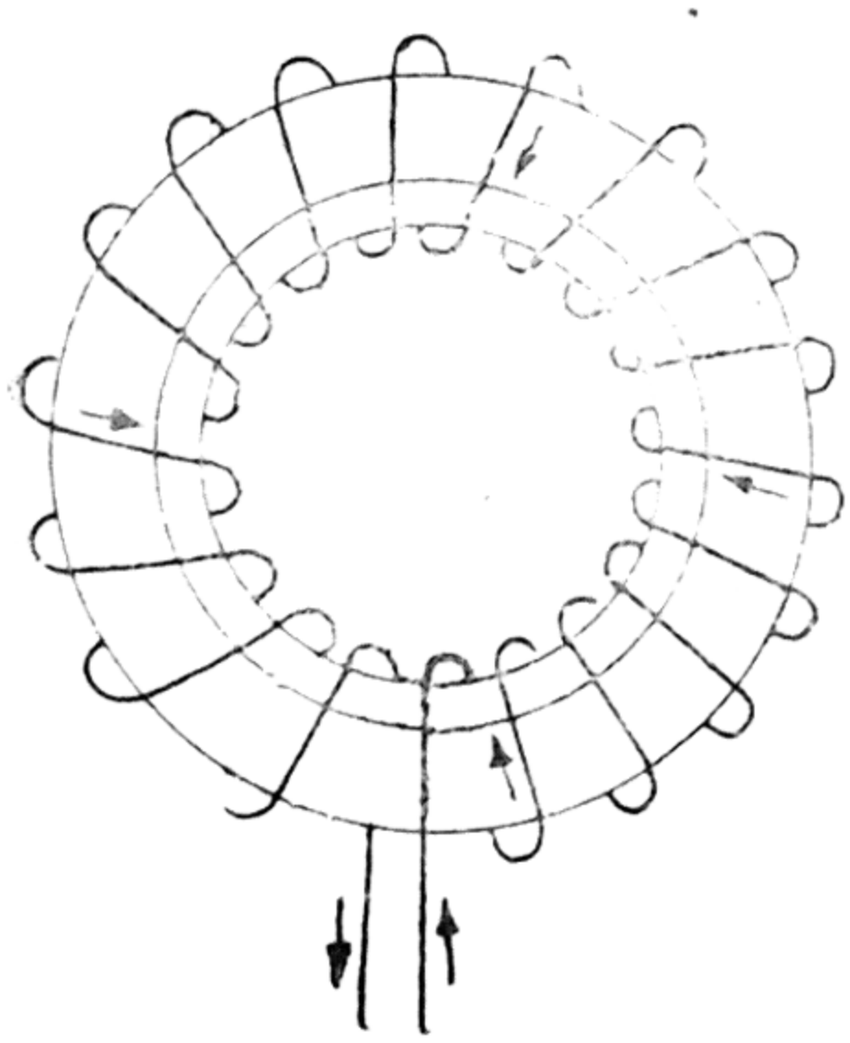


Fig. 8.31.

Relation (72) may be applied also to a solenoid of small radius bent round to form a *toroid* or *anchor ring*. In an ideal toroid, the field is confined entirely within the core and is sensibly uniform. The value of the magnetic field at any point on the mean circumferential line is given by

$$B=\mu_0 ni=\mu_0(N/2\pi R)\, i$$
$$=\mu_0 Ni/2\pi R, \qquad ...(74)$$

where N is the total number of turns and R is the mean radius of ring.

8.12. AMPERE'S CIRCUITAL LAW

We know that the lines of magnetic field are continuous and do not arise from any source in the way as lines of electric force originate on charges. These thus form loops without any beginning or end. This property may be used to go through the nature of magnetic field and for calculating the magnetic field in certain situations of high symmetry.

Let us use expression for the field round a long wire carrying a current i, *i.e.*, $B=\mu_0 i/2\pi r$. The magnetic lines are concentric around the wire in the plane of the paper if the wire is $\perp$ to this plane. The

line integral around the path of radius r, starting at any point and returning to the same point, is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int B dl = \int \frac{\mu_0 i}{2\pi r} \, dl = \frac{\mu_0 i}{2\pi r} \int dl$$

$$= (\mu_0 i / 2\pi r) 2\pi r = \mu_0 i. \qquad \ldots(75)$$

We have taken r outside the integral as it is constant for the particular circuital path. This relation is also true, whatever be the nature of the closed path. For this let us consider the diagram 8.32. The field **B** at any point P is in the plane of the paper and is $\perp$ to the radius vector **r** from the wire O. This **B**-vector makes an angle θ with an element $d\mathbf{l}$ of a closed path PQR encircling the wire, its component along $d\mathbf{l}$ is $B_1 = B \cos\theta$. From the triangle PST, we get

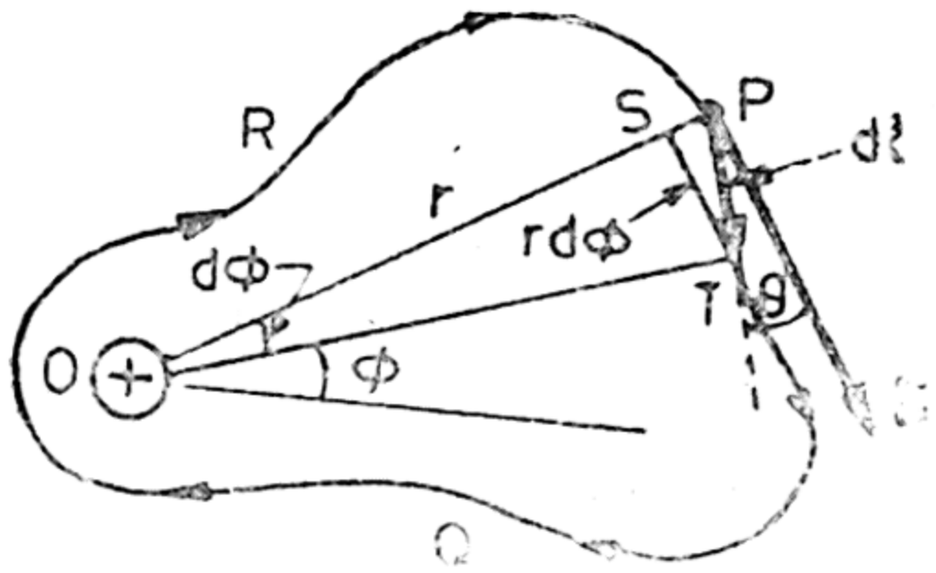


Fig. 8.32.

$$r d\phi = dl \cos\theta \text{ or } dl = r d\phi / \cos\theta.$$

$$\therefore \quad \mathbf{B}.d\mathbf{l} = B_1 dl = B \cos\theta \; dl = B r d\phi$$

$$= \frac{\mu_0 i}{2\pi r} \, r d\phi = \frac{\mu_0}{2\pi} \, i d\phi.$$

Hence the line integral of **B** around the closed path is

$$\oint \mathbf{B}.d\mathbf{l} = \frac{\mu_0 i}{2\pi} \int d\phi = \mu_0 i. \qquad \ldots(76)$$

Thus we see that *the line integral does not depend on the shape of the path or on the position of the wire with in it.* If the current is in opposite sign, the line integral would have the opposite sign. If any number of long straight conductors pass through the surface bounded by the path, the line integral equals μ_0 times the algebraic sum of the currents.

Fig. 8.33.

For the path, through which no current flows is chosen (Fig. 8.33), the line integral is zero, because the angle ϕ then has the same value at the start and finish of any round trip. Thus the conductor may contribute to the value of **B** but the line integral of such a field around the path not consisting this conductor is always zero.

Above results can be summarized as : *The line integral of the magnetic induction* **B** *around any closed path is equal to* μ_0 *times the net current across the area bounded by this path.* It is called *Ampere's*

circuital law. It plays the same role in magnetostatics that Gauss's law ($\int \mathbf{E}.d\mathbf{S}=q/\varepsilon_0$) played in electrostatics.

Ampere's Law and $\nabla\times\mathbf{B}$. In a region of distributed current flow, the total current $i=\int \mathbf{j}.d\mathbf{S}$, where $\mathbf{j}$ is the current density in an element $d\mathbf{S}$ of the surface bounded by the closed path. Hence we get

$$\oint \mathbf{B}.d\mathbf{l}=\mu_0 \int \mathbf{j}.d\mathbf{S}.$$

Using Stokes theorem, we have

$$\oint \mathbf{B}.d\mathbf{l}=\int \text{curl } \mathbf{B}.d\mathbf{S}=\mu_0 \int \mathbf{j}.d\mathbf{S},$$

or
$$\text{curl } \mathbf{B}=\mu_0 \mathbf{j}. \qquad \ldots(77)$$

This is the differential form of Eq. (76). This establishes a local relation between current density $\mathbf{j}$ and magnetic induction $\mathbf{B}$ at the same point. Here we have a new kind of situation which is quite different from electrostatics, where we had curl $\mathbf{E}=0$, or $\oint \mathbf{E}.d\mathbf{l}=0$.

Unlike the electrostatic field the magnetostatic field does have rotation hence cannot be conservative. At any point where $\mathbf{j}=0$, the value of $\nabla\times\mathbf{B}=0$. If a field exists at all, $\mathbf{j}$ can not be zero everywhere and $\nabla\times\mathbf{B}\neq 0$ everywhere. For the singly connected volumes which do not link any currents, it is possible to treat $\mathbf{B}$ as conservative in this restricted volume. One may then proceed to treat $\mathbf{B}$ in a manner analogous to electrostatics. This is the case with permanent magnets with no conventional currents.

Divergence of Magnetostatic field. Divergence of magnetic field $\mathbf{B}$ is defined as the flux of $\mathbf{B}$ through the surface S enclosing per unit volume. It can be obtained directly from the Biot-Savart law, as

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}\times\mathbf{r}}{r^3}\,dV, \qquad (\text{as } id\mathbf{l}=\mathbf{j}dV)$$

$$\because \quad \text{div } \mathbf{B}=\nabla.\mathbf{B}=\nabla.\frac{\mu_0}{4\pi}\int_V \frac{(\mathbf{j}\times\mathbf{r})}{r^3}\,dV.$$

Since the del operator is a function of the field point (test point) but the independent variable under the integral sign is a function of the source point hence the del operator may be moved to the right of the integral sign. Therefore,

$$\nabla.\mathbf{B}=\frac{\mu_0}{4\pi}\int_V \nabla.\left[\frac{\mathbf{j}\times\mathbf{r}}{r^3}\right]\,dV.$$

Using the vector identity

$\nabla.(\mathbf{A}\times\mathbf{B})=\mathbf{B}.(\nabla\times\mathbf{A})-\mathbf{A}.(\nabla\times\mathbf{B})$ with $\mathbf{A}=\mathbf{j}$ and $\mathbf{B}=\mathbf{r}/r^3$, we get

$$\nabla . \left(\mathbf{j} \times \frac{\mathbf{r}}{r^3} \right) = \frac{\mathbf{r}}{r^3} \cdot (\nabla \times \mathbf{j}) - \mathbf{j} . \left(\nabla \times \frac{\mathbf{r}}{r^3} \right).$$

The first term on the right is zero as **j** is a function of the source point, while the del operator derivates w.r.t. the field point.

As $$\nabla \times \frac{\mathbf{r}}{r^3} = \nabla \times \left[-\nabla \left(\frac{1}{r} \right) \right] = -\nabla \times \nabla \left(\frac{1}{r} \right).$$

Since the curl of the gradient of any function is identically zero hence the second term $\nabla \times (\mathbf{r}/r^3)$ is also zero. Consequently the entire integrand vanishes and

$$\nabla . \mathbf{B} = 0. \qquad \ldots (78)$$

Thus the vector **B** is found to be *solenoidal*. We have found relation (78) for magnetostatics but this can be found to hold true in general. Comparing it to the analogous equation in electrostatics ($\nabla . \mathbf{B} = \rho$) we can conclude that there is no magnetic analog of an electric charge. There are no *magnetic charges* from which lines of **B** can emerge. The magnetic flux

$$\Phi = \int_s \mathbf{B}\, d\mathbf{S} = \int_v \nabla . \mathbf{B}\, dV, \qquad \ldots (79)$$

Since $\nabla . \mathbf{B}$ is always zero, hence the flux over any closed surface is zero (*i.e., lines of* **B** *close into themselves and do not have sources or sinks of magnetic flux*). It shows that the magnetic charge known as the *magnetic monopole* can not exist as the lines of *B* crossing any closed surface around this pole will not be zero. Despite of intensive research for the existence of monopoles no experimental evidence has yet been found. If any source or sink of magnetism is ever found the law of Biot-Savart will have to be altered.

8.13. APPLICATIONS OF AMPERE'S LAW

Similar to the Gauss's law in electrostatics the Ampere's law is useful for the determination of magnetic field at any point.

(*a*) **Long Cylindrical Wire.** Let us consider a straight wire *AB* of radius *R*, along which a current *i* is flowing. The lines of magnetic induction are concentric circles centered in the wire in planes $\perp$ to the wire. Hence the field *B* at point *P* at a distance *r* from the axis of the wire is given by Ampere's law as

$$\oint \mathbf{B} . d\mathbf{l} = \mu_0 i,$$

$$B \,.\, 2\pi r = \mu_0 i \text{ or } B = \mu_0 i / 2\pi r. \qquad \ldots (80)$$

If $r < R$, *i.e.*, the point *P* is inside the wire. Symmetry suggests that **B** is tangent to the path of concentric circles. If the current is steady and therefore distributed uniformly through the cross-

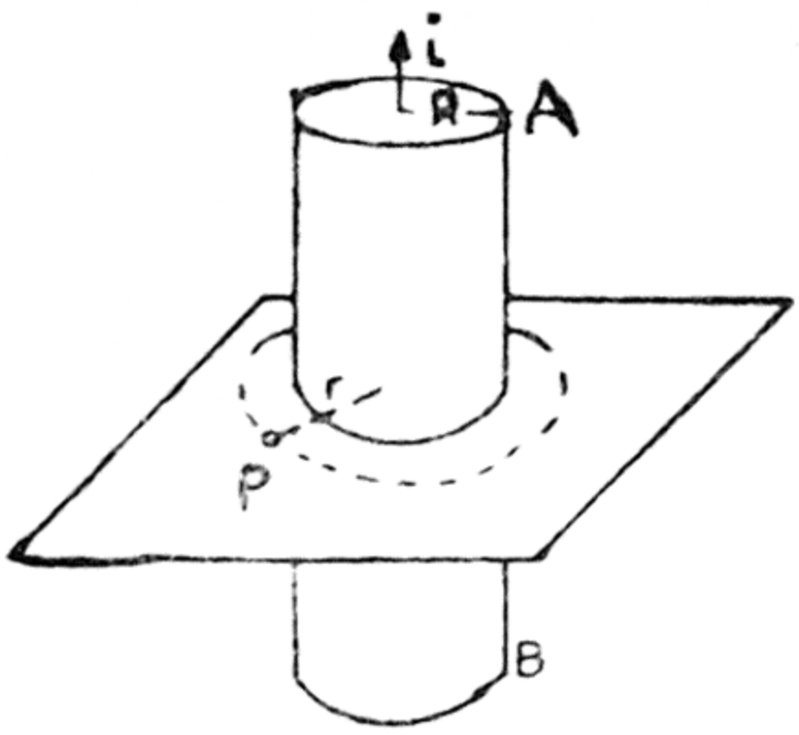


Fig. 8.34.

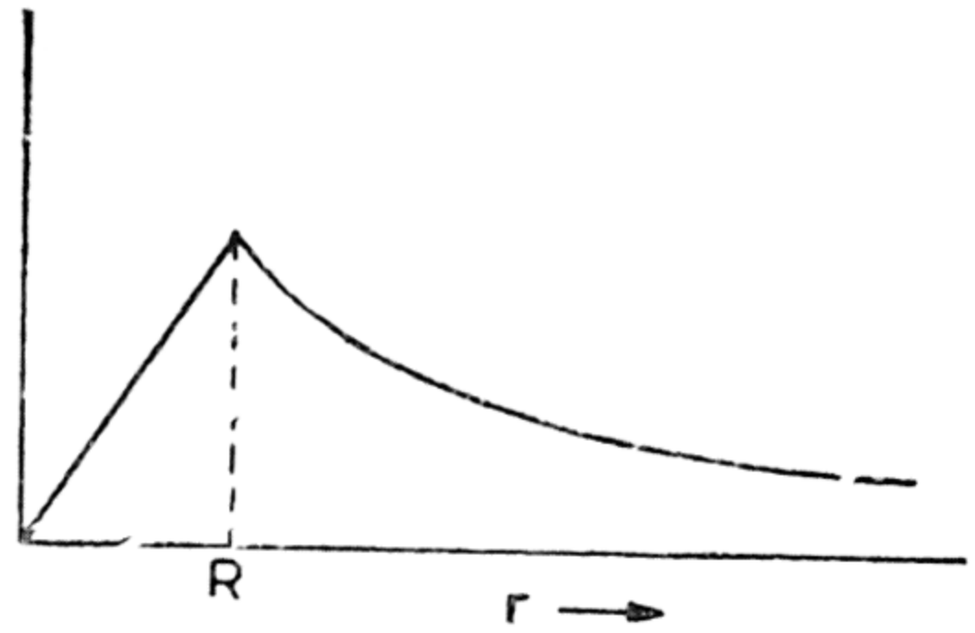


Fig. 8.35.

section of the wire, the current surrounded the path is ir^2/R^2. Thus from Ampere's law $\oint \mathbf{B}.d\mathbf{l} = \oint Bdl = \mu_0 i'$,

or $$B(2\pi r) = \mu_0 ir^2/R^2 \quad \text{or} \quad B = \mu_0 ir/2\pi R^2.$$

At the surface of the wire, $r = R$ and we get the same relation as obtained from Eq. (80). The graph between B and r is shown in Fig. 8.35.

(*b*) **Long Solenoid**. Let us consider a long straight solenoid ($L >> R$) having n turns per unit length. If the solenoid wire is closely spaced and carrying a current i_0. The field of a solenoid is uniform on its cross-section in the middle region of the solenoid. Ampere's law can thus easily be applied for the value of **B** inside the solenoid with in the middle region.

For the field at point inside the solenoid, let us consider a rectangular path $ABCDA$ as the path of integration. This path is particularly simple as :

(*a*) Outside the solenoid, along CD, we can assume field to be zero, it is many times weaker than that inside.

$$\therefore \quad \int_C^D \mathbf{B}.d\mathbf{l} = 0.$$

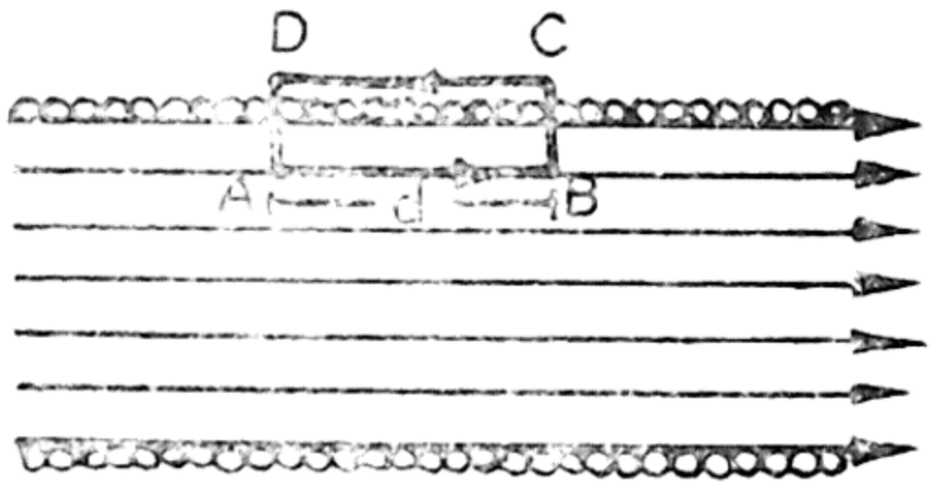


Fig. 8.36.

(*b*) Along BC and DA, **B** is either zero (for outside parts of BC and DA) or perpendicular to **B** (for the parts inside the solenoid).

$$\therefore \quad \int_B^C \mathbf{B}.\, d\mathbf{l} = \int_D^A \mathbf{B}.d\mathbf{l} = 0.$$

(*c*) Inside the solenoid **B** is constant and is along *AB*.

$$\therefore \qquad \int_A^B \mathbf{B}.d\mathbf{l}=B(AB)=Bd.$$

Let us now apply Ampere's law for *ABCDA*, we get

$$\oint \mathbf{B}.d\mathbf{l}=\int_A^B \mathbf{B}.\,d\mathbf{l}+\int_B^C \mathbf{B}.\,d\mathbf{l}+\int_C^D \mathbf{B}.\,d\mathbf{l}+\int_D^A \mathbf{B}.d\mathbf{l}=\mu_0 i$$

$$Bd=\mu_0 ndi_0 \text{ or } B=\mu_0 ni_0. \qquad \ldots(81)$$

This is same as obtained by Biot-Savart law (Eq. 72).

(*c*) **Toroid.** Let us consider an endless, solenoid in the form of a ring also called *toroid*, wound uniformly with *N* turns. The lines of force are circles in planes parallel to the paper with centres on the axis through *O*. For the path 1 of radius r_1, **B** at all the points will be tangent to the path.

Hence

$$\oint \mathbf{B}.d\mathbf{l}=B\,2\pi r_1=\mu_0 i=0,$$

since the current enclosed is zero.

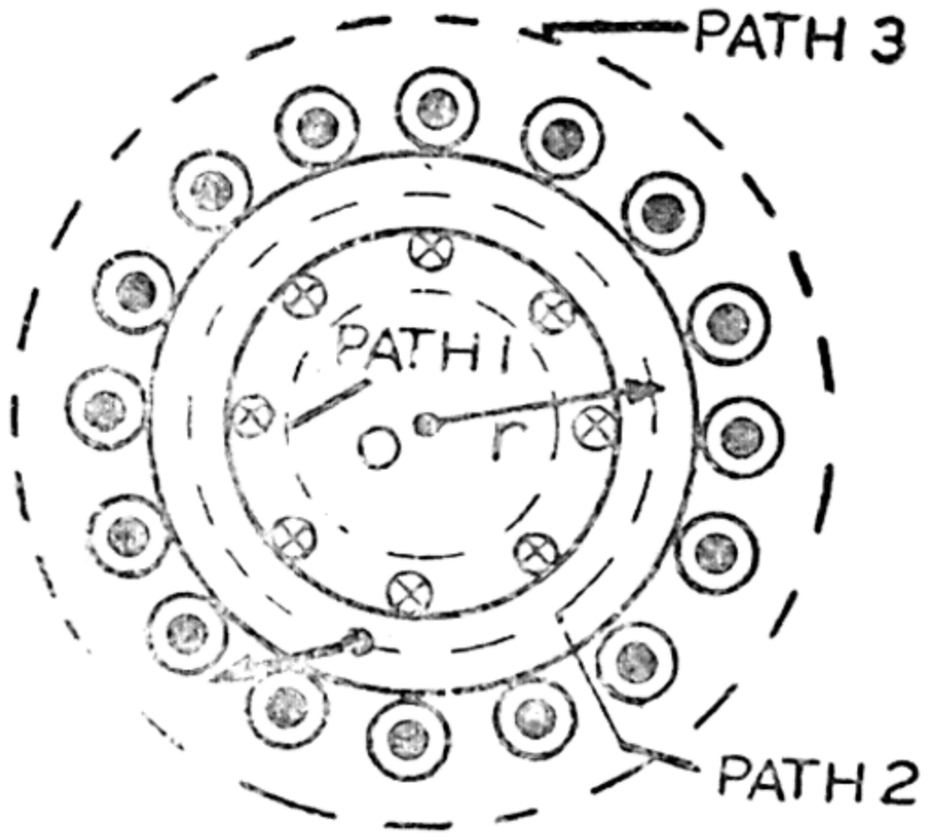


Fig. 8.37.

The line integral and thus $B=0$ at all points of the path 3, (*i.e.*, outside the toroid), as each turn of the winding passes twice through the area bounded by this path carrying equal currents in opposite directions and the net current *i* through the area is zero. Thus the field of the toroid is confined wholly to the space enclosed by the windings and the toroid will work as a solenoid bent into a circle. For the path 2, we have

$$\oint \mathbf{B}.d\mathbf{l}=B2\pi r=\mu_0 Ni \quad \text{or} \quad B=\mu_0 Ni/2\pi r. \qquad \ldots(82)$$

If the cross section of the toroid is sufficiently small, the variations in *r* can be neglected and $N/2\pi r$ can be written as *n*, the number of turns per unit length, then $B=\mu_0 ni$, which is as for a long solenoid.

8.14. VECTOR POTENTIAL A

According to the Biot-Savart law the magnetic induction **B** at any point *P* at a distance *r* from the current element of volume *dV* is

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}\times\mathbf{r}}{r^3}\,dV=\frac{\mu_0}{4\pi}\int_V \mathbf{j}\times\nabla\left(\frac{1}{r}\right)dV.$$

Using the vector identity

$$-\mathbf{j}\times\nabla\left(\frac{1}{r}\right)=\nabla\left(\frac{1}{r}\right)\times\mathbf{j}=\nabla\times\left(\frac{\mathbf{j}}{r}\right)-\left(\frac{1}{r}\right)\nabla\times\mathbf{j},$$

where the second term is zero because $\mathbf{j}$ is a function of the source point. Thus we have

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int_V \nabla\times\left(\frac{\mathbf{j}}{r}\right)dV=\nabla\times\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}}{r}dV.$$

Here curl operation is taken outside the integral since the integral is over the source point. To understand it more clearly let x, y, z be the coordinates of the field point P and x', y', z' be that of the source point (current element). Then above relation becomes

$$\mathbf{B}\,(x, y, z)=\nabla\times\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(x', y', z')}{r}\,dV'.$$

Here r is the function of x, y, z and x', y', z' as it is the distance between field and source points.

From above equations it is clear that $\mathbf{B}$ can be written as a curl of a vector $\mathbf{A}$, where

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}}{r}\,dV \quad \text{or} \quad \mathbf{A}\,(x, y, z)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}\,(x', y', z')}{r}\,dV'. \qquad ...(83)$$

The components of $\mathbf{A}$ vary as $1/r$, like the electric potential V, which does not diverge with in a charge distribution. As divergence of a curl of a vector is always zero and div $\mathbf{B}=0$ can be written as a curl of a vector and thus $\mathbf{A}$ is a vector. Due to these reasons $\mathbf{A}$ is known as **vector potential.**

The integral for $\mathbf{A}$ is easier to evaluate than the original equation for $\mathbf{B}$. Hence $\mathbf{B}$ is obtained by calculating first $\mathbf{A}$ and then making curl operation.

Vector Potential and Biot-Savart Law. Since the free magnetic poles do not exist, hence divergence of $\mathbf{B}$ is zero everywhere. Since divergence of curl of any vector $\mathbf{A}$ is always zero hence one can define $\mathbf{B}$ as a curl of $\mathbf{A}$. As curl of $\mathbf{B}$ is equal to $\mu_0\mathbf{j}$, therefore

$$\text{curl curl } \mathbf{A}=\mu_0\mathbf{j}.$$

Using the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla \nabla . \mathbf{A} - \nabla^2 \mathbf{A}$,

we get, $\nabla \nabla . \mathbf{A} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}$.

Since div $\mathbf{A} = (\mu_0/4\pi) \int \nabla . (\mathbf{j}/r)\, dV = 0$, hence

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}. \qquad \ldots(84)$$

It shows that **A** is a solution of the vector Poisson equation.

Comparing this equation with the Poisson's equation $\nabla^2 V = -\rho/\varepsilon_0$ and its solution, we get

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}}{r}\, dV. \qquad \ldots(85)$$

Consider a conductor carrying a current i. Due to a small element of length $\delta \mathbf{l}$ and volume δV, the vector potential at a distance r from this element is

$$\delta \mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{j}}{r} \delta V = \frac{\mu_0}{4\pi} \frac{\mathbf{j}}{r} S \delta l$$

$$= \frac{\mu_0}{4\pi} \; i \; \frac{\delta \mathbf{l}}{r},$$

($\because$ $\delta V = S \delta l$, $i = \mathbf{j}.\mathbf{S} = jS$, where **S** is the area of cross section of element whose plane is normal to **j**).

Since **B** = curl **A** hence the contribution of $\delta \mathbf{A}$ to the magnetic field **B** is $\delta \mathbf{B}$, given as

$$\delta \mathbf{B} = \nabla \times \delta \mathbf{A} = \nabla \times \frac{\mu_0}{4\pi} \; i \; \frac{\delta \mathbf{l}}{r}$$

$$= \frac{\mu_0}{4\pi} i \nabla \times \left(\frac{\delta \mathbf{l}}{r} \right)$$

$$= \frac{\mu_0}{4\pi} \; i \left[\frac{1}{r} \nabla \times \delta \mathbf{l} + \nabla \left(\frac{1}{r} \right) \times \delta \mathbf{l} \right]$$

$$= \frac{\mu_0}{4\pi} i \nabla \left(\frac{1}{r} \right) \times \delta \mathbf{l} = - \frac{\mu_0}{4\pi} \; i \delta \mathbf{l} \times \left(-\frac{\mathbf{r}}{r^3} \right)$$

$$= \frac{\mu_0}{4\pi} \; i \; \frac{\delta \mathbf{l} \times \mathbf{r}}{r^3}. \qquad \ldots(86)$$

It is the Biot-Savart law.

8.15. THE MAGNETIC DIPOLE

Consider a circular loop of wire of radius a and carrying a current i. To find **B**, first calculate **A** at a point P on the x-z—plane. The vector potential $d\mathbf{A}$ at this point due to an element $d\mathbf{l}$

$$dA=\frac{\mu_0}{4\pi}\ i\ \frac{dl}{r'}$$

Since dl is in x-y plane, hence dA is in x-y plane. For any given value of r', we can consider two symmetrical dl elements whose y-components add and whose x-components cancel. Thus total vector potential due to the whole loop

$$A=\frac{\mu_0}{4\pi}\ i\int_0^{2\pi}\frac{dl}{r'}\cos\phi$$

$$=\frac{\mu_0}{4\pi}\ i\int_0^{2\pi}\frac{(ad\phi)\cos\phi}{r'}$$

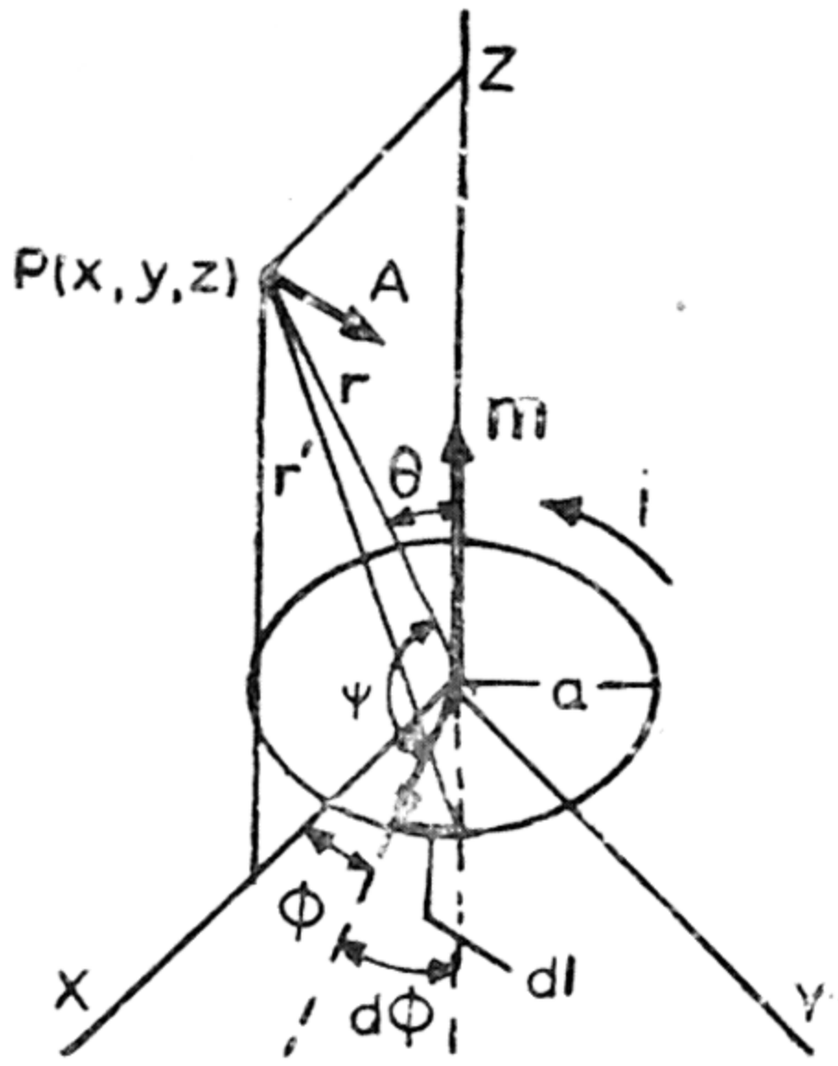


Fig. 8.38.

From the geometry of the figure

$$r'^2=r^2+a^2-2ar\cos\psi.$$

or $$\frac{1}{r'}=\frac{1}{r[1+a^2/r^2-(2a/r)\cos\psi]^{1/2}}\approx\frac{1}{r}\left[1-\frac{a^2}{2r^2}+\frac{a}{r}\cos\psi\right]$$

We have assumed here that $a<<r$.

To replace cos ψ, let us write **r.a** in two ways and compare $(x\mathbf{i}+z\mathbf{k}).(a\cos\phi\mathbf{i}+a\sin\phi\mathbf{j})=ra\cos\psi$ or $x\cos\phi=r\cos\psi$.

$$\therefore\quad\frac{1}{r'}=\frac{1}{r}\left[1-\frac{a^2}{2r^2}+\frac{ax}{r^2}\cos\phi\right]$$

and $$A=\frac{\mu_0 i}{4\pi}\int_0^{2\pi}\frac{a\cos\varphi}{r}\left[1-\frac{a^2}{2r^2}+\frac{ax\cos\phi}{r^2}\right]d\phi$$

$$=\frac{\mu_0}{4\pi}\ \frac{i\pi a^2}{r^3}\ x=\frac{\mu_0}{4\pi}\ \frac{i\pi a^2}{r^2}\ \sin\theta=\frac{\mu_0}{4\pi}\ \frac{m}{r^2}\ \sin\theta,$$

or $$\mathbf{A}=\frac{\mu_0}{4\pi}\ \frac{\mathbf{m}\times\mathbf{r}}{r^2}=\frac{\mu_0}{4\pi}\nabla\times\frac{\mathbf{m}}{r}\ ,\qquad\ldots(87)$$

where **m** is the dipole moment, given by $\mathbf{m}=i(\pi a^2)\mathbf{k}=i\mathbf{S}$. It is along the motion of the right hand screw if is rotated along the current flow.

The magnetic field **B** is given by

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\ \nabla\times\left(\nabla\times\frac{\mathbf{m}}{r}\right)=\frac{\mu_0}{4\pi}\left[\nabla\left(\nabla.\frac{\mathbf{m}}{r}\right)-\nabla^2\left(\frac{\mathbf{m}}{r}\right)\right]$$

$$=\frac{\mu_0}{4\pi}\nabla\left[\mathbf{m}.\nabla\left(\frac{1}{r}\right)\right].\qquad\ldots(88)$$

Here we have used relations $\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla.\mathbf{A})-\nabla^2\mathbf{A}$ and $\nabla^2(1/r)=0$ for $r\neq0$. The relation for **B** is same as obtained for electric field due to an electric dipole.

The relation (88) is of the same form as the expression for the electric field due to an electric dipole of dipole moment **p**

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\nabla\left[\mathbf{p}.\nabla\left(\frac{1}{r}\right)\right].$$

Thus the electric and magnetic dipoles give rise to similar fields. If we draw lines of force due to these dipoles, we find a fundamental difference —*The electric lines of flux leave and terminate on charges while the magnetic lines of flux are continuous closed loops*

For an arbitrary current loop the magnetic dipole moment is defined as $\mathbf{m}=i\mathbf{S}$. The positive direction of current flow is related to the positive direction of the surface by the usual right hand rule. To generalize the magnetic dipole moment to a volume distribution of stationary current, we first express the vector area. Let r be a vector from some convenient origin to a point on the arbitrary current loop C. The vector area of the shaded region due to an arc of length $d\mathbf{r}$ is

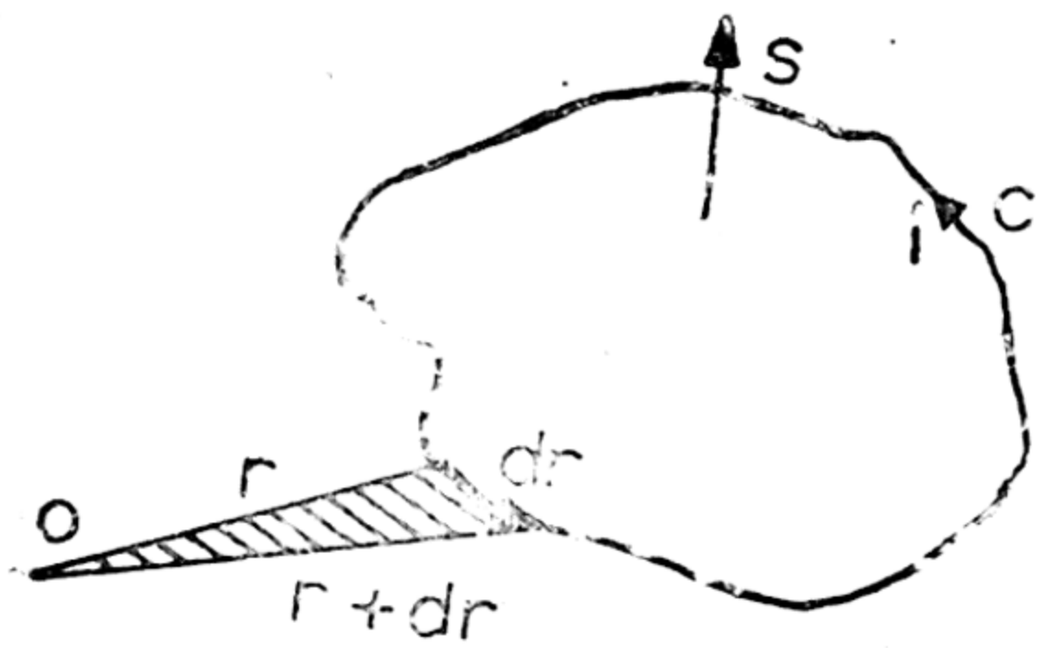


Fig. 8.39.

$$d\mathbf{S}=\tfrac{1}{2}\mathbf{r}\times d\mathbf{r}.$$

The area subtended by the contour C is

$$\mathbf{S}=\oint_c d\mathbf{S}=\tfrac{1}{2}\oint_c \mathbf{r}\times d\mathbf{r}.$$

The magnetic dipole moment of an arbitrary shaped loop is therefore given as

$$\mathbf{m}=\tfrac{1}{2}i\oint_c \mathbf{r}\times d\mathbf{r}.$$

For a volume distribution of current, current can be separated into a large number of infinitesimal closed current flow tubes. For any one flow tube, current $di=jdS$, where j is the current density and dS is the cross sectional area of this flow tube. A single flow tube thus contributes a dipole moment given as

$$d\mathbf{m}=\frac{di}{2}\oint_c \mathbf{r}\times d\mathbf{r}=\oint_c\left(\frac{dS}{2}\right)\mathbf{r}\times\mathbf{j}\,dr. \qquad \ldots(89)$$

Since $dSdr$ is the volume element dV, hence by summing over all current flow tubes (integrating over dS), we get the general relation for the magnetic dipole moment as

$$\mathbf{m}=\tfrac{1}{2}\int_c \mathbf{r}\times\mathbf{j}\;dV. \qquad \ldots(\ 0)$$

Torque on a magnetic dipole. If an electric dipole **p** is placed in an uniform electrostatic field **E**, it will experience a torque vector $\tau=\mathbf{p}\times\mathbf{E}$, but no translational force. Similar to it a magnetic dipole **m** experiences a torque vector $\tau=\mathbf{m}\times\mathbf{B}$, when placed in a

magnetic field **B**. The torque is such that it tends to align the dipole with the field. This relation has also been proved for rectangular loop dipole in article 8.8. The generalization to an arbitrary current loop is obtained by considering the loop as made up of a large number of infinitesimal square loops, each with a dipole moment $d\mathbf{m}$. The torque on each square loop dipole is $d\mathbf{m} \times \mathbf{B}$. Its integration over all the infinitesimal dipoles gives the relation for the total torque.

$$\tau = \int d\mathbf{m} \times \mathbf{B} = \int i\,(d\mathbf{S} \times \mathbf{B}). \qquad \ldots(91)$$

8.16. GALVANOMETERS

Instruments in which the magnetic effects are utilised for the detection and measurement of current, charge, potential difference, resistance and power, are termed as galvanometers. Ammeter, voltmeter, ohm-meters and wattmeters are the modified forms of the galvanometers, *e.g.*, an ammeter is a galvanometer in which a small resistance is connected in parallel with its coil. The main types of galvanometers are : *Moving coil dead beat or ballistic, Moving magnet, Unipivot, Vibration, String or Einthoven.*

Moving coil ballistic galvanometers are used for the measurement of charge. Unipivots are used in ammeters, voltmeters, and other portable instruments. Vibration galvanometers are used for the detection and measurement of small low frequency alternating currents. String galvanometers are used for the detection of very small intermittent rushes of current and recording them, particularly in physiological work.

The properties of a galvanometer to be considered, when it is being chosen, for a special purpose are as follows : (*a*) *Periodic time*, (*b*) *Damping*, (*c*) *Zero-keeping qualities*, (*d*) *Resistance*, (*e*) *Optical system*, (*f*) *Independence of its motion by external magnetic fields and* (*g*) *Sensitivity*

8.17. MOVING COIL GALVANOMETER

This type of galvanometer was first devised by Lord Kelvin in 1890 and was modified for laboratory use by D' Arsonval. The principle of a moving coil galvanometer is that when a current is passed through a coil, suspended freely in a magnetic field, it experiences a force in a direction given by Fleming's left hand rule. This force rotates the coil. The deflection in coil is measured, which gives us the strength of current.

A moving coil galvanometer consists of a rectangular flat coil (10 to 20 turns) of insulated fine Cu-wire wound on a light brass or aluminium frame. It is suspended between the poles of a permanent horse-shoe-shaped magnet by a fine conducting wire or thin flat strip of phosphor bronze, which also serves to conduct the current to and from the coil. The lower end of the wire is attached to a small spring (loosely wound helix) and the upper through the suspension to the torsion head at the top of the instrument. The

axis of rotation of the coil coincides with the principal axis of symmetry of the magnetic field. To increase the magnetic flux in the gap, a cylindrical soft iron core is arranged within the coil.

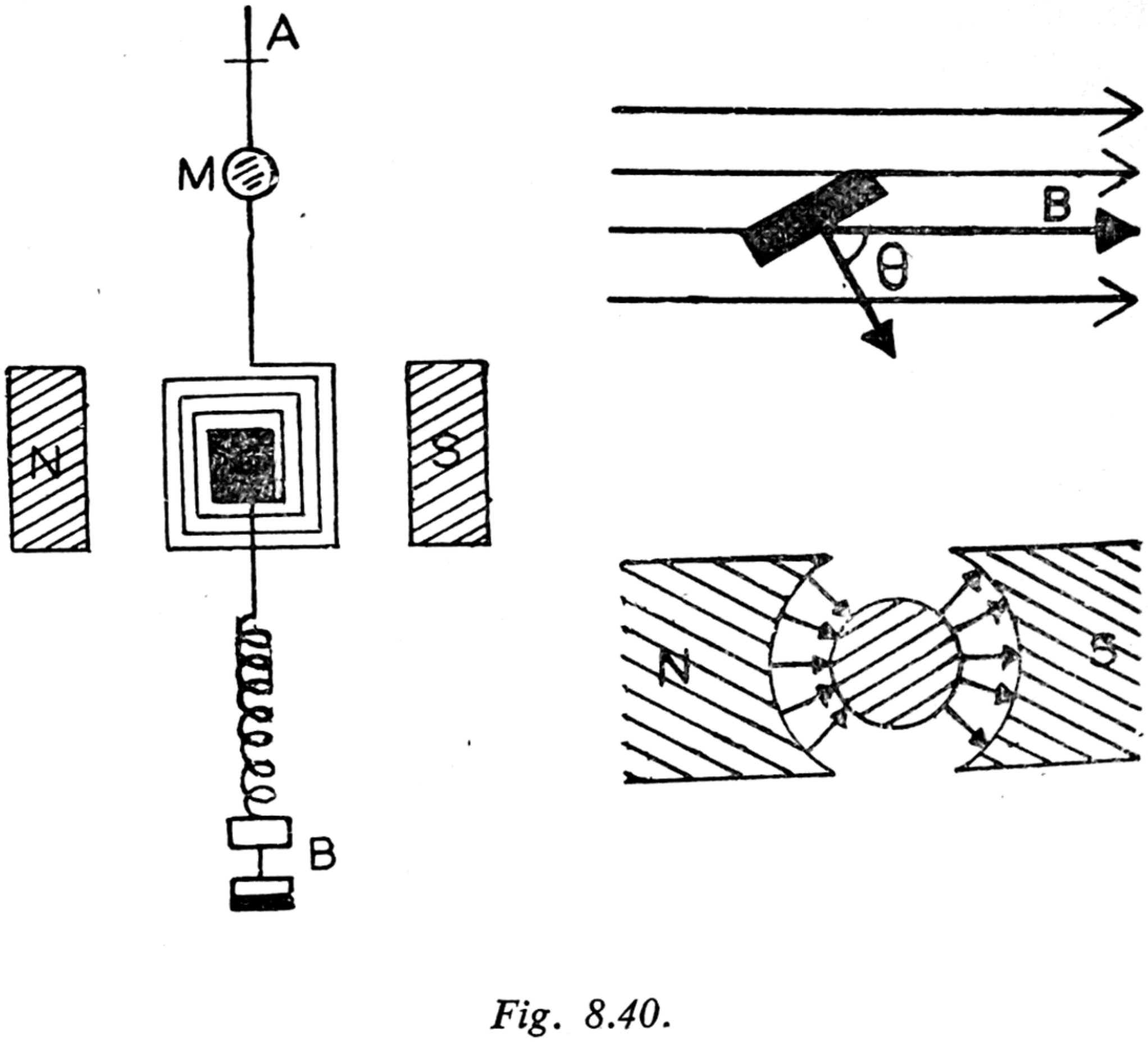


Fig. 8.40.

The pole pieces of the magnet are cylindrically concave in shape so that the lines of force are radial in the narrow annular gap and the magnetic field **B** is perpendicular to the coil surface, in its any position. Either a pointer is attached to the coil or mirror is attached to the suspension wire. The whole instrument is enclosed to a nonmagnetic metallic cover to protect it from air droughts. This cover has a central glass window to allow the light to go to and from the mirror, and three levelling screws at the bottom surface (base). Before using the galvanometer the coil is first released and the galvanometer is levelled properly, such that the coil swings freely between the pole pieces. The moving system is completely insulated from the metal case.

When a current i enters the coil of N turns through the upper end of suspension A and leaves through the lower end B of the spring, horizontal and oppositely directed side forces are exerted on its vertical sides. These forces produce a couple about a vertical axis through its centre given by the equation.

$$\tau = Ni\,(\mathbf{A}\times\mathbf{B}) \text{ or } \tau = Ni\,AB\sin\theta, \qquad \ldots(51)$$

where A is the area of the coil suspended in a mangnetic induction B and θ is the angle between the normal to the plane of the coil and the field direction.

The coil rotates in the direction of this couple and comes to rest in such a position that the restoring torque exerted by the suspension wire equals the deflecting torque due to the side forces,

Restoring torque $c\theta = $ *Deflecting couple* $NiAB \sin\theta$

or $$i = \frac{c}{NAB \sin\theta}\,\theta.$$

where c is known as torque constant of the material of suspension wire. As the field is made radial by making the pole pieces cylindrical, the plane of the coil always remains parallel to the magnetic field **B** or **A** $\perp$ **B**, thus we have

$$i = (c/NAB)\theta = K\theta, \qquad \ldots(92)$$

where K is called the *galvanometer constant*. Thus the deflection angle of the galvanometer is proportional to the current through it.

The deflection angle is observed with the aid of a beam of light reflected from a small mirror M cemented to the upper suspension, which is two times an angle θ through which the mirror is rotated. Reflected light is observed on a scale placed at a distance one metre from the galvanometer. If d is the deflection in millimeters of a light beam reflected by the galvanometer mirror on a scale 1 metre away, then,

$$i = K\,(d/2\times 10^{3}) = K'd,$$

where K' is called the figure of merit of the galvanometer. Currents of the order of 10^{-8} amp can be measured by a most sensitive galvanometer of this type.

Sensitivity of a Galvanometer. A galvanometer is said to be sensitive if for a given small current, the deflection θ is sufficiently large and is defined as $d\theta/di$. Using Eq. (87) we get

$$d\theta/di = NAB/c.$$

Thus we see that the higher sensitivity is achieved by increasing N, A and B and by decreasing c.

(*i*) *No of turns.* The higher sensitivity is achieved by winding more turns on the coil, this increases the resistance of the coil and is undesirable in many uses. This also increases the mass of the coil which is limited by the tensile strength of the suspension. More turns are allowed by employing oxidised aluminium wire in place of Cu-wire in some galvanometers.

(*ii*) *Area A.* To increase galvanometer sensitivity, the area cannot be increased indefinitely as it is limited by the separation

between the poles, without decreasing B. Only due to this reason, rectangular coil of the galvanometer is always suspended in such a way that its smaller side is along the magnetic field direction.

(*iii*) *Field B.* The magnetic field induction **B** is increased by using a powerful laminated magnet of cobalt steel and by placing a soft iron cylinder within the coil. This field **B** must be so large that the deflection does not change by outer magnetic fields, such as horizontal component of earth magnetic field, due to any small magnet placed in the neighbourhood of the coil.

(*iv*) *Torque constant c.* The torque constant c is decreased by using a thin and long suspension ($c = \pi\eta r^4/2l$). The suspension should be of such a material that it can support the coil and must have a great tensile strength. c is further decreased by using the suspension in the form of rectangular strip instead of the wire of circular cross section. The rectangular strip possesses an additional advantage of offering a comparatively large surface area for the dissipation of heat produced by the current.

Moving coils of copper or aluminium have a resistance temperature coefficient of about 0.4% per 1°*C*. To reduce it, a swamp resistance, in the form of a short fixed spiral of eureka wire of low temperature coefficient, is connected in series with the coil inside the instrument case. This swamp resistance is of greater value than that of moving coil, hence the change in the coil resistance with temperature produces small change in total resistance.

The advantages of these galvanometers are—

(*i*) The deflection of the coil is independent of direction and strengths of the external magnetic fields, hence does not require any adjustment in magnetic meridian and may be set up in any convenient position.

(*ii*) The coil moves in the radial field hence results the deflection directly proportional to the strength of the current passing through it. Thus an evenly divided scale can be used for measuring the current.

(*iii*) It can be made dead-beat or ballistic easily.

(*iv*) Its sensitivity can be increased without sacrificing accuracy.

(*v*) It is most useful for the detection of very small current of the order of 10^{-8} ampere.

(*vi*) It can be used as an ammeter and voltmeter.

8.18. AMMETERS AND VOLTMETERS

The ordinary ammeter and voltmeters are merely pivot galvanometers which have their sensitivities reduced in the proper manner. In pivot galvanometers the moving coil, instead of being suspended,

is pivoted between two jewel bearings. These galvanometers are very useful as may be used in any position and are conveniently portable.

When the galvanometer is used as a current measuring instrument, it is desirable that its resistance be very low compared with the total resistance in the circuit, so that the current in the circuit does not alter when the galvanometer is placed in series with source, etc., in the circuit. For this purpose the galvanometers are wound with a few turns of heavy wire to keep the resistance R_g as low as possible and a low resistance S is connected in parallel with the galvanometer. This small resistance in parallel with the galvanometer is known as *shunt*. The value of this shunt will depend upon the sensitivity of the galvanometer and thus upon the range I of an ammeter. If I_g is the total current through the galvanometer which can produce full scale deflection, then the resistance S of the shunt should be adjusted in such a way that the rest current $I-I_g$ passes through the shunt. At this stage

Potential across the shunt = Potential across the galvanometer coil

$$S(I-I_g)=I_g R_g$$

or

$$S=[I_g/(I-I_g)]\, R_g. \qquad \ldots(93)$$

When a galvanometer is to be used as a voltmeter, its resistance must be much larger than any resistance across which it is to be placed. Otherwise the voltmeter will decrease the current flow in the circuit and the *p.d.* measured will be smaller than its actual value *Ir*. Galvanometers used for this purpose are wound with many turns of fine wire. The total resistance is increased by connecting a high resistance R in its series. The value of this resistance will depend upon the range of the voltmeter, such that the potential across the system of galvanometer and resistance in series

$$I_g(R+R_g)=V \quad \text{or} \quad R=V/I_g-R_g. \qquad \ldots(94)$$

In ideal case, $S\to 0$ and $R\to\infty$ for the ammeter and voltmeter respectively.

8.19. DEAD-BEAT AND BALLISTIC

The galvanometers may be classified according to their use as *dead beat and ballistic types*. For the measurement of current, a steady deflection proportional to the current is required. The spot of light on the scale should not oscillate about its final position before coming to rest. A device, introduced to prevent this undesirable oscillation, is known as *damping* and the instrument as *dead-beat*. The damping is produced by (*a*) viscosity of air and (*b*) eddy currents set up when the coil is wound on a metallic frame. On the other hand, ballistic galvanometers measure *charge*, (*i.e.*, sudden discharge and not a continuous current). In this

case it is essential that the whole of the charge be passed through the coil before it begins to move, *i.e.*, the time of passage of charge must be smaller than the time occupied by the deflection. The coil therefore receives an impulse and a throw is observed. To achieve this a coil of high time period, at least 6 sec, is required. As time period $T=2\pi\sqrt{(I/c)}$, hence moving part should have a large moment of inertia I and the suspension should be as of smaller torque constant c. The damping should also be small as possible so that coil can swing freely.

However a moving coil ballistic galvanometer can be damped by connecting a resistance across its terminals, as the motion of the coil in a magnetic field will induce a current which will oppose the motion of the coil. The damping of a galvanometer can be varied by altering the resistance R of the closed circuit in which it is placed. Critical damping is obtained by choosing R so that, the galvanometer is just dead-beat and comes to rest without over-swing in a very short time.

Let us now investigate the condition, when a moving coil galvanometer can be used as dead-beat or a ballistic. Let θ be the deflection at any instant t when an instantaneous current i is passing through the coil at that instant. Different following couples act on the coil and keep the coil in equilibrium position.

(*i*) $I\,(d^2\theta/dt^2)$, *the retarding couple due to the moment of inertia* I *of the coil.* (*ii*) $a\,(d\theta/dt)$, *the retarding couple due to damping,* which is assumed to be proportional to the angular velocity $(d\theta/dt)$ of the coil, (*iii*) $NAB(d\theta/dt)=G\,(d\theta/dt)$, the *electromagnetic damping* due to induced current, (*iv*) $L\,(di/dt)$, the *opposite emf due to the self-inductance* L of the galvanometer and its circuit, (*v*) $NABi=Gi$, the *displacement couple* due to the current i. (*vi*) $c\theta$, the *restoring couple* due to the suspension fibre.

Assuming the factor due to self-inductance as negligible, the current due to the emf $\mathcal{E}$ applied to the circuit terminals

$$i=\frac{1}{R}\left[\ \mathcal{E}-G\frac{d\theta}{dt}\ \right]$$

where R is the total resistance, *i.e.*, the galvanometer resistance + external circuit resistance. Thus from Newton's law

$$I\frac{d^2\theta}{dt^2}+a\left(\frac{d\theta}{dt}\right)+c\theta=Gi=\frac{G}{R}\left[\ \mathcal{E}-G\frac{d\theta}{dt}\ \right]$$

$$I\frac{d^2\theta}{dt^2}+\left(\ a+\frac{G^2}{R}\ \right)\frac{d\theta}{dt}+c\theta=\frac{G\mathcal{E}}{R},$$

or

$$\frac{d^2\theta}{dt^2}+2b\,\frac{d\theta}{dt}+k^2\theta=f, \qquad \ldots(95)$$

where $2b=\frac{1}{I}\left(a+\frac{G^2}{R}\right)$, $k^2=\frac{c}{I}$ and $f=\frac{G\mathcal{E}}{IR}$.

When the galvanometer is used for the measurement of steady current, then $G\mathcal{E}/R$ will be applied continuously and final deflection θ_0 will be given by Eq. (92). When the galvonometer is used ballistically for charge measurement, the emf $\mathcal{E}=0$ and hence solution will be same as the complementary function of Eq. (95).

Let us now find the general solution of equation which is the sum of the complementary function and the particular integral. The complementary function is the solution of the equation

$$\frac{d^2\theta}{dt^2}+2b\frac{d\theta}{dt}+k^2\theta=0. \qquad \ldots(96)$$

Let a solution of this equation be $\theta=A_0e^{\alpha t}$. Substituting the value of θ and its derivatives in the above equation, we get

$$A_0e^{\alpha t}[\alpha^2+2b\alpha+k^2]=0.$$

As $A_0e^{\alpha t}$ cannot be zero, hence $\alpha^2+2b\alpha+k^2=0$

or
$$\alpha=-b\pm\sqrt{(b^2-k^2)}=-b\pm r$$

$\therefore$
$$\theta=A_1e^{(-b+r)t}+A_2e^{(-b-r)t}, \qquad \ldots(97)$$

where A_1 and A_2 are constants and $r=\sqrt{(b^2-k^2)}$.

The particular integral will be $\theta=f/k^2$ as Eq. (95) is satisfied by the substitution of $\theta=f/k^2$, $d\theta/dt=0$ and $d^2\theta/dt^2=0$ in it. Hence the total solution now becomes

$$\theta=e^{-bt}[A_1e^{rt}+A_2e^{-rt}]+f/k^2. \qquad \ldots(98)$$

When the instrument is used to measure steady current, the final deflection θ is obtained when time t is large and the equation (98) thus gives $\theta=\theta_0=f/k^2$. To find constants A_1 and A_2, consider that when $t=0$, $\theta=0$ and $d\theta/dt=0$, hence Eq. (98) with $f/k^2=\theta_0$ gives

$$0=A_1+A_2+\theta_0$$

and
$$d\theta/dt=0=-b[A_1+A_2]+r\,[A_1-A_2].$$

Solving these equations, we get

$$A_1=-(r+b)\theta_0/2r \text{ and } A_2=-(r-b)\theta_0/2r.$$

Thus equation (98) becomes

$$\theta=-e^{-bt}\left[\frac{r+b}{2r}e^{rt}+\frac{r-b}{2r}e^{-rt}\right]\theta_0+\theta_0. \qquad \ldots(99)$$

As $r=\sqrt{(b^2-k^2)}$. Hence there are three possible cases.

(*a*) **$b^2 > k^2$ or r is real.** Thus Eq. (99) indicates that the galvanometer deflection θ increases gradually from 0 to the steady value θ_0 and the motion is *nonoscillatory or dead-beat* and the galvanometer is known as *overdamped* (*b* large).

(*b*) **$b^2 = k^2$ or r is zero.** The galvanometer deflection returns to rest point in the minimum time and the velocity never changes in direction. The motion is just non-oscillatory and the galvanometer is known as *critical damped.* In this case

$$\left[\frac{1}{2I}\left(a+\frac{G^2}{R}\right)\right]^2=\frac{c}{I}.$$

If the circuit resistance is responsible for the appreciable damping, then *a* can be neglected and we get,

$$c=G^4/4IR^2.$$

(*c*) **$b^2 < k^2$ or r is imaginary.** Putting $r=(b^2-k^2)^{1/2}=i\sqrt{(k^2-b^2)}$ $=jp$, in Eq. (98), we have

$$\begin{aligned}\theta &= e^{-bt}[A_1e^{jpt}+A_2e^{-jpt}]+\theta_0\\ &= e^{-bt}[(A_1+A_2)\cos pt+j(A_1-A_2)\sin pt]+\theta_0\\ &= Ce^{-bt}\sin(pt+\phi)+\theta_0. \qquad \ldots(100)\end{aligned}$$

where $C=[(A_1+A_2)^2-(A_1-A_2)^2]^{1/2}$ and $\phi=\tan^{-1}(A_1+A_2)/j(A_1-A_2)$.

Eq. (100) shows that the motion of the galvanometer is simple harmonic, having time period

$$T=\frac{2\pi}{p}=\frac{2\pi}{\sqrt{(k^2-b^2)}}=2\pi\left[\frac{c}{I}-\frac{(a+G^2/R)^2}{4I^2}\right]^{-1/2} \qquad \ldots(101)$$

and of which amplitude dies away, leaving $\theta=\theta_0$.

For the galvanometers used for charge measurement there is no continuous source of emf, *i.e.*, $\mathcal{E}=0$ and thus $\theta_0=0$. Hence,

$$\theta=C\,e^{-bt}\sin[\sqrt{(k^2-b^2)}\,t+\phi]. \qquad \ldots(102)$$

The galvanometer deflection oscillates about its zero position and these oscillations are known as *underdamped oscillations* and the galvanometer is known as *oscillatory* or *ballistic.* Thus for ballistic motion of a galvanometer $b^2<k^2$, *i.e.*,

$$\frac{1}{21}\left[a+\frac{G^2}{R}\right]<\left(\frac{c}{I}\right)^{1/2} \qquad \ldots(103)$$

or *I*, *R* and *c* should be large and *a* and G^2 should be smaller.

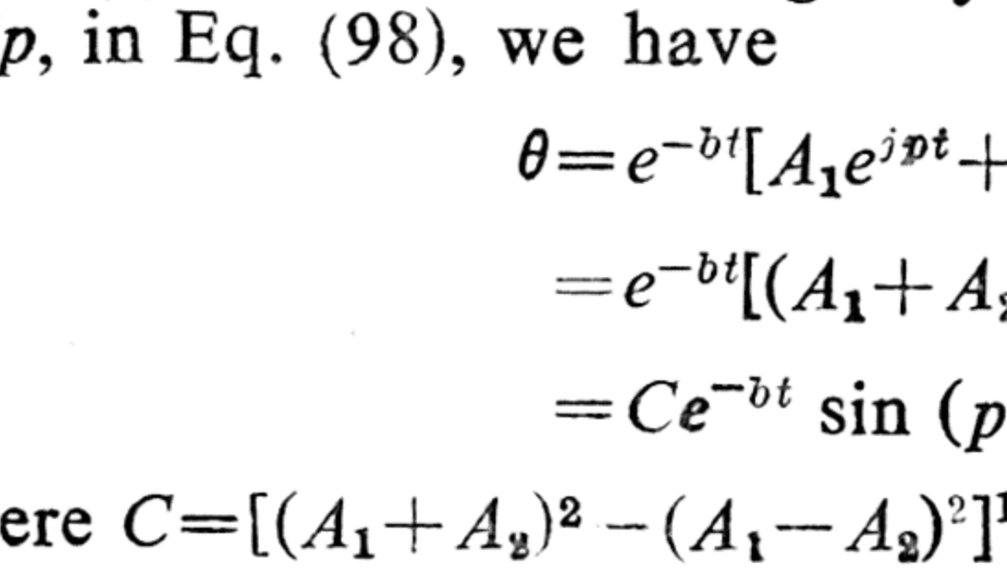


Fig. 8.41.

The motions of the galvanometer coil in above mentioned three cases are shown in Fig. 8.41.

8.20. BALLISTIC GALVANOMETER

Ballistic galvanometer is the instrument for the measurement of the charge sent through it in a short interval of time. For the correct measurement of charge, whole discharge should pass before the galvanometer has moved appreciably. For this, time of swing [$T=2\pi\sqrt{(I/c)}$ should be greater than the time of discharge. The damping should be small for oscillatory motion. The basic conditions with the B.G. are discussed in the preceding article.

Both moving magnet and moving coil types of ballistic galvanometer are used, but the latter are more common to accept. The principle and construction of such galvanometers are the same as those of dead-beat moving coil galvanometers, except that—

(*i*) The time period T is large and hence the moment of inertia I of the coil is large and torsional couple c is small. To increase I, copper wire coil is used and to decrease c, a thin and long phospher strip suspension is used.

(*ii*) The total damping is very small. The damping due to viscosity of air is very small. When coil circuit is closed, induced currents are set up in the coil circuit and the frame, which will damp oscillations of the coil. This type of damping is called *electromagnetic damping*. It can be reduced to zero by using non-conducting frame, such as of ivory, celluloid or bamboo.

Consider a galvanometer coil of turns N, area A, and moment of inertia I suspended from a torsion wire of torque constant c in a magnetic induction field **B**. Suppose a current pulse is sent through the coil. The current is at first zero, then rises to maximum and then comes to zero in such a short time Δt that the coil does not move through an appreciable angle while the current pulse exists but rotates only after current has ceased. During the current pulse the torque, at any instant when current is i, is given by $\tau=Ni\mathbf{A}\times\mathbf{B}$ which is $Ni\,AB$ when the plane of coil is parallel to the magnetic field, *i.e.*, $\mathbf{A}\perp\mathbf{B}$.

The angular impulse given to the galvanometer coil is equal to the change in angular momentum and is given by

$$NAB\int_0^{\Delta t} i\ dt=NBAq=I\omega, \qquad \ldots(104)$$

where ω is the angular velocity with which the coil begins to rotate.

The initial angular momentum $I\omega$ corresponds to an initial kinetic energy of the coil and is converted to elastic potential energy of the torsion fibre which is the energy spent in twisting suspension. Hence,

$$K.E.=\tfrac{1}{2}\,I\omega^2=P.E.=\int_0^{\theta_0} c\theta\,d\theta=\tfrac{1}{2}c\theta_0^{\,2}$$

or $$I\omega^2=c\theta_0^{\,2}. \qquad \ldots(105)$$

As we know that $T=2\pi\sqrt{(I/c)}$ or $I=cT^2/4\pi^2$, hence by eliminating ω from Eqs. (104) and (105) and then substituting this value of I, we get

$$N^2B^2A^2q^2=c^2T^2\theta_0^{\,2}/4\pi^2,$$

or $$q=\frac{T}{2\pi}\frac{c}{NAB}\theta_0=\frac{T}{2\pi}K\theta_0=K'\theta_0, \qquad \ldots(106)$$

where K' is a constant, known as *ballistic constant or ballistic reduction factor* and θ_0 is the first throw or ballistic throw. If θ_1 is measured in mm on the scale with the lamp and scale arrangement, then K' is defined as the number of charge required to produce one mm. deflection of light reflected from the mirror attached with the suspension wire on the scale placed at a distance of 1 meter. It is also known as *figure of merit* of galvanometer.

Correction of damping. In deducing the relation (106) we have assumed that the whole of the K.E. is used in twisting the suspension of the coil. But we know that there are damping forces, frictional damping and electromagnetic damping in overcoming of which some part of K.E. is used. Thus the swings get progressively shorter and shorter. If θ_1, θ_2, θ_3...be the successive swings to left and right, it is found experimentally that the ratio of the consecutive deflections is a constant quantity, *i.e.*

$$\theta_1/\theta_2=\theta_2/\theta_3=\theta_3/\theta_4=\ldots=d\ (\text{const.}) \qquad \ldots(107)$$

This constant is known as *decrement* and its logarithm as *logarithmic decrement* and is denoted as λ. Hence

$$\theta_1/\theta_2=\theta_2/\theta_3=\theta_3/\theta_4=\ldots e\lambda \qquad [As\ log_e\ d=\lambda \text{ or } d=e^{\lambda}]$$

For a complete vibration, $\frac{\theta_1}{\theta_3}=\frac{\theta_1}{\theta_2}\times\frac{\theta_2}{\theta_3}=d^2=e^{2\lambda}$.

Thus the value of decrement when coil completes a quarter of a vibration will be $e^{\lambda/2}$ and hence the first true deflection θ_0 is given by

$$\theta_0/\theta_1=e^{\lambda/2} \text{ or } \theta_0=\theta_1 e^{\lambda/2}=\theta_1\,(1+\gamma/2).$$

Here we have neglected higher order terms, as λ is very small.

Hence the correct relation for charge is

$$q=\frac{T}{2\pi}\frac{c}{NAB}\theta_1\left(1+\frac{\lambda}{2}\right). \qquad \ldots(108)$$

Measurement of ballistic reduction factor. The ballistic galvanometer, K of which is to be determined, is connected with a battery and a known capacitor with the help of Morse key, as shown in Fig. 8.42. The condenser of known capacity C is first charged by pressing knob A of the Morse Key. If the key is then released, the battery circuit is broken and the condenser is discharged through the ballistic galvanometer. If the deflection is out of scale use a rheostat as a potential divider. If θ_1 is the first throw of the galvanometer, with a potential difference V, then we get

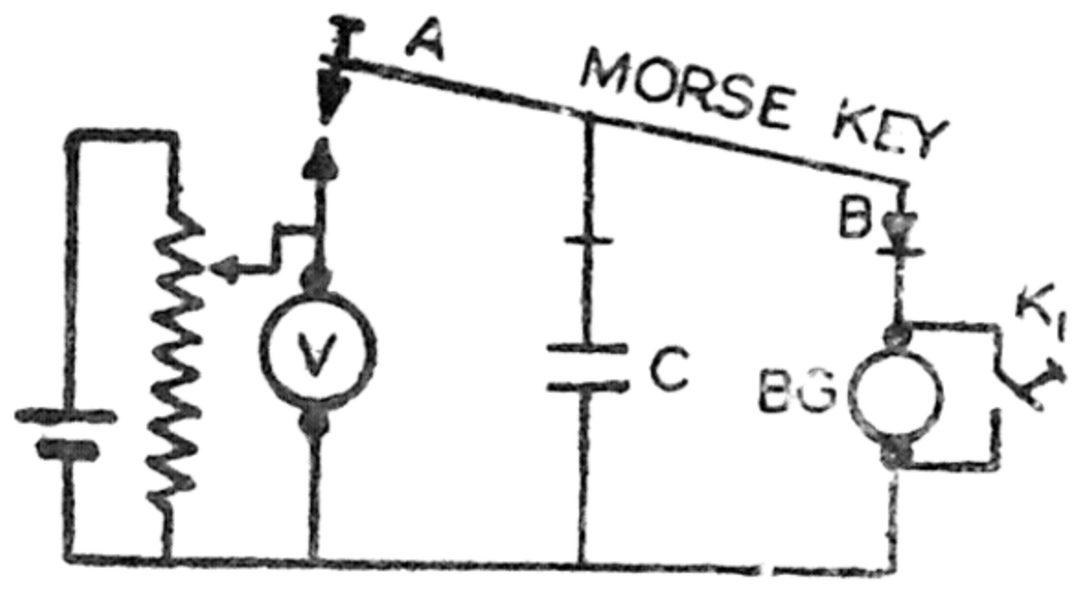


Fig. 8.42.

$$q = CV = K\theta_1 (1+\lambda/2) \text{ or } K = CV/\theta_1 (1+\tfrac{1}{2}\lambda) \ \textit{coulomb/radian}$$

λ can be calculated by observing first and eleventh throws of ballistic galvanometer coil and by using the relation

$$\frac{\theta_1}{\theta_{11}} = \frac{\theta_1}{\theta_2} \times \frac{\theta_2}{\theta_3} \cdots \frac{\theta_9}{\theta_{10}} \times \frac{\theta_{10}}{\theta_{11}} = e^{10\lambda}$$

$$\text{or } \lambda = \frac{1}{10} \log_e \frac{\theta_1}{\theta_{11}} = \frac{2.3026}{10} \log_{10} \frac{\theta_1}{\theta_{11}} = \frac{2.3026}{10} \log_{10} \frac{d_1}{d_{11}}.$$

where d_1 and d_{11} are the throws of the spot of light on the scale. Hence by substituting numerical values of C, V, λ and $\theta_1 (= d_1/2D)$, we can calculate ballistic reduction factor K.

Uses of Ballistic Galvanometer

(*a*) *Comparision of Capacitances.* The circuit is connected as shown in Fig 8.43, C_1 and C_2 are the capacitances to be compared. The plug in K_2 is first put to the left and the capacitor C_1 is charged with emf $\mathcal{E}$ by pressing knob A of Morse key. After releasing the knob, the capacitor C_1 is discharged through the ballistic galvanometer. If θ_1 is the first throw, then

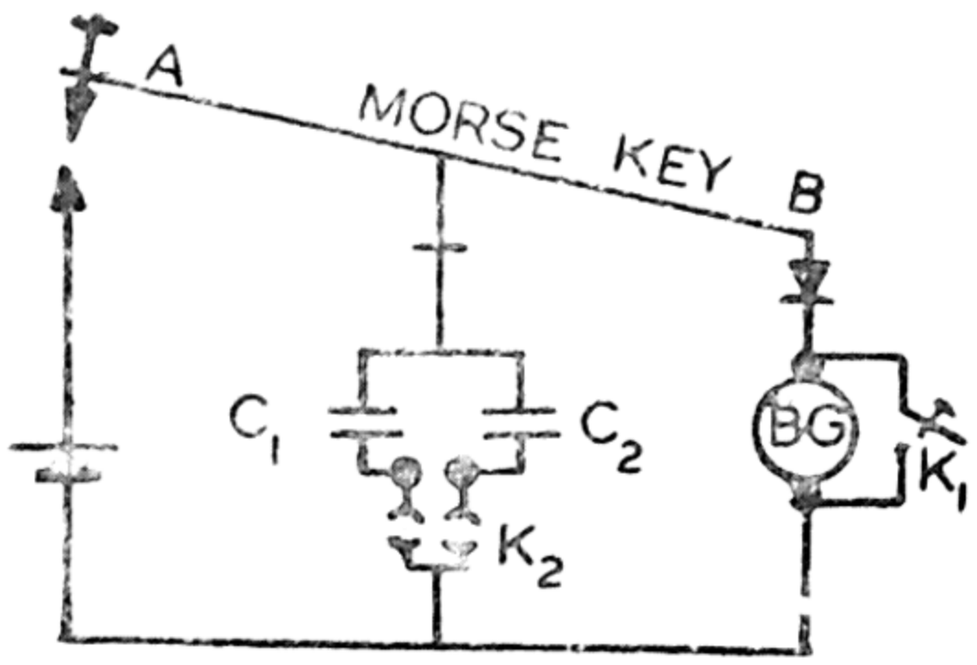


Fig. 8.43.

$$q_1 = C_1\mathcal{E} = K\theta_1 (1+\lambda/2).$$

Similarly the experiment is repeated with capacitor C_2 by pressing Morse key with the plug on the right in key K_2 and then releasing. If θ_2 is the first throw in this case, then

$$q_2 = C_2\mathcal{E} = K\theta_2(1+\lambda/2).$$

$$\therefore \ C_1/C_2 = \theta_1/\theta_2 = d_1/d_2, \qquad \ldots(109)$$

where d_1 and d_2 are the first throws of the spot light on the scale with capacitors C_1 and C_2 respectively.

(*b*) *Measurement of capacitance of a capacitor*—The connections are made as shown in Fig. 8.44, C is the unknown capacitor, R is a high resistance ($\sim 10000\,\Omega$) and r is a standard resistance of 1Ω. A steady current i flows through these resistances.

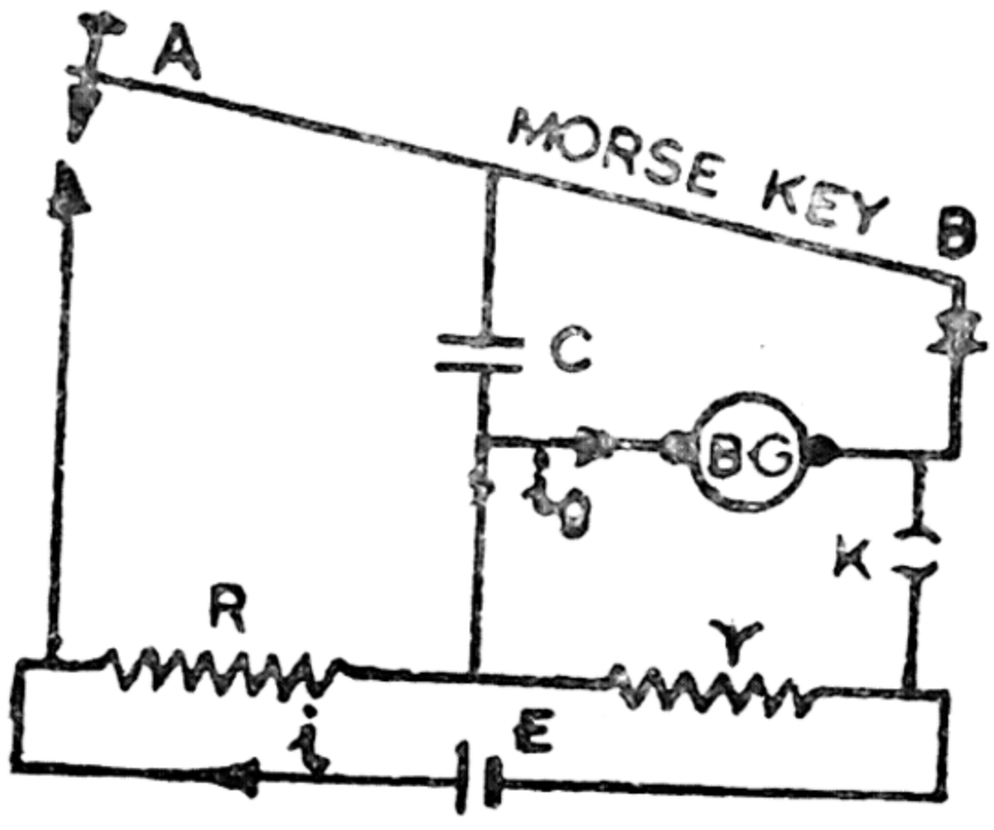


Fig. 8.44.

Initially key K is kept open and the knob A of the Morse key is pressed down. The capacitor C is now charged to a potential difference iR. This charge will be

$$q = C(iR).$$

If the knob A is released the charge of the capacitor will go to the ballistic galvanometer. If θ_1 is the first throw in the coil, then

$$q = CiR = \frac{c}{NAB} \cdot \frac{T}{2\pi} \theta_1 \left(1 + \frac{\lambda}{2} \right)$$

or

$$C = \frac{c}{NAB} \cdot \frac{T}{2\pi} \frac{1}{iR} \theta_1 \left(1 + \frac{\lambda}{2} \right) \quad \ldots(110)$$

For the values of c, N, A and B, a steady current i_g is allowed to pass through the galvanometer, by closing key K. If ϕ be the steady deflection, then

$$i_g = (c/NAB)\phi \text{ or } c/NAB = i_g/\phi.$$

If G be the galvanometer resistance, which is shunted by the resistance r, the current through the galvanometer is given by

$$i_g = [r/(r+G)]i = (r/G)i.$$

$\therefore$

$$c/NAB = r\ i/G\phi.$$

Substituting this value of c/NAB in Eq. (110), we get

$$C = \frac{T}{2\pi} \frac{1}{iR} \frac{ri}{G\phi} \theta_1 \left(1 + \frac{\lambda}{2} \right) = \frac{T}{2\pi} \frac{r}{RG} \frac{\theta_1}{\phi} \left(1 + \frac{\lambda}{2} \right) \quad \ldots(111)$$

Hence C can be calculated by measuring these quantities.

8.21. SENSITIVITY OF GALVANOMETER

(1) **Current sensitivity**. The current sensitivity of a galvanometer is defined as the deflection in millimeters produced on a scale placed at a distance of one metre from the galvanometer mirror, when a current of 10^{-6} ampere passes through the galvanometer coil. Thus,

Current sensitivity $S_i = d\theta/di = NAB/c$.

(2) **Charge Sensitivity.** It is defined as the deflection in millimeters produced on a scale placed at a distance of one metre from the galvanometer mirror, when a charge of 10^{-6} coulomb is allowed to pass through the galvanometer coil. Thus

$$\text{Charge sensitivity } S_q = \frac{d\theta}{dq} = \frac{NAB}{c}\,\frac{2\pi}{T}\,\frac{1}{(1+\lambda/2)} = \frac{NAB}{c}\,\frac{2\pi}{T}$$

$$= (2\pi/T) \times \text{current sensitivity } S_i.$$

Here $\lambda/2$ is neglected in comparison to 1 as λ is very small.

(3) **Voltage Sensitivity.** It is defined as the deflection on millimeters produced on a scale at a distance of one metre from the mirror by the potential of one microvolt. Thus

$$\text{Voltage Sensitivity } S_v = d\theta/dV = d\theta/Rdi = S_i/R.$$

Hence S_v is the current sensitivity divided by the circuit resistance.

(4) **The Meg-ohm Sensitivity.** It is the resistance in meg-ohms placed in series with the galvanometer in order to get a deflection of 1 mm on the scale of one metre apart by a potential difference of 1 volt.

(5) **Factor of Merit.** When the sensitivity has been reduced to allow for time period T and resistance R_g of the galvanometer, it is called **normal sensitivity** or **Factor of Merit.**

If the number of turns is increased to N by keeping area A constant, the length of the winding will increase N times but the cross sectional area of the wire must be necessarily reduced by N. Thus the resistance of the coil will correspondingly increase by N^2 ($\because R = \rho l/a$). As

$$S_i = \frac{NAB}{c} = \frac{NABT^2}{4\pi^2 I} = NT^2 \times \text{const.}$$

$$\propto R_g^{1/2}\, T^2.$$

$\therefore$ Factor of Merit $= 100\, D/T^2 R_g^{1/2}$,

where D is the deflection in millimeters on a scale at one metre from the coil, for current sensitivity due to 1μ A current.

Exercises

Example 1. *A 5.0 MeV proton is falling vertically downward in a uniform field of magnetic induction 1.5 webers/m², pointing horizontally from south to north. Find the force exerted on it.*

We know that the mass of proton $= 1.7 \times 10^{-27}$ kg and charge on it $= 1.6 \times 10^{-19}$ coul. Given that the kinetic energy of the proton.

$$K.E.=\tfrac{1}{2}mv^2=5MeV=5\times1.6\times10^{-13}\text{ joule}$$

$$\therefore\ \text{velocity } v=\sqrt{\left(\frac{2\ K.E.}{m}\right)}=\sqrt{\left(\frac{2\times8\times10^{-13}}{1.7\times10^{-27}}\right)}$$

$$=3.07\times10^{7}\text{ m/sec.}$$

The magnetic force $\mathbf{F}=q\mathbf{v}\times\mathbf{B}$ and its magnitude

$$F=qvB\sin\theta=(1.6\times10^{-19})\ .\ (3.07\times10^{7})\ (1.5)\sin90°$$

$$=7.37\times10^{-12}\text{ nt.}$$

The direction of **F** will be $\perp$ to **v** and **B** both. As **v** is in downward direction and **B** is from south to north horizontally, hence **F** will be in horizontal plane in west-east direction.

Example 2. *A stream of protons and deuterons in a vaccum chamber enters a unifarm magnetic field. Both protons and deuterons have been subjected to the same accelerating potential, hence the kinetic energies of the particles are the same. If the magnetic field is perpendicular and the protons move in a circular path of radius 15 cm, find the radius of the path traversed by the deuterons.*

If m_1 and m_2 are the masses of proton and deuteron respectively and v_1 and v_2 are their velocities.

$$K.E.=\tfrac{1}{2}m_1v_1^{\,2}=\tfrac{1}{2}m_2v_2^{\,2}.\qquad\ldots(i)$$

If r_1 and r_2 be the radii of the paths of protons and deuterons respectively in the magnetic field B, then

for protons $\qquad m_1v_1^{\,2}/r_1=Bqv_1$ or $v_1=Bqr_1/m_1$

and for deuterons $\qquad m_2v_2^{\,2}/r_2=Bqv_2$ or $v_2=Bqr_2/m_2$.

Substituting these values in the relation (i), we get

$$\tfrac{1}{2}m_1\,(Bqr_1/m_1)^2=\tfrac{1}{2}m_2\,(Bqr_2/m_2)^2\text{ or }r_1^{\,2}/m_1=r_2^{\,2}/m_2.$$

As deuteron is an isotope of hydrogen and consists of a proton and a neutron, hence $m_2=2m_1$. Therefore we have

$$r_2=r_1\sqrt{(m_2/m_1)}=0.15\sqrt{2}=0.212\text{ metre.}$$

Example 3. *A proton is shot with a speed of 1×10^6 m/sec along a line at an angle of 30° to the x-axis, parallel of which a uniform field of 0.20 wb/m² exists. Describe the motion of the proton.*

One can split the velocity vector into two components, one along x-axis and other along x-axis. Since the field is in the x-direction, the force $F_x=qv_xB\sin\theta=0$, and hence v_x is unaffected by the field. Thus the proton moves in the x-direction with a constant velocity $\qquad v_x=(1\times10^6)\cos30°=0.866\times10^6$ m/sec.

As **B** is $\perp$ to v_y the proton travels in a circle which is in y-z plane. Equating magnetic to centripetal force, we get

$$mv_v^2/r = Bqv_v$$

or
$$r = \frac{mv_v}{Bq} = \frac{1.67\times10^{-27}\times1\times10^{6}\times\frac{1}{2}}{0.20\times1.6\times10^{-19}} = 2.6\times10^{-2} \text{ m}$$

Hence time period

$$T = \frac{2\pi r}{v_v} = \frac{2\times3.14\times2.6\times10^{-2}}{1\times10^{6}\times\frac{1}{5}} = 3.26\times10^{-7} \text{ sec.}$$

During this time the proton will have travelled a distance $v_x t$ parallel to the x-axis. The distance

$$v_x t = 0.866\times10^{6}\times3.26\times10^{-7} = 0.282 \text{ meter.} = 28.2 \text{ cm.}$$

Thus the proton will follow a helix like path with radius 2.6 cm and pitch 28.2 cm.

Example 4. *A silver wire is taken in the form of a ribbon 0.5 cm wide and 0.1 mm thick. If a 2 ampere current passes through the ribbon parallel to its length and a 0.8 tesla magnetic field is applied to the current and along the thickness of the ribbon. Calculate the Hall emf produced.*

The atomic weight of silver is 108, its density is 10.5 gm/cm^3 and the number of atoms in a gm atom is 6.023×10^{23}. Hence the number of atoms per unit volume

$$n = 6.023\times10^{23}\times(10.5/108) = 5.85\times10^{22} \text{ cm}^{-3} = 5.85\times10^{28} \text{ per m}^3.$$

As v, B and d are perpendicular to each other, hence Hall emf

$$V_H = vBd = \frac{j}{nq}\,Bd = \frac{i}{nqA}\,Bd = \frac{iB}{nqt}.$$

$$= \frac{2\times0.8}{5.85\times10^{28}\times1.6\times10^{-19}\times0.1\times10^{-3}} = 1.7\times10^{-6} \text{ volt.}$$

Example 5. *An electron is moving with a velocity* $2\mathbf{i}+3\mathbf{j}$ *m/sec in an electric field of intensity* $3\mathbf{i}+6\mathbf{j}+2\mathbf{k}$ *and a magnetic field of* $2\mathbf{j}+3\mathbf{k}$ *Tesla. Find the magnitude and direction of the Lorentz force acting on the electron.*

Lorentz force $\mathbf{F} = q[\mathbf{E}+\mathbf{v}\times\mathbf{B}]$

$$= 1.6\times10^{-19}[3\mathbf{i}+6\mathbf{j}+2\mathbf{k}+(2\mathbf{i}+3\mathbf{j})\times(2\mathbf{j}+3\mathbf{k})]$$

$$= 1.6\times10^{-19}[3\mathbf{i}+6\mathbf{j}+2\mathbf{k}+9\mathbf{i}-6\mathbf{j}+4\mathbf{k}]$$

$$= 1.6\times10^{-19}[12\mathbf{i}+6\mathbf{k}] = 9.6\times10^{-19}[2\mathbf{i}+\mathbf{k}]$$

$\therefore$ $F = 9.6\times10^{-19}\sqrt{(4+1)} = 2.15\times10^{-18}$ newton.

It is in x—z plane, making an angle θ with the x-axis, where

$$\theta = \cos^{-1} 2/\sqrt{5}.$$

Example 6. *A proton, with initial velocity of* 5×10^6 *m/sec, passes through an electric field (transverse) of 200 volts/cm. Calculate the transverse deflection in travelling a distance of 1 meter.*

Let the proton direction as x and that of the electric field as y.

Time taken to travel a distance L along the x-axis

$$t=L/v=1/5\times10^6=2\times10^{-7}\text{ sec.}$$

Force acting on the proton due to electric field $=qE_y=mf_y$.

$\therefore$ Proton acceleration $f_y=qE_y/m$.

Hence the transverse deflection $y=\frac{1}{2}f_yt^2=\frac{1}{2}qE_yt^2/m$

$$=\tfrac{1}{2}\times1.6\times10^{-19}\times2\times10^4\times(2\times10^{-7})^2/1.6\times10^{-27}$$

$$=0.04\text{ meter.}$$

Example 7. *A deuteron travels in a circular path of radius 40 cm in a magnetic field of flux density 1.5 wb/m*^{2}. (*a*) *Find the speed of the deuteron,* (*b*) *the time required for it to make one half a revolution.* (*c*) *through what pd would the deuteron have to be accelerated to acquire this velocity* ?

Due to magnetic field **B**, the particle of mass m and of charge q moves in a circular orbit of radius R. The motion is given by the relation

$$mv=BqR \quad\text{or}\quad v=BqR/m.$$

As the deuteron is the nucleus of the isotope of hydrogen, consisting of one neutron and one proton, hence its mass is $2mu(=2\times1.66\times10^{-27}$ kg). Now substituting numerical values, we get

$$v=1.5\times1.6\times10^{-19}\times0.4/2\times1.66\times10^{-27}=2.89\times10^7\text{ m/sec.}$$

Time required for one half revolution

$$t=\pi R/v=3.14\times0.4/2.89\times10^7=4.34\times10^{-8}\text{ sec.}$$

If potential difference required to accelerate is V, then $qV=\frac{1}{2}mv^2$

or
$$V=mv^2/2q=2\times1.66\times10^{-27}\times(2.87\times10^7)^2/2\times1.6\times10^{-19}$$

$$=8.55\times10^6\text{ volts.}$$

Example 8. *Deuterons in a cyclotron describes a circle of radius 32.0 cm just before emerging from the D's. The frequency of the applied alternating voltage is 10 M Hz. Find* (*a*) *the magnetic flux density and* (*b*) *the energy and speed of the deuterons upon emergence.*

Frequency of the applied emf=cyclotron frequency $f=Bq/2\pi m$.

$\therefore$ Magnetic flux density $B=2\pi mf/q$

$$=2\times3.14\times2\times1.66\times10^{-27}\times10\times10^6/1.6\times10^{-19}$$

$$=1.30\text{ wb/m}^2$$

The maximum velocity $v_{max}=BqR/m=2\pi fR$

$=2\times3.14\times10\times10^6\times0.32=2.01\times10^7$ m/sec.

Hence the energy

$$E=\tfrac{1}{2}m(v_{max})^2=\tfrac{1}{2}\times2\times1.66\times10^{-27}\times(2.01\times10^7)^2$$
$$=4.22 \text{ MeV}.$$

Example 9. *Find the force on the portion of the current carrying wire shown in Fig. 8.45.*

The forces on the parts PQ and ST of the current carrying wire are zero as these are parallel to the direction of **B**. Situation is different along the arc QRS. Consider the small element of arc $d\mathbf{l}$. The force on this element is

$$d\mathbf{F}=id\mathbf{l}\times\mathbf{B},$$

or $$dF=i\,dl\,B\sin(180-\theta)$$
$$=i\,dl\,B\sin\theta.$$

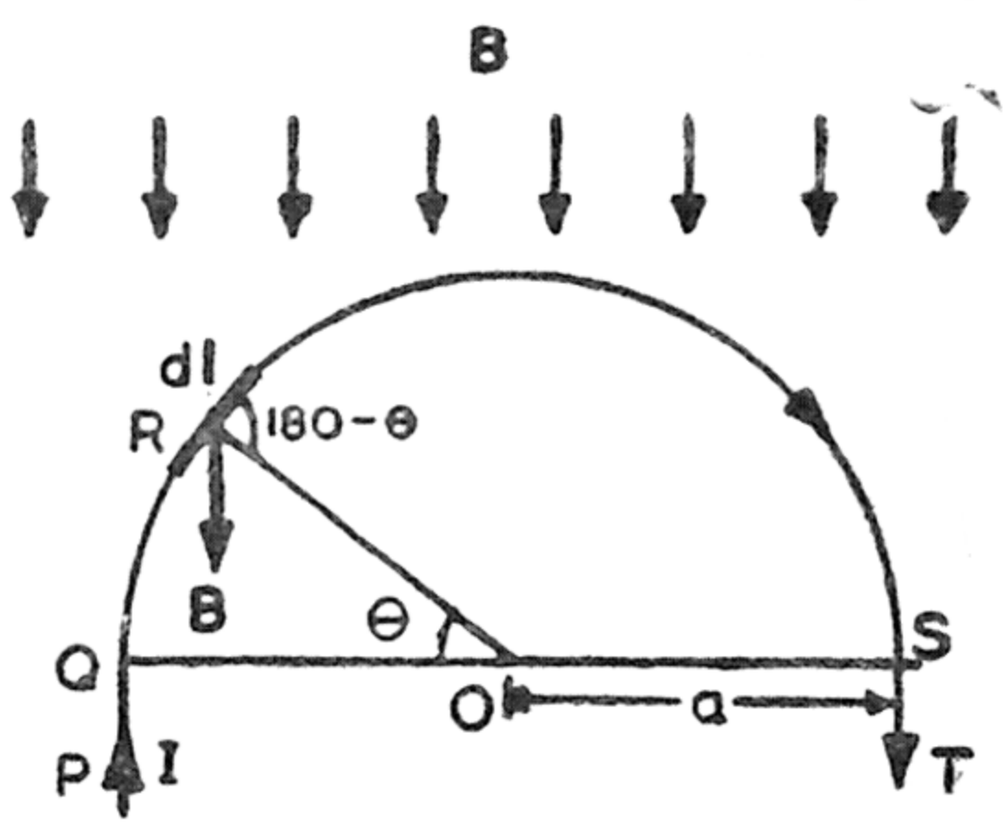


Fig. 8.45.

Its direction must be $\perp$ to the plane containing $d\mathbf{l}$ and **B**, *i.e.*, into the page. Each little element of arc will contribute a force in the same direction hence the total force

$$F=\int dF=\int iB\sin\theta\,dl=iB\int\sin\theta\,dl.$$

If $d\theta$ be the angle subtended by the element at O, then $dl=a\,d\theta$.

$$\therefore\quad F=iB\int\sin\theta\,a\,d\theta=iBa\int_0^{\pi}\sin\theta\,d\theta=2iBa.$$

This is the same force that a straight wire QS would experience.

Example 10. *Find the total force on the circuit (Fig. 8.46) due to the magnetic field acting in downward $\perp$ to the paper plane.*

The force on the straight wire portion $F_1=i(PR)B=iLB$.

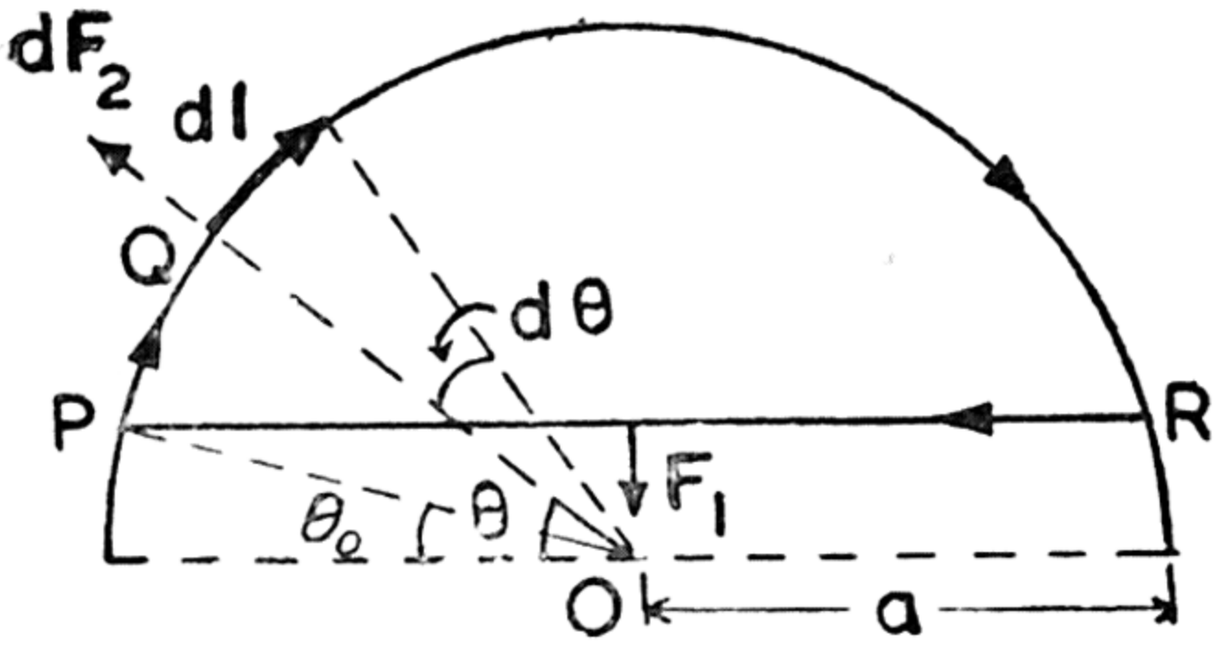


Fig. 8.46.

It is directed downward in the plane of the page. Consider a small element $d\mathbf{l}$ of the arc. The force on this element

$$d\mathbf{F}_2=iBdl.$$

It is directed along OQ. Since $d\mathbf{F}_2$, for various elements, are in different directions. The horizontal component of $d\mathbf{F}_2$ will be cancelled by an equal and opposite component due to a symmetrie element on the other side of the arc. Therefore the total force

$$F_2 = F_{2y} = \int dF_2 \sin\theta = \int iB \sin\theta \, dl.$$

As dl can be written as $ad\theta$, hence

$$F_{2y} = iaB \int \sin\theta \, d\theta = iaB \int_{\theta_0}^{180-\theta_0} \sin\theta \, d\theta$$

$$= 2iaB \cos\theta_0 = iBL. \qquad (\text{as } a\cos\theta_0 = L/2)$$

Thus, we see that F_{2y} is equal and opposite to F_1 and the total force is hence zero. It is true for any closed circuit in a uniform magnetic field.

Example 11. *Find an expression for the force acting between two current carrying coplanar parallel wires.*

Let us consider two coplanar parallel wires each of length L.

If the currents in these wires flow in the same direction then the magnetic field at point O_2 due to the current element $d\mathbf{l}_1$ of the first wire is given as

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \, i_1 \frac{d\mathbf{l}_1 \times \mathbf{r}_{12}}{r_{12}^3}.$$

In this magnetic field current element $d\mathbf{l}_2$ experiences a force

$$d\mathbf{F} = \frac{\mu_0}{4\pi} \, i_1 i_2 \frac{d\mathbf{l}_2 \times [d\mathbf{l}_1 \times \mathbf{r}_{12}]}{r_{12}^3}.$$

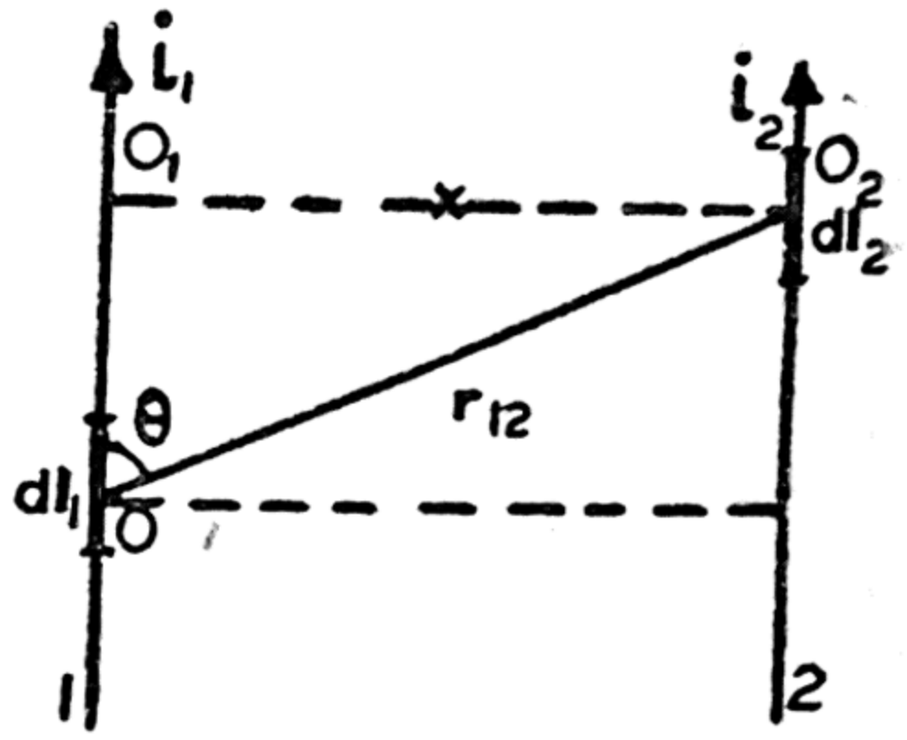


Fig. 8.47.

Therefore, the total force on the second wire due to the current in the entire length of the first wire

$$\mathbf{F} = \frac{\mu_0}{4\pi} \, i_1 i_2 \iint \frac{d\mathbf{l}_2 \times [d\mathbf{l}_1 \times \mathbf{r}_{12}]}{r_{12}^3}.$$

Using the geometry of the figure, the vector product $d\mathbf{l}_1 \times \mathbf{r}_{12}$ has a magnitude $dl_1(x^2 + l_1^2)^{1/2} \sin\theta$ and direction normal to the page in inward direction. Therefore $d\mathbf{F}$ must be perpendicular to $d\mathbf{l}_2$ and in the plane of paper towards the first wire.

$$F = \frac{\mu_0}{4\pi} \, i_1 i_2 \int dl_2 \int \frac{dl_1 \, (x^2 + l_1^2)^{1/2} \sin\theta}{(x^2 + l_1^2)^{3/2}}.$$

Force on a unit length of wire 2 due to the entire length of first wire

$$f = \frac{\mu_0 i_1 i_2}{4\pi} \, x \int_{-L/2}^{L/2} \frac{dl_1}{(x^2 + l_1^2)^{3/2}} = \frac{\mu_0 i_1 i_2}{2\pi x[1 + 4(x/L)^2]^{1/2}},$$

If these wires are of infinite length, then we have

$$f = \mu_0 \, i_1 i_2 / 2\pi x.$$

Example 12. *Prove that the torque acting on the flat current loop placed in a magnetic field is independent of the shape of the loop.*

Consider a flat loop of wire in xy plane. Magnetic field **B** is acting in yz plane and making an angle θ with the z-axis. Let us find the force on the small length dS_1 making an angle ϕ_1 with the y-axis. The force F_{1z} acting on this element due to field component B_y is

$$F_{1z}=iB_y\, dS_1 \sin \phi_1.$$

The force acting due to component B_z is

$$F_{1y}=iB_z\, dS_1.$$

Similarly forces on the element dS_2 due to field components B_y and B_z are

$$F_{2z}=iB_y\, dS_2 \sin \phi_2 \text{ and } F_{2y}=iB_z\, dS_2.$$

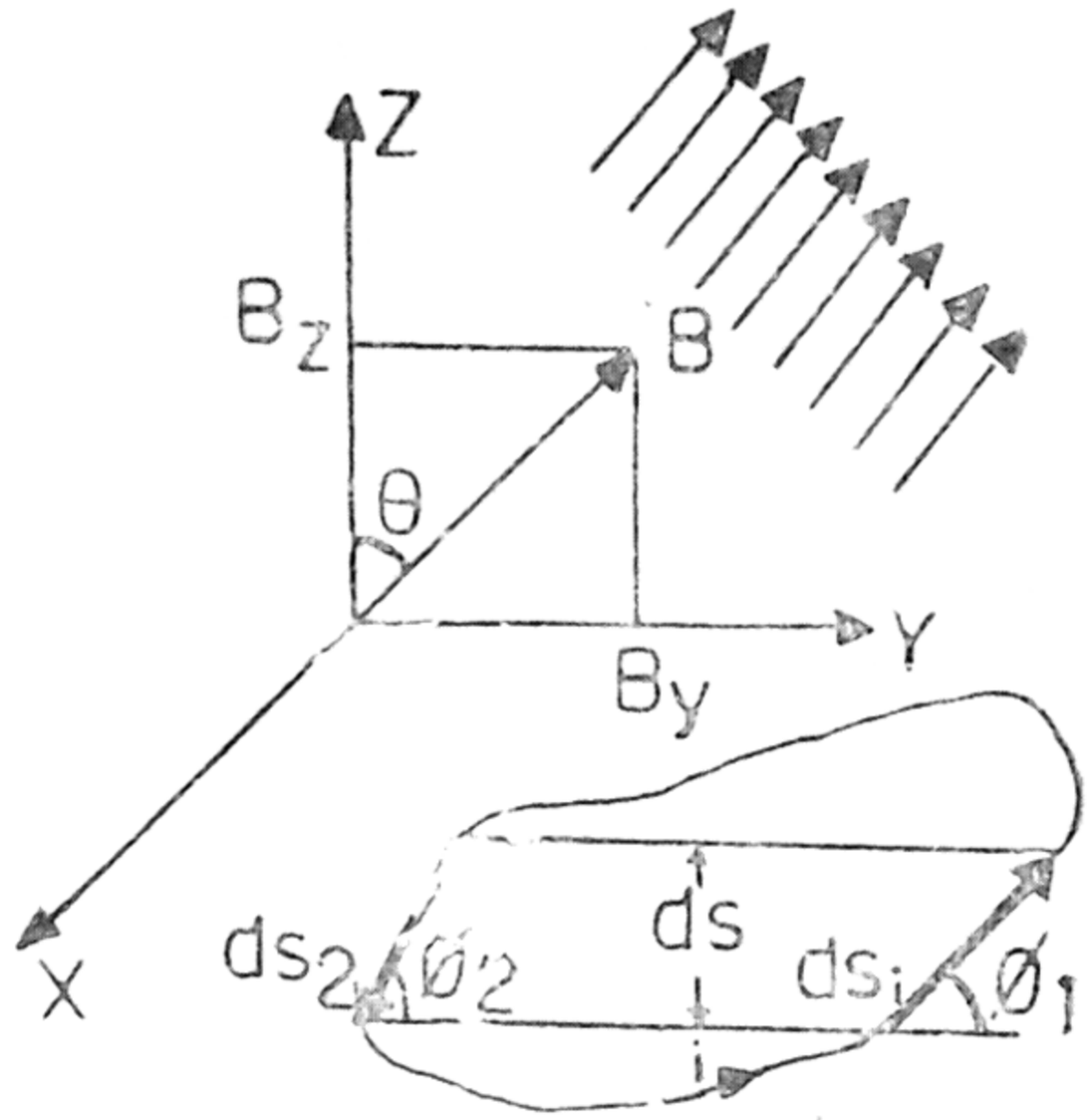


Fig. 8.48.

As elements dS_1 and dS_2 are very small, hence the lever arm of the forces F_{1y} and F_{2y} is very small and no torque will act due to these forces.

The forces F_{1z} and F_{2z} are equal as $dS_1 \sin \phi_1 = dS_2 \sin \phi_2 = dS$ and directed in the $-z$ and $+z$-directions respectively and hence constitute a couple. The moment of the couple or the torque.

$$d\tau = iB_y\, dS\,(y_1 - y_2) = iB_y \times \text{area of the strip in the loop.}$$

It is along x-axis as area of the strip is having vector direction along z-axis. If we divide the whole loop into large number of elements and consider oppositely placed pairs and find $d\tau$ for each pair, then the total torque acting on the loop

$$\tau = iB_y \text{ (area of the loop } A) = iBA \sin \theta$$

or $$\tau = i(\mathbf{A} \times \mathbf{B}).$$

It is same as obtained for a rectangular loop, hence the statement is correct.

Example 13. *A long straight vertical wire carries a current of 30 amperes directed upwards, Calculate the position of neutral point, if the horizontal component of earth's magnetic field is* 2×10^{-5} *web/m*^{2}.

We know that the magnetic field **B** due to a long straight wire carrying a current i at a distance r from the wire is given as

$$B = \mu_0 i / 2\pi r.$$

The direction of this field is perpendicular to the plane consisted of the wire and the distance vector **r**. Thus for a vertical wire, the field will be in horizontal direction. According to this problem, this is balanced by the horizontal component of earth's magnetic field B_H, therefore

$$B_H = \mu_0 i/2\pi r \qquad \text{or} \qquad r = \mu_0 i/2\pi B_H.$$

Substituting the numerical values, we get

$$r = 4\pi \times 10^{-7} \times 30/2\pi \times 2 \times 10^{-5}$$
$$= 0.30 \text{ meter} = 30 \text{ cm}.$$

Examples 14. *In the Bohr model of the hydrogen atom the electron circulates around the nucleus in a path of radius* 5.1×10^{-11} *metre at a frequency* ν *of* 6.8×10^{15} *rev./sec. (i) What value of B is set up at the centre of the orbit ? (ii) What is the equivalent magnetic dipole moment ?*

We know that the current is defined as the rate of flow of charge, therefore in terms of frequency ν current

$$i = e\nu = 1.6 \times 10^{-19} \times 6.8 \times 10^{15}$$
$$= 1.09 \times 10^{-3} \text{ amp.}$$

The electron circulating around the nucleus is equivalent to a current carrying circular loop, where the loop radius is equal to the radius of the Bohr orbit. Thus the magnetic field B at the centre of the orbit is given by

$$B = \frac{\mu_0 i a^2}{2(x^2 + a^2)^{3/2}} = \frac{\mu_0 i}{2a} \qquad (x=0)$$
$$= 4\pi \times 10^{-7} \times 1.09 \times 10^{-3}/2 \times 5.1 \times 10^{-11}$$
$$= 13.4 \text{ weber/meter}^2.$$

The current carrying circular loop is equivalent to a magnetic dipole, with magnetic dipole moment

$$p_m = NiA = 1 \times 1.09 \times 10^{-3} \times 3.14 \times (5.1 \times 10^{-11})^2$$
$$= 8.9 \times 10^{-24} \text{ amp. meter}^2.$$

Example 15. *The flat dielectric disc of radius a carries an excess charge on its surface of* σ *coul/m*^{2}. *Consider the disc to rotate around the axis passing through its centre and* $\perp$ *to its plane with angular speed* ω *rad/sec. If a magnetic field* **B** *is directed* $\perp$ *to the rotation axis, show that the torque which acts on the disc is* $\sigma a \pi B a^4/4$.

Let the disc be placed in the x-y plane with the magnetic field along y-axis and is rotated about the z-axis. Consider an annular ring of radius r and of thickness dr on this disc. Charge with in this ring

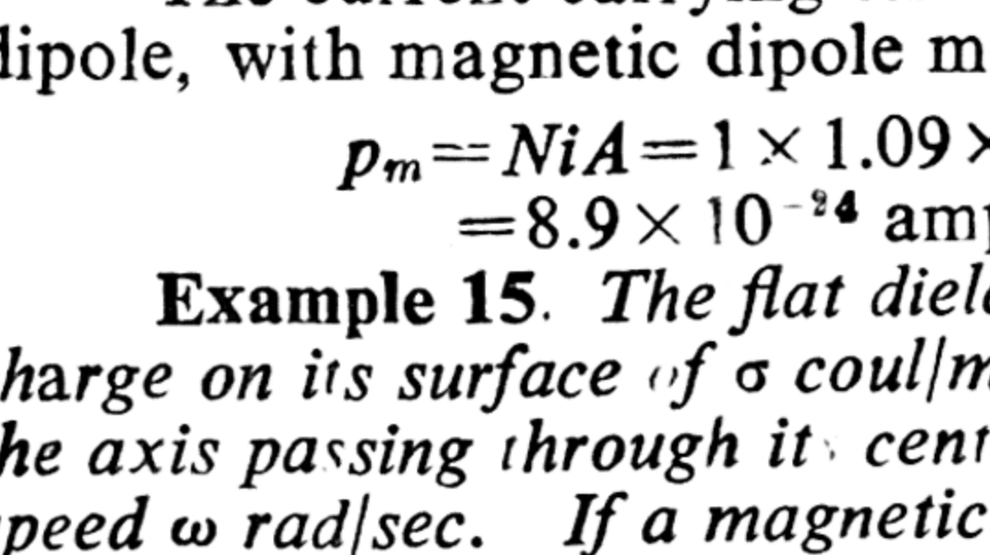


Fig. 8.49.

$$= \sigma 2\pi r dr.$$

As ring rotates with angular velocity ω, the current

$$i = dq/dt = \sigma 2\pi r dr(\omega/2\pi)$$
$$= \sigma \omega r dr.$$

As the torque acting on the current loop of area **A** placed in a magnetic field **B** is

$$\tau = i\mathbf{A} \times \mathbf{B}.$$

Hence the torque on this annular ring $d\tau = \sigma\omega r dr \pi r^2 B$

and $$\tau = \int_0^a \sigma\pi\omega B r^3 dr = \frac{\sigma\pi\omega B a^4}{4}.$$

It will be along x-direction.

Example 16. *A wire shaped to a regular hexagon of side 2 cm carries a current of 2 amp. Find the magnetic induction at the centre of the hexagon.*

Let each side of the hexagon is b and the corners are at a distance a from the centre. The magnetic field at O due to element AB is thus given by the relation

$$\mathbf{B} = \frac{\mu_0 i}{4\pi} \sum \frac{d\mathbf{l} \times \mathbf{r}}{r^3}.$$

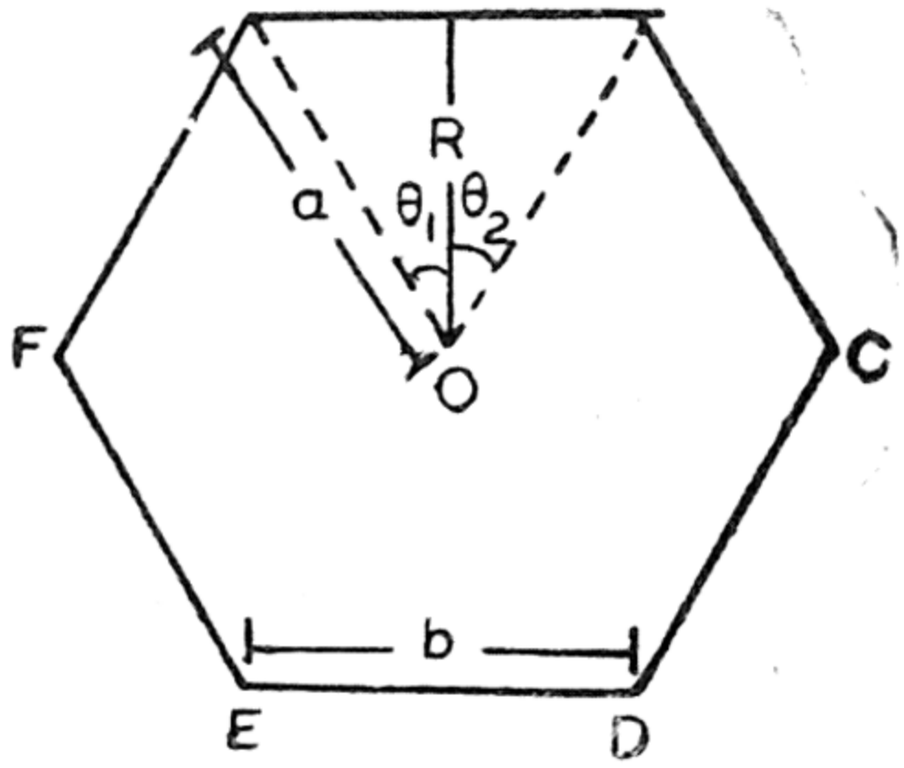


Fig. 8.50.

It is clear from the vector product that **B** points away from the reader in the direction normal to the plane of paper. It is the case for all six elements. Hence resultant field will be 6 times the magnetic field due to single current element. Therefore, the total field

$$B = 6 \frac{\mu_0 i}{4\pi} \sum \frac{dl \sin\theta}{r^2} = 6 \frac{\mu_0 i}{4\pi R} (\sin\theta_1 + \sin\theta_2).$$

Here $\theta_1 = \theta_2 = 30°$, $R = GO = a\sqrt{3}/2$.

$$\therefore \quad B = 6 \frac{\mu_0 i}{4\pi a\sqrt{3}/2}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{\sqrt{3}\mu_0 i}{\pi a}$$

$$= \frac{1.73 \times 4\pi \times 10^{-7} \times 2}{\pi \times 2 \times 10^{-2}} = 6.93 \times 10^{-5} \text{ wb/m}^2.$$

Example 17. *A wire carrying a current i is in the shape of a plane curve $r = f(\theta)$, (r, θ) being polar coordinates. Find the magnetic field at the pole. Deduce also the expression for the field at the focus of an elliptical wire carrying a current i.*

We know that

$$\mathbf{B} = \frac{\mu_0}{4\pi} i \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3} = \frac{\mu_0}{4\pi} i \int \frac{dl \sin\phi}{r^2} \wedge n$$

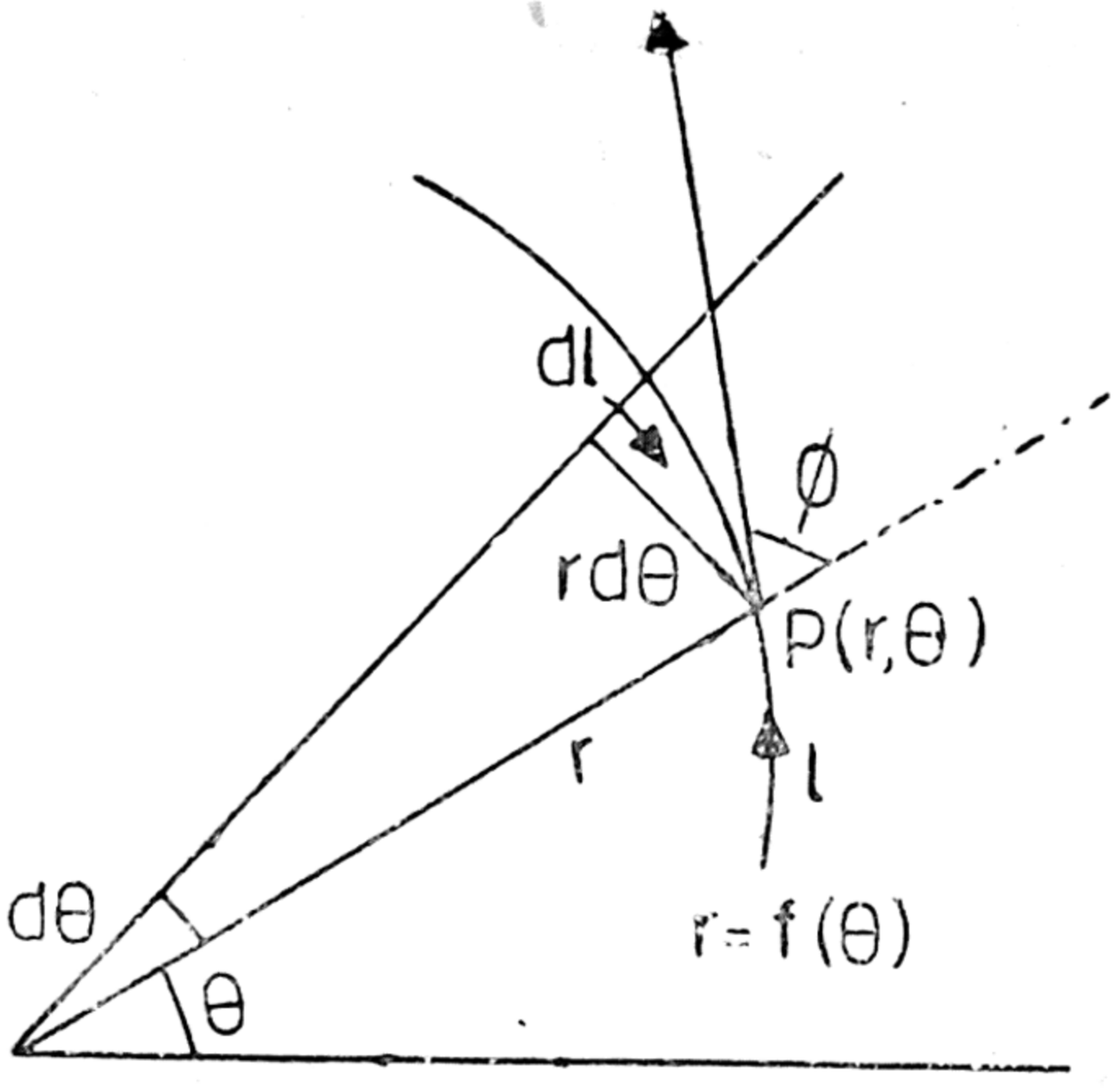


Fig. 8.51.

where ϕ is the angle between $d\mathbf{l}$ and $\mathbf{r}$.

As $\sin\phi = r d\theta/dl$.

$$B = \frac{\mu_0}{4\pi} i \int \frac{r d\theta}{r^2} = \frac{\mu_0}{4\pi} \int \frac{d\theta}{r}.$$

The integration being extended over the length of the wire.

In the second part of question current carrying wire is in the elliptical form, hence

$r = l/(1 + e\cos\theta)$.

$$\therefore \quad B = \frac{\mu_0 i}{4\pi l} \int_0^{2\pi} (1 + e\cos\theta)\, d\theta = \frac{\mu_0 i}{2l}.$$

Example 18. *Find the magnetic field intensity at a point P, which is at a distance b from the centre of a circular conductor of radius a and carrying a current i.*

By Biot-Savart law we have

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3}.$$

But the circle is a plane curve, hence by the result of example (16) we have

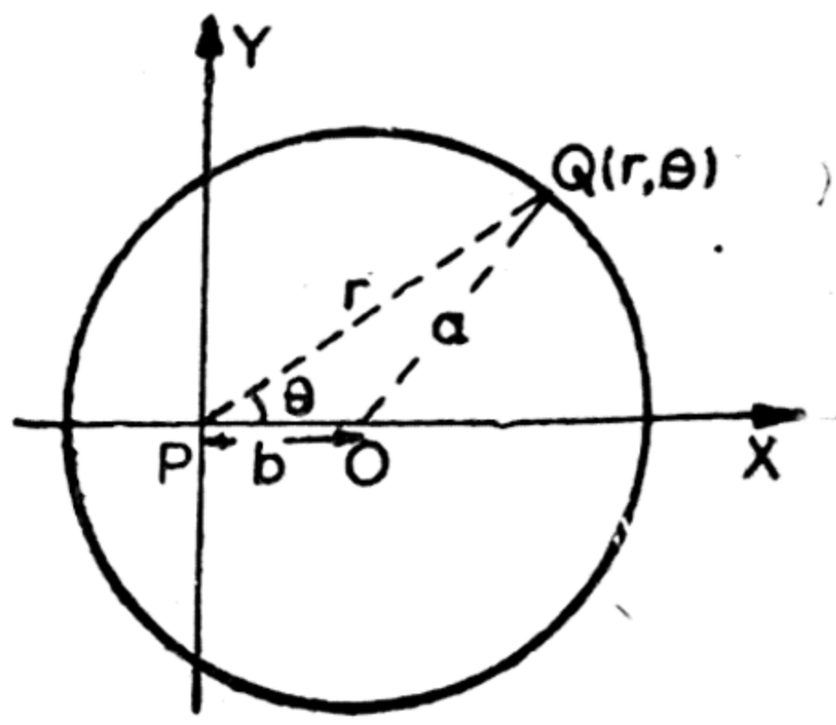


Fig. 8.52.

$$B = \frac{\mu_0}{4\pi} i \int_0^{2\pi} \frac{d\theta}{r}$$

Here point P is the origin. PO is along x-axis and Q is the point with polar coordinates r, θ on the circle. Thus from geometry of figure, we have

$$a^2 = r^2 + b^2 - 2br\cos\theta \quad \text{or} \quad r^2 - 2br\cos\theta + b^2 - a^2 = 0.$$

$$\therefore \quad r = b\cos\theta + \sqrt{(b^2\cos^2\theta - b^2 + a^2)}$$

or
$$\frac{1}{r} = \frac{1}{b\cos\theta + \sqrt{(a^2 - b^2\sin^2\theta)}} = \frac{\sqrt{(a^2 - b^2\sin^2\theta)} - b\cos\theta}{a^2 - b^2}$$

$$\therefore \quad B = \frac{\mu_0}{4\pi} i \int_0^{2\pi} \frac{\sqrt{(a^2 - b^2\sin^2\theta)} - b\cos\theta}{a^2 - b^2} d\theta$$

$$= \frac{\mu_0}{4\pi} \frac{a}{(a^2 - b^2)} i \int_0^{2\pi} \left(1 - \frac{b^2\sin^2\theta}{a^2}\right)^{1/2} d\theta = \frac{\mu_0}{4\pi} \frac{a}{a^2 - b^2} i f_e\left(\frac{b}{a}, 2\pi\right),$$

where f_e is an elliptic function.

Example 19. *Two str aight wires carry currents i_1 and i_2 enter into the page and are at a distance d apart. Find magnetic field at any point R and the force between these wires.*

The fields $\mathbf{B}_1$ and $\mathbf{B}_2$ due to the wires P and Q are in the plane of the paper as wires are $\perp$ to it and are $\perp$ to the radius vectors r_1 and r_2 respectively.

$$B_1=\mu_0 i_1/2\pi r_1$$

and $B_2=\mu_0 i_2/2\pi r_2$.

Taking x and y-components, we get

$$B_x=B_{1x}+B_{2x}$$

$$=\frac{\mu_0}{2\pi}\left(\frac{i_1}{r_1}\sin\theta_1+\frac{i_2}{r_2}\sin\theta_2\right)$$

and $B_y=B_{1y}+B_{2y}$

$$=\frac{\mu_0}{2\pi}\left(-\frac{i_1}{r_1}\cos\theta_1+\frac{i_2}{r_2}\cos\theta_2\right)$$

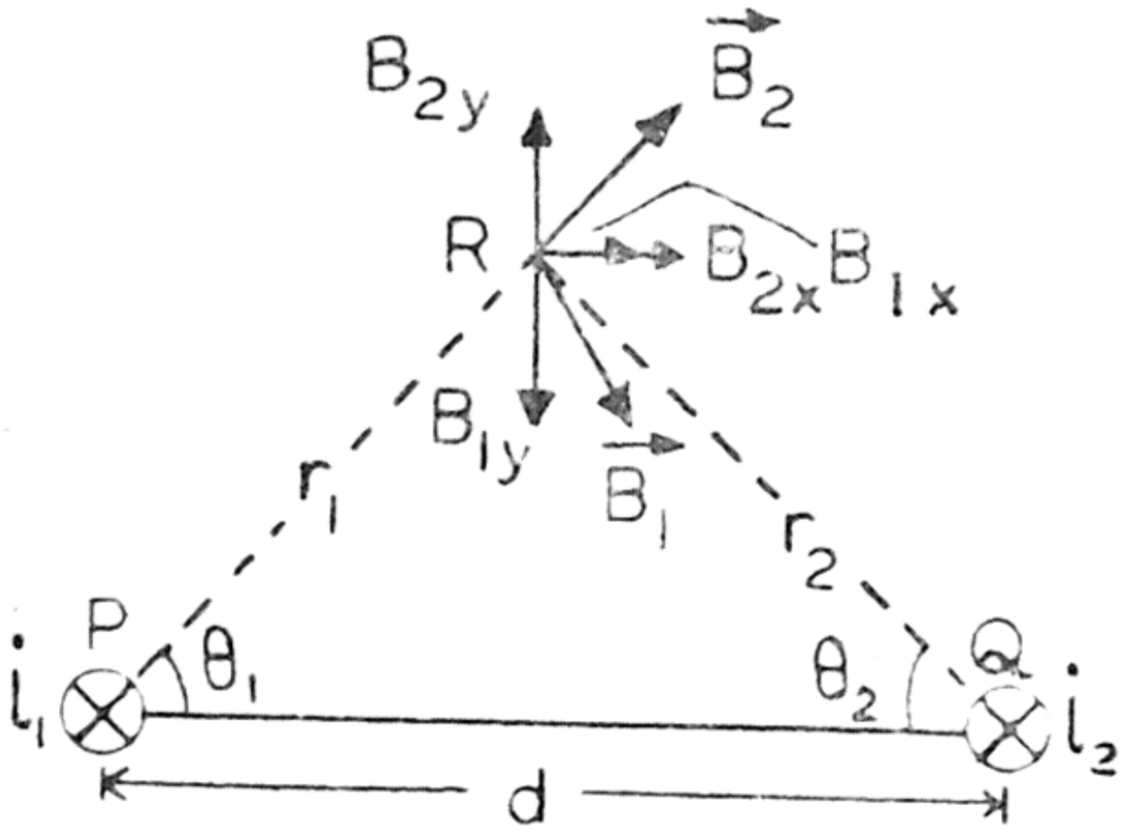


Fig. 8.53.

The resultant magnetic field will be of magnitude $B=\sqrt{(B_x^2+B_y^2)}$ and making an angle θ, with the x-axis, given by the relation $\theta=\tan^{-1} B_y/B_x$.

$$B=\frac{\mu_0}{2\pi}\left[\frac{i_1^2}{r_1^2}+\frac{i_2^2}{r_2^2}-\frac{2i_1 i_2}{r_1 r_2}\cos(\theta_1+\theta_2)\right]^{1/2}$$

The magnetic field at Q due to the current in wire P is

$$B=\mu_0 i_1/2\pi d.$$

As $\mathbf{B}$ is $\perp$ to wire Q, the force on a length L of wire Q is

$$\mathbf{F}=i_2\mathbf{L}\times\mathbf{B} \quad \text{or} \quad F=i_2 L\mu_0 i_1/2\pi d,$$

or the force per unit length $F/L=\mu_0 i_1 i_2/2\pi d$.

Example 20. *Two short solenoids, each of radius 6 cm, and length 13 cm, uniformly wound with 30 turns of wire per cm, are placed coaxially with a gap of 6 cm between them. If these are connected in series. Find the magnetic field at the centre of the gap due to the current of 1 ampere through the combination.*

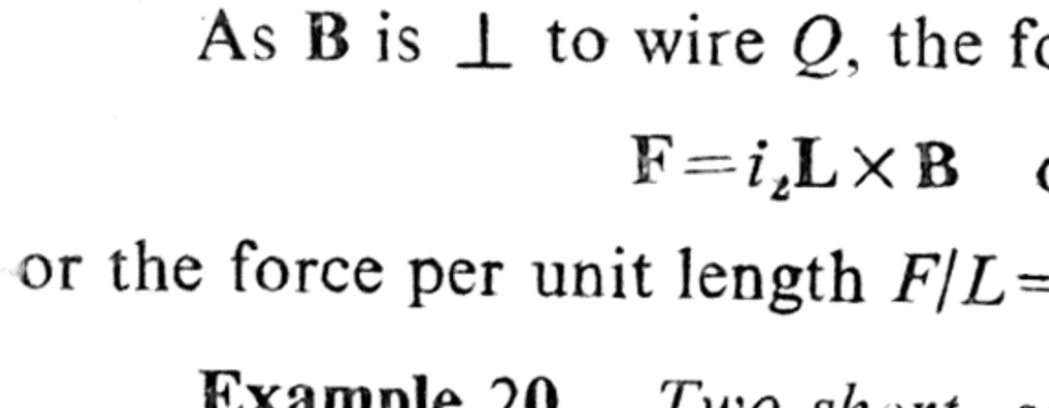


Fig. 8.54.

Magnetic field at any point P on the axis of the solenoid is given as

$$B_1=\tfrac{1}{2}\mu_0 ni\,(\cos\theta_1-\cos\theta_2)$$

The field is in the direction along the symmetry axis. The field will be same due to both the solenoids as they are identical, have been

placed symmetrically and the equal current is passing through them

$$\therefore \quad B=B_1+B_2=2B_1=\mu_0 ni\,(\cos\theta_1-\cos\theta_2)$$

$$=4\times 3.14\times 10^{-7}\times 3000\times 1\left[\frac{16}{\sqrt{(16^2+6^2)}}-\frac{3}{\sqrt{(3^2+6^2)}}\right]$$

$$=1.84\times 10^{-3}\ \text{web/m}^2.$$

Example 21. *A thin dielectric rod is bent into the form of a circle of radius a. If charge per unit length on the rod is λ and the circle is rotated round its axis at a speed of n rotations per second, find the value of* **B** *at point P on the axis at a distance b from the centre.*

The moving charge caused by the rotation of the circle constitutes the current. By the definition of current, the *flow of charge per sec through any area of cross section*

$$i=(2\pi an)\lambda.$$

This problem reduces to : **B** at a point on the axis of a circular coil carrying current i. We know from article 8.11 (b) that

$$B=\frac{\mu_0}{2}\frac{i\,a^2}{(a^2+b^2)^{3/2}}=\frac{\mu_0(2\pi an\lambda)a^2}{2(a^2+b^2)^{3/2}}$$

$$=\mu_0\pi a^3 n\lambda/(a^2+b^2)^{3/2}.$$

Example 22. *Derive an expression for* **B** *at a distance x from the axis of a long cylindrical wire of radius a and carrying a current* i_0.

The cross section of the wire, $\perp$ to its length, is shown in Fig. 8.55. Assuming that the current i_o is distributed uniformly over the cross section of the wire. Now consider a circular path of radius x round the axis of wire. By symmetry, we see that the **B** at any point over this path is tangential to it everywhere. Thus by Ampere's law

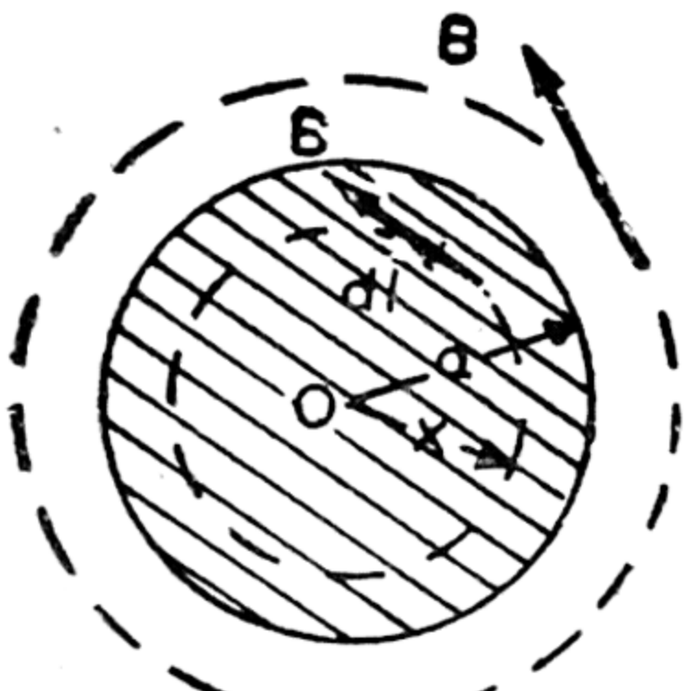


Fig. 8.55.

$$\oint \mathbf{B}.d\mathbf{l}=\mu_0 i$$

or $\oint B\,dl=B2\pi x=\mu_0$ (current enclosed by the path)

There are three possible cases—

(i) $x>a$, current i=total current i_0 passing through the whole wire.

$$\therefore \quad B2\pi x=\mu_0 i_0 \quad \text{or} \quad B=\mu_0 i_0/2\pi x.$$

(ii) $x<a$. The fraction of the current that passes through the enclosed path contributes to **B**.

$$\oint B\,dl=B2\pi x=\mu_0 i=\mu_0(i_0/\pi a^2)\pi x^2$$

or $$B=\mu_0 i_0 x/2\pi a^2.$$

(*iii*) $x=a$, $B=\mu_0 i_0 a/2\pi a^2=\mu_0 i_0/2\pi a$.

Example 23. *A long cable is constructed of a solid conductor of radius r in a coaxial hollow cylindrical conductor of inner and outer radii R_1 and R_2 respectively. The i_0 ampere current flows in the inner and outer conductors in opposite directions. Find the field induction* **B** *at any point at a distance x from the axis of the cable.*

Cross sectional view of the cable is shown in Fig. 8.56. Consider a circular path of radius x around the axis of the cable. Sym-

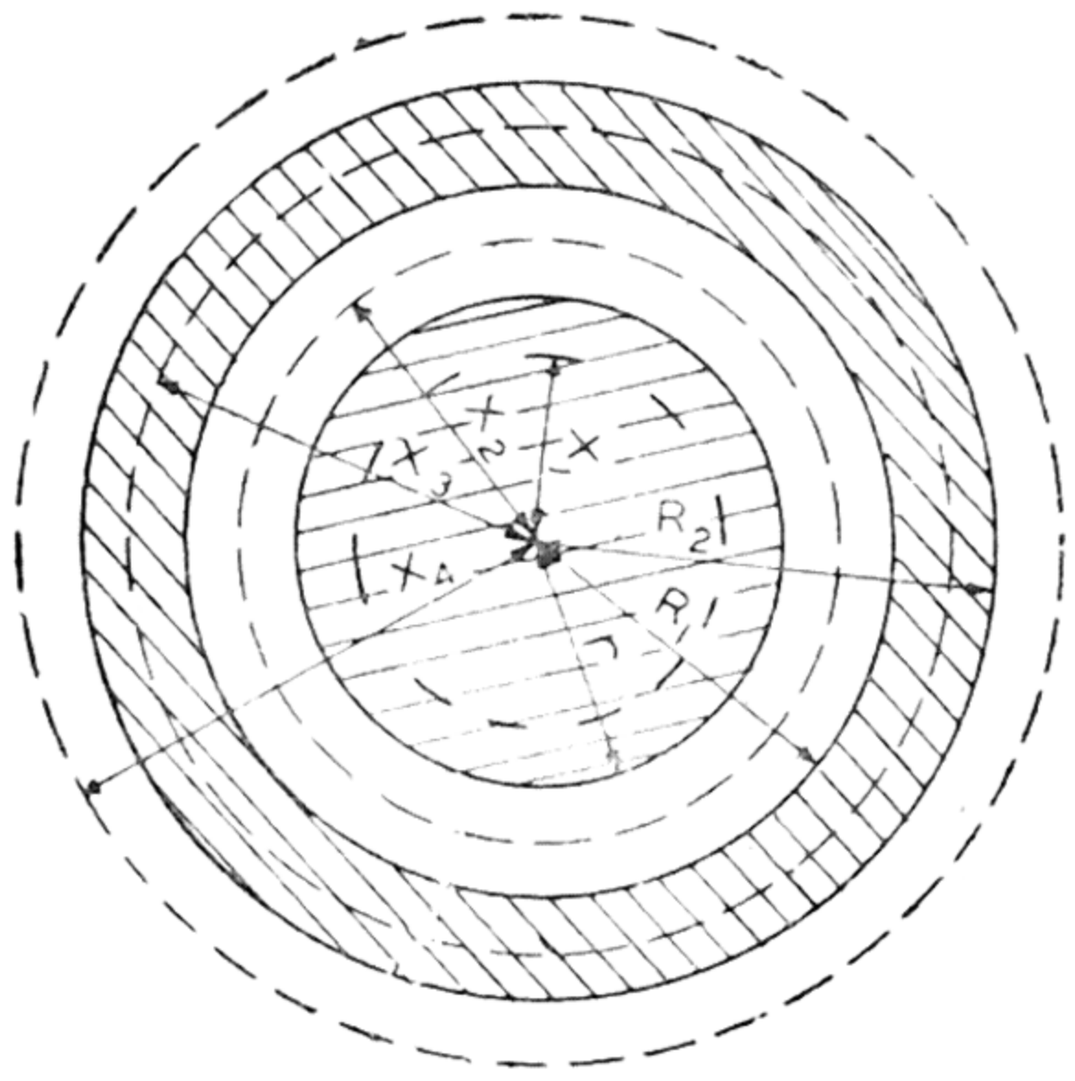


Fig. 8.56.

metry suggests that the magnetic field **B**, produced by the current flow, at any point over this path is tangential to it. *By* Ampere's law, we have

$$\int \mathbf{B}.d\mathbf{l}=\mu_0\times(\text{current enclosed by the path})$$

Four cases are possible—

(*i*) $x<r$:—In this case

current i=current enclosed by the path of radius x_1

$\therefore$ $$\int \mathbf{B}.d\mathbf{l}=\int Bdl=B2\pi x_1=\mu_0(i_0/\pi r^2)\pi x_1^2$$

or $$B=\mu_0 i_0 x_1/2\pi r^2.$$

(*ii*) $r<x<R_1$:—i=current enclosed by the path of radius x_2.

$\therefore$ $$\int \mathbf{B}.d\mathbf{l}=\int Bdl=B\,2\pi x_2=\mu_0 i_0$$

or $$B=\mu_0 i_0/2\pi x_2.$$

(*iii*) $R_1<x<R_2$:—In this case current enclosed by the path of radius x_3 is the difference of the current i_0 through the solid conductor and the current enclosed between R_1 and x_3, *i.e.*

$$i=i_0-\frac{i_0}{\pi[R_2^2-R_1^2]}\cdot\pi(x_3^2-R_1^2)=i_0\left[\frac{R_2^2-x_3^2}{R_2^2-R_1^2}\right].$$

$\therefore$ $$\int \mathbf{B}.d\mathbf{l}=B\,2\pi x_3=\mu_0 i_0(R_2^2-x_3^2)/(R_2^2-R_1^2)$$

or $$B=\mu_0 i_0(R_2^2-x_3^2)/2\pi x_3(R_2^2-R_1^2).$$

(*iv*) $x > R_2$:—i=current enclosed by the path of radius x_4

$$\therefore \qquad \int \mathbf{B}.d\mathbf{l} = \int B dl = B 2\pi x_4 = \mu_0 . 0$$

or $$B = 0.$$

Example 24. *A long horizontal rigidly supported wire carries a current i_a of 100 amperes. Directly above it and parallel to it, there is a fine wire that carries a current i_b of 20 amperes and weighs 0.073 newton per meter. How far above the lower wire should the second wire be kept if we wish to support it by magnetic repulsion.*

We know that the force between the two parallel current carrying wires is attractive if the currents are in the same direction and is repulsive if these are in the opposite directions. In the present problem, the repulsive force is balanced by the weight of the lower wire. Thus the magnetic repulsive force per unit length.

$$\frac{F}{l} = \frac{\mu_0}{2\pi} \frac{i_1 i_2}{R} = \frac{4\pi \times 10^{-7}}{2\pi} \times \frac{100 \times 20}{R}$$

$$= 0.073$$

or $$R = 5.48 \times 10^{-3} \text{ meter} = 5.48 \text{ mm}.$$

Example 25. *Copper wire is wound around a cylindrical solenoid of 6 cm diameter and 30 cm length. There are 5 turns per cm. The wire has a diameter of 0.163 cm and has a resistance of 0.01 ohm per metre. If this solenoid is connected across a 24 volt battery. What will be the magnetic field strength within the solenoid?*

The solenoid is long hence the magnetic field strength within the solenoid is given by

$$B = \mu_0 n i \text{ weber/meter}^2,$$

where n is the number of turns per meter and i is the current in amperes flowing in the solenoid of radius r and length L.

$$n = 5 \times 100 = 500 \text{ turnes/meter}$$

Total length l of the wire $= 2\pi r n L$

$$= 2\pi \times 0.03 \times 500 \times 0.3 = 9\pi$$

and the resistence R of this wire $= 9\pi \times 0.01 = 0.09\pi$ ohm.

$\therefore$ Current $i = V/R = 24/0.09\pi = 800/3\pi$ amp.

Substituting numerical values in the above relation, we get

$B = (4\pi \times 10^{-7}) \times 500 \times (800/3\pi) = 5.33 \times 10^{-2}$ weber/m^2.

Example. 26. *A capacitor charged upto 2 volts is discharged through a ballistic galvanometer having time period of 12 seconds and current sensitivity 2.2×10^{-8} amp/cm. If the first and eleventh throws of galvanometer are 9.6 cm and 8.0 cm respectively, calculate the capacitance of capacitor.*

The deflection θ in the ballistic galvanometer due to a charge q is given by the relation

$$q = \frac{T}{2\pi} \frac{c}{NAB} \theta(1 + \lambda/2)$$

where the logrithmic decrement λ is given as

$$\lambda=\frac{2.3026}{10}\log_{10}\frac{\theta_1}{\theta_{11}}=\frac{2.3026}{10}\log_{10}\frac{9.6}{8.0}=0.018$$

Current sensitivity $=c/NAB=2.2\times10^{-8}$ amp/cm

and the time period $T=12$ seconds.

Substituting numerical values, we get

$$q=\frac{12}{2\times3.14}\times2.2\times10^{-8}\times9.6\left(1+\frac{0.018}{2}\right)$$

$$=4.072\times10^{-7}\text{ coulomb.}$$

Since $q=CV$, hence the capacitance of capacitor

$$C=q/V=4.072\times10^{-7}/2=2.036\times10^{-7}\text{ farad.}$$

Oral Questions :—

1. In the relation $\mathbf{F}=q\mathbf{v}\times\mathbf{B}$, which pairs (**F**, **v** ; **F**, **B** ; **v**, **B**) are always perpendicular to each other ?

2. If a beam of electrons travels in a straight line in a certain region. Can we say that there is no electric field in the region, or no magnetic field?

3. A beam of electrons is deflected side ways in passing through a certain region. Is it due to (*a*) magnetic field, (*b*) electric field or (*c*) both of these ?

4. What type of motion do you expect, when the proton moves (*a*) along **B**, (*b*) $\perp$ to **B** or (*c*) along a line at an angle of $\theta°$ to the direction of **B** ?

5. What would be the transverse Hall field, if the charge carriers are electrons and holes of equal charge density ?

6. Why do you require electric and magnetic fields in the cyclotron ?

7. What is the value of the net force acting on a current carrying (*a*) rectangular or (*b*) circular loop placed in a uniform magnetic field ? What do you expect about the torque in each case ?

8. Does the work required to turn a current loop end for end in an external field constant and independent of the original orientation of the loop ?

9. Compare Coulomb's law and Biot-Savart law.

10. Why do you use Helmholtz golvanometer in place of tangent galvanometer ?

11. What determines which method one should use in calculating **B**, Ampere's law or the Biot-Savart law ?

12. What do you mean by the relations : curl $\mathbf{B}\neq0$ and div $\mathbf{B}=0$?

13. In electronics, the wires carrying equal and opposite currents are often twisted together to reduce magnetic field effects. What is the basis for this procedure.

14. How do you distinguish over-damped, critically-damped and underdamped oscillations ?

15. How can one use ballistic galvanometer as dead beat galvanometer ?

Problems :—

1. The magnetic induction B in a certain region is 2 wb/m² and along +ve x-direction. If *abcdef* prism is placed in this field in such a way that surface *bcde* is in x-y plane and *abef* in y-z plane, The sides *bc* and *ed* along x-axis and each of length 30 cm., *ab* and *ef* along z-axis and each of length 40 cm. If the height $af=be=cd=30$ cm. Find the magnetic flux across each surface.

2. The television tube is oriented so that the electrons of energy 12 keV move horizontally from south to north. The vertical component of the earth's magnetic field ($B=5.5\times10^{-5}$ wb m²) points down. (*a*) In what direction will beam deflect ? (*b*) What is the acceleration of given electrons ?
[(*a*) *East*, (*b*) 6.3×10^{14} *m/sec*²]

3. A proton, a deuteron and an α-particle have equal kinetic energies. Compare the radii of their paths when a normal magnetic field is applied.
[$r_d=\sqrt{2}r_p=\sqrt{2}r^{\alpha}$]

4. An electron of energy 1000 e*V* describes a circle in a field of magnetic induction 200 gauss. Calculate the radius of the cricle.
[0. 563 cm]

5. An electron is accelerated through a potential difference of 100 *V* and then enters a region where it is moving $\perp$ to a magnetic field B=0.20 wb/m². Find the radius of the circular path. Repeat this problem for a proton.
[*0.0167 cm, 0.7 cm*]

6. A proton travelling at a speed of 10^5 m/sec enters a region of space where there is a uniform magnetic field of 0.1 tesla. Its velocity makes an angle of 30° with the direction of **B**. Describe quantitatively the path of the proton.
[*Helix of radius 0.32 cm and pitch 5.7 cm*]

7. A Cu strip 2.0 cm wide and 1.0 mm thick is placed in a magnetic field $B=1.5$ wb/m². If a current of 200 amp is set up in the strip, find the Hall potential difference across the strip.
[$2.2\times10^{-5}V$]

8. An electron initially travelling with velocity 3×10^7 m/sec enters in an electric field of 18 volts/cm. If the transverse deflection is 2mm in traversing a length of 10 cm. Find the value of e/m. [2×10^{11} coul/kg.]

9. A particle having a mass of 0.5 gm carries a charge of 2.5×10^{-8}C. The particle is given an initial horizontal velocity of 6×10^4 m/sec. Find the magnitude and direction of the minimum magnetic field that will keep the particle moving in a horizontal direction. [*3.27 wb/m², $\perp$ to* **v**]

10. An ion of mass **M** and charge q is accelerated by pd V and allowed to enter a field of magnetc induction **B**. In the field it moves in a semi-circle, striking photographic plate at a distance x from the slit. Show that the mass **M** is given by $M=B^2qx^2/8V$

11. Electric and magnetic fields are applied mutually $\perp$ to each other. Show that a charged particle will follow a straight path which is $\perp$ to both these fields, if its velocity is E/B in magnitude.

12. Calculate the magnetic field in which the cyclotron Dee's should be placed to accelerate protons, the frequency applied being 8×10^6 cycles/sec.
[*0. 524 weber*]

13. Deuterons are accelerated in a fixed frequency cyclotron to a maximum dee orbit radius of 88cm. The magnetic field is 1.4 wb/m². Calculate the energy of the emerging deuteron beam and the frequency of the dee voltage. What change in magnetic flux density is necessary if doubly ionized helium ions are accelerated. [*36.3 MeV, 10.67 Mc/sec. 1.39 wb/m²*]

14. Calculate the magnetic field in which the cyclotron dees should be placed to accelerate protons, the frequency applied being 8×10^6 c/s.
[*0. 524 web/m²*]

15. Calculate the current which, flowing through a Helmholtz galvanometer having coils each of 25 turns and 10 cm radius, produces a deflection of 45^0. The earth's horizontal magnetic intensity is 0.3 oersted.
[*0.18 amp*]

16. A solenoid of length 20 cm and radius 2 cm is closely wound with 200 turns. Calculate the magnetic field intensity at the centre of an end of the solenoid. The current in the winding is 5 amperes.
[3.14×10^{-3} *web/m²*]

17. A current i flows through a wire bent as shown in Fig. 8,57. Calculate the force exerted on the wire, if it is placed in a uniform magnetic field B. The field is acting $\perp$ to the plane of the wire and away from the reader. [$2i\,B(l+R)$]

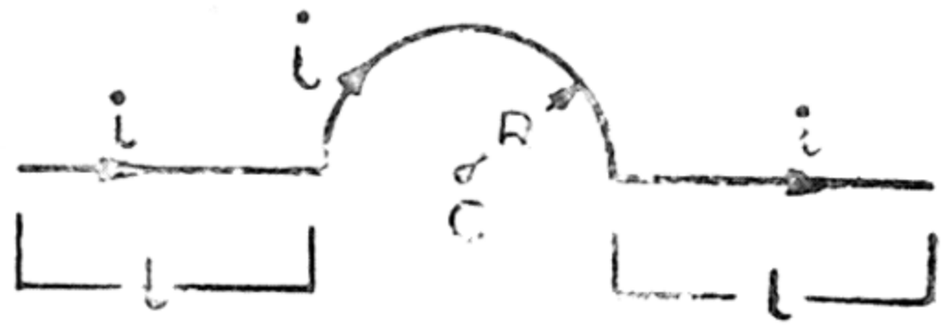


Fig.. 8.57.

18. Find the resultant force in the are AC subtending an angle $3\pi/2$ at the centre carrying a current I and is placed in a uniform field B directed into page.
[*zero*]

19. In the above problem, find the torque on the loop as a result of B. Repeat if the field is in the plane of the page directed from left to right. Find the direction of the torque vector. [*0*, $\frac{3}{4}\pi b^2 iB$ *in y—direction*]

20. The dielectric sphere of radius b spins with angular speed ω about the y-axis. If it carries a uniform surface charge σ coul/m² and a magnetic field **B** exists in the x-direction. Show that the torque acting on it is $\tau = \frac{4}{3}\pi\omega b^4 B$ along z-direction.

21. If a curreat i flows through a circuit in the shape of a regular hexagon of side a. Find the induction at the centre of the hexagon.
[$\sqrt{3}\mu_0 i/\pi a$]

22. Find the expression for the magnetic induction at the centre of an elliptical wire carrying a current i.

[$\mu_0 iL/4A$, *where L is the perimeter and A the area of the ellipse*]

23. Show that the magnetic field at the centre of a square of side $2a$ and carrying a current i is $\sqrt{2}\mu_0 i/a\pi$.

24. Two infinite cylindrical conductors are placed parallel to one another at a distance $2a$ apart. They carry equal and opposite currents. Show that in the equatorial plane the gradient of the magnetic field is greatest at a distance $a/\sqrt{3}$ from the plane through the axes of the cylinders.

25. A solenoid of square cross section has n turns per unit length and carries current i. The cross sectional dimension is a. If the solenoid is very long, find the axial magnetic induction at its centre. [$\mu_0 ni$].

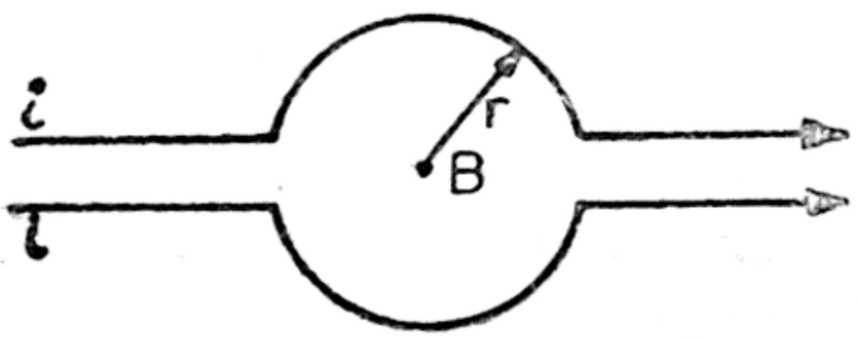


Fig. 8.58

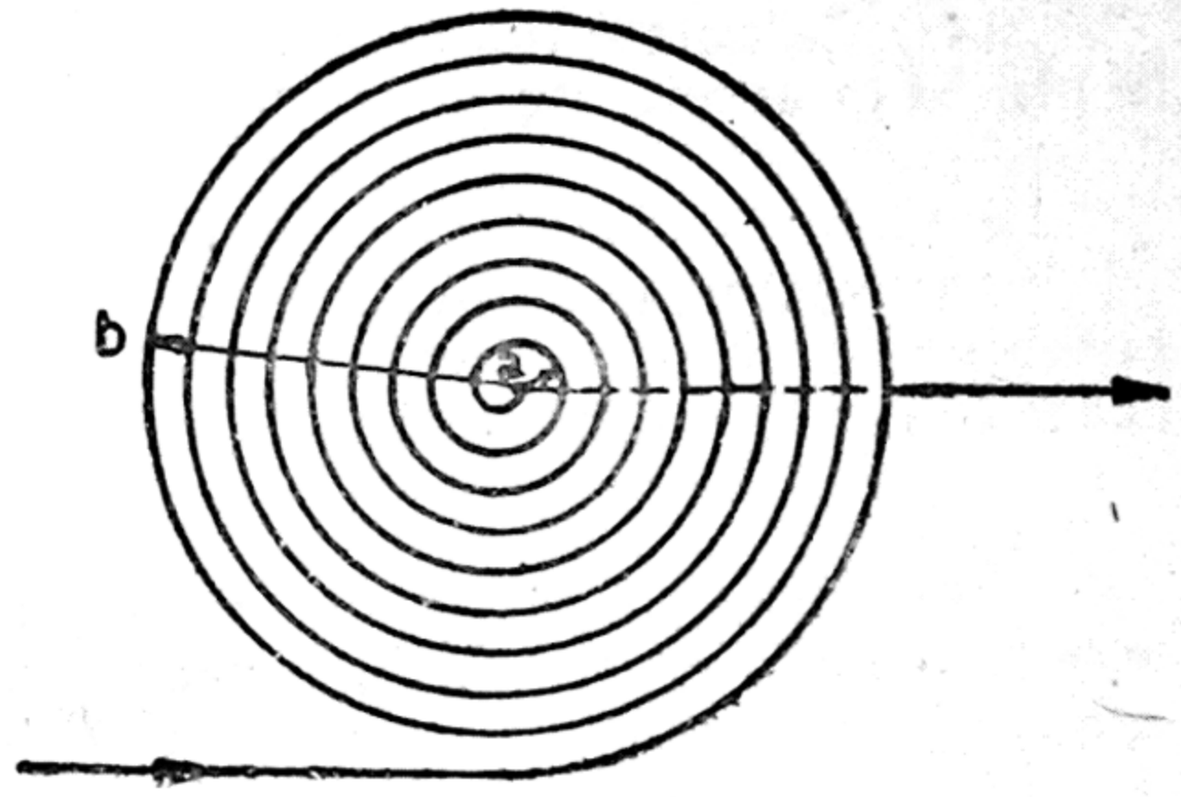


Fig. 8.59

26. A pair of infinitely long thin wires carry equal currents *i*. They are bent as shown in Fig. 8.58. Find the magnetic induction **B** at the centre of the circular parts ?

27. A flat coil is wound so that it contains N number of turns per unit distance along its radius, Prove that the magnetic induction at the centre of such a oil of inside radius *a*, outside radius *b* and carrying a current *i* is $\frac{1}{2}\mu_0 iN \log (b/a)$.

28. Two long straight parallel wires are kept in free space at a distance of 1.0 meter and 1.0 ampre current is flowing in the same direction in erch wire. Calculate the direction and magnitude of the mutual force per meter between the two wires. What will be the difference if the currents flow in mutually opposite directions. [2×10^{-7} *newton*]

29. The resistance of a moving coil ballistic galvanomter is 125 ohms. A steady potential difference of 0.00025 volt deflects the spot of light on the scale through 15 cm. The suspended system has an undemped oscillation of periodic time 10 sec. When a condenser is discharged through the galvanometer, the corrected first throw is 5 cm. Find the quantity of charge discharged. If the condenser was charged to 1 volt, what is its capacity ?
[*1,06 uf*]

9

Electromagnetic Induction

9.1. FARADAY'S EXPERIMENTS

In 1831 Michael Faraday in England performed a series of experiments and discovered that if a closed circuit moved across a magnetic field, a current flowed even though there were no batteries present. **In one of his experiments**, he took one closed circuit consisting of a coil and a galvanometer. When a bar magnet was moved in the neighbourhood of this coil, following observations were obtained.

(*i*) If this magnet is moved towards the coil with its north pole facing the coil, the galvanomoter deflects, the electric current is thus produced in the circuit.

(*ii*) If this magnet is moved away from the coil, the galvanometer again deflects but in opposite direction.

(*iii*) If the south pole end instead of the north pole end of the magnet is towards the coil the deflections are reversed.

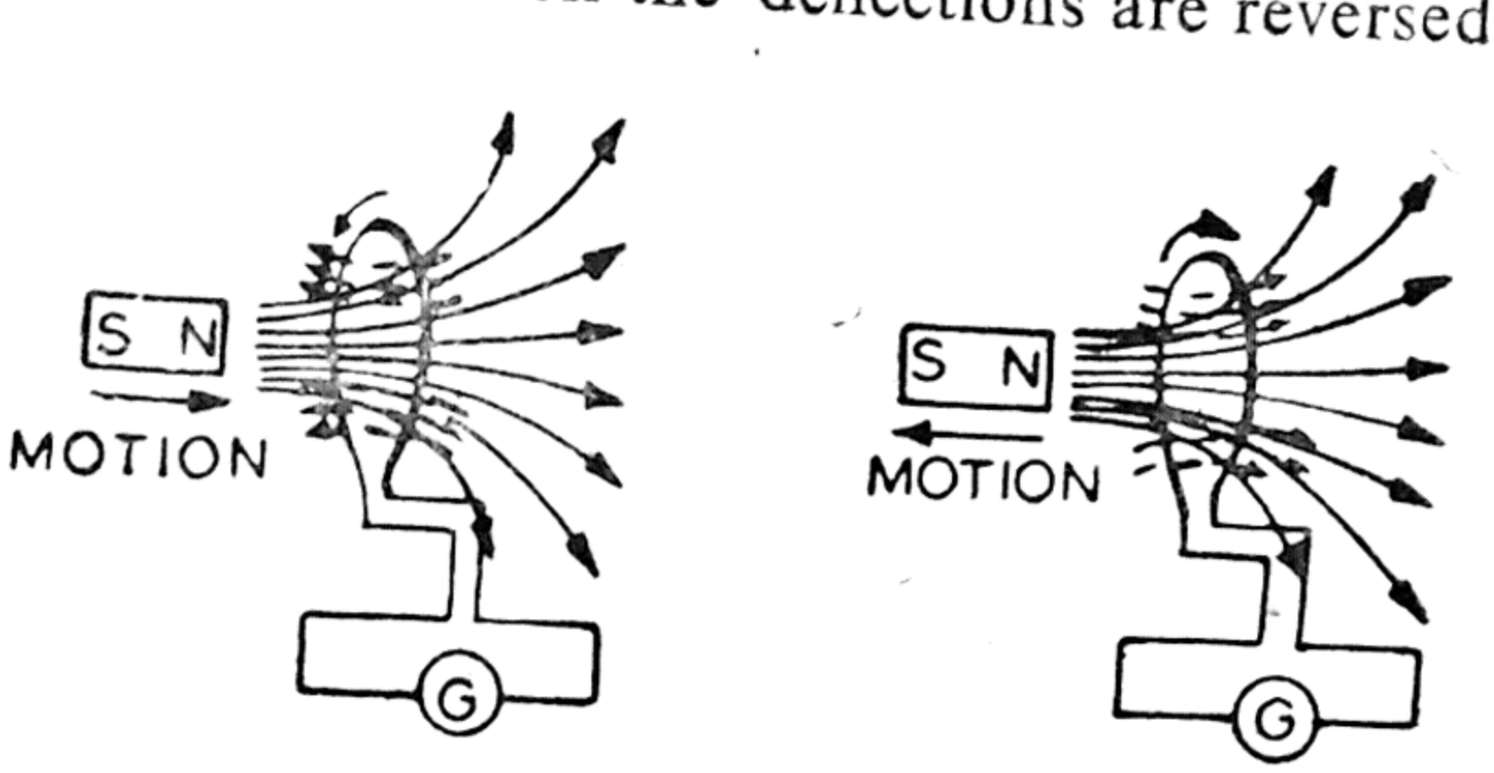


Fig. 9.1.

(*iv*) The galvanometer also shows deflection if the magnet is kept stationary and the coil is moved.

(*v*) The deflection in galvanometer is found to be proportional to the relative speed of the magnet and the coil.

(*vi*) The galvanometer shows no deflection when the magnet is held stationary with respect to the coil.

In his another experiment, two coils were placed close together, but at rest with respect to each other. One coil was connected to a galvanometer, and the other to a battery. When the battery circuit, known as *primary circuit*, was closed, there was a sudden deflection in the galvanometer circuit, known as *secondary circuit.* Following results were observed.

(*i*) When the primary circuit is closed or broken, the galvanometer needle deflects momentarily. The directions of deflection are opposite to each other.

(*ii*) The galvanometer also shows deflection if the current in the primary circuit is varied. *This deflection depends upon the rate of change of current and not the value of the current.*

(*iii*) The current in the secondary circuit is momentarily, and does not exist when the current in the primary is fully estabilished or reduced.

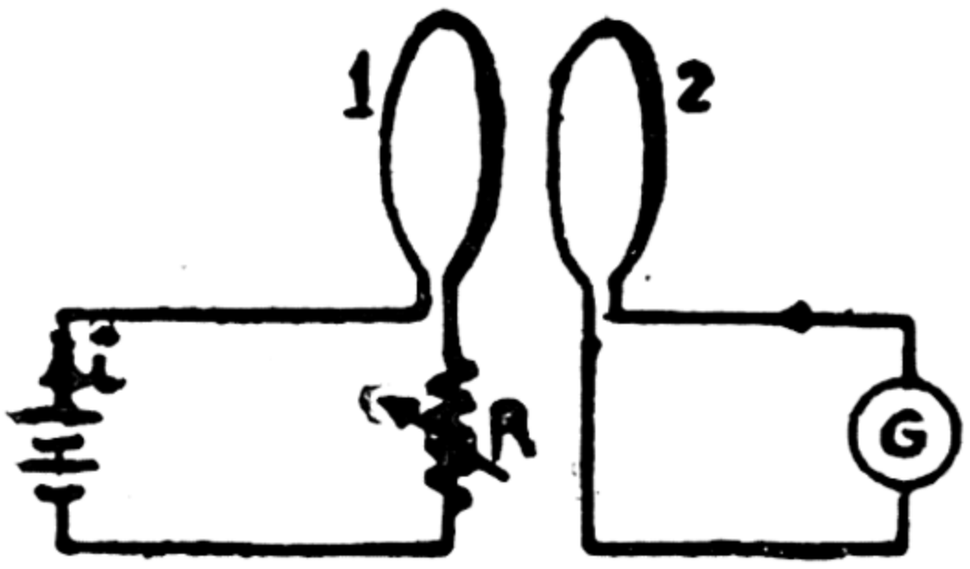


Fig. 9.2

(*iv*) When there is a relative motion between these coils, the deflection also exists.

The deflection in galvanometer in above experiments is due to the current developed in the secondary circuit, this current is called an *induced current* and the emf giving rise to such current is called an *induced electromotive force.* This phenomenon was given the name *electromagnetic inducion* by Faraday. *This is due to the change of magnetic flux through the closed circuit due to the relative motion of the circuit and the magnetic lines of force produced by either the magnet or the primary coil.*

9.2. FARADAY'S LAWS OF INDUCTION

From his experimental results, Faraday gave two laws :

(*i*) *When the flux of magnetic induction through a circuit is changing an electromagnetic force is induced in the circuit.*

(*ii*) *The magnitude of this emf is equal to the negative rate of change of the flux, i.e.,*

$$\mathcal{E} = -d\Phi/dt.$$

This equation is known as *Faraday's law of induction* also known as *Neumann's law.* It is found to be independent of the way in which the flux is changed, the circuit may be moved or destroyed or the value of **B** at various points inside the circuit may be changed. If Φ is the instantaneous value of the magnetic flux, then $\Phi = \int \mathbf{B}.d\mathbf{S}$.

The induced emf $\mathcal{E}$ around the closed path C is given by the line integral of the induced electric field **E** around C, *i.e.*,

$$\mathcal{E} = \oint_C \mathbf{E}.d\mathbf{l}. \qquad \ldots(2)$$

If the circuit C be bounded by an open surface S, then

$$\therefore \quad \mathcal{E} \oint_c \mathbf{E} \cdot dl = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{S}. \qquad \ldots(3)$$

Thus the Faraday's law tells us that *the line integral of the electric intensity around any fixed closed path is equal to the time rate of decrease of flux of magnetic induction through the surface enclosed by the curve.*

In addition to Eq. (3), there are other equivalent statements of Faraday's law. Since $\mathbf{B} = \text{curl } \mathbf{A}$, hence

$$\oint_c \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = -\frac{d}{dt}\oint_c \mathbf{A}.d\mathbf{l} \qquad \ldots(4)$$

by using Stoke's law. This relation shows that the induced emf can be calculated directly from the vector potential $\mathbf{A}$.

The *different form* of the Faraday's law is obtained by using Stoke's law to replace line integral into surface integral,

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_s \mathbf{B} \cdot d\mathbf{S}$$

or

$$\int_s \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) d\mathbf{S} = 0.$$

Since S can be an arbitrary surface hence the integrand must be zero and we have

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t. \qquad \ldots(5)$$

This equation relates the space derivatives of $\mathbf{E}$ at a particular point to the time rate of change of $\mathbf{B}$ at that point, and shows that the electric field induced by $\mathbf{B}$ is not *as the electrostatic field for which the curl is zero.*

9.3. INDUCED ELECTRIC FIELDS

The induced electric field, which is non electrostatic in nature produced in a loop due to the changing magnetic flux is given by

$$\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt. \qquad \ldots(6)$$

It is exactly analogous to the circuital law of Ampere,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i \qquad \ldots(7)$$

Comparison of Eqs. (6) and (7) shows that the time rate of change of flux corresponds to the current i and that the lines of electric force due to changing magnetic induction form closed curve similar to the lines of magnetic force produced by a current ins wire. The equation for $\mathbf{B}$ due to an element $d\mathbf{l}$ carrying current i is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\, i\, \frac{d\mathbf{l} \times \mathbf{r}}{r^3}. \qquad \ldots(8)$$

The corresponding equation for calculating **E** due to an element $d\mathbf{l}$ in which the flux is changing at a rate $d\Phi/dt$ will thus be

$$d\mathbf{E} = -\frac{1}{4\pi}\frac{d\Phi}{dt}\frac{d\mathbf{l}\times\mathbf{r}}{r^3}. \qquad \ldots(9)$$

Hence the *induced electric fields* set up by varying magnetic fields are geometrically identical to the magnetic fields set up by the corresponding steady currents. The lines of force of an induced electric field are like the lines of a **B**-field as they do not originate or terminate on electric charges. Thus the number of lines of force emerging from any closed surface is equal to the number entering the surface and hence for any closed surface

$$\oint \mathbf{E} \cdot d\mathbf{A} = 0, \qquad \ldots(10)$$

which is same as that for the **B**-vector.

E-field can be calculated from Eq. (9), the analog of the Biot law, by exactly the same procedure. It is unnecessary to repeat the analysis. We can directly get **E**—field around the long straight solenoid as

$$E = -(1/2\pi r)\ d\Phi/dt. \qquad \ldots(11)$$

If a wire coincides with any lines of electric force due to changing magnetic induction, the electric field exerts a force on the free electrons in the direction everywhere tangent to the wire. A current is thus produced. The line integral of **E** around such a path gives us an emf.

9.4. LENZ'S LAW

The negative sign in Faraday's law indicates that the direction of the induced emf is such as to tend to oppose the change that produces it. Thus if we increase the flux through a circuit, the induced emf tends to cause currents in such a direction as to decrease the flux. If we thrust one pole of a magnet into a coil, induced emf tends to cause currents in such a direction that the magnetic field produced by it tends to repel the pole. Thus the *direction of the induced emf and of induced current is such as to oppose the cause producing it.* It is known as *Lenz's law.*

The cause of the current may be the motion of the conductor in a magnetic field or the change of magnetic field. In the first case, the direction of the induced current in the moving conductor is such that the force acted upon the conductor is opposite to its motion. In the second case, the induced current sets up a magnetic field which within the area bounded by the circuit is (*a*) in the direction opposite to the original field if this is increasing and (*b*) in the direction of the original field if it is decreasing. Thus we see that *the induced field does not oppose the magnetic field but opposes the change in it.*

Since this law refers to induced currents, hence we must have a closed circuit. If a conductor does not form a closed circuit, we

can usually think in terms of what would happen if it was closed and in this way find the direction of the induced emf.

When a conductor moves in a magnetic field the direction of its motion along with the directions of the magnetic field and the emf is given by the Fleming's right hand rule. It states that *if the thumb and the first two fingers of the right hand are spread out in such a way that they are at right angles to one another. If the index (or fore) finger represents the direction of the magnetic field and the thumb the direction of motion of the conductor, then the central finger will give the direction of the induced emf and thus of the induced current.*

When a magnet is moved before the coil, either towards or away from it, a work is always required. Lenz's law is no violation of the law of conservation of energy. Here energy is converted from one form into another. The mechanical energy produce induced current by electromagnetic induction. *Thus mechanical energy is converted into electrical energy.* Lenz's law may be considered as the electrical analogue of Newton's law of action and reaction or as an electrical application of the general principle of Le Chatellier, which states that *if any state of equilibrium is disturbed a reaction is induced in such a sense as to oppose the change.*

9.5. PROOF OF FARADAY'S LAW

Helmholtz and a little later Kelvin showed that Faraday's law was a direct consequence of the principle of the conservation of energy. Consider a straight conductor PQ which slides on a pair of horizontal rails separated by a distance l and are connected to a source of emf $\mathcal{E}_0$. If the magnetic field induction **B** is perpendicular to the plane of this circuit and in inward direction. Due to field **B** and the current i in the circuit the conductor PQ experiences a force **F** of magnitude iBl to the right. Because of this it slides with a velocity **v** at a particular instant t and thus work is being done at the rate Fv. By the principle of conservation of energy, power supplied by the emf $\mathcal{E}_0$ is equal to the power loss in heat generation + power loss in doing work.

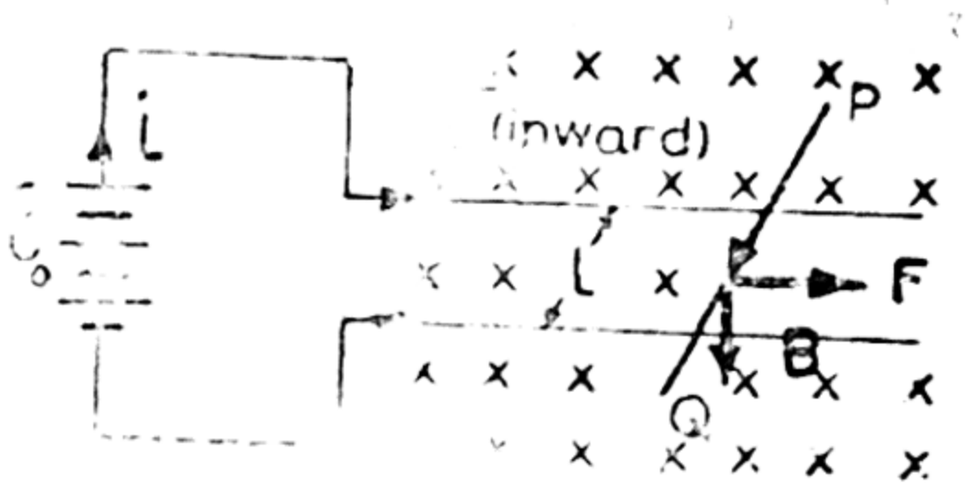


Fig. 9.3.

i.e.,
$$\mathcal{E}_0 i = I^2 R + Fv = i^2 R + iBlv$$
or
$$i = (\mathcal{E}_0 - Blv)/R, \qquad \ldots(12)$$

Thus we see that the current is less than the original value it is due to $-Blv$. the emf induced by the motion of the conductor in a magnetic field **B**. We have assumed here that the

resistance of the closed circuit is so large or current i is so small that it has a negligible effect on the value of **B**. The induced emf can be written as

$$\mathcal{E} = -Blv = -Bl\,(dx/dt) = -B\,(dA/dt) = -d\Phi/dt \quad ...(13)$$

If **v** is arbitrarily oriented with respect to **l**, then the only component of **v** which is $\perp$ to **l** contributes $\mathcal{E}$. Thus $\mathcal{E}$ is proportional to $\mathbf{l} \times \mathbf{v}$. As $\mathcal{E}$ is a scalar quantity, hence

$$\mathcal{E} = \mathbf{B}\,(\mathbf{l} \times \mathbf{v}). \quad ...(14)$$

9.6. MOTIONAL ELECTROMOTIVE FORCE

Consider a conducting rod PQ of length l, parallel to x-axis, in a uniform magnetic field **B**, along z-axis. The rod is moving along y-axis with velocity **v** and a charge q within it is acted on by a force **F**, known as Lorentz force and having value $q\,(\mathbf{v} \times \mathbf{B})$.

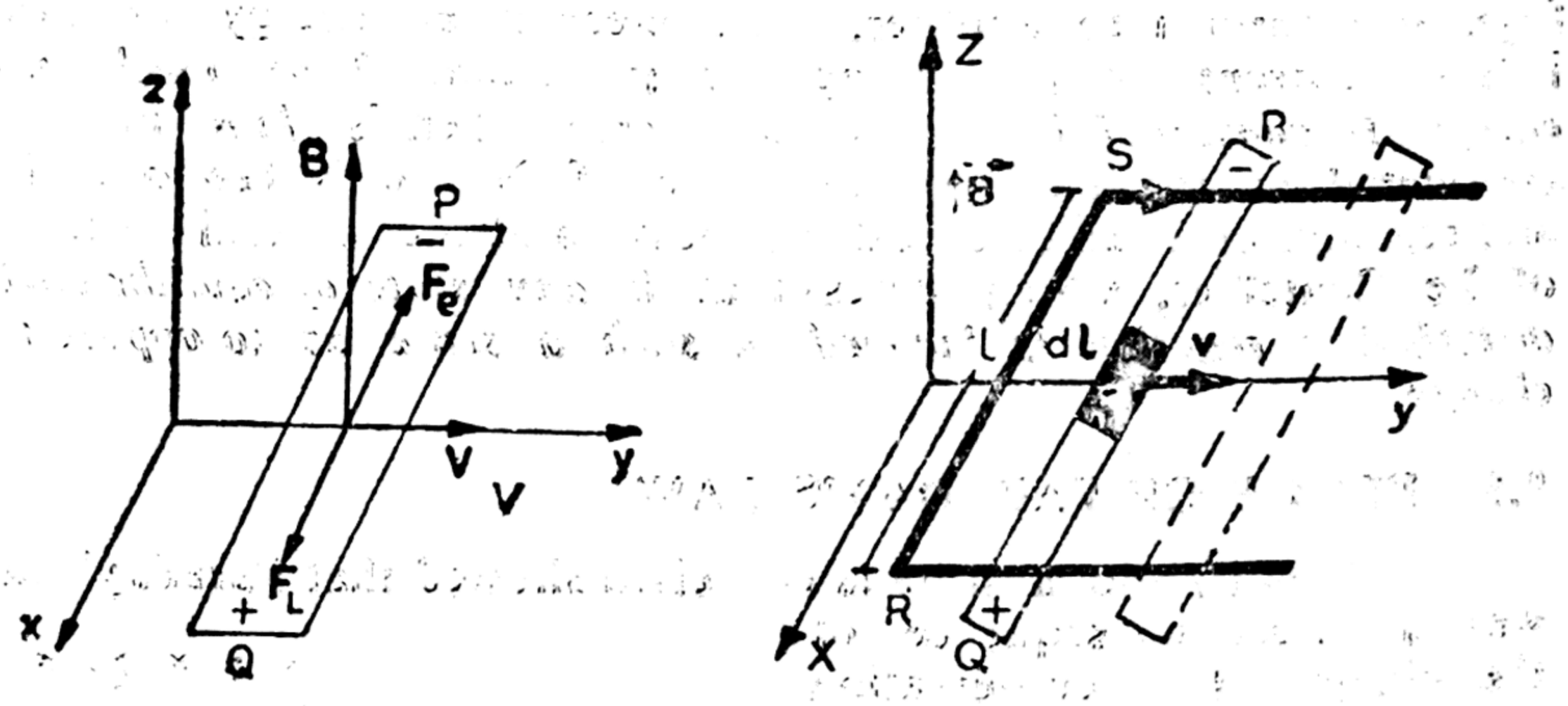


Fig. 9.4.

It is in the direction $\perp$ to the plane containing **v** and **B** hence along x-axis. Thus the free electrons in the rod will move along $-$ve x-axis, thus leaving the end Q positively charged. This accumulation of excess charges at the ends continues till it establishes an electrostatic field $\mathbf{E}_e$ which opposes the displacement of charges due to the Lorentz force. Electric field $\mathbf{E}_e$ (produced due to separated $+$ve and $-$ve charges in the rod) at every point inside the rod thus balances $\mathbf{E}_L$, the field due to Lorentz force **F**.

$$\mathbf{E}_e = -\mathbf{E}_L = -\mathbf{F}/q = -(\mathbf{v} \times \mathbf{B}). \quad ...(15)$$

This is the expression for the electric field produced due to the motion of a conductor in a uniform magnetic field.

Let us now *consider a conducting rod PQ sliding along h stationary U-shaped conductor placed in a uniform magnetic field.* As the U-shaped conductor is stationary, hence the force acting on the free charge carriers of it is zero. But due to the motion of the conducting rod PQ in the magnetic field **B**, the opposite charges inside this rod are separated and are collected at the ends ($+$ve at Q

and —ve at P). As the circuit is closed through RS, the electric current will flow. This flow reduces the strength of the charges at the ends of the rod, but compensated immediately by the further displacement of free electrons caused by the magnetic force $\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$. Hence so long as the rod is in motion, a current flows through the closed circuit $PQRS$. Thus a moving rod in a $\perp$ magnetic field produces some kind of emf. This emf is known as *motional induced emf*, and the current produced by it as *induced current*.

Consider a small element $d\mathbf{l}$ of a moving conducting rod. The motional electromotive force

$$\mathcal{E}=-\int\mathbf{E}_e.d\mathbf{l}=\int\mathbf{E}_L.d\mathbf{l}=\int(\mathbf{v}\times\mathbf{B}).d\mathbf{l}. \qquad ...(16)$$

As $(\mathbf{v}\times\mathbf{B}).d\mathbf{l}=\mathbf{B}.(d\mathbf{l}\times\mathbf{v})$

$$\therefore \quad \mathcal{E}=\int\mathbf{B}.(d\mathbf{l}\times\mathbf{v}). \qquad ...(17)$$

As $d\mathbf{l}\times\mathbf{v}$ is the area swept per unit time by the length dl and hence $\mathbf{B}.(d\mathbf{l}\times\mathbf{v})$ is the flux of induction through this area. Therefore the *motional emf is equal to the flux of induction cut by the conductor per unit time.* If the length is l and $\mathbf{l}$, $\mathbf{v}$ and $\mathbf{B}$ are mutually $\perp$ to each other then

$$\mathcal{E}=Blv. \qquad ...(18)$$

Let us consider few cases :

(*a*) *Circuit moving in a magnetic field.* Motion may be due to the translation or rotation of the circuit as a whole or the circuit may have a movable arm which slides back or forth relative to the remaining part of the circuit. In any case the periphery or its part moves in a magnetic field.

Let A_1 and A_2 be the locations of the curve, moving with velocity $\mathbf{v}$, at the time t and $t+dt$ respectively. If the circuit is taken +ve in counter-clockwise direction, its area should point towards the reader. Thus due to the motion of the circuit, the area $PQRS$ $(=d\mathbf{S})$ is added to the elementary length $d\mathbf{l}$ of this circuit in time dt.

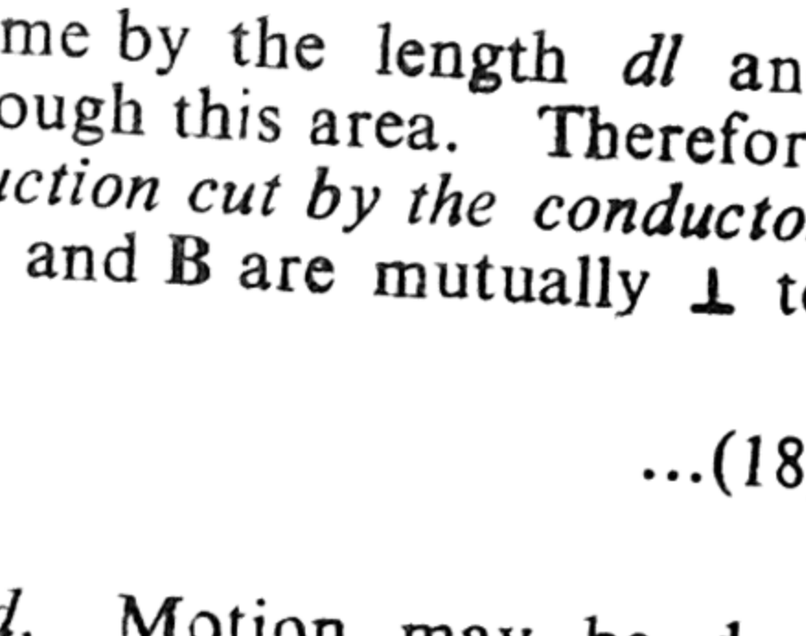


Fig. 9.5.

$$\therefore \quad d\mathbf{S}=d\mathbf{l}\,v\,dt \quad \text{or} \quad d\mathbf{S}=-d\mathbf{l}\times\mathbf{v}\,dt.$$

The negative sign is due to the fact that +ve direction of $d\mathbf{S}$ is towards the reader while cross product points in opposite direction. Thus Eq. (17) becomes

$$\mathcal{E}=-\int\mathbf{B}.(d\mathbf{S}/dt). \qquad ...(19)$$

This integral represents the negative of the flux of $\mathbf{B}$ through the area added to the circuit per unit time on account of the motion of the periphery. Thus *the motional emf around the closed circuit of any shape is equal to the rate of decrease of magnetic induction through the circuit due to the periphery.*

(*b*) *Circuit is stationary and the field source is moving.* Consider two frames S and S', the latter be moving with a velocity **v** in the y-direction relative to the former frame. The circuit will appear to be stationary in frame S' and to be moving with velocity **v** relative to the frame S. A stationary magnetic field source in frame S will appear to be moving with a velocity $\mathbf{v}'$ $(=-\mathbf{v})$ relative to the frame S'. The electric field due to Lorentz force in frame S' is given by

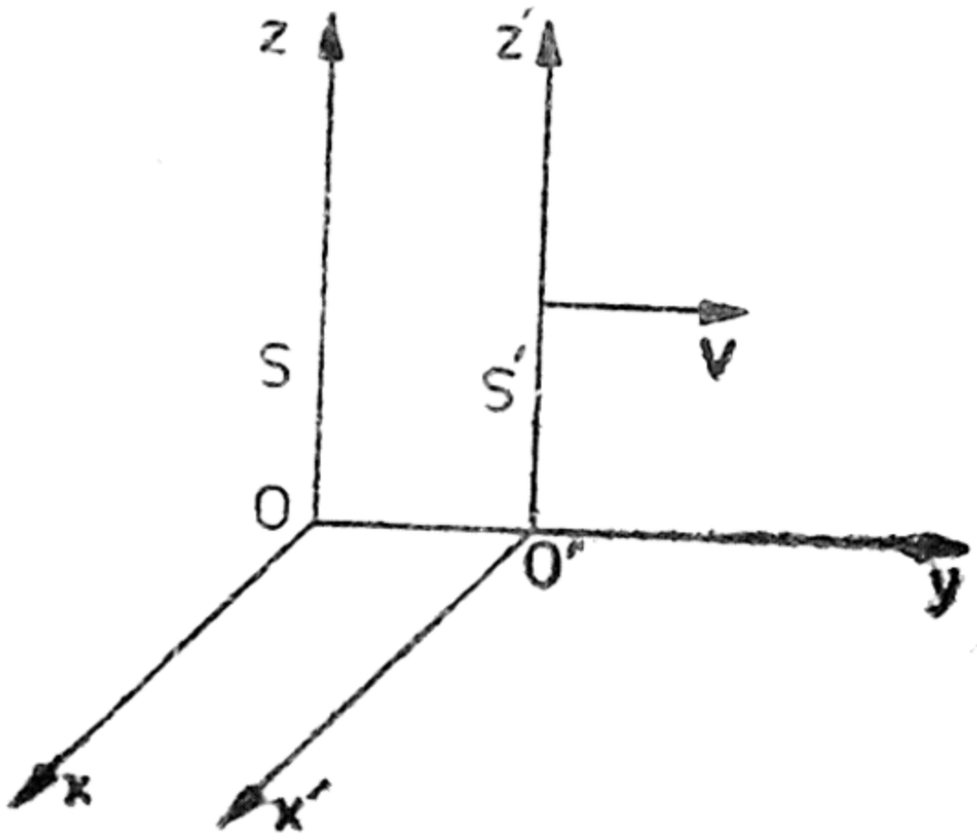


Fig. 9.6.

$$\mathbf{E}' = -(\mathbf{v}' \times \mathbf{B}') = \mathbf{v} \times \mathbf{B}'. \quad \ldots(20)$$

This will produce an emf in frame S'

$$\mathcal{E}' = \int \mathbf{E}' . d\mathbf{l}' = \int -(\mathbf{v}' \times \mathbf{B}') . d\mathbf{l}'.$$

$$= -d\Phi'/dt. \quad \ldots(21)$$

This is the same result as obtained for moving circuits.

(*c*) *The circuit is moving relative to the observer and the magnetic induction* **B** *is changing at every point with the time* (*the most general case*). Referring again to Fig. 9.5, if the circuit remains fixed in the position A_1, the emf is due to the change in **B** and is equal to the rate of decrease of flux of induction. In addition, we have a motional emf in the case of moving circuit equal to the rate of decrease of the flux due to the motion of the periphery. Thus the total emf is equal to the total rate of decrease of flux of magnetic induction through the circuit due to the change in **B** and to the motion of the circuit, *i.e.*,

$$\mathcal{E} = -\int_s \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{S} + \oint_c (\mathbf{v} \times \mathbf{B}) . d\mathbf{l}. \quad \ldots(22)$$

In the first term the integral of $\partial \mathbf{B}/\partial t$ is taken over the original surface S. We have shown earlier in this article that the motional emf contribution is $-\int \mathbf{B}.(\partial \mathbf{S}/\partial t)$, hence the generalization of Faraday's law can be written as

$$\mathcal{E} = -\frac{d}{dt}\int_c \mathbf{B}.d\mathbf{S}. \quad \ldots(23)$$

If there are N closed circuits arranged in series with linked fluxes $\Phi_1, \Phi_2, \Phi_3, \ldots \Phi_N$ respectively, the total induced emf

$$\mathcal{E} = -\left[\frac{d\Phi_1}{dt} + \frac{d\Phi_2}{dt} + \ldots \frac{d\Phi_N}{dt}\right].$$

If the closed circuits occupy exactly the same position (the case of a coil of N turns closely wound), then $\Phi_1 = \Phi_2 = \ldots \Phi_N = \Phi$ (say). Thus we get

$$\mathcal{E} = -N(d\Phi/dt). \quad \ldots(24)$$

9.7. INDUCED CHARGE

Whenever the magnetic flux linked with a coil of N turns changes, an emf is induced in the coil. If the coil is a closed circuit, a current flows and the charge is carried by the electrons. If R be the resistance of the circuit of which the coil is a part, the induced current

$$i = \frac{\mathcal{E}}{R} = -\frac{N}{R}\frac{d\Phi}{dt}. \qquad \ldots(25)$$

Hence δq, the quantity of electricity set in motion in time δt, is given by

$$\delta q = i\delta t = -\frac{N}{R}\frac{d\Phi}{dt}\delta t = -\frac{N}{R}\delta\Phi$$

$$\therefore \quad \text{charge } q = -\frac{N}{R}\int_{\phi_1}^{\phi_2} d\Phi = \frac{N}{R}\left[\Phi_1 - \Phi_2\right]. \qquad \ldots(26)$$

Thus, we see that the total quantity of induced electricity is independent of the rate at which the flux changes, but depends upon the resistance of the circuit and the total change in flux $(\Phi_1 - \Phi_2)$.

9.8. ROTATING LOOP IN A STATIONARY MAGNETIC FIELD (A Simple Dynamo).

The simple law of induction is the basis on which all dynamos or machines, for creating an emf are based. The principle of these instruments is the principle of rotation of a current carrying loop in a magnetic field.

Let us consider a rectangular loop of length a and width b rotating with uniform angular velocity ω about its axis in a uniform

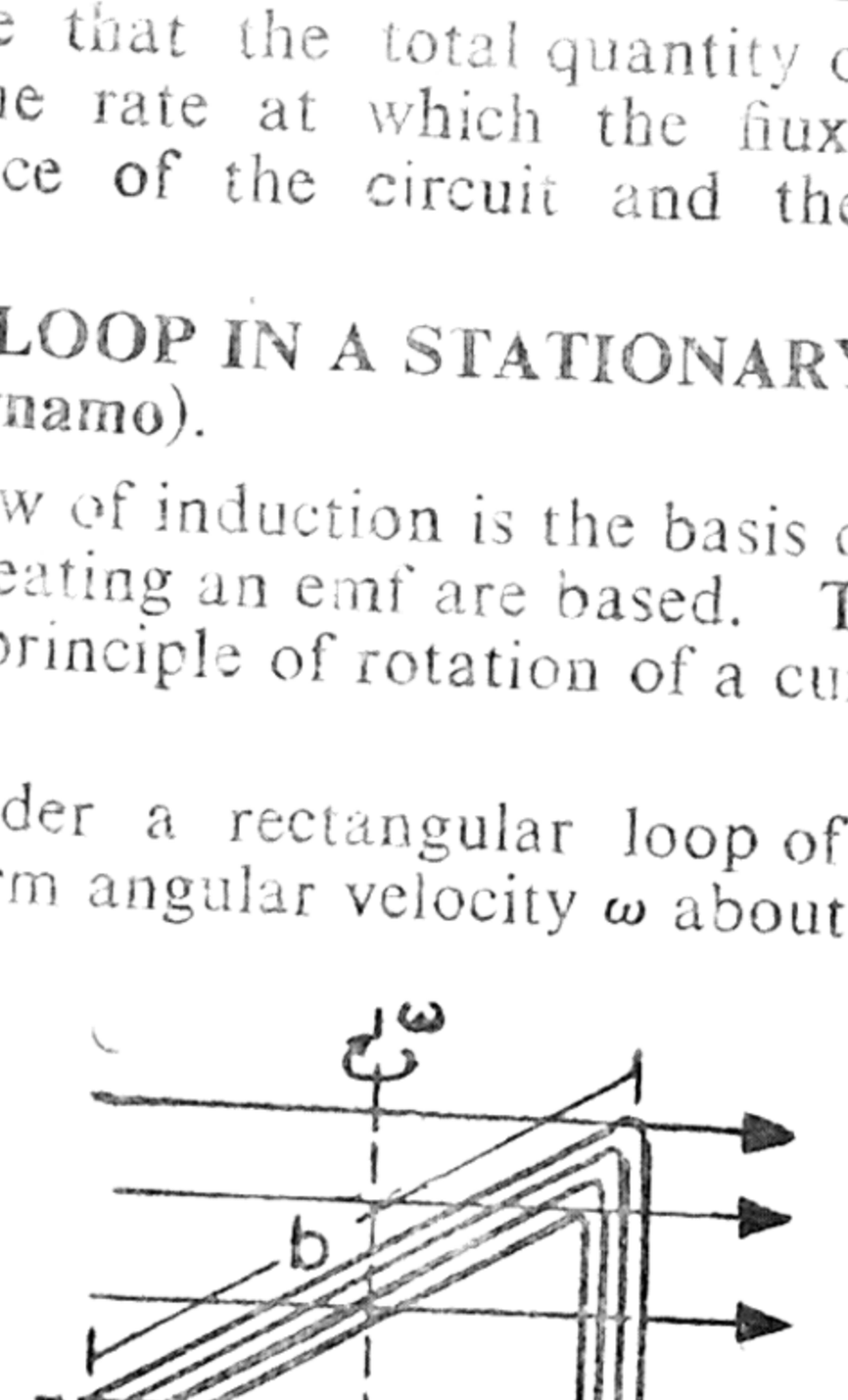


Fig. 9.7.

magnetic field **B**. If at a particular time t, θ is the angle which the normal to the plane of the loop makes with **B**, which is $\perp$ to the axis of rotation of the coil, then the magnetic flux through the coil of area $A(=ab)$ is

$$\Phi = \mathbf{B}.\mathbf{A} = BA\cos\theta = AB\cos\omega t. \qquad \ldots(27)$$

On account of the motion of the loop through the field, an emf induced in the coil.

$$\mathcal{E} = -\frac{d}{dt}\int \mathbf{B}.d\mathbf{S} = -\frac{d\Phi}{dt} = BA\,\omega\sin\omega t. \qquad \ldots(28)$$

If number of turns in the loop is N, then the total induced emf, as each turn contributes the same flux is

$$\mathcal{E} = NBA\omega\sin\omega t.$$

Thus, the emf varies sinusoidally with the time. The maximum emf occurs, when $\sin\omega t = 1$, and $\mathcal{E} = NBA\omega = \mathcal{E}_{max}$.

$$\therefore \qquad \mathcal{E} = \mathcal{E}_{max}\sin\omega t. \qquad \ldots(29)$$

From above relations it is clear that both flux and the induced emf reverse their senses relative to the circuit twice in each revolution. If the rotation is uniform, the one is maximum when the other is zero and vice versa. The emf is maximum, when $\theta = 90°$ or $270°$ and is zero when $\theta = 0°$ or $180°$. The emf depends upon area and no. of turns and not on its shape. It depends not on the flux through the loop but on its rate of change. The rotating loop can be utilized as the source in an external circuit by making connections to *slip rings* which rotate with the loop. *Carbon brushes* against the rings are connected to the output terminals. The instantaneous terminal voltage on open circuit is the instantaneous emf, known as ***alternating emf.*** The variation of $\mathcal{E}$ with t is as shown in Fig. 9.8(*b*).

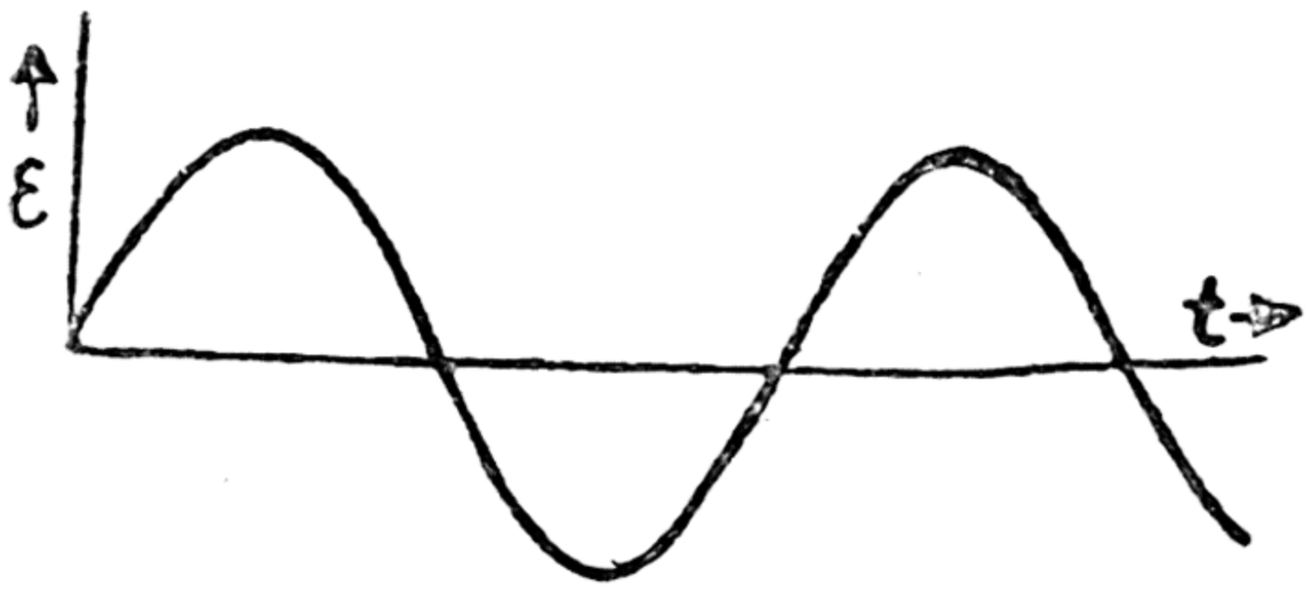


Fig. 9.8.

The instantaneous values of I, $\mathcal{E}$, ...are generally represented by small letters *i.e.*, ..., thus Eq. (29) becomes

$$e = e_{max}\sin\omega t. \qquad \ldots(30)$$

9.9. MEASUREMENT OF MAGNETIC FIELD

(*a*) **Search Coil Method.** The search coil is a closely wound small coil of about 50,100 or 200 turns of fine wire and about 1 sq. cm in cross-sectional area. This coil in combination with the properly calibrated ballistic galvanometer or Grassot fluxmeter is used for the measurement of magnetic induction of several *tesla*.

The search coil is connected in series with a ballistic galvanometer through a rheostat and a secondary coil of a standard sole-

noid (which is used to calibrate ballistic galvanometer). The primary of the solenoid is connected through a reversing key with a battery, rheostat, and an ammeter in series. First of all the primary circuit of the solenoid is kept off and the search coil is inserted in the magnetic induction **B**, to be measured. If the coil is placed with its plane $\perp$ to a magnetic induction, the magnetic flux linked with the coil of turns N, each of area A is $\Phi = BAN$. If the coil is not normal, then $\Phi = NB.A = NBA \cos\theta$. If the coil is first inserted with its plane $\perp$ to **B** and then removed out quickly, the flux through the coil thus decreases rapidly from BAN to zero. During the time that the flux is decreasing, an emf of short duration is induced in the coil and a kick is imparted to the ballistic galvanometer. The amount of charge passing through galvanometer

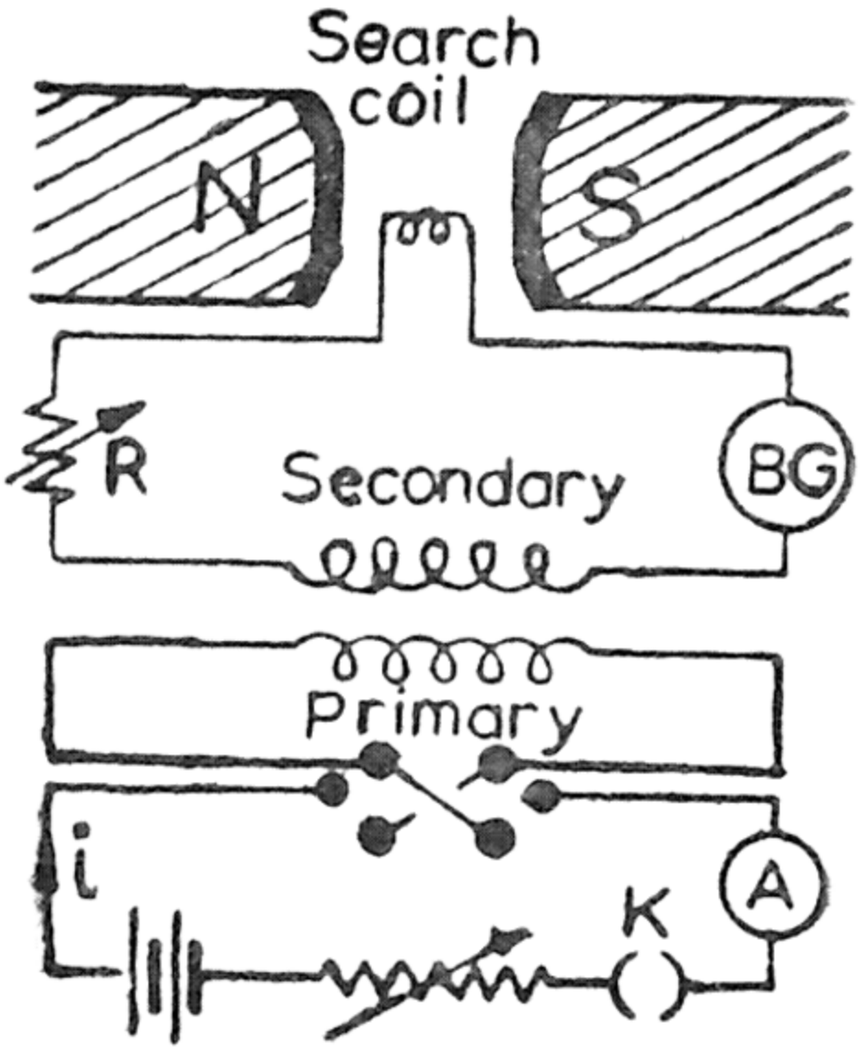


Fig. 9.9.

$$q = \int_0^t i\,dt = \int_0^t \frac{\mathcal{E}}{R}\,dt = \int_\Phi^0 \left(-\frac{1}{R}\right) d\Phi = \frac{\Phi}{R} = \frac{BAN}{R}.$$

If θ is the first throw of the galvanometer due to this charge, then

$$q = BAN/R = K\theta(1 + \tfrac{1}{2}\lambda), \qquad \ldots(31)$$

where K is the ballistic constant and λ is the logarithmic decrement of ballistic galvanometer.

To calibrate the galvanometer by means of a standard flux, a known current i ampere flowing through the primary coil of the solenoid is reversed by means of the reversing key and the corresponding throw of the galvanometer is observed. Magnetic field at the middle region of the solenoid of n_1 turns per unit length due to the current i through it is $B_S = \mu_0 n_1 i$. Hence the total flux linked with the secondary coil, wrapped in the middle of the solenoid, of n_2 turns is $\Phi = n_2 A' B_S = \mu_0 n_1 n_2 i A'$, where A' is the area of each turn of the secondary coil. When the current in the primary of solenoid is reversed, the magnetic flux linked with the secondary changes to $-\mu_0 n_1 n_2 i A'$. The net change in flux $2\mu_0 n_1 n_2 i A'$ thus produces a kick in the galvanometer. If θ' is the corresponding first throw, then

$$q = 2\mu_0 n_1 n_2 i A'/R = K\theta'(1 + \tfrac{1}{2}\lambda). \qquad \ldots(32)$$

Thus from equations (31) and (32), we get

$$\frac{BNA}{2\mu_0 n_1 n_2 i A'} = \frac{\theta}{\theta'} \quad \text{or} \quad B = \frac{2\mu_0 n_1 n_2 i A'}{NA} \cdot \frac{\theta}{\theta'}. \qquad \ldots(33)$$

In this way B may be calculated. Here the deflection θ is maximum for the particular orientation of the coil, when the coil was $\perp$ to the field direction. Thus the magnitude and direction of an unknown field can both be found by this method. This method is used to measure strong magnetic fields, *e.g.*, the field between pole pieces of powerfull magnet.

(*b*) **Earth Inductor Method.** Earth inductor is an instrument usedfor the measurement of earth's magnetic field. It has a coil consisting of a large number of turns of insulated *Cu*-wire wound over a wooden or ebonite frame. The coil is fitted in a wooden frame and can be rotated quickly by means of a handle about an axis passing through the centre of the coil and lying in its plane. The rotation is confined to 180° exactly by means of stops provided in the frame. The frame itself can be rotated about an axis which is $\perp$ to the axis of rotation of coil. Thus the coil can be adjusted vertical or horizontal as desired. The ends of the coil wire are connected to two binding screws on the frame.

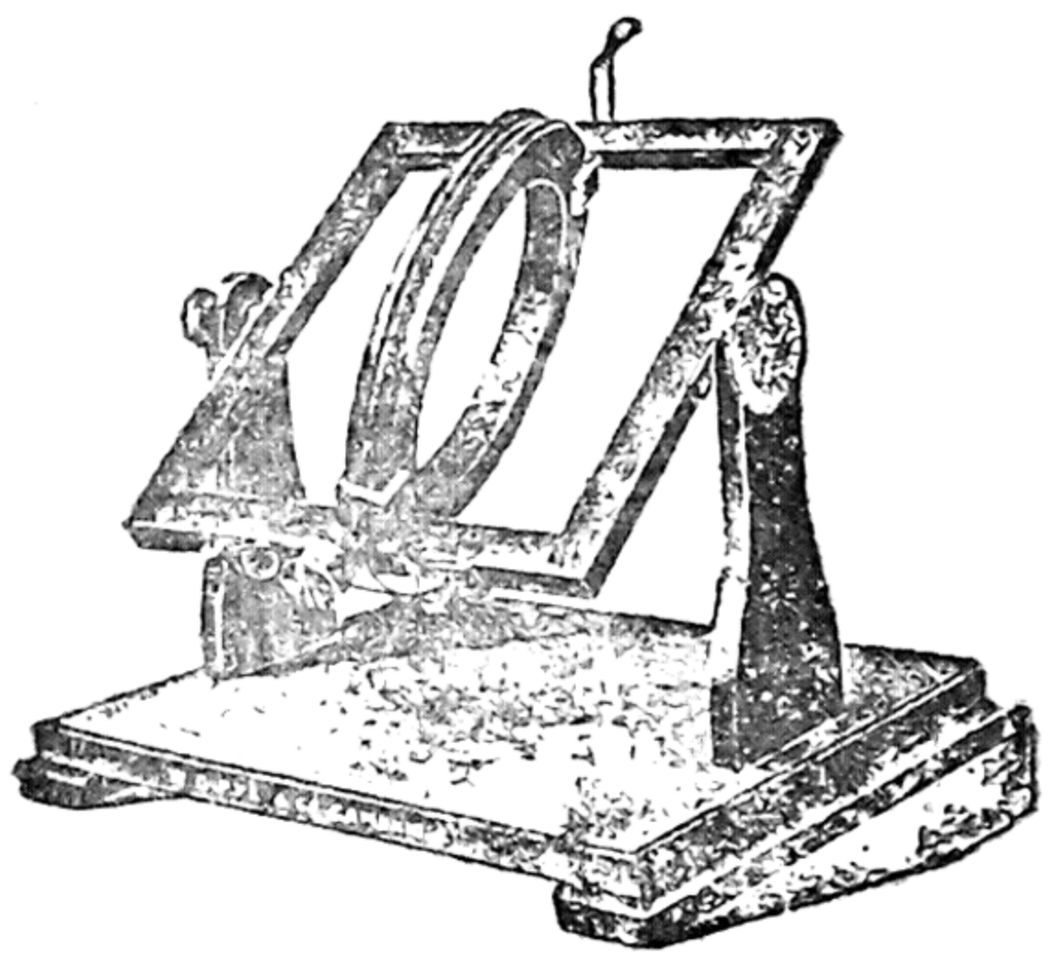


Fig. 9.10.

It works on the principle of the rotation of a coil in the magnetic field. It is generally used for the determination of the horizontal component of earth magnetic field H, vertical component V and angle of dip ϕ. To find H, the frame is kept vertical such that H acts along the normal of the plane of coil. The ends of the inductor coil are connected with the ballistic galvanometer, the coil is rotated through an angle 180° by the handle and the first throw in the galvanometer is noted.

If at any instant t the coil makes an angle θ with H, then the flux linked with the coil of n turns,

$$\Phi = n(\mathbf{A} \cdot \mathbf{H}) = nAH \cos\theta.$$

$$\therefore \text{ Induced current} = \frac{\text{Induced emf}}{R} = -\frac{1}{R} \cdot \frac{d\Phi}{dt}.$$

Thus the charge given to the galvanometer coil

$$q = \int_0^t i\,dt = -\int_0^{\pi} \frac{nAH}{R}\, d(\cos\theta) = -\frac{nAH}{R}\Big[\cos\theta\Big]_0^{\pi}$$

$$= 2nAH/R.$$

If θ_H is the first throw in galvanometer, then we have

$$q = K\theta_H\left(1 + \frac{\lambda}{2}\right) = \frac{2nAH}{R} \quad \text{or} \quad H = \frac{K\theta_H R}{2nA}\left(1 + \frac{\lambda}{2}\right). \quad \ldots(34)$$

To eliminate K and λ, the circuit is made as shown in Fig. 9.9. except that the search coil is replaced by the earth inductor and N- and S-poles are replaced by the earth's magnetic field. The experiment can be performed as given earlier in the search coil method. If θ be the first throw in the ballistic galvanometer, when the current i in the primary of the solenoid of n_1 turns per unit length is reversed, then we have

$$2\mu_0 n_1 n_2 i A'/R = K\theta(1+\lambda/2), \qquad \ldots(35)$$

where n_2 is the number of turns of the secondary of the solenoid and A' the area of each turn.

Equations (34) and (35) give

$$H = \frac{\mu_0 n_1 n_2 i A'}{nA} \cdot \frac{\theta_H}{\theta} \text{ weber/m}^2. \qquad \ldots(36)$$

To determine vertical component of earth's magnetic field V, the coil of the earth inductor is laid horizontally with the axis of rotation along the magnetic meridian. Coil is now rotated through 180° and the first deflection in galvanometer coil is noted, if it is θ_V then

$$V = K\theta_V R(1+\tfrac{1}{2}\lambda)/2nA. \qquad \ldots(37)$$

Proceeding as above, the combination of equations (35) and (37), we get

$$V = \frac{\mu_0 n_1 n_2 i A'}{nA} \cdot \frac{\theta_V}{\theta} \text{ weber/m}^2. \qquad \ldots(38)$$

The angle of dip ϕ, thus can be given by

$$\tan\phi = \frac{V}{H} = \frac{\theta_V}{\theta_H} \quad \text{or} \quad \phi = \tan^{-1}\frac{\theta_V}{\theta_H}. \qquad \ldots(39)$$

(*c*) **Grassot Fluxmeter.** This instrument is really a special type of ballistic galvanometer in which both mechanical damping and restoring torque are made negligibly small but the electromagnetic damping is relatively large. The coil of small cross section and of fairly large open rectangular structure is suspended from a spring S support by means of a single thread of slik and hangs with its parallel sides in the air gap between the pole pieces of permanent magnet NS. This silk fibre has negligible restoring torque and the flat spiral saves it from effects of shocks. The current is led into the coil by spirals of very thin, annealed silver strips which induce very little mechanical damping of the motion. These leads are connected to the terminals T_1 T_2 to which a search coil may be con-

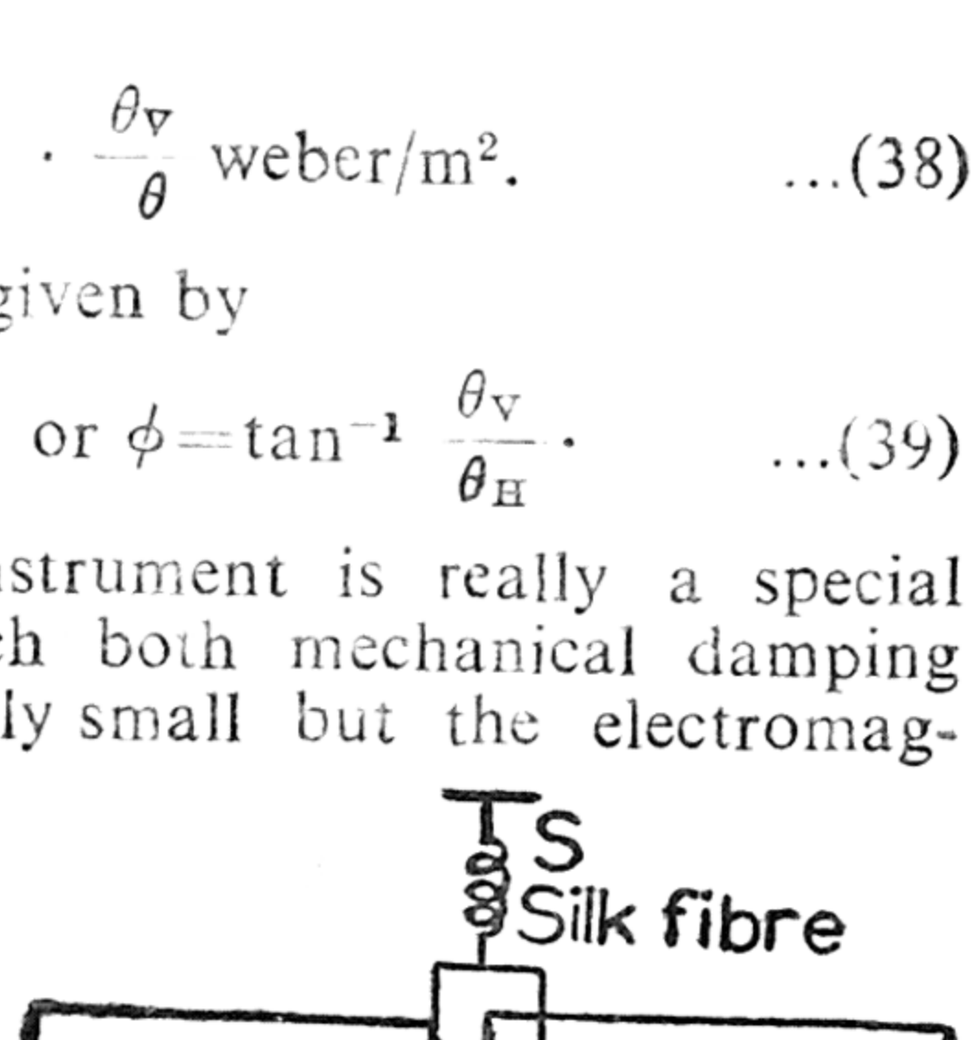


Fig. 9.11.

nected. The search coil is small and usually mounted within a flat brass container with an insulated handle. With the help of this handle the search coil may be inserted in or withdrawn from the region of unknown magnetic field. The deflection of the coil of the fluxmeter is indicated by a pointer moving over a calibrated scale attached with the coil or by the lamp and scale arrangement.

When a search coil, of turns n and of area of cross section of each turn A, is inserted in the magnetic induction **B**, then the flux changes from 0 to $\Phi(=nAB)$. Due to this change in flux, an induced emf $\mathcal{E}$ is produced in the circuit. This induces the current

$$i=\mathcal{E}/R.$$

where R is the total resistance of the circuit.

Due to this current a deflecting couple will act upon the coil of fluxmeter and thus will impart an angular momentum. If I is the moment of inertia of the fluxmeter coil and ω its angular velocity, then

$$i\,n'B'A'=I(d\omega/dt)$$

$$\therefore \qquad \mathcal{E}=iR=(IR/n'B'A')\,d\omega/dt. \qquad \ldots(40)$$

Here n' = number of turns in the coil of fluxmeter, A' = area of its each turn and B' = magnetic field of permanent magnet.

Actually the induced emf, which produces angular acceleration is the difference between the emf due to the rate of change of the flux in the search coil and the emf produced in the coil of fluxmeter due to its rotation, *i.e.*,

$$\mathcal{E}=d\Phi/dt-n'\,A'B'\omega=d\Phi/dt-n'A'B'\,d\theta/dt. \qquad \ldots(41)$$

Combining Eqs. (40) and (41), we get

$$\frac{IR}{n'\,B'A'}\,\frac{d\omega}{dt}=\frac{d\Phi}{dt}-n'\,A'B'\,\frac{d\theta}{dt}.$$

On integrating, we get

$$\int_0^t \frac{d\Phi}{dt}\,dt-n'A'B'\int_0^t \frac{d\theta}{dt}\,dt=\frac{IR}{n'\,B'A'}\int_0^t \frac{d\omega}{dt}\,dt$$

or

$$\int_0^{\phi} d\Phi-n'A'B'\int_0^{\theta} d\theta=\frac{IR}{n'\,B'A'}\int_0^{\theta} d\omega.$$

Integration limit on the right hand side is from 0to θ, as the coil is at rest before and after the change in flux.

$$\therefore \qquad \Phi-n'A'B'\,\theta=0 \quad \text{or} \quad \Phi=n'A'B'\,\theta=k\theta.$$

Hence the unknown magnetic induction

$$B=\frac{\Phi}{nA}=\frac{k}{nA}\theta. \qquad \ldots(42)$$

This relation shows that the total change of flux in a search coil which depends upon B, is proportional to the deflection of coil of fluxmeter.

This instrument is portable but not so sensitive as a ballistic galvanometer. It has the *following advantages over the ballistic galvanometer* : (*i*) The deflection is independent of time and is same whether the time taken for the change of flux be a fraction of a second or as much as one or two minutes, (*ii*) The deflection is independent of moment of inertia of the coil, hence large M.I. is not required. (*iii*) The coil is suspended by a silk fibre hence the mechanical couple is reduced to a smaller value. The induction of the search coil circuit is unimportant and may be quite large with negligible effect upon the deflection.

9.10. TIME VARYING MAGNETIC FIELDS

Upto this time we have considered emfs induced by the relative motion of magnets and coils. In this section we want to see the effect of magnetic field varying with time. Such a field may be produced by flowing alternating current through the winding of an electromagnet. The field B so produced also varies with time. If a conducting loop is placed in such a time varying field, the flux through the loop will change and an induced emf will be produced in the loop. This induced emf will set a motion in the charge carriers and thus will induce a current.

In macroscopic form we can say that this varying magnetic field B sets up an induced electric field E at all the points around the loop. This field will exert a force F on a test charge q_0. E at various points of the loop, from symmetry, must be tangent to the loop. Thus the electric lines of force that are set up by the varying magnetic field are the concentric circles. The work done in carrying a test charge q_0 around the circle of radius R is given by

$$W = \oint \mathbf{F} \cdot d\mathbf{l} = \oint q_0 \mathbf{E} \cdot d\mathbf{l} = q_0 E 2\pi R.$$

By the definition of an emf, we get

$$\text{emf } \mathcal{E} = W/q_0 = E 2\pi R = \oint \mathbf{E} \cdot d\mathbf{l}.$$

From Faraday's law of induction, we get

$$\mathcal{E} = -d\Phi/dt \qquad \therefore \quad -d\Phi/dt = \oint \mathbf{E} \cdot d\mathbf{l}. \qquad \ldots(43)$$

As close line integral of $\mathbf{E} \cdot d\mathbf{l}$ is not zero, hence we can say that the electric fields associated with changing magnetic fields are non-conservative, while the fields associated with stationary charges are conservative.

9.11. THE BETATRON

The cyclotron principle is inapplicable for the acceleration of electrons to high energies because of the large relativistic increase of mass at low energies. The first important machine for producing

high energy electrons was the betatron. The first betatron was constructed by Kerst at the University of Illinois in 1940.

The action of the betatron depends on the same fundamental principle as that of the transformer in which an alternating current applied to a primary coil induces an alternating current usually with higher or low voltage in the secondary coil. In the betatron secondary coil is replaced by a doughnut shaped vacuum chamber. Electrons, produced in the doughnut from a hot filament, are given a preliminary acceleration by the application of an electric field having potential difference of 20 *k V* to 70 *kV*. When an alternating magnetic field is applied parallel to the axis of the tube, two effects are produced, (1) *an electromotive force is produced in the electron orbit by the changing magnetic flux that gives an additional energy to the electrons* ; (2) *a radial force is produced by the action of magnetic field whose direction is perpendicular to the electron velocity which keeps the electron moving in a circular path*. Instead of spiralling, as in the cyclotron, conditions are arranged such that the increasing magnetic field keeps the electrons in a circular orbit of constant radius.

Using Faraday's law of induction, the work done on an electron of charge e in one revolution

$$W = e\mathcal{E} = e\,d\Phi/dt.$$

If F is the tangential force acting on the electron, the work done W can also be expressed as

$$W = 2\pi RF.$$

$$\therefore \qquad F = \frac{e}{2\pi R}\,\frac{d\Phi}{dt}. \qquad \ldots(44)$$

For an electron moving in a circular orbit

$$Bev = Mv^2/R \quad \text{or} \quad Mv = BeR.$$

This relation indicates that the magnetic field B at the orbit must increase as the electron energy increases otherwise radius R will not be constant. If R is kept constant, then

$$\frac{d}{dt}(Mv) = eR\,\frac{dB}{dt}. \qquad \ldots(45)$$

From Newton's second law, the rate of change of momentum is equal to the force. Hence on combining equations (44) and (45), we get

$$\frac{d\Phi}{dt} = 2\pi R^2\,\frac{dB}{dt} = 2\,\frac{d}{dt}(\pi R^2 B). \qquad \ldots(46)$$

This relation shows that the rate of change of flux Φ within the orbit of radius R is always twice what it would have been if the magnetic field intensity were uniform throughout the orbit. This relation is known as the *betatron condition*. Since the induced emf

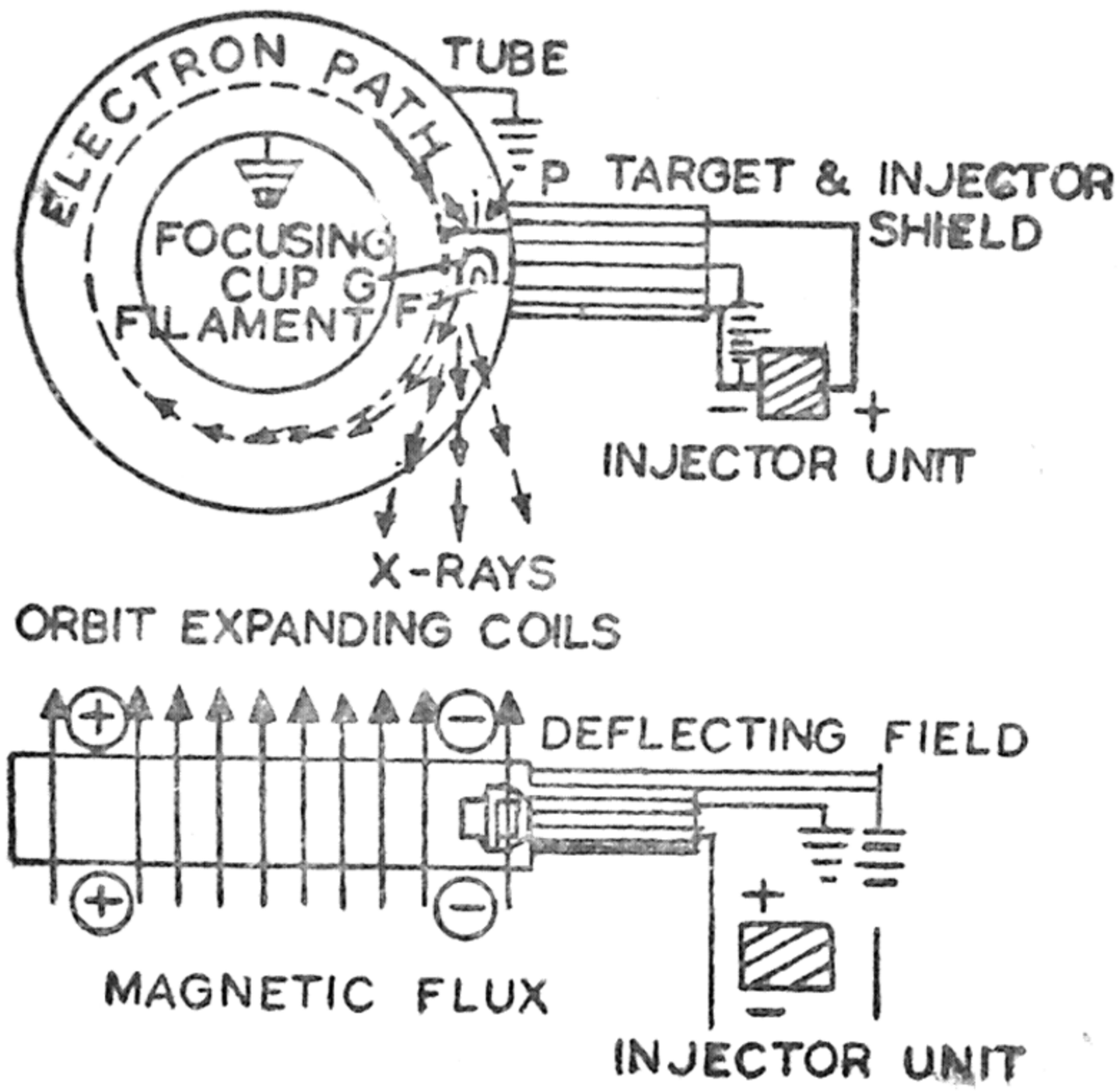


Fig. 9.12.

is determined by the rate of change of flux, the iron core is laminated as in a transformer and alternating potential at 50 or 150 cycles is used to produce varying magnetic field. The electrons are injected at an instant when the magnetic field is just rising from its zero value in the first quarter cycle. The increasing magnetic field induces a potential within the doughnut which increases the energy of the electrons. When the field strength passes its maximum value and starts to increase, the direction of the induced emf is changed and the electrons start to slow down. This effect is avoided by removing the electrons from their stable orbit by passing a pulse of current through an auxiliary coil when the field reaches its positive maximum. These high energy electrons leave the path tangentially to strike a target which then emits *X*-rays. The emission from the machine consists therefore of a succession of short pulses of *X*-radiations, each occurring at the peak of the a.c. maximum. *Only the first quarter of each a.c. cycle is usefully employed in acceleration.*

The energy of electrons can be estimated from the time average of the energy given to the electrons and the total number of revolutions made by the electrons. If the flux variation is given by the relation $\Phi = \Phi_0 \sin \omega t$, the time during which acceleration takes

place will be $\pi/2\omega$. Hence the energy gained by the electron per turn when the flux changes will be

$$E=e\mathcal{E}=ed\Phi/dt=e\omega\,\Phi_0\cos\,\omega t.$$

Thus, during the acceleration period, its average value will be

$$E_{av}=\frac{\int_{t=0}^{t=\pi/2\omega} e\omega\Phi_0\cos\,\omega t\,dt}{\int_{t=0}^{t=\pi/2\omega} dt}=\frac{e\omega\Phi_0(1/\omega)}{\pi/2\omega}=\frac{2}{\pi}e\omega\Phi_0.$$

For most of the time electrons travel with a velocity close to the velocity of light and therefore the total distance L travelled during the acceleration process will be $c\pi/2\omega$. If R is the radius of the orbit, the number of revolutions N will be given by

$$N=L/2\pi R=c/4\omega R. \qquad ...(47)$$

Hence the total energy gained by electrons

$$E=E_{av}\times N=ec\Phi_0/2\pi R. \qquad ...(48)$$

The energy of electrons can also be estimated by the help of relativistic equation for energy

$$K.E.=pc=BeRc. \qquad ...(49)$$

where B is the magnetic field at the acceleration. The energy obtained is therefore limited by the radius and the peak strength of the magnet of the orbit.

Most of the modern betatrons are operated from a 50 cycles/sec, a.c. source. If the length of the circular path within the doughnut of a betatron is supposed to be 3 metres and the energy gained by the electron in each revolution 400 *eV*, the electrons will then have a total energy of 167 *MeV*. In practice the energy of electrons can be varied from 10 *MeV* to 100 *MeV* by applying the orbit shifting magnetic field at different times during the quarter cycle in which the field is increasing.

At a certain limit the energy gain is balanced by the radiation loss (because of the centripetal force, the circulating electrons are expected to emit radiations). Schwinger (1946) suggested that the energy radiated per revolution was proportional to E^4/R. Thus the radiation loss is very serious at very high energies and this device hence cannot be used for acceleration of electrons to a very large energies.

9.12. MUTUAL INDUCTANCE

As illustrated in the article 9.1, a changing current through one coil can produce an induced emf in a neighbouring coil. It is due

to the change in flux in the second coil. The first coil in which current passes is known as *primary coil* and the other coil, in which induced emf is set up is known as *secondary coil.* Let n_1 and n_2 be the number of turns in primary and secondary respectively. A current in coil 1 sets up a magnetic lines of force and some of these lines pass through the second coil 2. If flux Φ_2 is result of the magnetic field arising from current i_1 in coil 1. Since $B \propto i_1$, hence we have total flux associated with n_2 turns of secondary

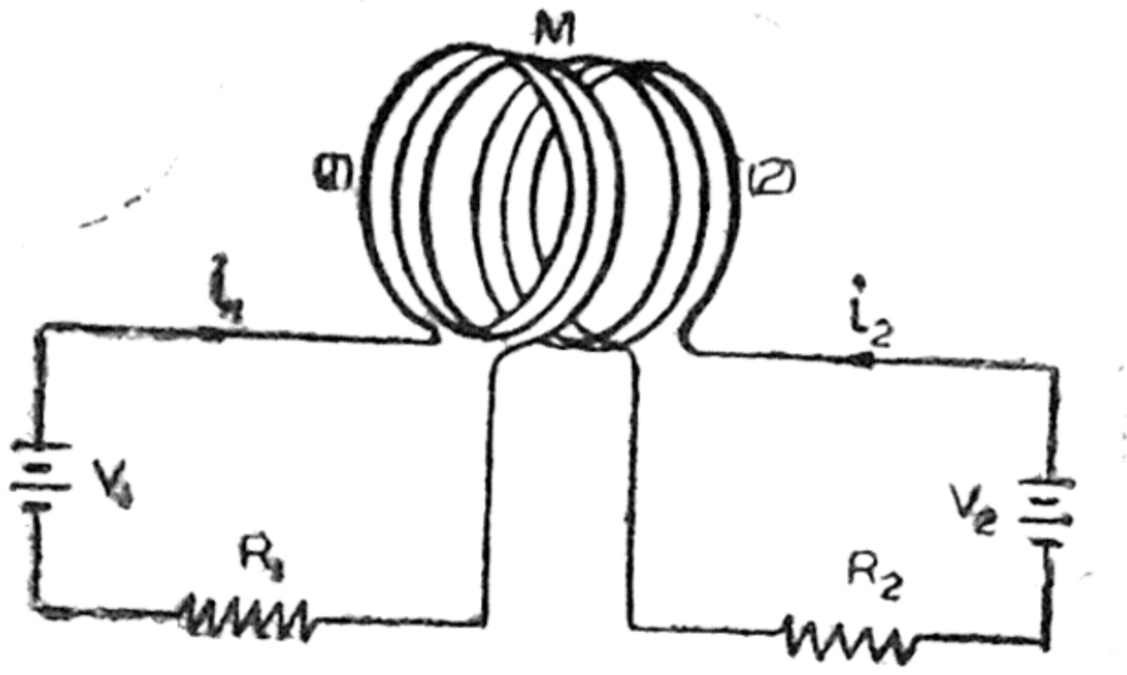


Fig. 9.13.

$$n_2\Phi_2 \propto i_1 \quad \text{or} \quad n_2\Phi_2 = Mi_1. \qquad \ldots(50)$$

The *constant of proportionality M* depends upon the geometrical and coil construction factors as ; the closeness of two coils, their orientations and sizes and the number of turns in primary. This is known as the *mutual inductance* or *coefficient of mutual inductance of the two coils.* Actually mutual inductance is the property of the two coils regarding the mutual action of the magnetic fields.

If $\mathcal{E}_2$ is the induced emf in the secondary due to variation of $n_2\Phi_2$, then

$$\mathcal{E}_2 = -\frac{d}{dt}\left[n_2\Phi_2\right] = -\frac{d}{dt}\left[Mi_1\right] = -M\frac{di_1}{dt},$$

or
$$M = -\mathcal{E}_2/(di_1/dt). \qquad \ldots(51)$$

Thus from relations (50) and (51) we can define M as :

(*i*) *The mutual inductance of two coils is numerically equal to the magnetic flux linked with one coil due to the unit current in the other.*

(*ii*) *It can also be defined as the emf induced in one coil when the rate of decrease of current in the other coil is unity.*

The unit of $M = \dfrac{\text{unit of } \mathcal{E}}{\text{unit of } (di/dt)} = \dfrac{\text{volt}}{\text{amp/sec}} = \text{volt sec/ampere.}$

This unit is given the special name *henry*, in honour of Joseph Henry.

A Reciprocity Theorem. The mutual inductance between two coils is used to evaluate the effect of changing the current in one coil on the behaviour of the current in the second coil. Let us represent the induced emf in the secondary coil due to the change of current i_1 in the primary coil as

$$\mathcal{E}_{21} = M_{21}(di_1/dt), \qquad \ldots(52)$$

where M_{21} represents the mutual inductance of coil 2 with respect to coil 1.

This induced emf will produce current i_2 in the secondary coil which will also produce induce emf in the primary coil. Thus

$$\mathcal{E}_{12} = M_{12}(di_2/dt), \qquad \ldots(53)$$

where M_{12} is the mutual inductance of coil 1 with respect to coil 2.

The force experienced by an element $\delta \mathbf{l}_2$ of the secondary carrying current i_2 and assumed to be placed in a magnetic field $\mathbf{B}_1$ (due to current i_1 in primary) is given by

$$\delta \mathbf{F}_1 = i_2 \delta \mathbf{l}_2 \times \mathbf{B}_1. \qquad \ldots(54)$$

∴ Amount of work done in displacing this element by a distance $\delta \mathbf{r}$

$$\delta W_2 = i_2(\delta \mathbf{l}_2 \times \mathbf{B}_1).\delta \mathbf{r} = i_2 \delta \Phi_{21}, \qquad \ldots(55)$$

where $\delta\Phi_{21}$ is the flux with in the volume of the parallelopiped of $\delta \mathbf{l}_2$, $\mathbf{B}_1$ and $\delta \mathbf{r}$.

This relation is same for the motion of the secondary coil. Thus δW_2 represents the work done in displacing a secondary coil by a distance $\delta \mathbf{r}$ and $\delta\Phi_{21}$ the additional flux linked with the secondary by virtue of the current in primary.

The total work done in bringing the secondary coil from far apart to a point where the flux linkage is Φ_{21}

$$W_2 = \int dW_2 = \int i_2 \, d\Phi_{21} = i_2 \Phi_{21}. \qquad \ldots(56)$$

Similarly the work done in bringing the primary keeping the secondary fixed

$$W_1 = \int i_1 d\Phi_{12} = i_1 \Phi_{12}, \qquad \ldots(57)$$

where Φ_{12} is the flux linked with the primary. Since the final states are equivalent regardless of the way, hence

$$W_1 = W_2 \quad \text{or} \quad i_1 \Phi_{12} = i_2 \Phi_{21}.$$

$$\therefore \quad i_1(M_{12} i_2) = i_2(M_{21} i_1) \quad \text{or} \quad M_{12} = M_{21}. \qquad \ldots(58)$$

Hence the mutual inductance of the coils remains same whether the current is flowing in coil 1 or in 2. This is known as *reciprocity theorem of mutual inductances.*

If $M_{12} = M_{21} = M$ (say), then the amount of work done and thus the mutual potential energy of the coils is

$$W = W_1 = W_2 = M i_1 i_2. \qquad \ldots(59)$$

The reciprocity theorem can also be proved as :

The current i_1 to the primary coil produces a magnetic flux Φ_{12} linking the secondary coil.

$$\Phi_{12} = \int \mathbf{B}_1 . d\mathbf{S}_2 = \int (\nabla \times \mathbf{A}_1) . d\mathbf{S}_2,$$

$$\Phi_{12}=\oint_2 \mathbf{A}_1.d\mathbf{l}_2 =\oint\left(\frac{\mu_0 i_1}{4\pi}\oint_1\frac{d\mathbf{l}_1}{r}\right).d\mathbf{l}_2$$

$$=\frac{\mu_0 i_1}{4\pi}\oint_1\oint_2\frac{d\mathbf{l}_1.d\mathbf{l}_2}{r}=M_{12}i_1.$$

or
$$M_{12}=\frac{\mu_0}{4\pi}\oint_1\oint_2\frac{d\mathbf{l}_1.d\mathbf{l}_2}{r}. \qquad ...(60)$$

This is known as *Neumann's formuia.*

Since r is the distance between a point on the primary to a point on the secondary. The integral as a whole is symmetrical with respect to subscripts, *i.e.*, the subscripts 1 and 2 may be interchanged without interchanging the end result.

$$\therefore \qquad M_{12}=M_{21}.$$

Measurement of Mutual Inductance. In this article we are discussing the *dc* method only. Few more methods (*ac*) are given in chapter (12). Here P and S are the two coils whose mutual inductance M is to be determined. By short circuiting resistance r, a suitable current i is allowed to pass through the primary coil P. If the positions 1 and 2 of the commutator C are connected, then the induced current will flow of the circuit consisting secondary coil S.

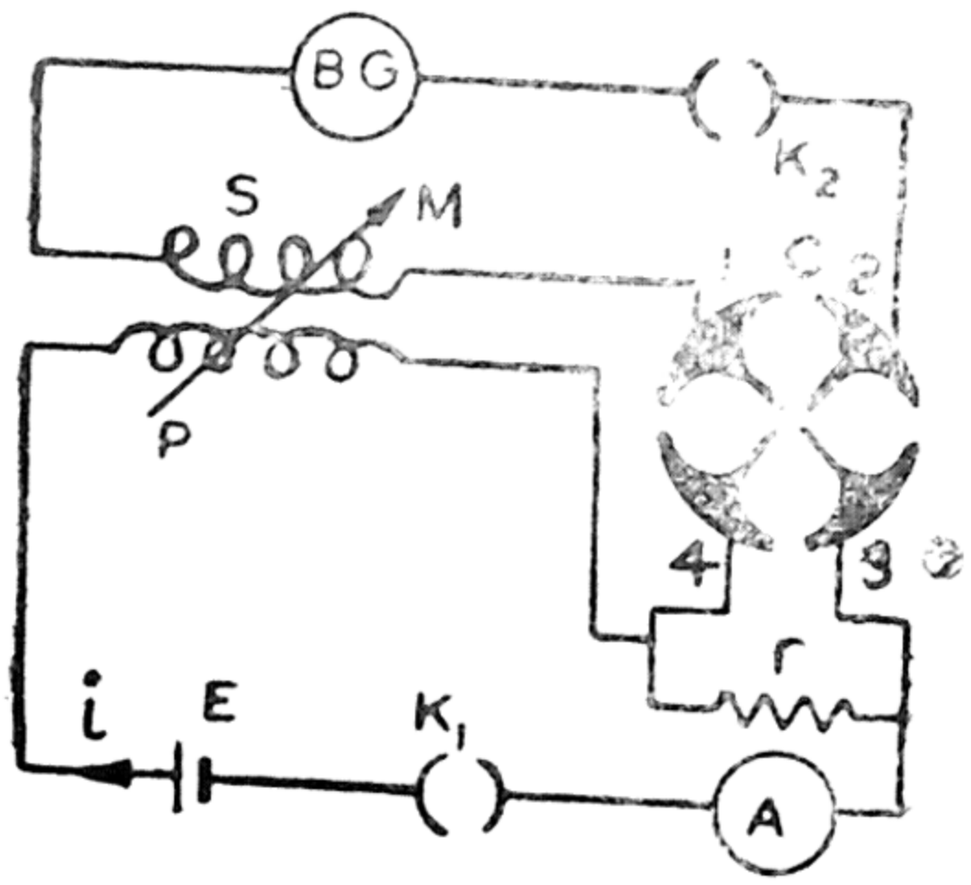


Fig. 9.14.

Induced emf $\mathcal{E}=-M(di/dt)$.

$\therefore$ Induced current

$$=\mathcal{E}/R=-(M/R)\,di/dt,$$

where R is the resistance in the secondary circuit.

Hence the total charge through the ballistic galvanometer

$$q=\int_0^t\left(-\frac{M}{R}\right)\frac{di}{dt}dt=-\frac{M}{R}\int_0^i di=-\frac{Mi}{R}.$$

The −ve sign may be omitted as it gives the direction of flow of charge. If θ be the first throw in the ballistic galvanometer, then

$$q=\frac{Mi}{R}=K\frac{T}{2\pi}\theta\left(1+\frac{\lambda}{2}\right). \qquad ...(61)$$

To eliminate K and λ, the contacts between 1 and 4 and that between 2 and 3 are made In this way small resistance r ($\sim 0.2\ \Omega$) is placed in the primary circuit and a steady current is now allowed

to pass in the primary. The current will be same, as very small resistance r does not effect its value. The *pd* across resistance r thus sends a steady current ir/R through the galvanometer and a steady deflection θ' is observed. Then

$$ir/R = K\theta'. \qquad \ldots(62)$$

From relations (61) and (62), we get

$$M = r\frac{T}{2\pi}\ \frac{\theta}{\theta'}\left(1+\frac{\lambda}{2}\right). \qquad \ldots(63)$$

Thus mutual inductance M can be calculated.

9.13. SELF INDUCTANCE

Faraday induction is of equal importance in a single circuit. If a current flows in a single coil, the current causes a flux through the same coil. If the current changes, the flux also changes and emf is induced in the coil. The direction of the induced emf is such as to oppose the change of current in the circuit. It is given by Lenz's law, this is against the current when current is increasing and in the direction of current when the current is decreasing. The phenomenon (the production of induced emf in the circuit when the current through it changes) is known as *self induction* and the induced emf as *self induced emf*.

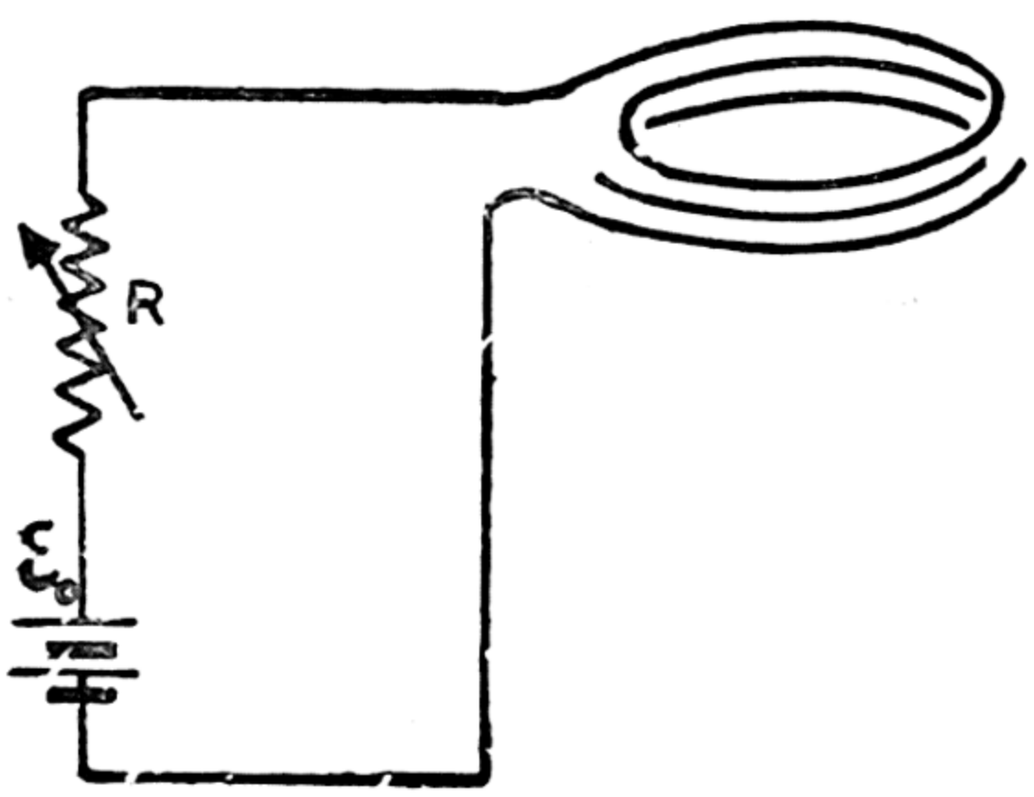


Fig. 9.15.

By analogy with the definition of mutual inductance, the self inductance L may be defined as

$$L = N\Phi/i, \qquad \ldots(64)$$

where N is the number of turns in the coil and Φ is the flux per turn in the coil due to current i.

From the Faraday's law of induction, we have

$$\text{induced emf } \mathcal{E} = -d(N\Phi)/dt = -Ldi/dt. \qquad (65)$$

When a coil is connected to a battery, the current will flow through the coil and a work will be done on the coil by the battery. If the current is changing with the rate di/dt, this results in an emf $L(di/dt)$. This opposes the current flow through the coil. Thus to keep the current same, power P is applied to the inductor.

$$\text{Power } P = \mathcal{E}i = iL\,(di/dt).$$

$\therefore$ Total work done by the battery on the coil is

$$W=\int Pdt=L\int_0^I idi=\frac{1}{2}LI^2. \qquad \ldots(66)$$

After the current has reached its final steady value, $di/dt=0$, the power input is zero. The energy that has been supplied to the inductor (coil) is stored as potential energy. When the circuit is opened the magnetic field around the inductor collapses and the energy is returned to the circuit. Relation (66) is similar to the expression for the mechanical kinetic energy ($\frac{1}{2}mv^2$) and thus shows that the *self induction L plays the same role in an electrical circuit as the mass or inertia does in mechanical motion.*

Self inductance L may be defined, from relation (64), (65) or (66), as

(*i*) *The flux linked with the circuit when unit current flows in it.*

(*ii*) *The induced emf produced in the circuit when the rate of change of current in it is unity.*

(*iii*) *The twice the work done in establishing the magnetic flux associated with the unit current in the circuit.*

Its unit is *henry* as that of the mutual inductance.

Measurement of 'L' by Rayleigh's bridge Method. Lord Rayleigh gave a method for the measurement of self inductance of a coil using the principle of Wheatstone's bridge. The coil whose self-inductance is to be measured, with a very small resistance r in series, is connected in the unknown arm CD. This resistance may be short-circuited by a key K_3. A sensitive ballistic galvanometer along with the lamp and scale arrangement is placed in galvanometer arm BD.

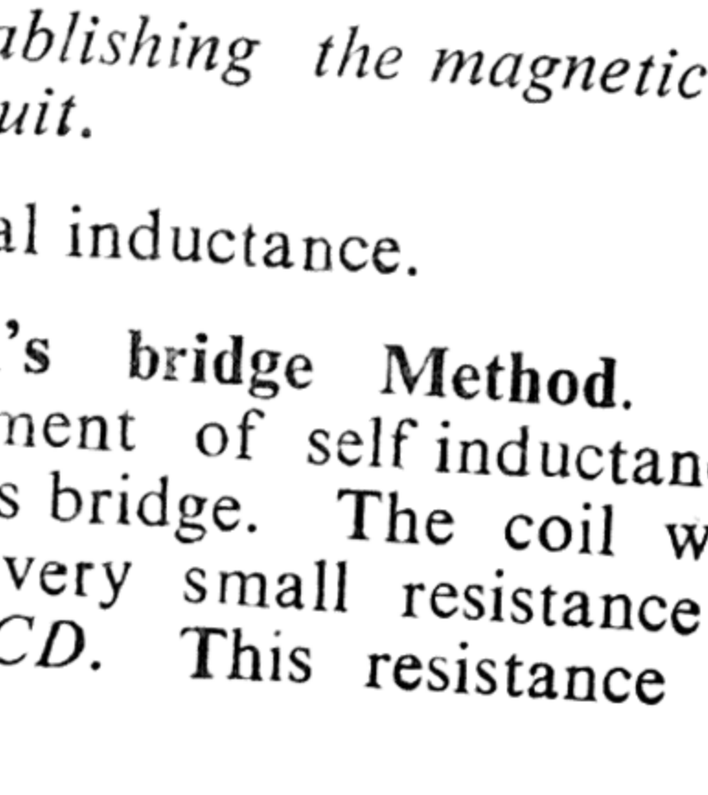


Fig. 9.16.

Initially keeping the key K_1 closed, P is made equal to Q and the resistance R is adjusted to a value for which galvanometer coil does not deflect while pressing key K_2. In this case galvanometer current is zero and the inductance is not effective.

If galvanometer key K_2 is closed first and then the battery key K_1, galvanometer will observe the induced emf produced in CD arm due to the growth of current through the coil. This extra emf ($\mathcal{E}=-Ldi/dt$) causes proportionate currents in every branch of the circuit. Let the current in the galvanometer arm be $kL\,(di/dt)$. It is for the time in which current

in CD arm reaches to its steady value i_0 (say). Hence a impulse of charge q will pass through the galvanometer.

$$q=\int_0^t kL\,\frac{di}{dt}\,dt=\int_0^{i_0} kL\,di=kLi_0.$$

If θ is the first throw of the galvanometer coil due to this charge, then

$$q=kLi_0=\frac{T}{2\pi}\,K\theta\left(1+\frac{\lambda}{2}\right). \qquad ...(67)$$

To eliminate k and i_0, a small resistance r is connected in series by taking out the plug from key K_3. This small resistance will not affect the steady current i_0 and the distribution of currents in the arms of the bridge and will introduce an additional emf ri_0 in the arm CD. This will produce a steady currents kri_0 in the galvanometer. If the corresponding steady throw is θ', then

$$kri_0=K\theta'. \qquad ...(68)$$

From relations (67) and (68), we get

$$L=r\,\frac{T}{2\pi}\,\frac{\theta}{\theta'}\left(1+\frac{\lambda}{2}\right). \qquad ...(69)$$

9.14. CALCULATION OF SELF-INDUCTANCE

(*a*) **A long Solenoid.** Let us consider a long solenoid of length l, total number of turns N, and of area of cross-section A. If a current i is passing through it, then the magnetic field inside the solenoid is

$B=\mu_0(N/l)\,i$ and flux $\Phi=BA=\mu_0 NiA/l.$

$\therefore$ Total magnetic flux linked with N turns $=\mu_0 iA(N/l).N$

The induced emf around N turns

$$\mathcal{E}=-\frac{d}{dt}\left(\frac{\mu_0 iN^2A}{l}\right)=-L\,\frac{di}{dt}$$

or $$L=\mu_0 N^2A/l=\mu_0 n^2\,Al. \qquad ...(70)$$

where $n(=N/l)$ is the number of turns per unit length.

Now if a second short coil (*known as secondary*) of m turns is wound on the solenoid, insulated from the first, the flux linked with the secondary due to the current i in the primary is

$$\Phi=BmA=\mu_0 nmAi=Mi \text{ (By definition)}$$

or Mutual inductance $M=\mu_0 nmA.$ ...(71)

If the solenoid is wound on the core of a material of constant permeability μ, then we have

$$L=\mu n^2\,Al \text{ and } M=\mu nmA. \qquad ...(72)$$

(*b*) **Long Parallel Wires.** Let P and Q be two parallel wires placed at a distance d apart in air. If a be the radius of either wire and the currents in the wires are each equal to i and are in opposite directions. The magnetic field at any point O, at a distance x from one wire due to the current in the wires is

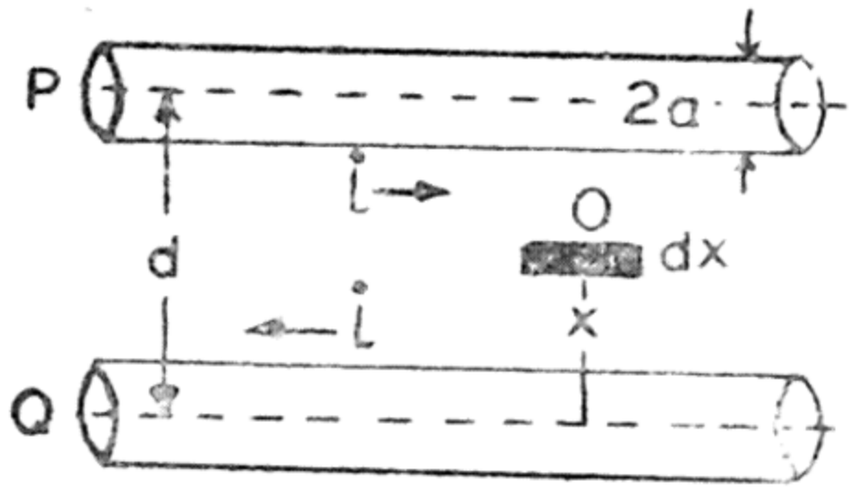


Fig. 9.17.

$$B=\frac{\mu_0}{2\pi}\left[\frac{i}{x}+\frac{i}{d-x}\right].$$

The flux through a small element of unit length and of width dx is

$$d\Phi=Bdx=\frac{\mu_0}{2\pi}\left[\frac{1}{x}+\frac{1}{d-x}\right]idx.$$

Hence the total flux passing through the entire area between two wires of unit length, is

$$\Phi=\frac{\mu_0 i}{2\pi}\int_a^{d-a}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx=\frac{\mu_0 i}{2\pi}\left[\log_e\frac{d-a}{a}-\log_e\frac{a}{d-a}\right]$$

$$=\frac{\mu_0 i}{\pi}\log_e\frac{d-a}{a}.$$

$\therefore$ Self inductance per unit length,

$$L=\frac{\Phi}{i}=\frac{\mu_0}{\pi}\log_e\frac{d-a}{a}. \qquad \ldots(73)$$

(*c*) **Co-axial Cylinders.** An important method of transmitting radio-frequency alternating currents over a distance is by means of a cable, a pair of coaxial cylinders of radii a and b ($<a$). The current in these cylinders is exactly equal in magnitude but in

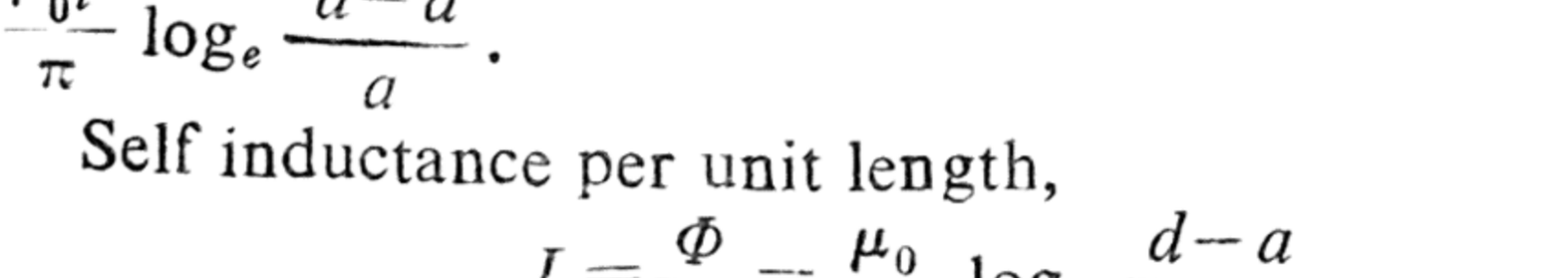


Fig. 9.18.

opposite direction. Due to this opposite current the field outside the cable is zero and thus the cable is non-radiating.

The magnetic field at a distance r from the axis and within the region between two cylinders is

$$B=\mu_0 i/2\pi r.$$

Hence the flux through the area between r and $r+dr$ and of length unity is

$$d\Phi=(\mu_0 i/2\pi r)\,dr.$$

$\therefore$ Total flux between the unit length of the cylinders

$$\Phi=\int_b^a \frac{\mu_0 i}{2\pi}\frac{dr}{r}=\frac{\mu_0 i}{2\pi}\log_e\frac{b}{a}.$$

Hence the self inductance per unit length

$$L=\frac{\mu_0}{2\pi}\log_e\frac{b}{a}. \quad \ldots (74)$$

If the *inner cylinder is solid*, the above result is not valid. Since the current i is distributed uniformly over the cross-section of area πa^2. The field B is $\mu_0 ir/2\pi a^2$ inside the inner cylinder and is $\mu_0 i/2\pi r$ outside to it. The value of the magnetic flux between unit length of the cylinders

$$d\Phi'=\int_r^b B\,dr=\frac{\mu_0 i}{2\pi a^2}\int_r^a r\,dr+\frac{\mu_0 i}{2\pi}\int_a^b\frac{dr}{r}$$

$$=\frac{\mu_0 i}{4\pi a^2}(a^2-r^2)+\frac{\mu_0 i}{2\pi}\log_e\frac{b}{a}. \quad \ldots(75)$$

Since this flux does not link the entire current i, but the fraction of a current that is in an annular ring of area $2\pi r dr$. Hence the equivalent flux linkage

$$d\Phi=\frac{2\pi r dr}{\pi a^2}\,d\Phi'=\frac{\mu_0 i}{\pi a^2}\left[\frac{(a^2-r^2)}{2a^2}\,r dr+\left(\log_e\frac{b}{a}\right)r dr\right]$$

$\therefore$ Induction per unit length

$$L=\frac{1}{i}\int_0^a d\Phi=\frac{\mu_0}{8\pi}+\frac{\mu_0}{2\pi}\log_e\frac{b}{a}. \quad \ldots(76)$$

Hence first term is the internal inductance of the inner solid cylinder.

9.15. COEFFICIENT OF COUPLING OF THE TWO CIRCUITS

Let us now calculate mutual inductance between two circuits in terms of the self inductance of each circuit alone. This is the case of maximum flux linkage, when all the flux through one circuit links the other. For this two coils of equal area (but not necessarily of equal number of turns) should be placed next to each other. Calling coils 1 and 2, we can write for the mutual inductance

$$M_{12}=N_1\Phi_{12}/i_2 \text{ and } M_{21}=N_2\Phi_{21}/i_1.$$

Similarly the self inductance of the coils is given by

$$L_1=N_1\Phi_1/i_1 \text{ and } L_2=N_2\Phi_2/i_2.$$

If all the flux of coil 2 links coil 1, then

$$\Phi_{12}=\Phi_2.$$

or the flux Φ_{12} in coil 1 due to the current i_2 in coil 2 is the same as the flux Φ_2 in coil 2 due to the same current i_2. Similarly

$$\Phi_{21}=\Phi_1.$$

Since $M_{12}=M_{21}=M$ (say), hence we have

$$M_{12}M_{21}=M^2=N_1N_2\Phi_1\Phi_2/i_1i_2=L_1L_2,$$

or
$$M_{max}=\sqrt{(L_1L_2)}. \qquad \ldots(77)$$

This is the maximum possible value of M as the total flux associated with one coil links with the other. In general only a fraction K_2 of Φ_2 passes through the coil 1. Hence $\Phi_{12}=K_2\Phi_2$.

Similarly only a fraction K_1 of Φ_1 is linked with coil 2, *i.e.*, $\Phi_{21}=K_1\Phi_1$.

$$\therefore \qquad M_{21}M_{12}=M^2=N_1N_2K_1K_2\Phi_1\Phi_2/i_1i_2=K_1K_2\Phi L_1L_2$$

or
$$M=K\sqrt{(L_1L_2)}, \qquad \ldots(78)$$

where K is a number, depending on the geometry of the coils and their relative positions and having value between 0 and 1.

9.16. COMBINATION OF INDUCTANCES

If the several inductances are in series and are arranged so that there are no interactions through mutual inductance. The induced voltage across such n isolated inductances, is given as

$$-L_1\frac{di}{dt}-L_2\frac{di}{dt}-L_3\frac{di}{dt}-\ldots L_n\frac{di}{dt}=-L\frac{di}{dt},$$

or
$$L=L_1+L_2+L_3+\ldots L_n=\Sigma L_i. \qquad \ldots(79)$$

If these inductances are arranged in parallel, then we have

$$\mathcal{E}_1=\mathcal{E}_2=\mathcal{E}_3=\mathcal{E}_4\ldots\ldots=\mathcal{E},$$

or
$$-L_1\frac{di_1}{dt}=-L_2\frac{di_2}{dt}=-L_3\frac{di_3}{dt}=\ldots\ldots=\mathcal{E}=-L\frac{di}{dt}.$$

As
$$\frac{di}{dt}=\frac{di_1}{dt}+\frac{di_2}{dt}+\frac{di_3}{dt}+\ldots=-\left(\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}+\ldots\ldots\right)\mathcal{E},$$

hence the equivalent inductance L is given by

$$\frac{1}{L}=\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}+\ldots \qquad \ldots(80)$$

The relations (79) and (80) show that the *non-interacting inductances* can be combined as the *non-inductive resistances.*

If the flux from one inductance links another, mutual inductance term becomes important. The voltage across the pair of such coils, connected in series, will thus be given as

$$\mathcal{E}=-L_1\frac{di}{dt}-L_2\frac{di}{dt}\pm 2M\frac{di}{dt}. \qquad \ldots(81)$$

Here the mutual inductance occurs twice, once for the voltage induced in the second coil by current changes in the first and once for the converse. This mutual interaction may increase or decrease the flux due to the self induction, hence the equivalent inductance of the pair of coils in series is

$$L=L_1+L_2\pm 2M. \qquad \ldots(82)$$

9.17. ENERGY STORED IN A MAGNETIC FIELD

We know that the current **j** is related with the electric field **E** as $\mathbf{j}=\sigma\mathbf{E}$, where σ is the conductivity of the medium. This electric field intensity is the sum of two terms :— ∇V due to accumulation of charge and $-\partial\mathbf{A}/\partial t$ due to the changing magnetic field

$$\therefore \qquad \mathbf{j}=\sigma\mathbf{E}=\sigma(-\nabla V-\partial\mathbf{A}/\partial t).$$

Consider an element in the form of a rectangular parallelepiped of length dl and area of cross section dS. If **j** is along $d\mathbf{l}$, a charge jdS will go from left hand to right hand face in one second and thus will set a pd of $-\nabla V$. $d\mathbf{l}$ across these faces. The power supplied by the source to this element of volume $d\tau$

$$dP=-\nabla V\,.d\mathbf{l}\,j\,dS=-\nabla V\,.\mathbf{j}d\tau=(\mathbf{E}+\partial\mathbf{A}/\partial t).\ \mathbf{j}d\tau.$$

$$P=\int_\tau\left[\mathbf{E}.\,\mathbf{j}+\frac{\partial\mathbf{A}}{\partial t}\cdot\,\mathbf{j}\,\right]d\tau=\int_\tau\frac{j^2}{\sigma}d\tau+\int\frac{\partial\mathbf{A}}{\partial t}.\,\mathbf{j}d\tau. \qquad \ldots(83)$$

The first term is the *Joule losses*. We are mainly concerned with the second term. Let us write latter as P_m or dW_m/dt.

(*a*) **In terms of B**— We know that curl $\mathbf{B}=\mu_0\mathbf{j}$, hence

$$\frac{dW_m}{dt}=\frac{1}{\mu_0}\int_\tau\frac{\partial\mathbf{A}}{\partial t}\cdot(\nabla\times\mathbf{B})\,d\tau$$

$$=\frac{1}{\mu_0}\int_\tau\left[\mathbf{B}.\left(\nabla\times\frac{\partial\mathbf{A}}{\partial t}\right)-\nabla.\left(\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{B}\right)\right]d\tau$$

$$=\frac{1}{\mu_0}\int_\tau\mathbf{B}.\frac{\partial\mathbf{B}}{\partial t}\,d\tau+\frac{1}{\mu_0}\int_s\left(\mathbf{B}\times\frac{\partial\mathbf{A}}{\partial t}\right).\,d\mathbf{S}. \qquad \ldots(84)$$

Here we have used relation $\nabla.(\mathbf{A}\times\mathbf{B})=(\nabla\times\mathbf{A}).\mathbf{B}-(\nabla\times\mathbf{B}).\mathbf{A}$ and the divergence theorem for the conversion of volume integral into surface integral.

If we choose S to be a spherical surface at infinity. As is shown for current loop, **A** falls off as $1/r^2$ and hence **B** as $1/r^3$ at large distances. Since the surface area $S\propto r^2$, hence the surface intergral decreases as $1/r^3$ and thus vanishes

$$\therefore\ \frac{dW_m}{dt}=\frac{1}{\mu_0}\int_\infty\mathbf{B}.\frac{\partial\mathbf{B}}{\partial t}\,d\tau=\frac{1}{2\mu_0}\frac{d}{dt}\int_\infty B^2d\tau.$$

Setting $W_m=0$, when $\mathbf{B}=0$. we get $W_m=\frac{1}{2\mu_0}\int_\infty B^2d\tau$ ...(85)

$\therefore$ Energy density $dW_m/d\tau=B^2/2\mu_0$. ...(86)

These relations show the similarity with the electrostatic energy in terms of **E**. As magnetic energy varies as B^2, hence the total energy due to several fields is not just the sum of energies calculated for separate fields.

(*b*) **In terms of j and A**—Using relation **B**=curl **A**, Eq. (85) becomes

$$W_m = \frac{1}{2\mu_0}\int_\infty \mathbf{B}.(\nabla \times \mathbf{A})d\tau.$$

Using the vector identity and the divergence theorem as used before, we have

$$W_m = \frac{1}{2\mu_0}\int_\infty \mathbf{A}.(\nabla \times \mathbf{B})d\tau + \frac{1}{2\mu_0}\int_\infty (\mathbf{A}\times\mathbf{B}).\ d\mathbf{S}.$$

The surface integral vanishes. As $\nabla \times \mathbf{B} = \mu_0\,\mathbf{j}$, hence

$$W_m = \tfrac{1}{2}\int_\infty (\mathbf{j}\ .\ \mathbf{A})d\tau,$$

where τ is any volume consisting all regions where $\mathbf{j} \neq 0$.

$\therefore$ Energy density $dW_m/d\tau = \frac{1}{2}(\mathbf{j}.\ \mathbf{A})$. ...(88)

(*c*) **In terms of i and** Φ—With filamentary currents one can replace $\mathbf{j}\,d\tau$ by $i\,d\mathbf{l}$, where $d\mathbf{l}$ is an element of the circuit carrying the current i, in Eq. (87). Thus we have

$$W_m = \tfrac{1}{2}\int i d\mathbf{l}.\mathbf{A} = \tfrac{1}{2}\ i \oint \mathbf{A}.d\mathbf{l} = \tfrac{1}{2}\ i\ \Phi. \qquad ...(89)$$

The directions of i and Φ are related by right hand screw rule.

(*d*) **In terms of** i **and L**—Since the magnetic flux can be replaced by the product of the self inductance L and the currents i, hence

$$W_m = \tfrac{1}{2}(Li) = \tfrac{1}{2}Li^2.$$

It can also be proved as given in section (9.13). For two circuits carrying currents i_1 and i_2, we have

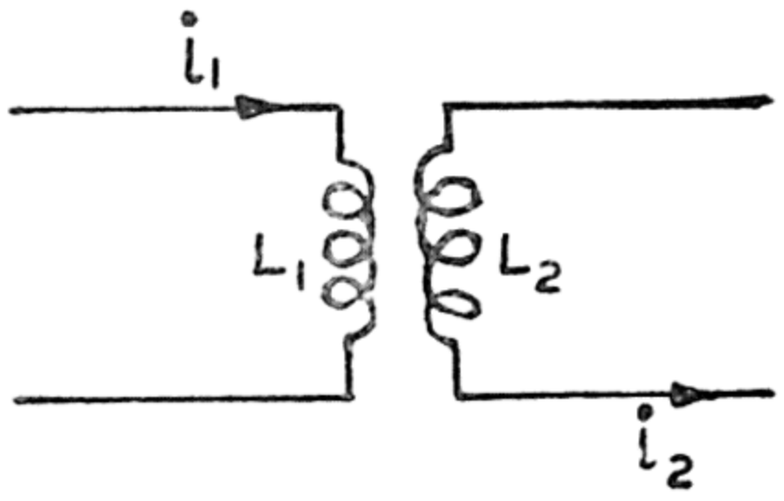


Fig. 9.19.

$$dW_m = -\mathcal{E}_1 dq_1 - \mathcal{E}_2 dq_2$$

$$= \left[\ L_1\frac{di_1}{dt} \pm M\frac{di_2}{dt}\ \right]dq_1 + \left[\ L_2\frac{di_2}{dt} \pm M\frac{di_1}{dt}\ \right]dq_2$$

$$= L_1 i_1 di_1 \pm M i_1 di_2 + L_2 i_2 di_2 \pm M i_2 di_1$$

$$= L_1 i_1 di_1 + L_2 i_2 di_2 \pm M d(\ i_1 i_2\ )$$

$\therefore$ Total energy stored $W_m = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$. ...(90)

The first two terms are the self energies and the third term is the interaction energy due to mutual inductance.

9.18. MAGNETIC FORCE BETWEEN TWO CIRCUITS

Consider two loops carrying currents i_1 and i_2 in the same direction. The loops tend to move toward each other and are fixed in the space only if the magnetic forces are balanced by equal

and opposite mechanical forces. We assume a small virtual displacement of one coil, without rotation. This displacement is very slow so the K.E. can be assumed as zero. For simplicity we neglect the Joule losses and thus have

Work done by the sources = Increase in magnetic energy + Mechanical work.

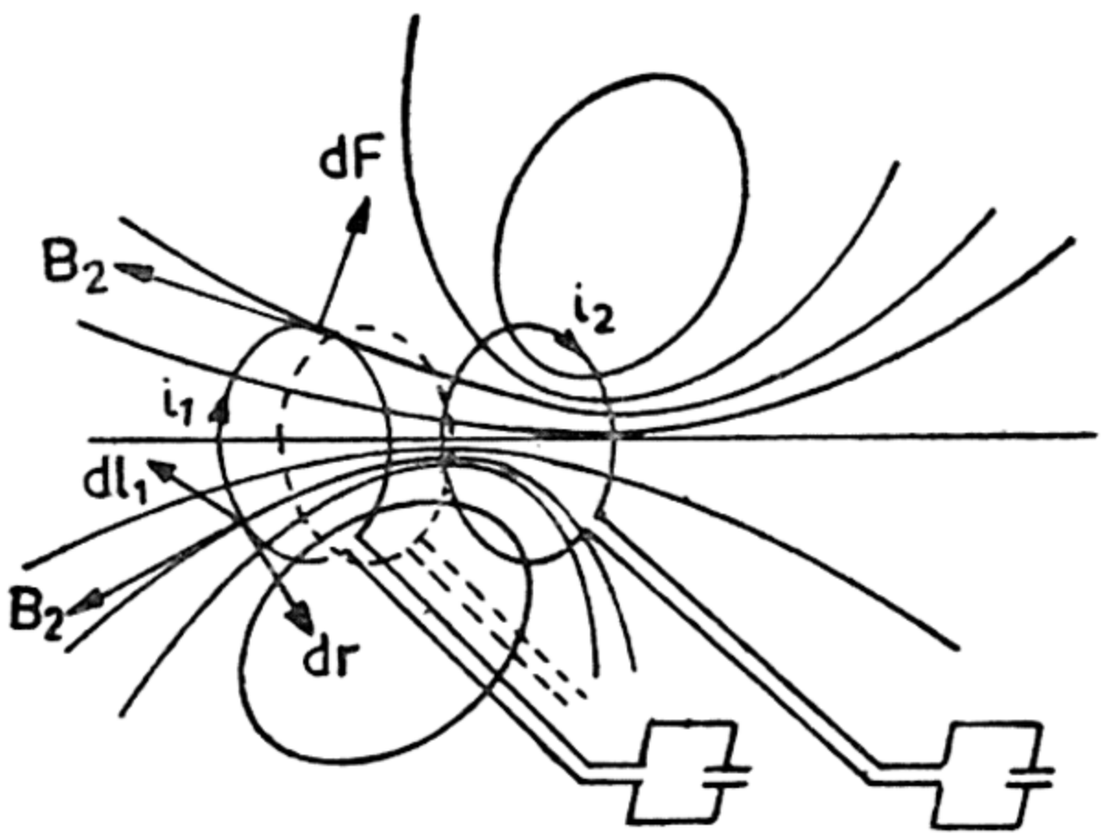


Fig 9.20.

The force between these coils can be found in two ways.

1. Currents are kept constant. Assumed that the loop 1 is allowed to move a distance *dr* toward loop 2. As the currents are kept constant, only the interaction energy changes and the self energies remain constant. Hence eqn. (60) gives.

$$dW_m = i_1 i_2 dM = i_1 d\Phi_{21} = i_2 d\Phi_{12},$$

where Φ_{12} is the flux originating in 1 and linking 2. Since Φ_{12} and Φ_{21} increase in this case, hence dW_m is positive and the magnetic energy increases.

In loop 1, Φ_{21} increases and the induced emf is set in such a way as to oppose a magnetic field $\mathbf{B}_2$ (originating in 2 and linking 1). It will tend to oppose the current i_1. To keep the current constant, the source voltage must be increased at each instant by $d\Phi_{21}/dt$. The source in loop 1 therefore supplies an energy

$$dW_{21} = i_1 (d\Phi_{21}/dt)\, dt = i_1\, d\Phi_{21} = i_1 i_2\, dM.$$

Similarly the amount of energy supplied by the source in loop 2

$$dW_{12} = i_1 i_2 dM.$$

$\therefore$ Total energy supplied by the sources $dWs = 2 i_1 i_2\, dM$ ...(91)

Since $i_1 i_2\, dM$ of energy is stored as a *magnetic energy*, hence the remainder $i_1 i_2\, dM$ would have gone into mechanical energy. If $\mathbf{F}_{21}$ is the force exerted by loop 2 on loop 1, then

$$\text{Mechanical work done } (dW_m)_{\text{constant } i} = \mathbf{F}_{21}.\, d\mathbf{r}. \qquad \ldots(92)$$

As dW_m is +ve, hence to keep the scalar product +ve, $\mathbf{F}_{21}$ must point toward coil 2, like $d\mathbf{r}$. It is right as we have assumed the motion of coil 1 toward 2.

Let us write relation (92) in the form of equation 8.59. Using Eq. (60), we get

$$\mathbf{F}_{21}.d\mathbf{r} = \frac{\mu_0}{4\pi} i_1 i_2 d\left(\oint_1 \oint_2 \frac{d\mathbf{l}_1 . d\mathbf{l}_2}{r}\right) = \frac{\mu_0}{4\pi} i_1 i_2 \oint_1 \oint_2 d\left(\frac{d\mathbf{l}_1 . d\mathbf{l}_2}{r}\right).$$

In the process of moving coil 1 by a distance $d\mathbf{r}$, both $d\mathbf{l}_1$ and $d\mathbf{l}_2$ remain unchanged and thus differential is to be taken of $1/\mathbf{r}$.

$$\therefore\ d\left(\frac{d\mathbf{l}_1.d\mathbf{l}_2}{r}\right)=d\mathbf{l}_1.d\mathbf{l}_2\left(\frac{-dr}{r^2}\right)=-\ d\mathbf{l}_1.d\mathbf{l}_2\ \frac{\hat{r}_{21}.dr}{r^2}$$

and
$$\mathbf{F}_{21}.d\mathbf{r}=-\frac{\mu_0}{4\pi}\,i_1i_2\oint\oint_2\frac{d\mathbf{l}_1.d\mathbf{l}_2}{r^2}\hat{r}_{21}.d\mathbf{r}.$$

Since dr on the right is independent of $d\mathbf{l}_1$ and $d\mathbf{l}_2$, hence

$$\mathbf{F}_{21}=-\frac{\mu_0}{4\pi}\,i_1i_2\oint_1\oint_2\frac{(d\mathbf{l}_1.d\mathbf{l}_2)}{r^2}\hat{r}_{21}. \qquad \ldots(93)$$

It is same as Eq. (:9) of chapter 8.

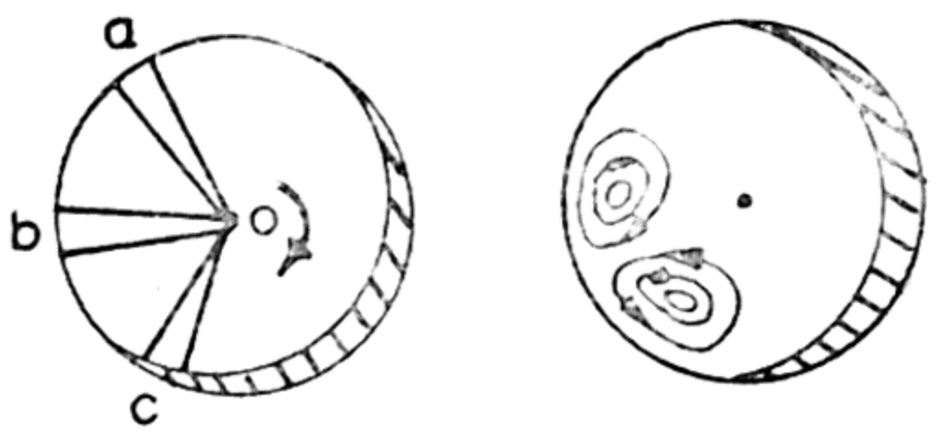
2. **Fluxes are kept constant:** Instead of currents if the fluxes are assumed constant, then during the vertical displacement the induced emf is zero. Hence no energy is supplied by the sources to the system, except Joule losses which are very negligible. Thus the mechanical work is accompanied by a decrease in magnetic energy.

$$\therefore\qquad \mathbf{F}_{21}.d\mathbf{r}=-(dW_m)_{constant\ \phi}.$$

9.19. EDDY CURRENT

Focault in 1895 discovered that whenever a conductor is placed in a region of varying magnetic field, induced currents are set up by induction. These induced currents circulate throughout the volume of the metal. Because of their general circulatory nature these are referred to as *eddy currents*. The direction of circulation of these currents is governed by Lenz's law.

If a magnetic field is applied to the portion of a rotating disc in the direction $\perp$ to the plane of the disc. Reference to Fig. 9.21(*a*), element *ob* is moving across the field and has an emf induced in it. Elements *oa* and *oc* are not in the field but, with all other elements which are outside the field, provide return conducting paths along which charges displaced along *ob* can return from *b* to *o*. A general eddy current circulation is thus set up in the disc as shown in Fig. 9.21(*b*).



(*a*) (*b*)

Fig. 9.21.

To find the direction of the eddy current, Arago performed a simple experiment. A *Cu*-disc was pivoted horizontally and a magnetic needle was suspended just above the disc, without touching it. When the disc was rotated rapidly, the needle was found rotating in the direction of the disc. If the direction of rotation of the disc was

reversed, the needle was observed as rotating in reverse direction. By applying Fleming left-hand rule, it is clear that the motion of the needle is as due to the current sets up in the disc.

Eddy currents are formed in the interior of the iron cores of the rotating armatures of motors and dynamos and also in the cores of transformers. These eddy currents are very undesirable both because of the energy which they dissipate and because of the flux which they themselves set up. These currents are nearly, although not completely, eliminated by the use of a laminated core, instead of a single solid piece of iron, Fig. 9.22(*a*). Thus core consists of thin

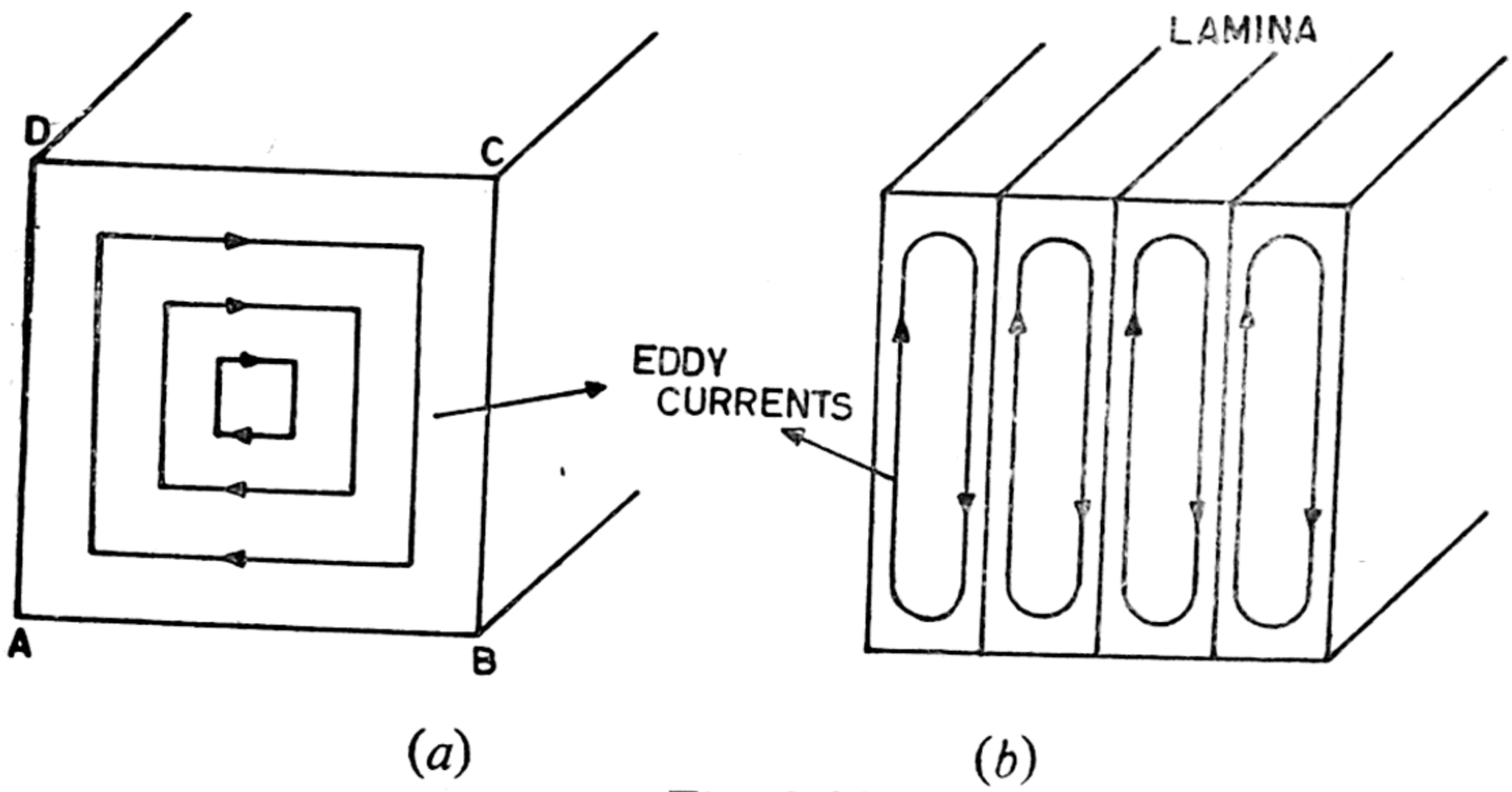


Fig. 9.22.

sheets of lamina, insulated from each other shown by curved lines, Fig. 9.22(*b*). In this way the eddy currents are restricted to the individual laminations instead of flowing across the whole core. The resulting length of path is greatly increased with consequent increase in resistance. Hence the induced currents and their heating effects are minimized.

Applications. (*i*) *Electromagnetic Damping.* An important application of the eddy current is made in galvanometers to make them dead beat. For this the coil is wound on a light metal frame. When the coil and frame rotate in the field of a permanent magnet, the eddy currents are set up in the frame. These currents produce a force which opposes the motion of the frame. The motion of the frame is thus damped. This damping is known as electromagnetic damping. The oscillations of a moving ballistic galvanometer are stopped by short circuiting the coil, where the induced current is set up in the coil itself.

(*ii*) *Induction Furnace.* Heating effect of eddy currents is used in induction furnaces, in which the substance to be heated is placed in a rapidly changing magnetic field. Heat produced by eddy current in the metal is so much that melting takes place. This process is used in extracting the metal from ore. Now the heating effect of eddy current is used for the heating of tissues.

(*iii*) *Induction Motor.* A metallic cylinder, when placed in a rotating magnetic field, rotates due to the eddy currents set up in it. The rotation is as given by Lenz's law. This is the principle of an induction motor.

(*iv*) *Electric Brakes.* When a strong magnetic field is suddenly applied to a rotating conducting drum, the eddy currents set up in the drum exert a torque which stops the motion of the drum. This principle is used in stopping electric trains.

9.20. DYNAMO

The principle of the dynamo has been discussed in article 9.8, but the actual form of the machine was not described. If the two ends of the rotating coil are connected to a pair of slip rings, the emf is enabled to produce an alternating current in the external circuit. To produce a current in one direction a device called *commutator* is used. It consists of a *split ring AB*, to each half to which is connected one end of the rotating coil. Two brass conductors *C* and *D*, called *brushes*, bear upon the split ring at opposite ends of a diameter. As the direction of the emf induced in the coil changes sign, each segment of the split ring moves on to the other brush, and hence the emf always acts in one direction in the external circuit. As shown in Fig. 9.23, the emf and current in the external circuit are constant in direction but not in value. Thus we get pulsating direct current.

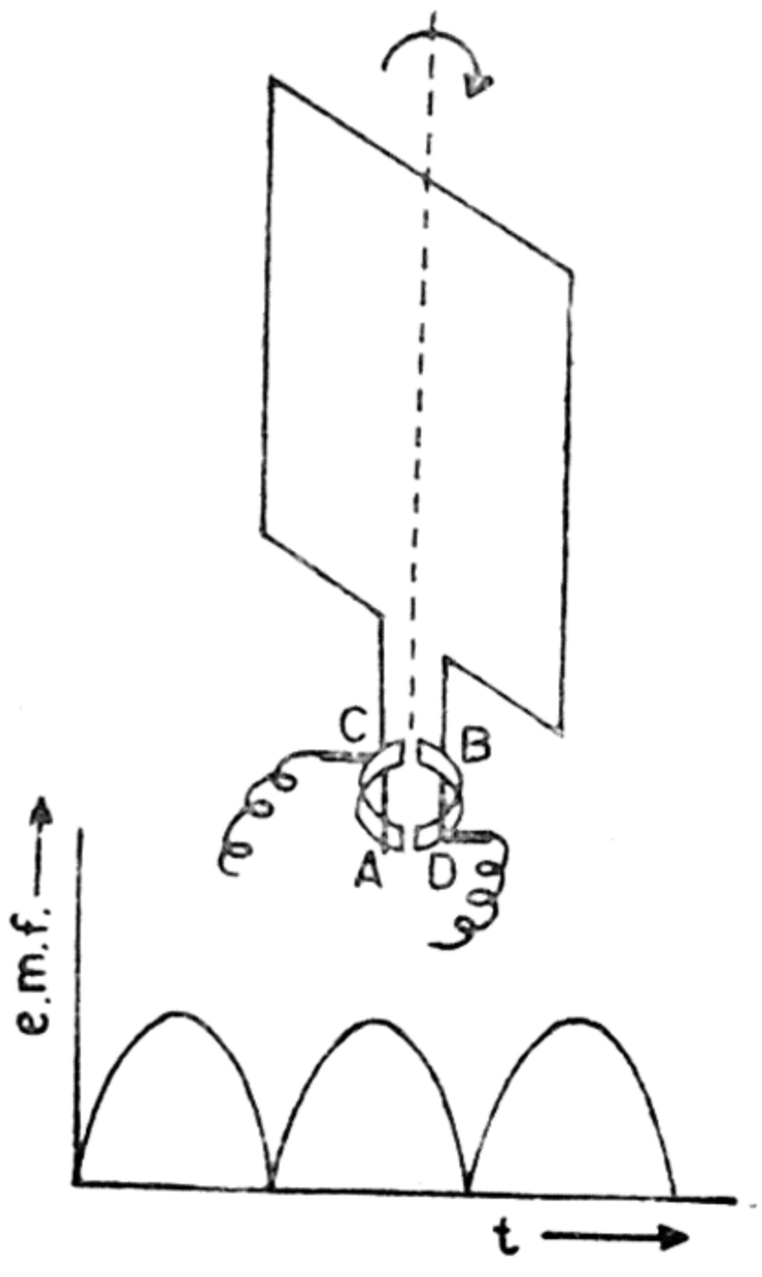


Fig. 9.23.

Continuous or direct current. Consider a pair of coils whose planes are at right angles to each other. As shown in Fig. 9.24, the ends *e* and *h* are joined to the commutator section *A*, and *f* and *g* are joined to section *B*. At the time of rotation of the coils, it will be seen that when the one coil is producing zero emf the other coil is producing its maximum emf. The resultant output emf is shown by thick curve in Fig. 9.24. It shows that the emf is not constant in value but varies between the limits *OA* and *OB*.

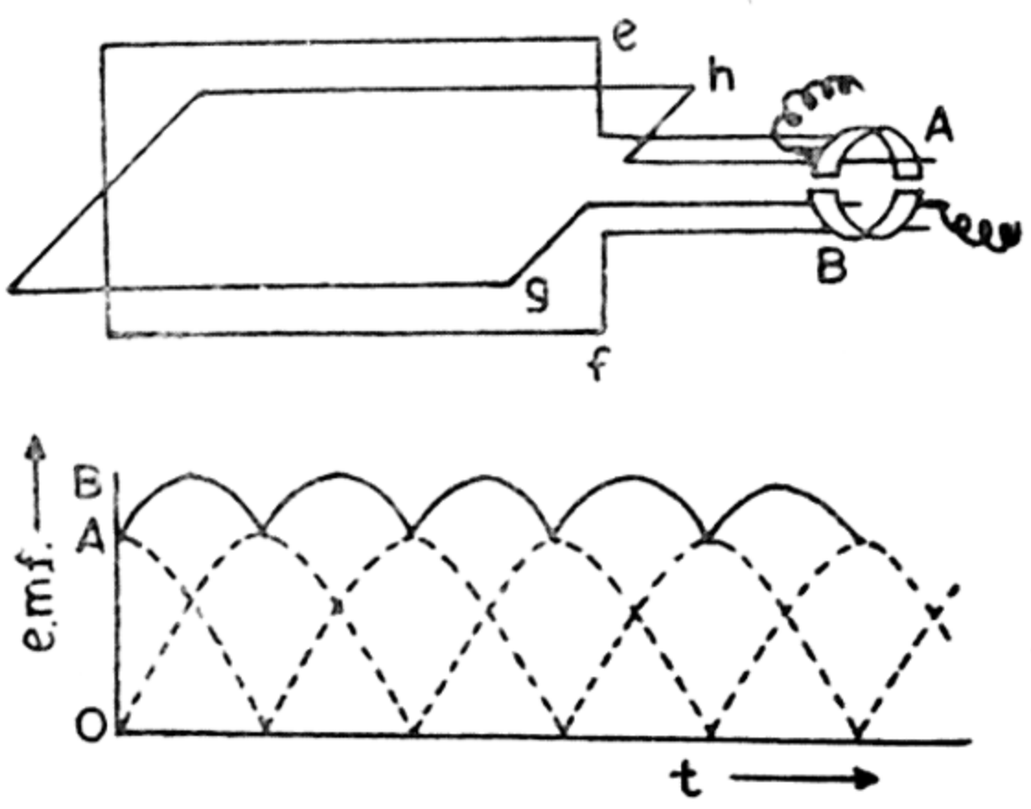


Fig. 9.24.

By increasing the number of conductors round the circumference of the *armature* (the rotating part of the dynamo), the emf and current curves may be made much smoother. There are many forms of armature winding. Let us discuss an armature of eight conductors

four coils). These conductors lie in slots *abcdefgh* in an iron core and are connected to the four commutator bars *ABCD* at the front of the armature, as shown by the continous lines, and to each other at the back of the armature, as shown by the dotted lines. The conductors *b*, *c* and *d* are descending, hence the emf is directed from back to front, represented by the dot in a circle. The conductors *f*, *g*, and *h* are ascending, hence the emf is directed from front to back, represented by the cross in a circle. Current enters the armature by the brush *E* and passing through paths *EAadDgbCF* or *EAfcBheCF* exists to the external circuit through brush *F*.

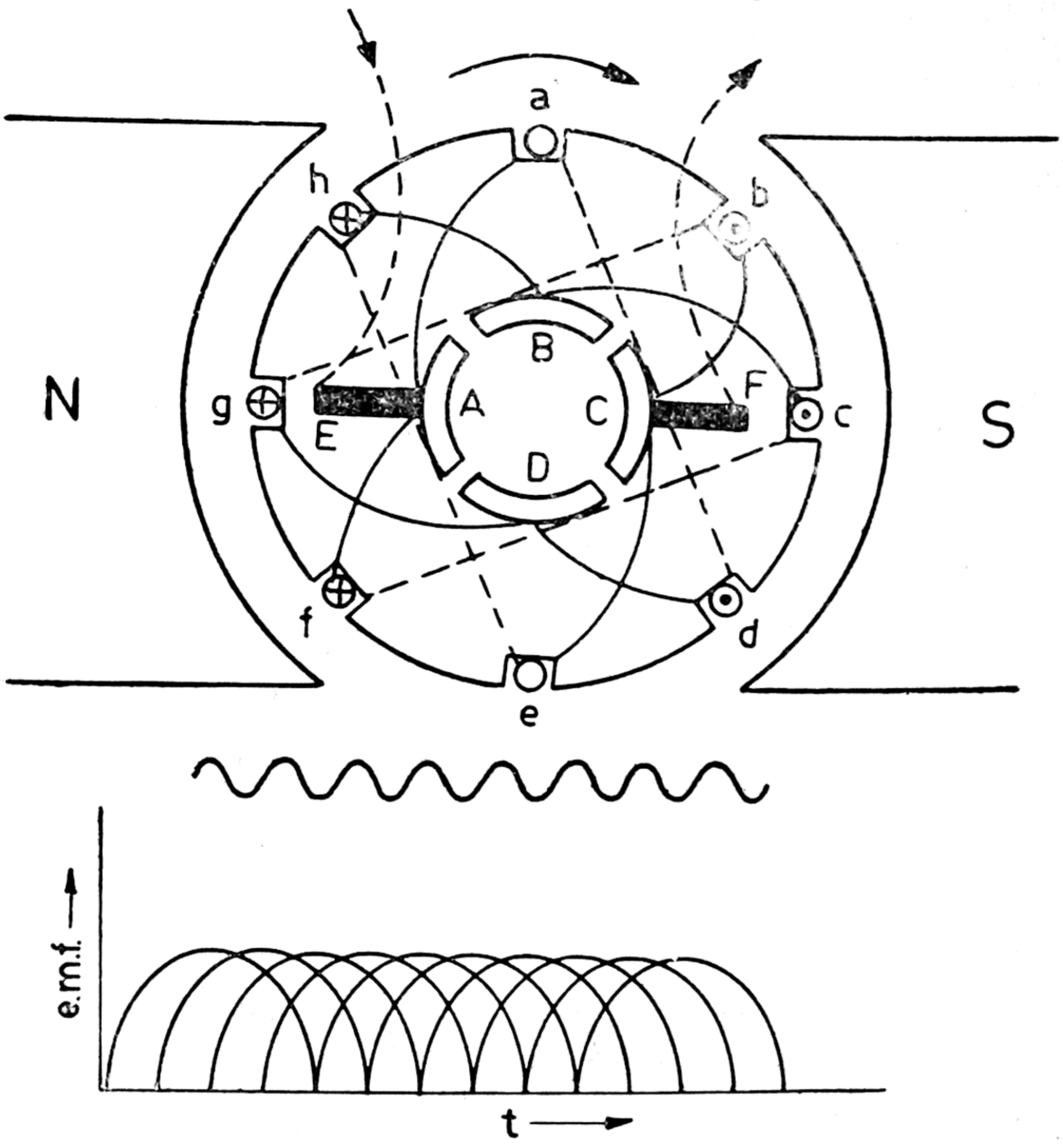


Fig. 9.25.

The way in which the mean voltage is increased and the ripple voltage is reduced by the superposition of a number of waves progressively different in phase is shown in Fig. 9.25. By increasing the number of conductors, we can get nearly constant potential, the armature from its shape is called a *drum armature*.

There are mainly two types of armature windings.

(1) *Lap*, (2) *Wave winding*. The difference in these types consists merely in the arrangement of the end connections at the

front of the armature. The length of the front connections is called the *front pitch* y_F and that of the back the *back pitch* y_B. These pitches are measured in terms of armature conductors. The two conductors which join two commutator bars constitute a *winding element*. The distance between the first and last conductors in an element is called the *resultant pitch* y.

$y = y_B - y_F$ for a lap winding,

$y = y_B + y_F$ for a wave winding.

Since the end connection of the consecutive conductors point alternately to right and left, hence the partial pitches y_B and y_F must be odd numbers. For number of poles p and number of conductors n the pole pitch is n/p.

Field Magnets. In small dynamos, the magnetic field is produced by an electromagnet. If the current required to excite the magnet is drived from a separate source, the dynamo is said to be *separately excited*. If the current is taken from the dynamo itself, the dynamo is said to be *self excited*. The former is used in the large dynamos. In most dynamos the current for exciting the field magnets is drived from the machine itself, but the ways are different.

(*a*) If the field winding is such that the armature and field coils are in series with the external circuit, this is called *series winding*. It is applicable for incandescent lamps, not for charging of accumulators. As the current flowing increases, the emf increases because the increasing current produces greater magnetic field and lines of force are cut by each conductor in every revolution.

(*b*) If the field winding is such that the external circuit and the field coils are in parallel, this is called *shunt winding*. It is particularly of much use for charging accumulators. In this case the emf of the machine drops as the current increases in the external circuit. As the whole emf of the armature is applied directly to the ends of the shunt coil, its resistance prevents an excessively great current flowing in it. The magnetisation therefore falls off and the e.m.f. thus decreases.

(*c*) As the emf rises with increasing current in series wound dynamo, and the emf decreases in shunt wound dynamo, hence the

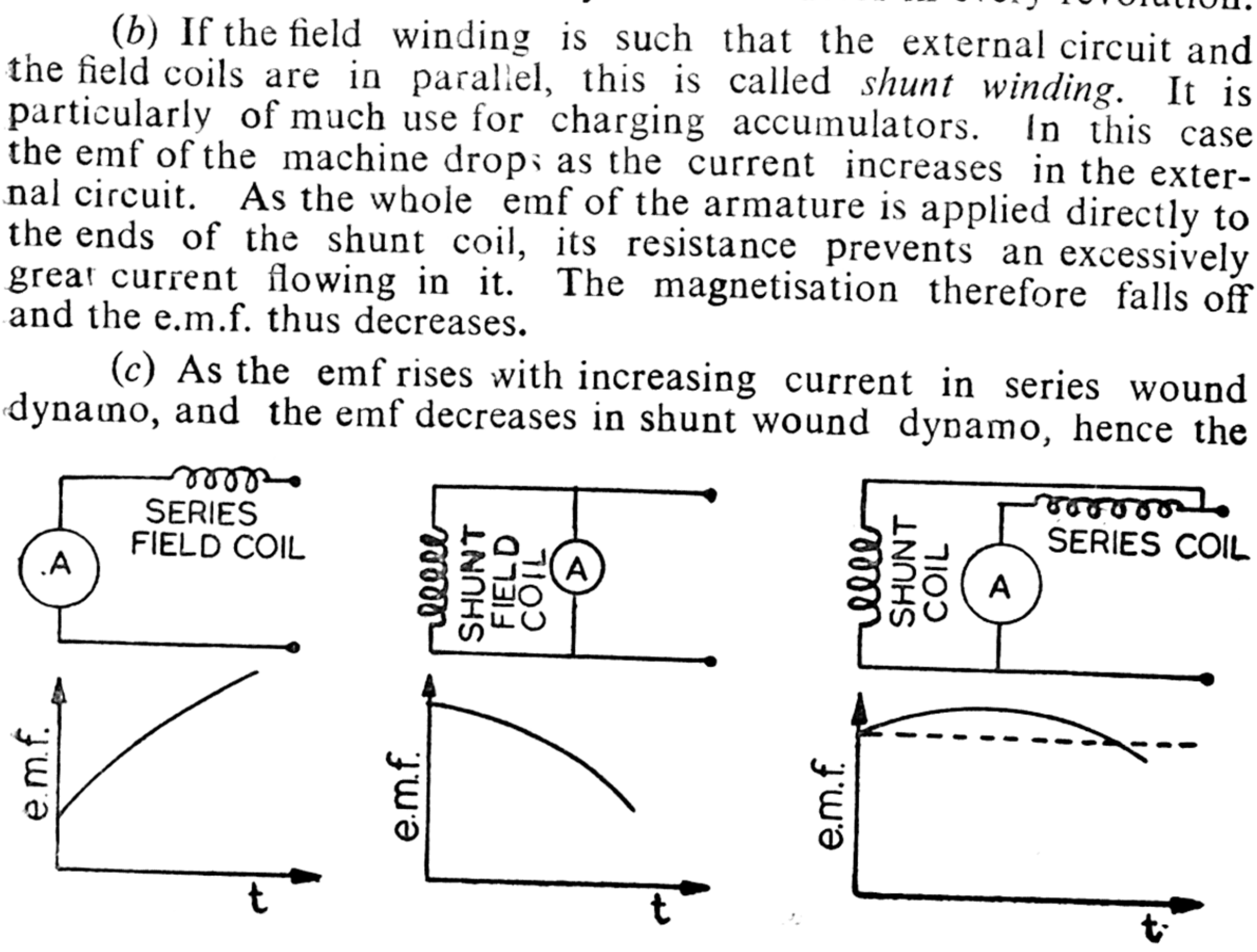


Fig. 9.26.

combination of these two may give a constant emf for all external currents. Such an arrangements is called *compound winding*. The arrangements with their characteristic curves are shown in Fig. 9.26.

Brushes. Two main types of brushes are generally used in dynamos : (*a*) The *metal brush* consists of thin sheet, wire or gauze of brass or copper, compressed into rectangular form. Its one edge bears against the commutator. (*b*) The *carbon brush* is made of compressed graphite. It rests normally on the commutator, hence does not require resetting if the direction of rotation is reversed.

As the commutator segment leaves the commutator brush, a back emf induced by the self inductance of the coil causes a sparking. This is reduced by using *carbon brushes*, as the area of the segment in contact with the brush diminishes, the increasing resistance makes the current change more smoothly through zero. The *second method* to reduce sparking is the alteration of the direction of the lines of magnetic induction produced by the current flowing through the armature. The main difficulty is the difficulty of adjusting the brushes to every variation in current. This problem is reduced by applying an external field from interpoles, which are connected with the armature and produce the field as required.

If the machine is with more than two poles, the number of brushes must be equal to the number of poles. If Φ be the total flux of magnetic induction from each pole, and p the number of poles, then the flux cut by each conductor per second is Φpf, where f is the frequency of revolution of the armature. If n is the number of conductors between successive brushes, then the emf between successive brushes is Φpfn.

9.21. ALTERNATORS

In a simple alternator (*alternating current dynamo*) a coil rotates in a magnetic field. The ends of the coil are attached to two insulated brass rings (say S_1 and S_2), called *slip rings*. Two pieces of metal called brushes touch slip rings, one rests on each. Alternative emf is observed between two terminals of the brushes T_1 and T_2. This alternator is of no use for practical purposes, since the speed revolution of the armature should be great.

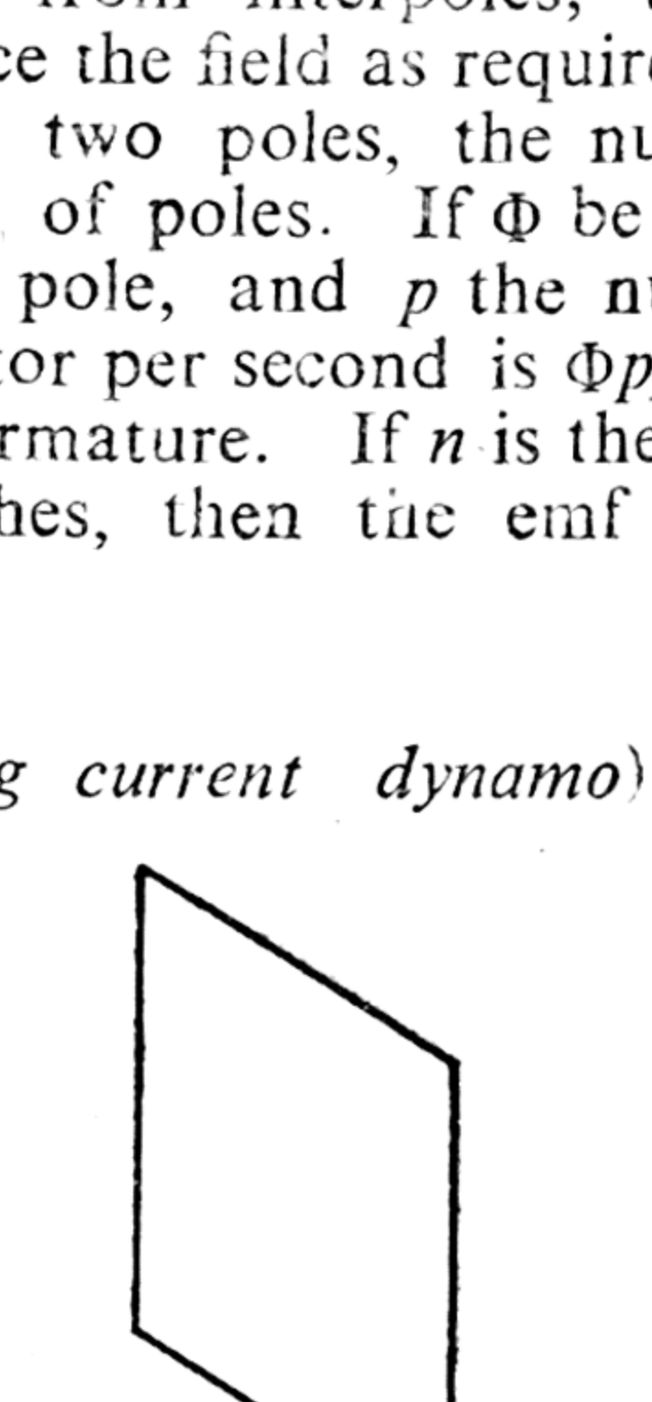


Fig. 9.27.

An arrangement, where the armature coils are on the outside and stationary and the poles rotate inside them, is very much used for generating high voltage alternating current. The stator is of iron to decrease the reluctance of the magnetic circuit and

is laminated to reduce eddy currents set up by the rotating poles. The poles are excited by coils on the rotor carrying direct current through slip rings. In large generators an electromagnet is used as a field magnet.

The alternators differ from d.c. generators in construction as :

(1) The former require no commutator. However slip-rings are used instead of it.

(2) In the former the magnets usually rotates (and are often called the *rotor*) while the armature is called the *stator*.

(3) The former are nearly always *separately excited*.

(4) The alternator windings are nearly always of the *open coils* type, *i.e.*, the armature windings come to terminals and the circuit is not complete until the external circuit is closed.

Emf in a Generator. When the armature rotates under the poles of an external field system, or the field is rotating keeping the armature stationary, the magnetic flux linked with the armature changes and thus emf is induced. If the direction of induced current, as given by Fleming's right hand rule, is positive during the first half turn of the armature, it will be negative during the next half turn. In this way alternating current is produced by the generator (Fig. 9.27)

If N be the number of turns and A the area of the armature, the magnetic flux Φ linked with the armature if the normal makes an angle θ with the direction of magnetic field B is given by

$$\Phi = NAB \cos \theta.$$

If the coil rotates with an angular velocity ω, then the induced emf

$$e = -\frac{d\Phi}{dt} = -\frac{d}{dt}(NAB \cos \theta) = -\frac{d}{dt}(NAB \cos \omega t)$$

$$= NAB\,\omega \sin \omega t = \mathcal{E}_0 \sin \omega t.$$

If the magnetic field B is in horizontal direction, the induced emf is zero when the armature is vertical $\omega t = 0$, it is maximum ($e = \mathcal{E}_0$) when the armature is horizontal $\omega t = \pi/2$, it is again zero when $\omega t = \pi$, maximum ($e = -\mathcal{E}_0$) when $\omega t = 3\pi/2$ and is again zero when the armature returns to its original position $\omega t = 2\pi$.

In India the frequency of ac supply is 50 cycles/sec.

In the Fig. 9.28, there are eight poles and the frequency of alternation is four times the frequency of revolution. The number of magnetic poles is considerable. The number of complete alternations of emf per revolution of the rotor is half the number of

poles in the rotator. It is because the turns in the coils are in series and oppositely wound. Hence a coil is passing a *N*-pole at the same time adjacent coil is passing a *S*-pole. The emf's produced all act the same round the stator circuit. As the poles are excited through slip rings, hence the rotor circuit is broken and the current in the external circuit is broken accordingly.

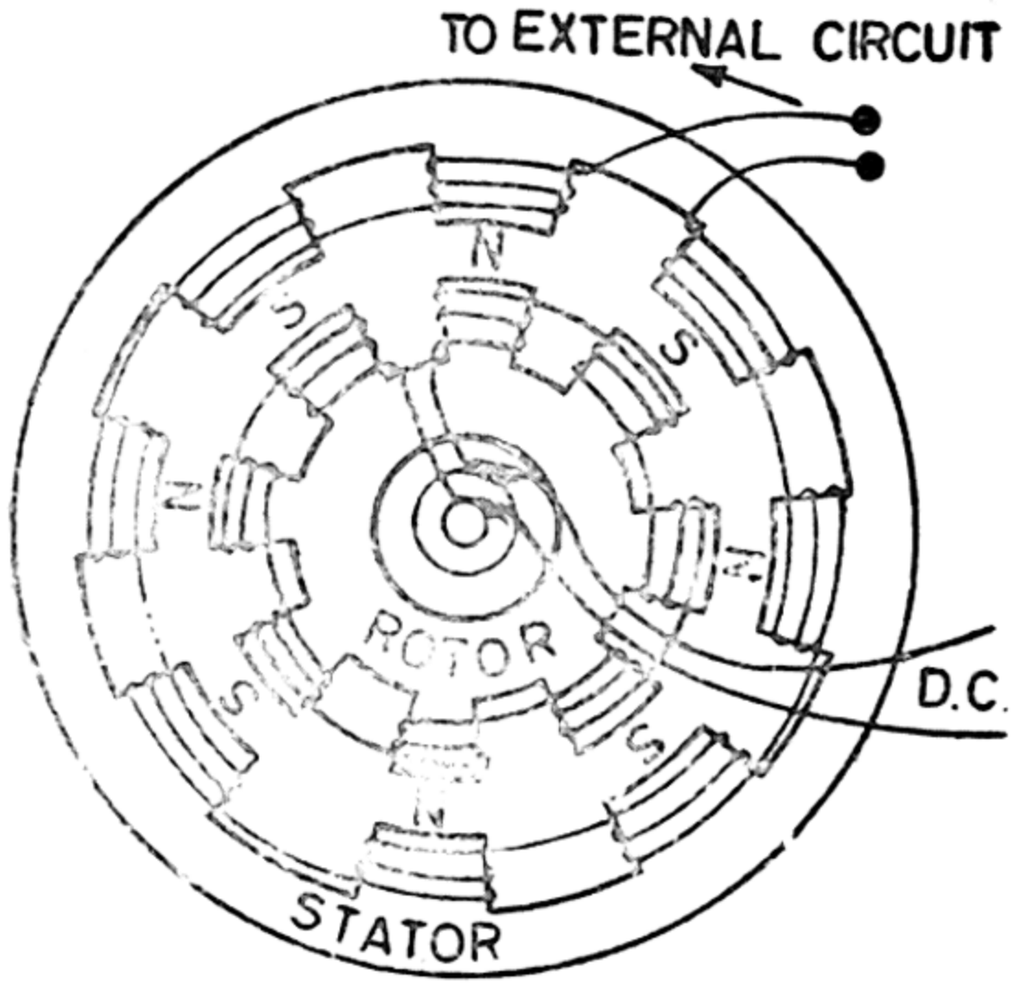


Fig. 9.28.

The above type of generator is called **single phase generator.** Two phase and three phase alternators are generally designed. **In a two phase alterator,** there are two coils placed at right angles to each other so that emf in one is maximum when it is zero in the other. Thus phase difference of 90° is maintained between the two. The number of turns and area of cross section of each coil are the same, therefore the amplitude and frequency of the generated emf's will be the same. The emf in the two coils are given by

$$v=V_0 \sin \omega t,\ v=V_0 \sin (\omega t+\pi/2). \qquad ...(94)$$

In a **three phase generator,** there are three coils inclined at 120° to one another, and are rotated at constant speed in a uniform magnetic field. The emf's generated will all be sinusoidal and of same frequency and of equal maximum values. The emf of coil *B* will lag behind that of *A* by $\frac{1}{3}$ of a period and that of coil *C* will lag by a further $\frac{1}{3}$ of a period ($\frac{2}{3}$ in all). Hence emf's are given by

$$v=V_0 \sin \omega t,\ v=V_0 \sin (\omega t+2\pi/3),\ v=V_0 \sin (\omega t+4\pi/3). \qquad (95)$$

As one complete cycle of emf is generated by the passage of one pair of poles, hence for $2n$ poles (or n pairs) on the rotor, n cycles per revolution will be generated. If m is the number of revolutions per sec, emf of frequency nm cycles/sec will be generated.

The polyphase sys em has the following advantages.

(1) In polyphase alternators the total power does not fluctuate, while in a single phase generator the current fluctuates.

(2) The output power of a polyphase alternator is always greater than that of a single phase generator of the same size.

(3) The polyphase alternators are used for power generation as they are superior for transmission and distribution purposes,

Distribution of three phase alternating current. The number of terminals to carry current of 3-phases, is reduced to four terminals. The starting point of all the three windings is made common and is called the *neutral.* In this way power can be handled by three heavy

transmission wires and one light one, instead of six heavy wires would require in three separate single-phase systems. There are following two ways of connecting these wires :

(*i*) *Star connection.* The common point of the three wires is the star point and the connecting wires are as the prongs of a star. The total current at star point is zero. A wire, known as *neutral wire* is connected with this point and three phase wires with the ends of the prongs. Single phase current can thus be obtained by connecting a load between any of the three phase wires and the neutral. If V_0 is the amplitude of the voltage, the voltage difference between two phase wires

$$\mathbf{V}=\mathbf{V}_2-\mathbf{V}_1 \ (=\mathbf{V}_3-\mathbf{V}_2=\mathbf{V}_3-\mathbf{V}_1)$$

or
$$V=V_0 \sin(\omega t+2\pi/3)-V_0 \sin \omega t$$

$$=2V_0 \cos(\omega t+\pi/3) \sin(\pi/3)$$

$$=\sqrt{3}\ V_0 \cos(\omega t+\pi/3). \qquad \ldots(96)$$

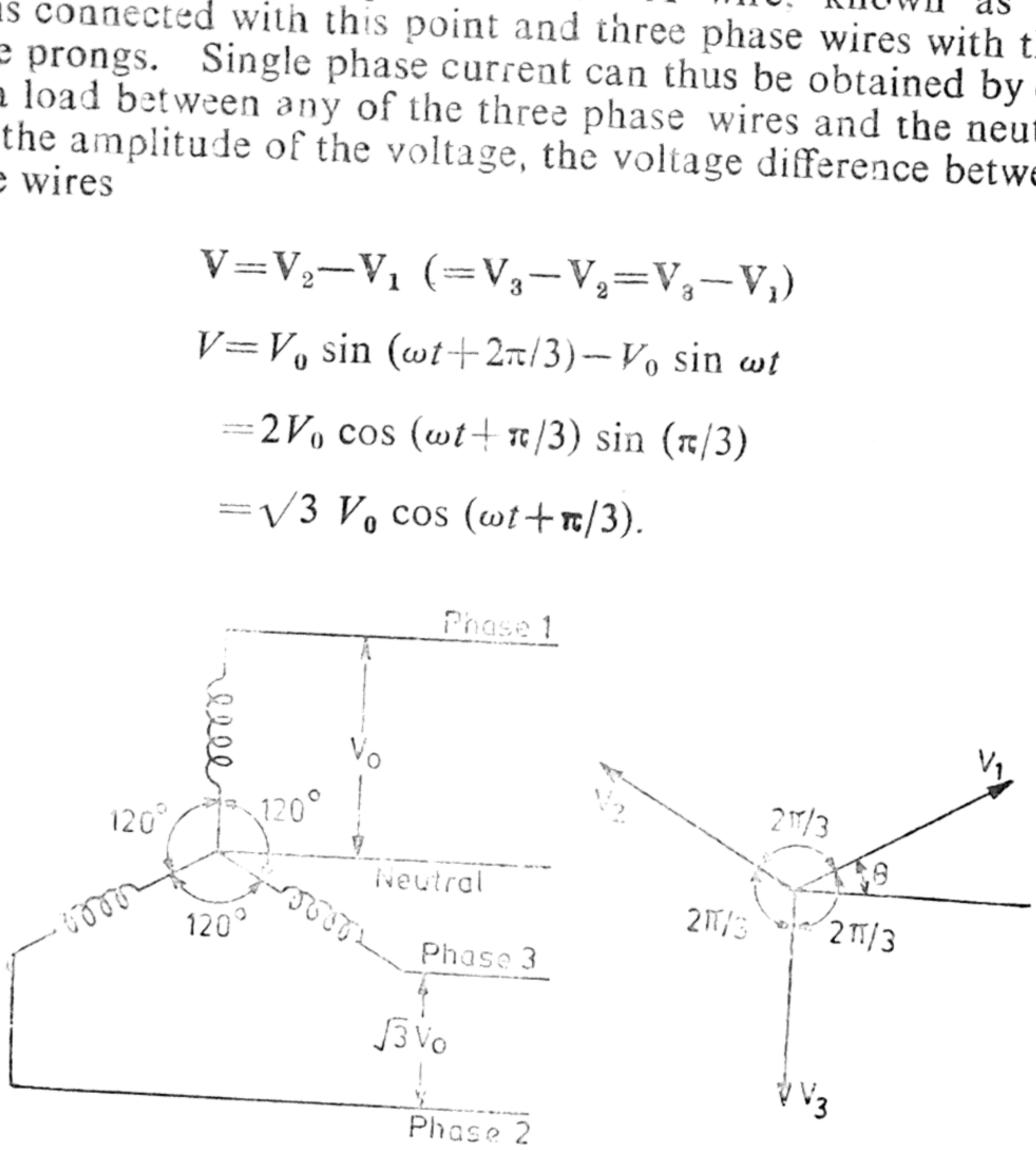


Fig. 9.29

Thus the voltage amplitude is $\sqrt{3}\ V_0$ and the phase difference is 30°. Thus for a house supply of 230 volts, one line wire and a neutral wire is connected to the supply. For power supply in factories 400 volts, three line wires and a neutral wires are connected to factory. Since no current passes through the neutral wire, hence the line current is same as that of phase current.

The line currents are equal to the phase currents in the star connected arrangements and the line emf is $\sqrt{3}$ times the phase emf. The power is given as

$$\text{Power } W=3\ V_{phase}\ I_{phase} \cos\phi=\sqrt{3}\ VI \cos\phi.$$

(*ii*) *Delta connection.* In this type of connections three windings are joined in series. Hence the *line voltage*, *i.e.*, the difference of voltage between two phase wires is same as that across each winding, which is known as *phase voltage*. The line current

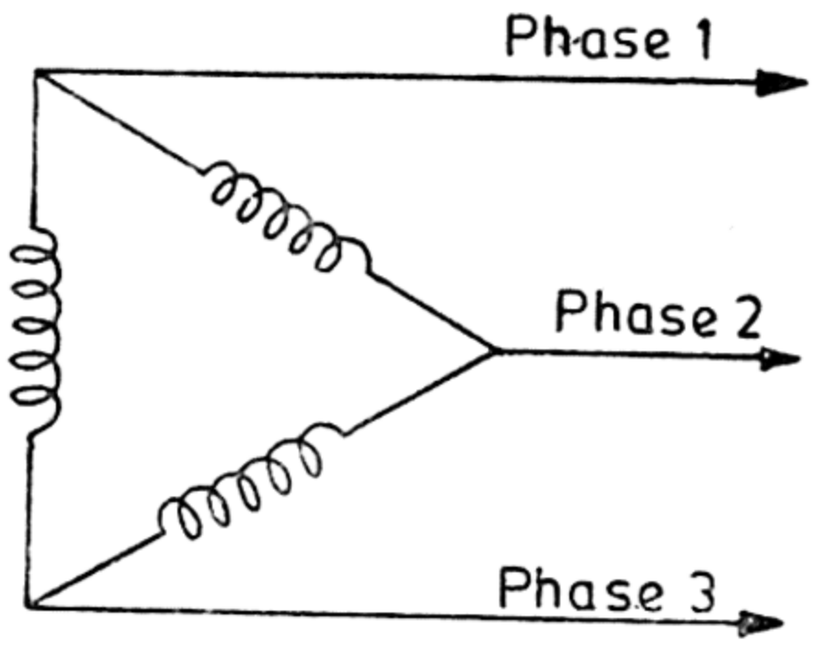


Fig. 9.30

$$\mathbf{I}=\mathbf{I}_2+\mathbf{I}_1$$
$$\mathbf{I}=I_0 \sin (\omega t+2\pi/3)-I_0 \sin \omega t$$
$$=2I_0 \cos (\omega t+\pi/3) \sin(\pi/3)$$
$$=\sqrt{3}I_0 \cos (\omega t+\pi/3). \quad ...(97)$$

Thus the line current is equal to $\sqrt{3}$ times the phase current.

In this way we see that the power consumption in both the cases is same.

9.22. DC MOTORS

A motor is a device to convert electrical energy into mechanical energy. It is based on the principle that a force acts on a conductor carrying a current and placed in a magnetic field. The principle may be better understood if reference is made to the article 8.8. From that the torque on a coil of N turns, carrying a current i, placed in a magnetic field B and enclosing an area A is given by the relation

$$\tau=iN\,AB \sin \theta, \quad ...(98)$$

where θ is the angle which the normal to the plane of the coil makes with the direction of magnetic field.

The ends of the coil are connected with each half of the split ring AB and current is allowed to pass through the brushes B_1 and B_2, bear upon the split ring at opposite ends of a diameter. When the plane of the coil is vertical, *i.e.* $\theta=0$, no torque acts on the armature (coil). As the armature has sufficient inertia and the frictional forces are small, the angular momentum gained earlier carries it through the zero torque position. After the vertical position the current passes in opposite direction due to split ring, and hence a torque acts in the same direction. Thus we see that *the armature rotates in the same direction with two positions of no torque in each revolution.* In this way electrical energy is converted into mechanical energy.

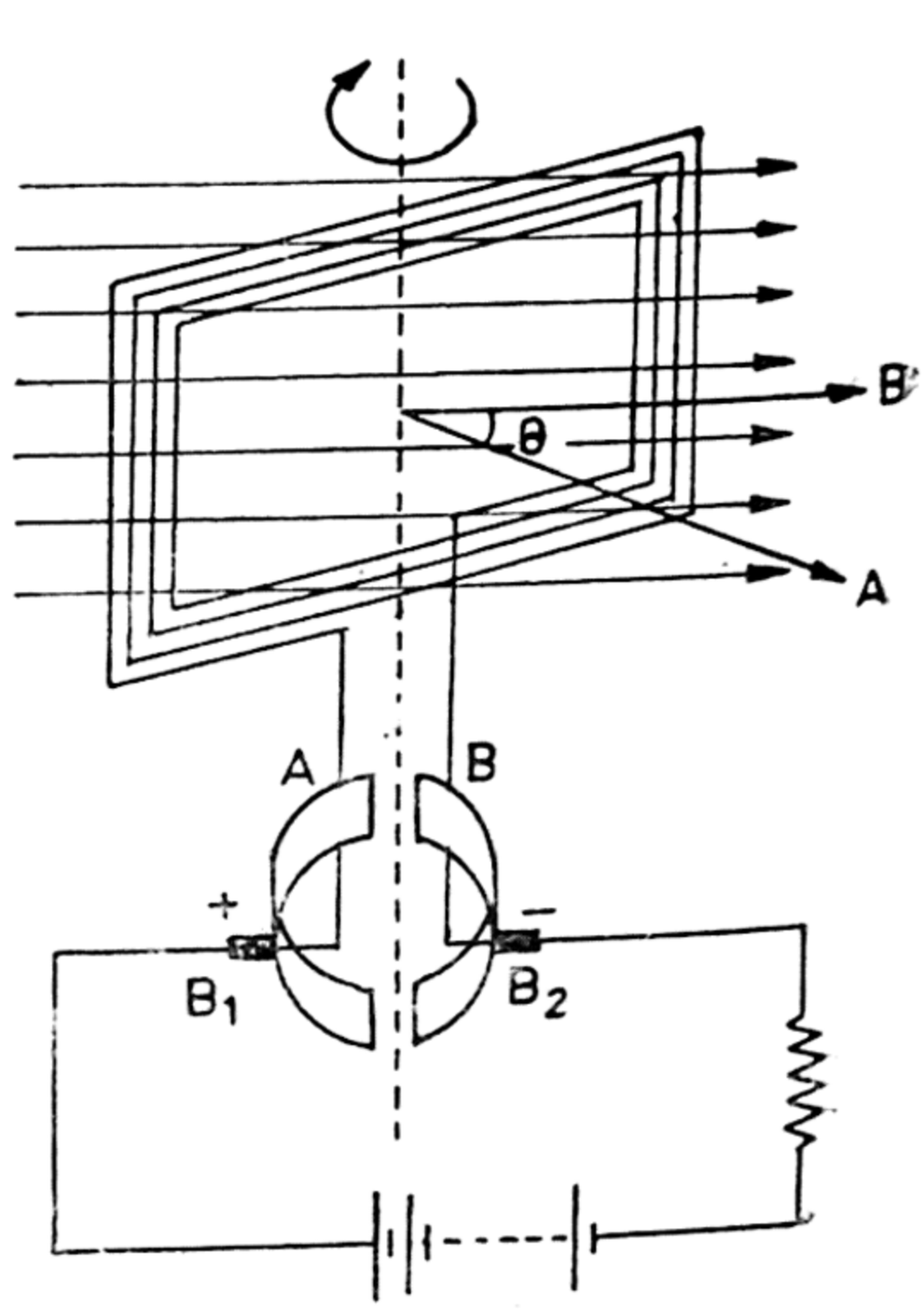


Fig. 9.31

The motion of motor with a single coil and two poles is not continuous. To make the

motor smooth running the single coil is replaced by several coils set at equal inclination to each other. A continuous drum winding may be preferred for this purpose. Refer to Fig. 9.28, dynamo with the eight armature conductors, let us apply some source of emf to the brushes. If current enters E and leaves F, the current in b, c and d will now be from back to front and in f, g and h from front to back. On applying left hand rule, it will be seen that b, c and d experience an upward force and f, g and h downward force. These forces will cause rotation in anti-clockwise direction, *i.e.* in the opposite direction to that in which it was previously driven by mechanical means. Thus we see that if *a current be supplied by external means to a dynamo the machine will run as a motor, the rotation being in opposite direction when running as a motor to that in which it, as a dynamo, was driven.*

Back e.m.f. in armature. When the armature is rotating, each conductor is cutting across magnetic lines of force and therefore emf is produced in it. This induced emf is proportional in value to the rate of change of number of lines of force, irrespective of the fact that the motion is due to this same magnetic field. To find its direction. apply the right hand law to the conductors b, c and d (of Fig. 9.28), it will be seen that emf due to cutting magnetic lines of force is from front to back and in f, g and h from back to front. Thus we get an emf in the opposite direction to the applied emf. It tends to reduce the current in the armature, hence is known as back *emf*.

In the beginning, the armature is at rest and there is no back emf. Hence the current depends upon the applied source and the resistance of the armature. As the armature picks up speed, the back e.m.f. increases and the current is consequently decreased. *This decrease is very much useful as, it prevents any injury in the insulation of the armature.*

If V be the potential applied at the brushes and $\mathcal{E}$ the induced back e.m.f., given by

$$\mathcal{E} = BAN\, d\theta/dt = BAN\omega, \qquad \ldots(99)$$

then the current through an armature of resistance R

$$i = (V - \mathcal{E})/R.$$

Hence the torque acting over the coil

$$\tau = BANi = BAN(V - \mathcal{E})/R. \qquad \ldots(100)$$

The mechanical energy produced per sec = The rate of working

$$= \tau\, d\theta/dt = BANi\omega = \mathcal{E}i.$$

As $V - \mathcal{E} = iR$, or $Vi = \mathcal{E}i + i^2R$,

Energy supplied/sec = Mechanical energy obtained/sec + Energy loss/sec.

$$\therefore \text{ Electrical Efficiency } \eta = \frac{\text{Mechanical energy obtained}}{\text{Electrical energy supplied}} = \frac{\mathcal{E}i}{Vi}$$
$$= \text{Back emf./Applied emf.} \quad ...(101)$$

η is maximum, when $\mathcal{E} = V$, but current becomes zero in this case and thus no work will be done. Work output is maximum when $\mathcal{E}i$ = maximum, or $\mathcal{E}\ (V - \mathcal{E})\ R$ = maximum,

$$\therefore \quad \frac{d\mathcal{E}}{R}\ (V - \mathcal{E}) - \frac{\mathcal{E}}{R} - d\mathcal{E} = 0 \quad \text{or } \mathcal{E} = \frac{V}{2} \quad ...(102)$$

Hence *for maximum outpnt of work the back emf should be half of the applied emf.* However this condition is not practicable, as the armature current will be too great to burn the coil.

Motor Starter. As the current is too great to burn the coil in the very beginning, some high resistance is required in series with the armature. This resistance may be cut out step by step until the full speed is attained and the back emf is sufficient to keep the current in the armature down to a safe value. Such a variable resistance is known as *starting resistance* and also as *starter*. One form of starting resistance for a shunt motor is shown in Fig. 9.32. When the motor is at rest the contact arm *AO* is pulled

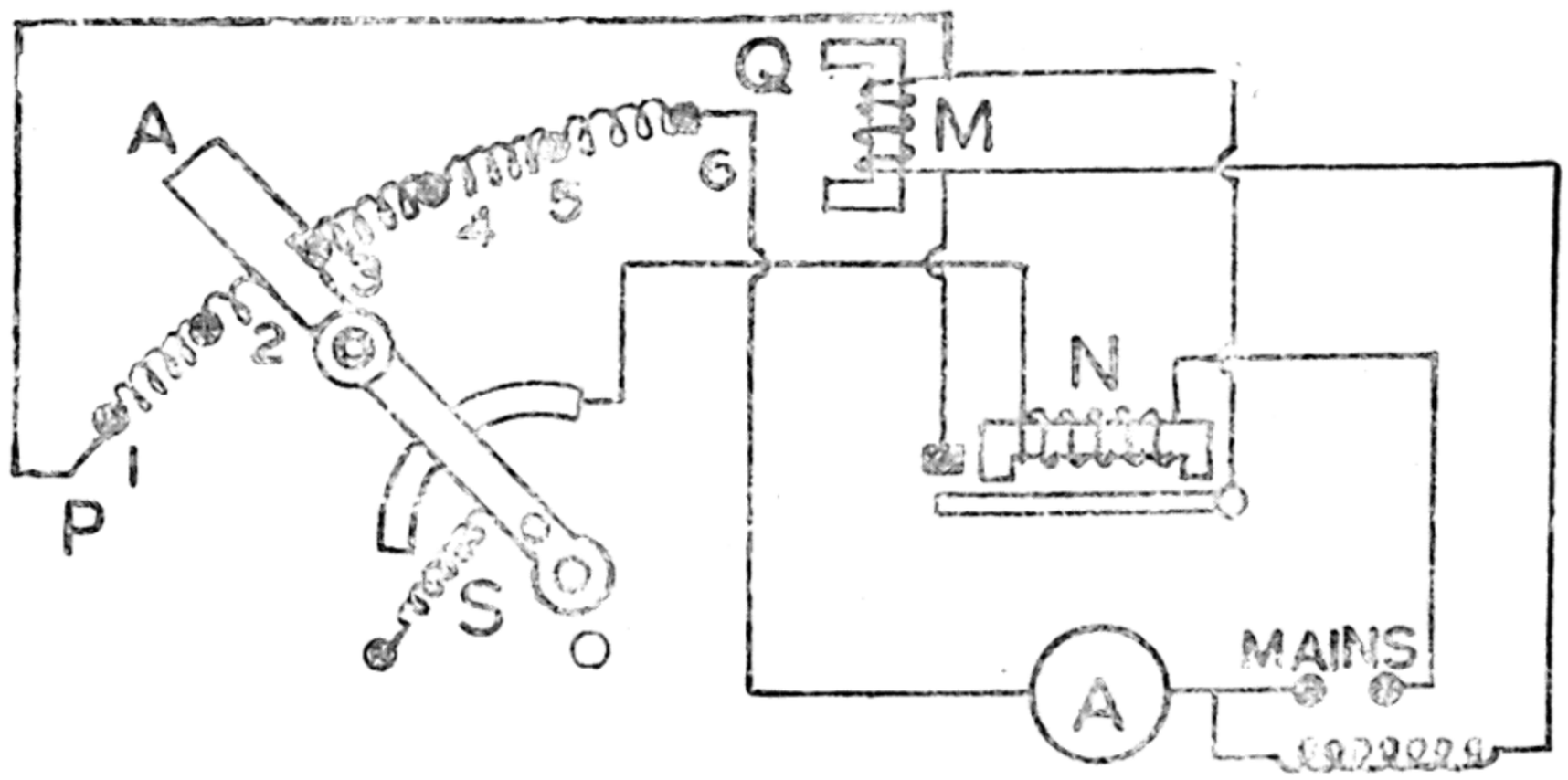


Fig. 9.32.

back by the spring *S* and the armature and field circuits are incomplete. When the arm is raised, it first makes whole resistance *PQ* in circuit. *AO* is moved forward step by step as the armature increases. Finally all the resistance coils are cut out and the armature arm comes in contact with the poles of the electromagnet *M*, which is magnetised by the current also flowing in the field coils of the motor.

If the electric supply stops suddenly the electromagnet loses its magnetisation and the contact arm *AO* is pulled back by the spring and thus cuts off the motor from the mains. This avoids the burning out of the armature due to the large induced emf produced in opposite direction and is thus known as *no volt release*.

Another safety device which the starter contains is the another electromagnet *N*, called the *over load release*. If the armature current becomes too great, the armature of this electromagnet is pulled up, thus short circuiting the electromagnet *M*. Hence the arm *AO* is released and goes back. In this way the supply current is automatically cut off, when it becomes too great. The starter is only used for large motors, not in small motors, as in latter case they gain speed rapidly.

The direction of rotation of motor can be reversed by reversing either the armature current or the field current, but not both as they will produce the rotation in the same direction as before.

Motor used as a brake. The modern electric trains and tram cars demand powerful emergency brakes in addition to the mechanical brakes. These emergency brakes, also named as *magnetic brakes*, consist of an electro-magnet suspended by two strong springs just over the rails. When a current flows, the electromagnet is energised and the shoe is attracted and makes contact with the rails.

To apply the brake, the driver disconnects the motor from the mains and the resistance coil is placed in the circuit. Owing to the momentum of the tram or train, the armature is still driven and the motor acts as a dynamo. The current produced by this dynamo brings the train to rest. Thus we see that the K.E. of the train is used to stop it.

Series, Shunt and Compound Wound Motors. Like dynamos, the motors are also of three types. In the *series wound*, torque is proportional to current^2. As current decreases when speed increases, hence the series motor exerts large torque at low speed and is used for cranes, lifts, sewing machines, etc. In *shunt wound*, the resistance of the field coils is high and field current is independent of the load, hence the *motor acts as a constant speed machine*. The torque is proportional to current. This type of motor is generally used in fans, blowers and other machines where constant speed is required. The *compound wound motor* possesses the advantages of both the series and shunt wound motors. It is used when a large initial torque is required and then in order to maintain speed, the series part of the field winding is automatically cut off, when the motor reaches a certain speed.

9.23. WATTMETER

Wattmeter is an instrument to measure the power consumed in a circuit in watts. In direct current circuits power may be measured by an ammeter and voltmeter. The *dynamometer type of instrument* is also used for this purpose. In this a low resistance

moving coil B is inserted in series with the main supply circuit, while the fixed coils AA have high resistance and are in parallel with the load. The current through coils AA is proportional to the voltage across the load, whilst the current through coil B is same as through the load. As the current I is passing through coil B suspended in the magnetic field due to coils AA through which current proportional to V is flowing, hence the torque acting on the coil B is given by

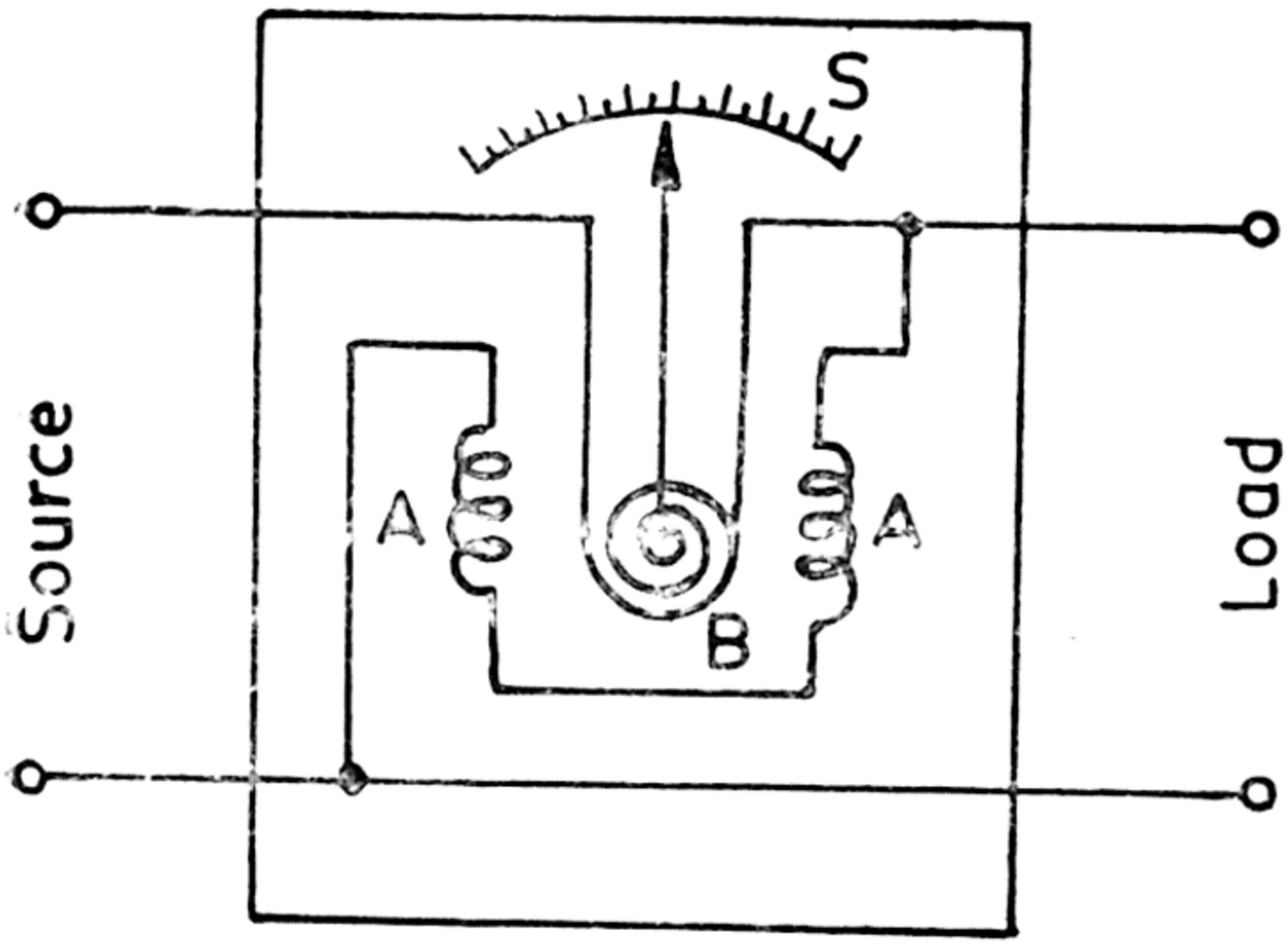


Fig. 9.33.

$$\tau \propto VI.$$

This torque produces a deflection ϕ in the coil, given by

$$VI \propto \phi. \qquad \ldots(103)$$

As VI is the power consumed. Hence the deflection of the coil B measures the power consumed. The scale over which the needle attached with the coil swings can thus be calibrated to read in watts.

A dynamometer type of A.C. wattmeter is suitable for low frequency (< 1 *kc/s*) circuits. A low resistance coil C is placed in the series with the *ac* supply and the load. As the main current i passes through it, hence is named as *current coil.* A high resistance coil P, also having a high non-inductive resistance R in series, is connected in parallel with the supply. This coil is named as *voltage coil or pressure coil.* The current i' passing through this coil is proportional to the *pd.* across the load. The mechanical force or couple between the two coils is proportional to the mean value over a complete cycle of the product ii'.

Owing to the absence of iron in the neighbourhood of the coils the emf across the voltage coil is in phase with the current through it. The pointer is attached to the voltage coil and arranged to move over a scale directly calibrated in watts.

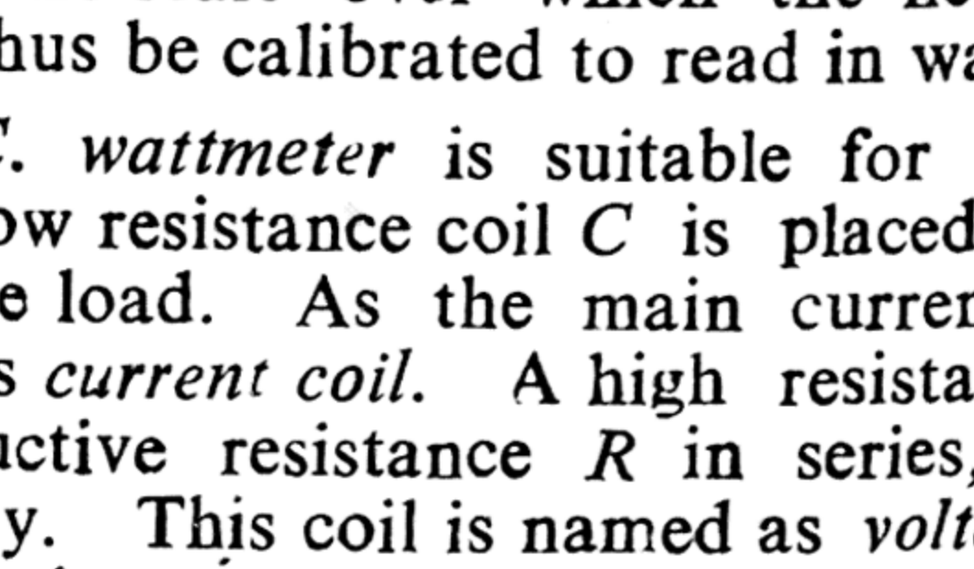


Fig. 9.34.

Due to some self inductance L of the voltage coil, current in it is not in phase with the emf strictly, but lags behind by an angle θ, given by $\theta = \tan^{-1} \omega L/R$. The error is very small if $R \gg \omega L$. If the supply emf is represented by $e = \mathcal{E}_0 \sin \omega t$, then the load current will be of the form

$$i = I_0 \sin (\omega t - \phi), \qquad \ldots(104)$$

where cos ϕ is the power factor of the circuit in which power is required. The current through the voltage coil is given by

$$i' = \frac{\mathcal{E}_0}{\sqrt{(R^2+\omega^2L^2)}} \sin(\omega t - \theta).$$

Hence the instantaneous value of power

$$ii' = \frac{\mathcal{E}_0 I_0}{\sqrt{(R^2+\omega^2L^2)}} \sin(\omega t-\theta)\sin(\omega t-\phi)$$

$$= \frac{\mathcal{E}_0 I_0}{2\sqrt{(R^2+\omega^2L^2)}} \left[\cos(\phi-\theta) - \cos(2\omega t-\theta-\phi).\right]$$

$\therefore$ The mean value over the complete cycle $= \dfrac{\mathcal{E}_0 I_0 \cos(\phi-\theta)}{2\sqrt{(R^2+\omega^2L^2)}}$

If $L=0$, then $\theta=0$ and the mean value of $ii' = \dfrac{\mathcal{E}_{rms}.I_{rms}.}{R}\cos\phi$,

which is proportional to the true power.

$$\therefore \quad \frac{\text{Indicated power}}{\text{True power}} = \frac{\mathcal{E}_0 I_0 \cos(\phi-\theta)}{2\sqrt{(R^2+\omega^2L^2)}} \times \frac{2R}{\mathcal{E}_0 I_0 \cos\phi}$$

$$= \frac{R}{\sqrt{(R^2+\omega^2L^2)}} \times \frac{\cos(\phi-\theta)}{\cos\phi}$$

$$= \sec\phi\cos\theta \times [\cos\phi\cos\theta + \sin\phi\sin\theta]$$

$$= \cos^2\theta + \tfrac{1}{2}\tan\phi\sin 2\theta. \qquad \ldots(105)$$

As this ratio is greater than 1, hence wattmeter should be calibrated to indicate the true power consumed in the circuit.

9.24. TRANSIENT CURRENTS

(*a*) **RC.—Series Circuit.** A capacitor C may be charged or discharged through a resistor R. **Charging** is done by connecting a battery of emf $\mathcal{E}$, capacitor C and a resistor R in series. If i is the instantaneous value of current at an instant t through R, the potential difference across the capacitor will be $\mathcal{E}-Ri$. If q is the instantaneous charge on the capacitor at this instant t, then we have

$$\mathcal{E}-Ri = q/C \qquad \text{or} \qquad \mathcal{E}-q/C = Ri.$$

Since $i = dq/dt$, hence

$$\mathcal{E}-\frac{q}{C} = R\frac{dq}{dt} \text{ or } \frac{dq}{\mathcal{E}-q/C} = \frac{dt}{R}.$$

On integrating, assuming $\mathcal{E}$ and C as constants, we get

$$-C\log_e(\mathcal{E}-q/C) = t/R + \text{const.}$$

Since when $t=0$, $q=0$, hence const$=-C\log_e \mathcal{E}$ and therefore

$$\log_e(\mathcal{E}-q/C)=-t/CR+\log_e\mathcal{E}.$$

$$\therefore \quad \log_e \frac{\mathcal{E}-q/C}{\mathcal{E}} = -\frac{t}{CR} \text{ or } \frac{\mathcal{E}-q/C}{\mathcal{E}} - e^{-t/CR}$$

or $$q=C\mathcal{E}(1-e^{-t/CR})=q_0(1-e^{-t/CR}). \qquad ...(106)$$

where $q_0=C\mathcal{E}$ = maximum charge associated with the capacity

The current $$i=\frac{dq}{dt}=\frac{C\mathcal{E}}{CR}e^{-t/CR}=\frac{\mathcal{E}}{R}e^{-t/CR}. \qquad ...(107)$$

The current and charge are therefore both exponential functions of time. At a time $t=RC$, the current has decreased to $1/e^{th}$ of its initial value. The product RC is known as the *time constant* or the *relaxation time* of the circuit. It is the time taken for the capacitor to acquire 63% of its maximum charge.

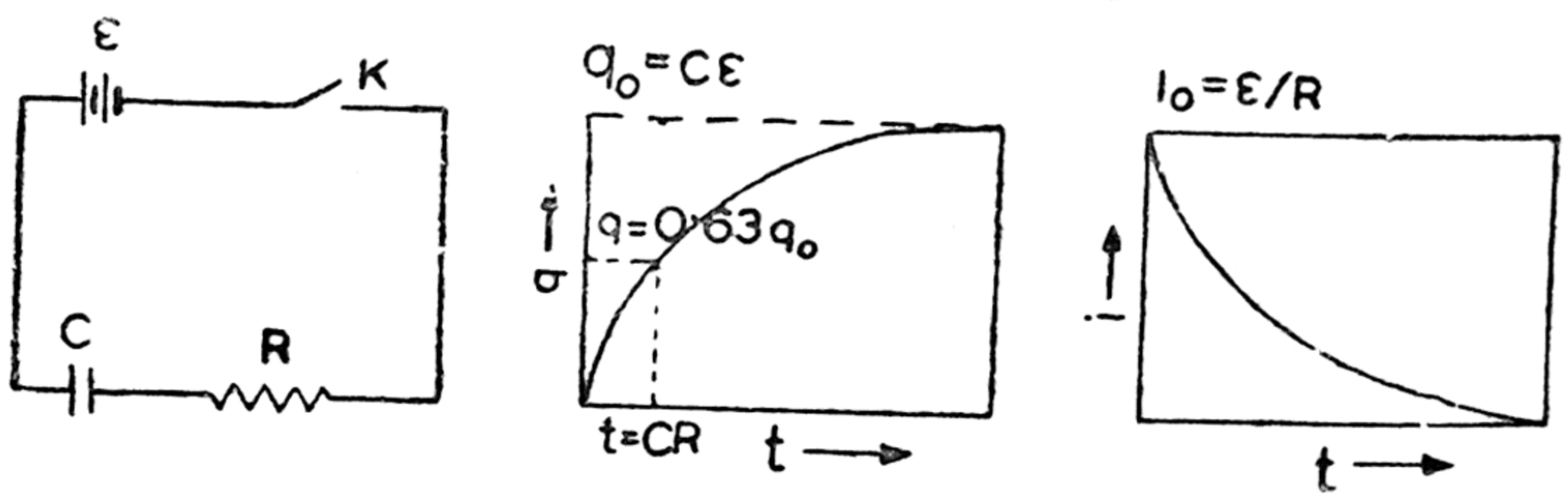


Fig. 9.35.

Eq. (106) and Fig. 9.35 show that the charge of a capacitor takes time to reach to its maximum value. It can be explained as:

When the circuit is closed, the initial current of $\mathcal{E}/R$ will set up in the circuit. Initially the capacitor experiences no potential difference as initial charge is zero. The flow of current through the resistor starts to charge the capacitor. Due to this charge, there will be a potential difference q/C across the capacitor. Since the sum of the potential differences (across the resistance and across the capacitor) must always equal $\mathcal{E}$, the applied emf, hence the potential difference across the resistor must decrease by the same amount q/C, which will reduce the charging current. Thus the charge of the capacitor builds up and the charging current decreases until the capacitor is fully charged. In time $t \gg RC$ or practically equal to $4RC$, the capacitor is fully charged for all practical purposes.

If a capacitor of capacitance C having initial charge q_0 is allowed to **discharge** through a resistance R, by connecting the latter in series. Let i be the instantaneous current in the circuit and q the charge on the capacitor at any instant t the emf equation in this case will be

$$iR+q/C=0 \quad \text{or} \quad R\,dq/dt+q/C=0.$$

$$\frac{dq}{q}=-\frac{1}{RC}dt \quad \text{or} \quad \log_e q=-\frac{t}{RC}+\text{const.}$$

As $q=q_0$ at $t=0$, hence const$=\log_e q_0$,

$$q=q_0 e^{-t/RC}. \qquad \ldots(108)$$

Current during discharge $i=\frac{dq}{dt}=-\frac{q_0}{RC}e^{-t/RC}$. $\qquad \ldots(109)$

Since $q_0=C\mathcal{E}$ and $\mathcal{E}/R=$Initial current i_0, hence

$$i=-i_0 e^{-t/RC}. \qquad \ldots(110)$$

The current thus falls exponentially. The initial current is due to the charge on the condenser. In the close circuit it is in the direction opposite to that in the case of charging. Eq. (107) and (110) show that the charging or discharging is more rapid for smaller values of RC.

Let us now calculate the *energy dissipated* in the resistance during the charging or discharging of a capacitor. The energy dissipated in time δt is given by

$$\delta W=i^2R\delta t=[(q_0/RC)e^{-t/RC}]^2R\delta t.$$

$$\therefore \quad W=\frac{q_0^2}{RC^2}\int_0^\infty e^{-2t/RC}\,dt=\frac{1}{2}\frac{q_0^2}{C}=\frac{1}{2}q_0\mathcal{E} \qquad \ldots(111)$$

This is independent of R. Eq. (111) shows that half of the total energy supplied by the battery is gained by the capacitor and rest half is dissipated in the resistor.

(*b*) **LR—Circuit.** Let us consider a circuit consisting of a battery, a coil of self inductance L, a resistor R and a key connected in series. If at time $t=0$ the key is closed so that the battery is in circuit. The current at any instant t flowing in this circuit is given by

$$\mathcal{E}-L(di/dt)=Ri. \qquad \ldots(112)$$

where $L(di/dt)$ is the emf induced in the circuit at this instant t due to the rate of increase of current across the inductance L, and R is the total resistance in the circuit.

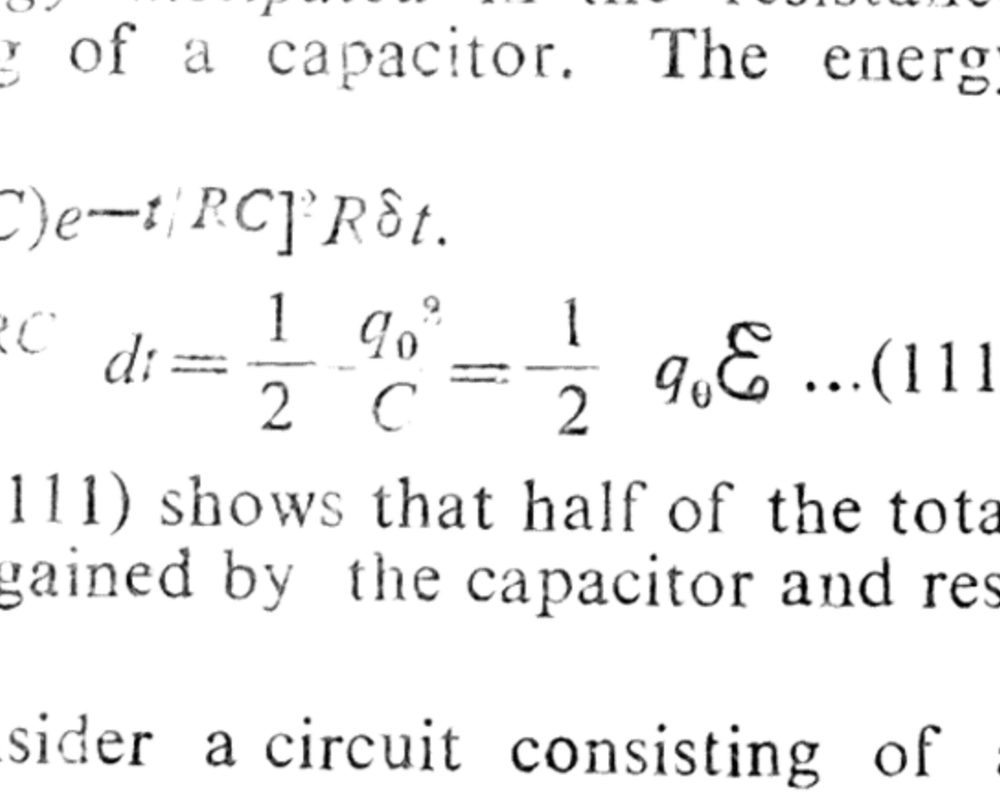


Fig. 9.36.

$$\frac{L\,di}{\mathcal{E}-Ri}=dt \quad \text{or} \quad L\int_0^i\frac{di}{\mathcal{E}-Ri}=\int_0^t dt$$

$$\therefore \quad t=-\frac{L}{R}\log_e(\mathcal{E}-Ri)+\frac{L}{R}\log_e\mathcal{E}=-\frac{L}{R}\log_e\frac{\mathcal{E}-Ri}{\mathcal{E}}$$

or $\qquad 1-Ri/\mathcal{E}=e^{-Rt/L}.$

$$\therefore \quad i=\frac{\mathcal{E}}{R}\left(1-e^{-Rt/L}\right)=i_0\left(1-e^{-Rt/L}\right). \qquad ...(113)$$

This is the relation for the current at any instant t. Here $i_0=(\mathcal{E}/R)$ is the steady value, the maximum value, of current when $t=\infty$. For $R=0$, current will not be infinite but $(\mathcal{E}/L)\ t$, obtained by direct integration of Eq. (112).

The rate of growth of current $\dfrac{di}{dt}=\dfrac{R}{L}\ i_0e^{-Rt/L}=\dfrac{R}{L}\left(i_0-i\right)$.

Thus we see that the rate of growth of current decreases, as $i\to i_0$ and for any value of current i depends on R/L. The greater the value of L/R (the ratio known as *inductive time constant*), the smaller will be rate of growth of current.

The time constant L/R is denoted by λ and is expressed in seconds if L and R are in henrys and ohms respectively.

If $t=\lambda$, then $i=i_0(1-e^{-1})=0.63\ i_0$.

Hence time constant is the time taken for the current to rise to 0.63 times its maximum value.

Decay of Current. After the current attains steady value i_0, the battery is removed by connecting the common terminal S with B. The current will fall and the self inductance of the coil will oppose this fall of current. During this stage, if i current is flowing at any instant t in the circuit then

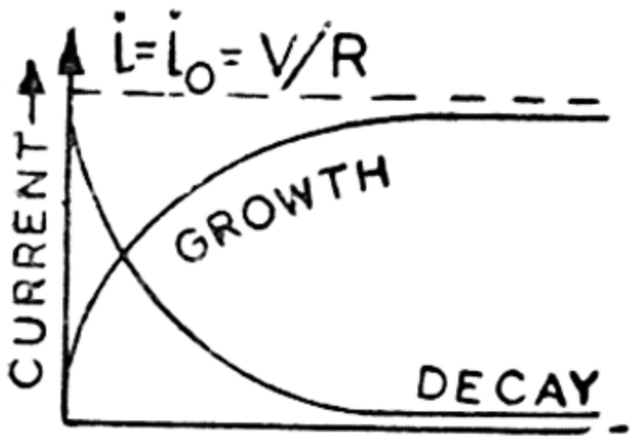


Fig. 9.37.

$$-L\frac{di}{dt}=Ri \quad \text{or} \quad dt=-\frac{L}{R}\frac{di}{i}.$$

On integration, we get

$$\int_{i_0}^{i} L\frac{di}{i}=-R\int_0^t dt \text{ or } L\log_e\frac{i}{i_0}=-Rt$$

$$\therefore \qquad i=i_0e^{-Rt/L}. \qquad ...(114)$$

Here we see that *the greater the value of time constant λ, the more slowly is the decay of current.*

From above results we see that the effect of the inductance is to prevent the current from falling instantaneously to zero. If the battery is suddenly open-circuited, a large induced emf $-L(di/dt)$ may cause a spark across the point at which the circuit is broken. *With large inductances very large voltages may arise in this way which can damage the insulation, if the circuit is broken suddenly.*

When the coil is switched on, the induced emf opposes the rise of current. The energy is supplied by the battery to do work against this emf. The total work done in bringing the current to its final value is stored as the magnetic field of the coil, which is taken back in the form of heat and light when the current decays,

(*c*) **LC-Circuit.** Let us consider a circuit consisting of capacitor C, a inductance L and a battery E in series. The resistance of the circuit is assumed to be negligible. First of all the common terminal S of the two way key is connected with the battery terminal A, and the *capacitor is charged*. When the battery is removed from the circuit, the charge begins to flow from one plate to another via inductance L.

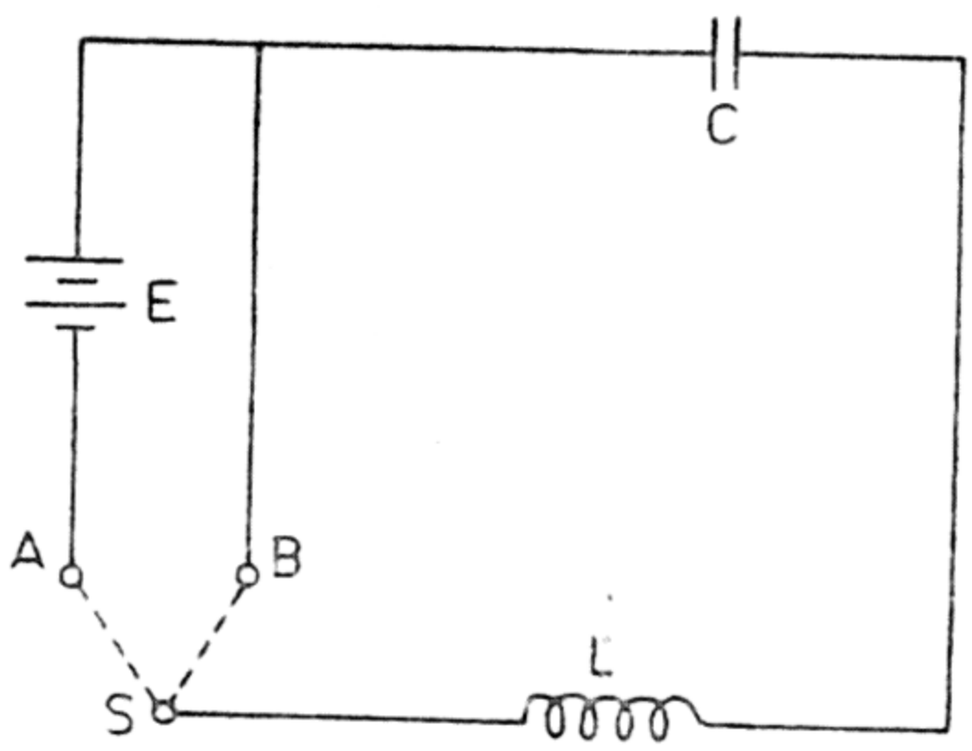


Fig. 9.38.

Let, at any instant t during discharge, q be the charge on the capacitor and i the current in the circuit. The *pd* across the capacitor is q/C and the induced emf set up in the inductance is $L\,(di/dt)$. Hence the total effective emf in the circuit

$$= q/C - L\,di/dt = 0. \qquad \ldots(115)$$

As $i = -dq/dt$, the rate of decrease of charge, hence

$$\frac{q}{C} + L\,\frac{d^2q}{dt^2} = 0 \text{ or } \frac{d^2q}{dt^2} + \frac{q}{LC} = 0,$$

or $$d^2q/dt^2 + \omega^2 q = 0 \qquad (\text{where } \omega^2 = 1/LC). \qquad \ldots(116)$$

This is the linear differential Eq. of second order with solution

$$q = Ae^{\alpha t}, \qquad \ldots(117)$$

where A and α are arbitrary constants, can be evaluated. Differentiating Eq. (117) with respect to t, we get

$$dq/dt = A\alpha e^{\alpha t} \text{ and } d^2q/dt^2 = A\alpha^2 e^{\alpha t}.$$

Substituting these values in relation (110), we get

$$A\alpha^2 e^{\alpha t} + \omega^2 A e^{\alpha t} = 0 \quad \text{or} \quad Ae^{\alpha t}(\alpha^2 + \omega^2) = 0.$$

As $Ae^{\alpha t}$ cannot be zero, otherwise solution will be zero, hence

$$\alpha^2 + \omega^2 = 0 \text{ or } \alpha = \pm j\omega, \qquad \text{where } j = \sqrt{(-1)}.$$

Thus we get two solutions of Eq. (116) corresponding to two values of α. Since the Eq. is linear, any linear combination of these two solutions will also be its solution. Hence the most general solution is

$$q = A_1 e^{j\omega t} + A_2 e^{-j\omega t}, \qquad \ldots(118)$$

where A_1 and A_2 are arbitrary constants.

Using initial conditions : at $t = 0$, $q = q_0$ and $i = 0$, we get

$$q = q_0 = A_1 e^{\circ} + A_2 e^{\circ} = A_1 + A_2,$$

and $$i = dq/dt = 0 = j\omega A_1 e^{\circ} - j\omega A_2 e^{\circ} = j\omega\,(A_1 - A_2).$$

Solving these two, we get $A_1 = A_2 = q_0/2$.

$$\therefore \quad q = \tfrac{1}{2} q_0 e^{j\omega t} + \tfrac{1}{2} q_0 e^{-j\omega t} = \tfrac{1}{2} q_0 (e^{j\omega t} + e^{-j\omega t}).$$
$$= q_0 \cos \omega t = q_0 \sin (\omega t + \pi/2), \qquad \ldots(119)$$

This is the equation of the discharge of the capacitor through the inductance. As this equation represents a simple harmonic motion, hence the discharge of the capacitor is *oscillatory and simple harmonic*. The periodic time of the discharge is given by

$$T = 2\pi/\omega = 2\pi\sqrt{LC},$$

and the frequency of oscillation is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}\right)} \qquad \ldots(120)$$

It is obvious from Eq. (119) that the electrical oscillations are of constant amplitude q_0. It is the ideal case, as we have assumed zero resistance of the circuit. In actual practice, coil itself possesses a finite resistance, due to which the energy is dissipated gradually and the oscillations die away ultimately (will be discussed latter).

The production of oscillations can be explained by Fig. 9.39. In the begining when the capacitor is fully charged and the *pd* between the plates is V_m, the electric field is set up between the plates and the energy stored is $q_0^2/2C$. When the circuit is closed, the capacitor starts to discharge through the inductance L and causing current to flow. As the current rises from zero it builds up a magnetic field around the inductance. At a later instant, represented by Fig. 9.39 (*b*), the capacitor is completely discharged and *pd* between its plates is decreased to zero. The current becomes maximum I_m the energy stored is $\frac{1}{2} LI_m^2$ At this time the whole electrical energy associated with the capacitor is completely transferred to the *magnetic field energy* linked with the inductance coil. The magnetic field now decreases, inducing an emf in the inductor in the same direction as the current. The current therefore persists, with diminishing magnitude, until the magnetic field is disappeared and the capacitor is again charged to the same potential. The charges so appeared on the capacitor are of opposite polarity. The process now repeats itself in the reversed direction. If there is no loss of energy, the above process continues and the process is known as *electrical oscillation*.

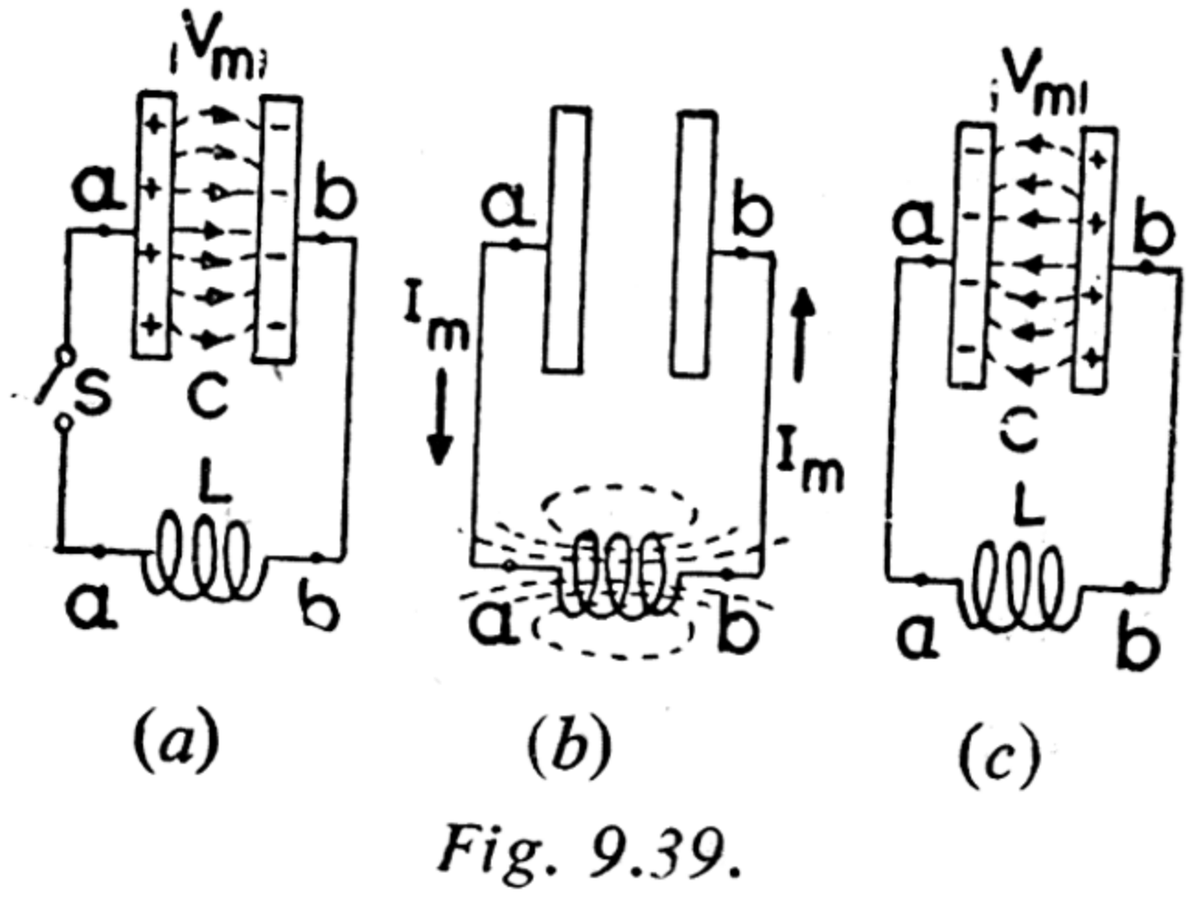


Fig. 9.39.

At any instant the sum of the energy associated with the magnetic field of the inductor $\frac{1}{2} Li^2$ and that associated with the

capacitor $q^2/2C$ is constant and is equal to the energy of the capacitor, when fully charged. *The electrical oscillations in an LC circuit can be very well compared with the oscillations of a mass on a spring.*

(*d*) **LCR-Circuit.** Let us consider a circuit consisting of capacitor of capacitance C, a coil of inductance L, a resistor of resistance R and a battery E in series, as shown in Fig. 9.40. When the switch S is thrown over to A, the capacitor begins to charge. If at any instant t, i is the instantaneous value of current in the circuit and q is the charge on the capacitor, then we have effective emf in the circuit

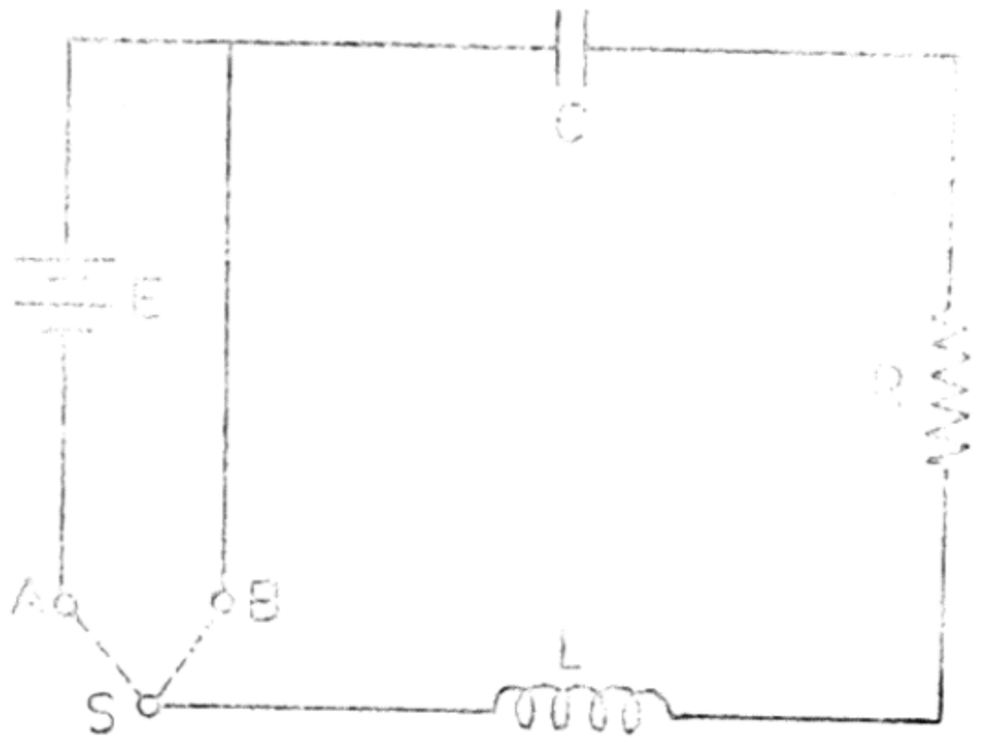


Fig. 9.40.

$$\mathcal{E} - L\, di/dt - q/C = Ri.$$

As $t = dq/dt$ and $di/dt = d^2q/dt^2$, hence we have

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{\mathcal{E}}{L}.$$

Putting $R/L = 2k$ and $1/LC = \omega^2$, we have

$$\frac{d^2q}{dt^2} + 2k\frac{dq}{dt} + \omega^2\left(q - \frac{\mathcal{E}}{L\omega^2}\right) = 0. \qquad \ldots(121)$$

Let us now replace quantity $q - \mathcal{E}/L\omega^2$ by a new variable x. Then we have $dq/dt = dx/dt$ and $d^2q/dt^2 = d^2x/dt^2$, hence Eq. (121) becomes

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0. \qquad \ldots(122)$$

Let its solution be $x = Ae^{\alpha t}$,

$\therefore$ $dx/dt = A\alpha e^{\alpha t}$ and $d^2x/dt^2 = A\alpha^2 e^{\alpha t}$.

Substituting these values in Eq. (122), we get

$$A\alpha^2 e^{\alpha t} + 2kA\alpha e^{\alpha t} + \omega^2 Ae^{\alpha t} = 0$$

or

$$Ae^{\alpha t}[\alpha^2 + 2k\alpha + \omega^2] = 0.$$

As $Ae^{\alpha t}$ cannot be zero otherwise solution will be zero, hence

$$\alpha^2 + 2k\alpha + \omega^2 = 0 \quad \text{or} \quad \alpha = -k \pm \sqrt{(k^2 - \omega^2)}.$$

Therefore the most general solution of Eq. (122) becomes

$$x = A_1 e^{[-k + \sqrt{(k^2 - \omega^2)}]t} + A_2 e^{[-k - \sqrt{(k^2 - \omega^2)}]\, t}.$$

As $\mathcal{E}/\omega^2 L = C\mathcal{E} \approx q_0 =$ final steady charge on the capacitor,

$\therefore$ $$q = q_0 + A_1 e^{[-k + \sqrt{(k^2 - \omega^2)}]\, t} + A_2 e^{[-k - \sqrt{(k^2 - \omega^2)}]\, t} \qquad \ldots(123)$$

Using initial conditions : at $t=0$, $q=0$ and $i=0$, we get

$$0=q_0+A_1+A_2 \text{ or } A_1+A_2=-q_0$$

and $$0=A_1\,[-k+\sqrt{(k^2-\omega^2)}]+A_2\,[-k-\sqrt{(k^2-\omega^2)}].$$

Combining these two Eqs. we get

$$A_1=-\frac{q_0}{2}\left[\,1+\frac{k}{\sqrt{(k^2-\omega^2)}}\,\right]$$

and $$A_2=-\frac{q_0}{2}\left[\,1-\frac{k}{\sqrt{(k^2-\omega^2)}}\,\right].$$

$$\therefore \quad q=q_0-\tfrac{1}{2}q_0[1+k/\sqrt{(k^2-\omega^2)}]\;e^{[-k-\sqrt{(k^2-\omega^2)}]\,t}$$

$$-\tfrac{1}{2}q_0\,[1-k/\sqrt{(k^2-\omega^2)}]\;e^{[-k-\sqrt{(k^2-\omega^2)}]\,t}. \qquad \text{...(124)}$$

There are three possible cases :

(1) $k^2>\omega^2$. In this case both the square roots are real and the charge gradually increases and acquires its steady value q_0 as shown by the lower curve in Fig. 9.41. The charge is known as *over damped or dead beat.*

(2) $k^2=\omega^2$. In this case the coefficients of the exponential terms of Eq. (124) becomes infinite. To simplify it let $\sqrt{(k^2-\omega^2)}=h$, where h is very small. Then Eq. (123) reduces to

$$q=q_0+A_1e^{(-k+h)t}+A_2e^{(-k-h)t}$$

$$=q_0+e^{-ht}\,[A_1\,(1+ht+\ldots\ldots)+A_2(1-ht+\ldots\ldots)]$$

$$=q_0+e^{-kt}\,(P+Qt), \qquad \text{...(125)}$$

where $P=A_1+A_2$ and $Q=h(A_1-A_2)$.

Applying initial conditions : *at* $t=0$, $q=0$ *and* $i=0$, we have

$$0=q_0+P \quad \text{or} \quad P=-q_0$$

and $$0=-kP+Q \quad \text{or} \quad Q=kP=-kq_0.$$

$$\therefore \quad q=q_0-q_0e^{-kt}\,(1+kt). \qquad \text{...(126)}$$

Thus charge increases without any oscillations and called *critically damped.*

(3) $k^2<\omega^2$. In this case $\sqrt{(k^2-\omega^2)}$ is imaginary. Let us put it as $j\beta$, where $\beta=\sqrt{(\omega^2-k^2)}$. Thus Eq. (124) reduces to

$$q=q_0-\frac{q_0}{2}\left(\,1+\frac{k}{j\beta}\,\right)e^{(-k+j\beta)t}-\frac{q_0}{2}\left(\,1-\frac{k}{j\beta}\,\right)e^{(-k-j\beta)t}$$

$$=q_0-\tfrac{1}{2}\,q_0e^{-kt}\,[(e^{j\beta t}+e^{-j\beta t})+(k/j\beta)\;(e^{j\beta t}-e^{-j\beta t})]$$

$$=q_0-q_0e^{-kt}\,[\cos\beta t+(k/\beta)\sin\beta t].$$

Let us replace quantity k/β by tan θ, where θ is another constant.

$$q=q_0\,[1-(\omega/\beta)\,e^{-kt}\cos\,(\beta t-\theta) \qquad \ldots(127)$$

This relation shows that the charge is *damped oscillatory*, as shown by the upper curve in Fig. 9.41. It fluctuates about the steady value q_0. The current at any instant t in the circuit can be obtained by differentiating Eq. (127) as

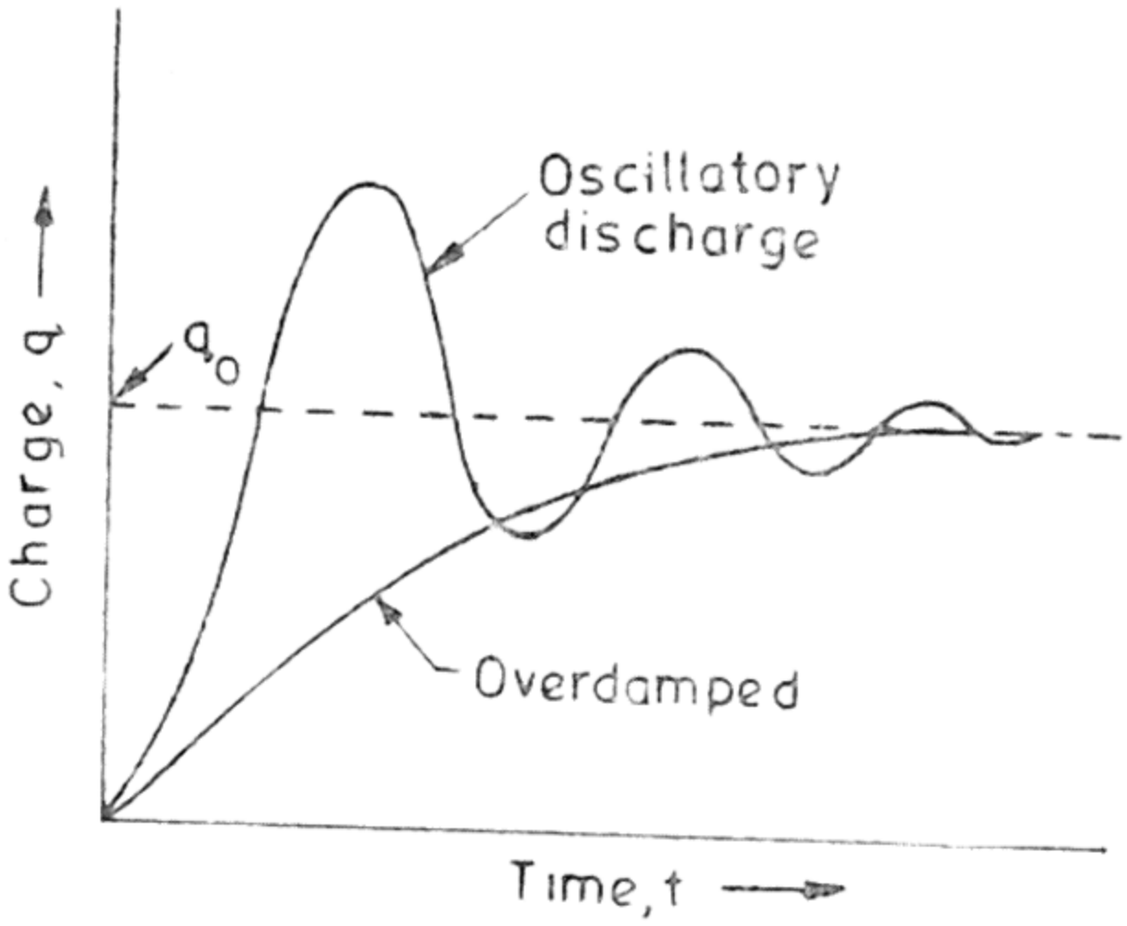


Fig. 9.41.

$$i=\frac{dq}{dt}=q_0\,\frac{\omega}{\beta}\left[ke^{-kt}\cos\,(\beta t-\theta)+\beta e^{-kt}\sin\,(\beta t-\theta)\right]$$

$$=q_0\,\frac{\omega^2}{\beta}\,e^{-kt}\left[\sin\,\theta\cos\,(\beta t-\theta)+\cos\,\theta\sin\,(\beta t-\theta)\right]$$

$$=q_0\,\frac{\omega^2}{\beta}\,e^{-kt}\sin\,\beta t. \qquad \ldots(128)$$

This is also the relation for damped oscillations.

Discharging. Consider a case when in the circuit (9.40) the capacitor is first charged and then the battery is removed from the circuit and the capacitor is allowed to discharge through the inductance and resistance. If the initial charge be q_0 and at any instant t during the discharge let q be the charge left on the capacitor and i the current in the circuit. Thus we have

$$q/C-Ldi/dt=Ri. \qquad \ldots(129)$$

As the charge q is decreasing with time, hence $i=-dq/dt$.

Therefore, $$L\,\frac{d^2q}{dt^2}+R\,\frac{dq}{dt}+\frac{q}{C}=0$$

or $$\frac{d^2q}{dt^2}+\frac{R}{L}\,\frac{dq}{dt}+\frac{q}{LC}=0.$$

On assuming $R/L=2k$ and $1/LC=\omega^2$, we get

$$\frac{d\,q}{dt}+2k\,\frac{dq}{dt}+\omega^2q=0. \qquad \ldots(130)$$

As obtained earlier the most general solution is

$$q=A_1e^{[-k+\sqrt{(k^2-\omega^2)}]\,t}+A_2e^{[-k-\sqrt{(k^2-\omega^2)}]\,t} \qquad \ldots(131)$$

Applying the initial conditions : *at* $t=0$, $q=q_0$ *and* $i=0$, we get

$$q_0=A_1+A_2,$$

and $$0=A_1[-k+\sqrt{(k^2-\omega^2)}]+A_2[-k-\sqrt{(k^2-\omega^2)}]$$

$$\therefore \quad A_1=\frac{q_0}{2}\left[1+\frac{k}{\sqrt{(k^2-\omega^2)}}\right] \text{ and } A_2=\frac{q_0}{2}\left[1-\frac{k}{\sqrt{(k^2-\omega^2)}}\right] \quad ...(132)$$

Substituting these values in Eq. (131). we get

$$q=\tfrac{1}{2}q_0[1+k/\sqrt{(k^2-\omega^2)}]\ e^{[-k+\sqrt{(k^2-\omega^2)}]\ t}$$

$$+\tfrac{1}{2}q_0[1-k/\sqrt{(k^2-\omega^2)}]\ e^{[-k-\sqrt{(k^2-\omega^2)}]\ t}. \quad ...(133)$$

There are three possible cases :

(1) $k^2>\omega^2$. $\sqrt{(k^2-\omega^2)}$ is thus real and positive. As t increases, the exponential terms decrease and becomes zero at $t=\infty$ (upper part of Fig. 9.42). This discharge is known as ***over damped, non oscillatory or dead beat.***

(2) $k^2=\omega^2$. As the coefficients of the exponential terms approach infinity, charge can be calculated as done earlier and we get,

$$q=q_0(1+kt)\ e^{-kt}, \quad ...(134)$$

which shows the non-oscillatory discharge. This discharge is known as *critically damped.*

(3) $k^2<\omega^2$. In this case $\sqrt{(k^2-\omega^2)}$ is imaginary. Let $\sqrt{(k^2-\omega^2)}=j\sqrt{(\omega^2-k^2)}=j\beta$. Thus Eq. (133) reduces to

$$q=\frac{q_0}{2}\left(1+\frac{k}{j\beta}\right)e^{(-k+j\beta)t}+\frac{q_0}{2}\left(1-\frac{k}{j\beta}\right)e^{(-k-j\beta)t}$$

$$=q_0\ e^{-kt}[\cos\beta t+(k/\beta)\sin\beta t].$$

Putting $k/\beta=\tan\theta$ and solving, we get

$$q=q_0\ (\omega/\beta)e^{-kt}\cos(\beta t-\theta). \quad ...(135)$$

This is the equation of the damped oscillations, hence the discharge is *damped oscillatory.* Thus the condition for oscillatory discharge is

$$k^2<\omega^2 \text{ or } -\frac{R^2}{4L}<\frac{1}{LC}$$

or $$R<2\left(\frac{L}{C}\right)^{1/2}.$$

The frequency of the discharge

$$f=\frac{1}{T}=\frac{\beta}{2\pi}=\frac{1}{2\pi}\sqrt{\left(\frac{1}{LC^2}-\frac{R^2}{4L^2}\right)}$$

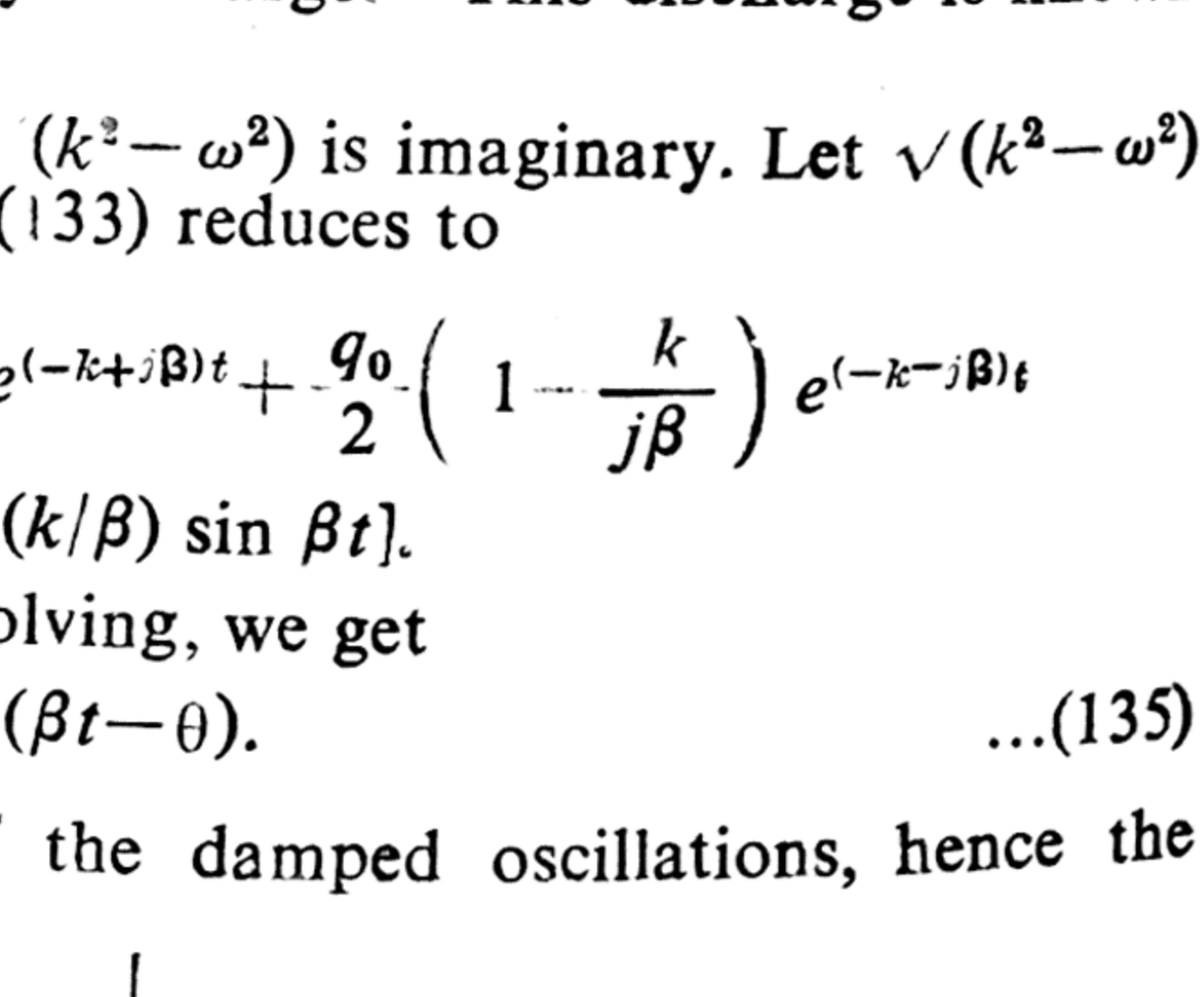


Fig. 9.42.

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}\right)}, \text{ if } R \text{ is very small.}$$

With the proper electronic arrangement, energy can be fed into a R-L-C circuit at the same rate at which it is dissipated from resistor, (*i.e.*, radiation losses i^2R). In effect a negative resistance is inserted in the circuit so that its total resistance is zero and the circuit thus oscillates as the ideal L-C circuit and thus emitting electromagnetic waves continuously.

9.25. MEASUREMENT OF HIGH RESISTANCE BY LEAKAGE METHOD

If a capacitor of capacitance C and initial charge q_0 is connected across a resistance R, the charge leaks through the resistance. The charge remaining on the capacitor at any instant t is given by

$$q = q_0\, e^{-t/CR}$$

or
$$R = -\frac{t}{C}\cdot\frac{1}{\log_e (q/q_0)} = \frac{t}{C}\cdot\frac{1}{\log_e (q_0/q)}. \quad \ldots(136)$$

Resistance R can be found by first charging a known capacitor to a charge q_0 and then by allowing this charge to leak through the resistance R for a known time t. The ratio q/q_0 is determined experimentally by a ballistic galvanometer.

If the capacitor is given a charge q_0 and then immediately discharged through a ballistic galvanometer, then

$$q_0 \propto \theta_1,$$

where θ_1 is the first throw of the galvanometer.

Next the capacitor is given the same change q_0 and then after being allowed the charge to leak through the same ballistic galvanometer. If ϕ_1 is the first throw of the galvanometer in this case, we have

$$q \propto \phi_1$$

$$\therefore \quad q_0/q = \theta_1/\phi_1.$$

and
$$R = \frac{t}{C}\cdot\frac{1}{\log_e (\theta_1/\phi_1)}$$

$$= \frac{t}{2.3026\ C \log_{10} (\theta_1/\phi_1)}. \quad \ldots(137)$$

For the purpose of charging and discharging of a capacitor Morse key is used. The clectrical connections are made as shown in

Fig. 9.43. By joining a and b, and pressing the Morse key the capacitor C is first charged to a constant pd. When the capacitor is fully charged, Morse key is released so that the whole of the charge will go to galvanometer. The first throw, θ_1 is thus noted.

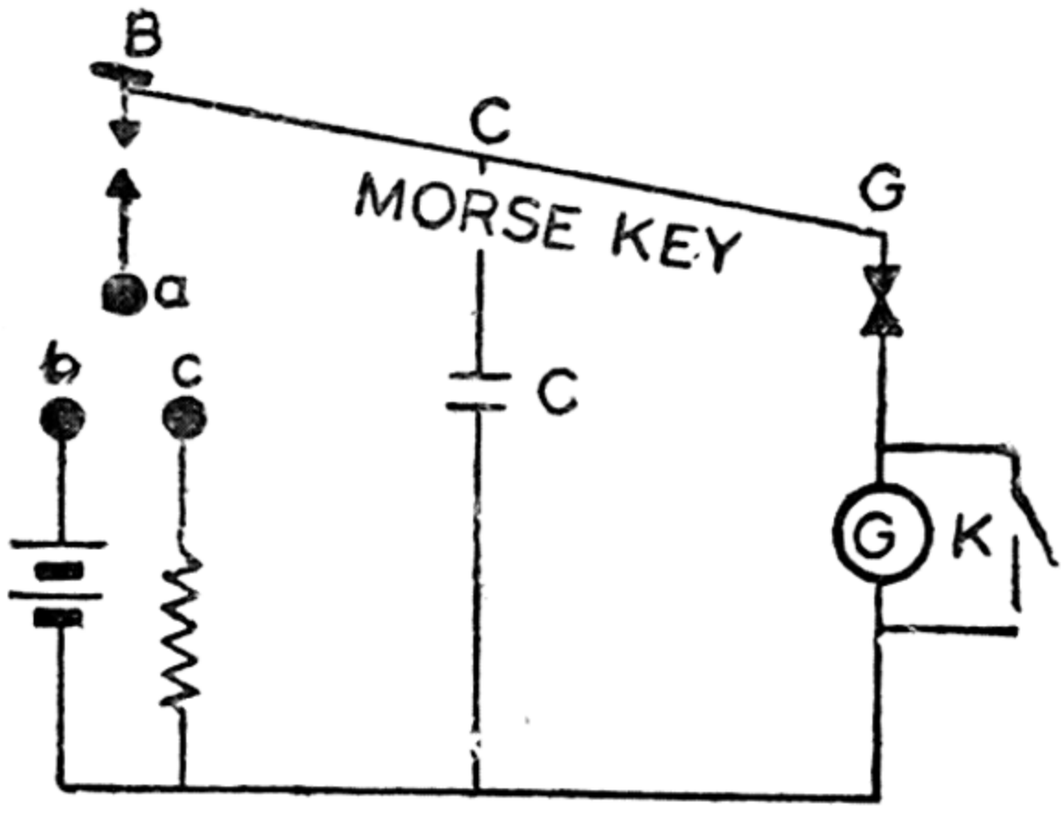


Fig. 9.43.

Again the capacitor is charged to the same potential by pressing the Morse key, then keeping the knob B pressed, plug from gap ab is inserted in the gap ac and the charge is allowed to leak through the resistance R, say for 5 or 10 second. Now the Morse key is released and the first throw ϕ_1 is noted. Resistance R is now calculated by Eq. (137).

In practice, capacitor is slightly defective, has a self leakage resistance due to imperfection of its dielectric or of its insulated mounting. This self leakage resistance R_2 can be determined by allowing the charge of leak without the external resistance. The resistance determined in the first part of the experiment is really the actual resistance R_1 in parallel with R_2. Since $1/R=1/R_1+1/R_2$, the value of R_1 can thus be determined.

9.26. COMPARISON OF CAPACITANCES, DE SAUTY'S DC METHOD

Two capacitors whose capacitances C_1 and C_2 are to be compared form the two adjacent arms AB and AD of a Wheatstone bridge. The remaining two arms consist of non inductive variable high resistances R_1 and R_2 respectively. A high resistance galvanometer G is inserted between B and D and a bettery of high emf between A and C through a two way key or a Morse Key. The resistances R_1 and R_2 are so arranged that there is no deflection of the galvanometer on moving the connection of a to b or c. In both cases, at balance, the pd across the galvanometer will then be zero. The V_1 and V_2 the pds across the capacitors C_1 and C_2 respectively at any instant t can be given by equations

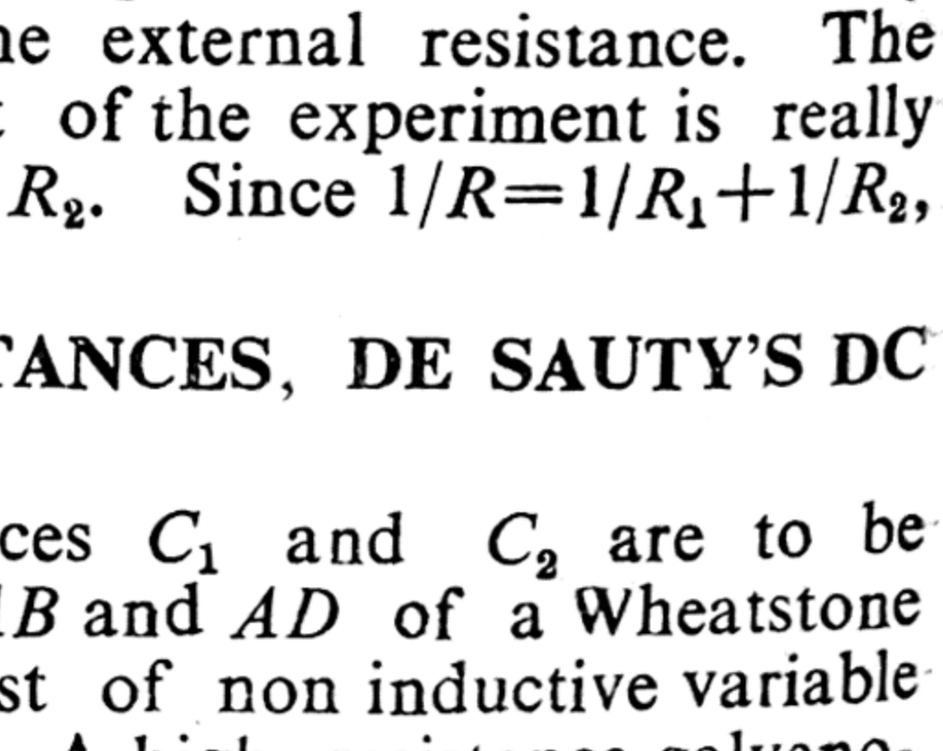


Fig. 9.44.

$$V_1=\frac{q_1}{C_1}=\frac{(q_0)_1}{C_1}\quad e^{-t/R_1C_1}$$

$$=V_0e^{-t/C_1R_1}$$

and $$V_2=\frac{q_2}{C_2}=\frac{(q_0)_2}{C_2}\quad e^{-t/C_2R_2}$$

$$=V_0e^{-t/C_2R_2}$$

At the time of balance, B and D are at same potential at any instant t, hence

$$V_1 = V_2 \quad \text{or} \quad C_1R_1 = C_2R_2$$

or
$$C_1/C_2 = R_2/R_1. \qquad \ldots(138)$$

The same condition for the balancing of the bridge would also result if the process of charging of the capacitors is considered.

Exercises

Example 1. *A Cu-rod of length L rotates at angular velocity* ω *in a uniform magnetic field* **B**. *Evaluate the emf developed between two ends of the rod.*

Consider a short portion of this rod of length dl at a distance l from centre O. Its velocity $v = \omega l$. Since **v** is perpendicular to **B**, which is $\perp$ to the plane of paper, hence a force **F** will act upon this element dl along its length and will produce an electric field **E** along it.

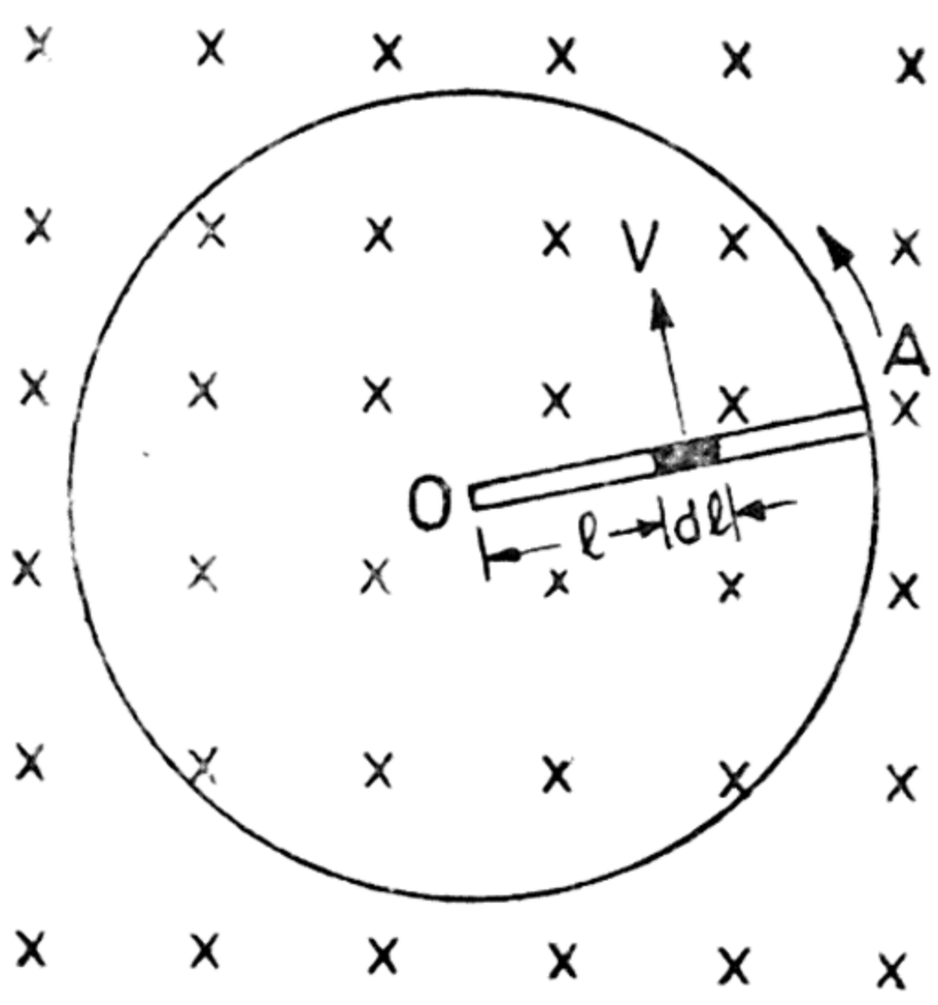


Fig. 9.45.

Electric field $\mathbf{E} = \mathbf{v} \times \mathbf{B}$,

or $E = vB = \omega lB$.

This electric field will produce an emf

$$\mathcal{E} = \int \mathbf{E}.d\mathbf{l} = \int Edl$$

$$= \omega B \int_0^L ldl = \frac{1}{2}\omega BL^2.$$

Example 2. *A metal disc of radius R rotates with an angular velocity* ω, *about an axis perpendicular to the plane of the disc and passing through its centre. If a uniform magnetic field* **B** *is acting along this axis, find the value of emf induced between the rim and the centre of the disc.*

Consider the radial segment of length dr at a distance r from the centre of the disc, rotating with velocity ω. If at any instant t this element makes an angle θ with y-axis, then in time dt, this segment will rotate through an angle $d\theta$.

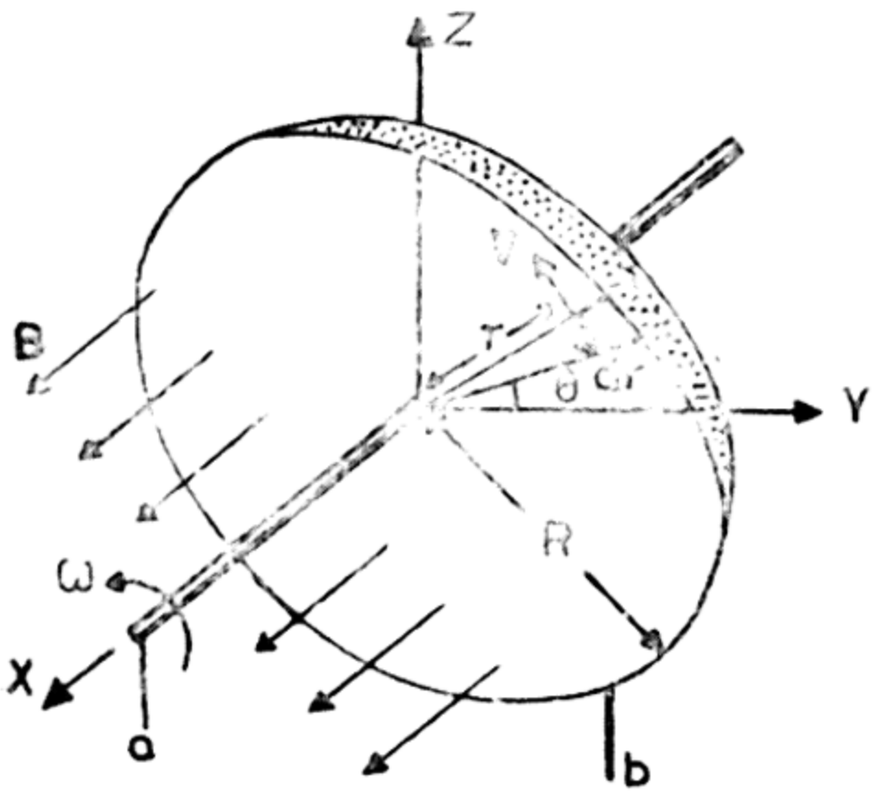


Fig. 9.46.

$\therefore$ Area swept by this segment in time $dt = rd\theta.dr$

and the flux through this area

$$d\Phi = \mathbf{B}.d\mathbf{A} = BdA = Brdrd\theta.$$

$\therefore$ Total change of flux in time dt due to the rotation of the disc

$$d\Phi' = \int_0^R Brdrd\theta = \frac{1}{2}BR^2\, d\theta.$$

Hence the induced emf between the points $r=0$ and $r=R$

$$\mathcal{E}=d\Phi'/dt=\tfrac{1}{2}BR^2(d\theta/dt)=\tfrac{1}{2}BR^2\omega.$$

Example 3. *The magnetic flux through the coil perpendicular to its plane and directed into the paper is varying according to the relation*

$$\Phi=(4t^2+5t+12)\ \textit{milliwebers.}$$

Calculate, (a) the magnitude of the emf induced in the loop when time $t=3$ sec., and (b) what is the direction of the current through the resistance R ?

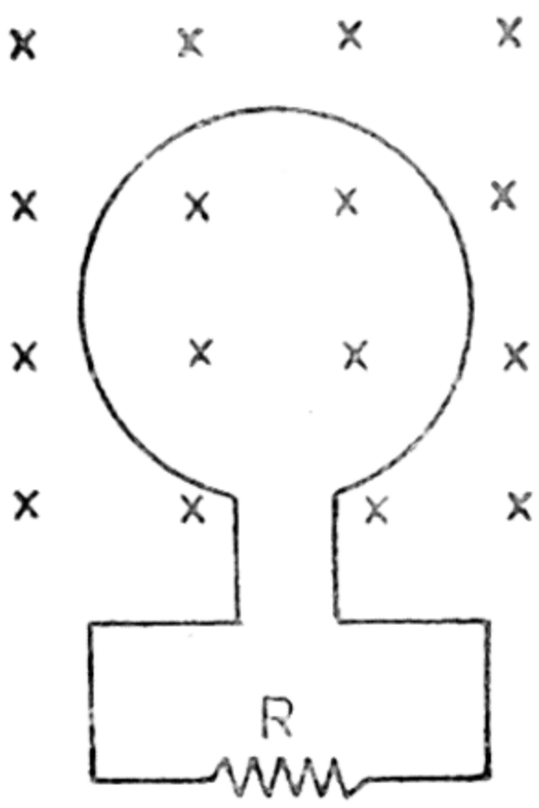


Fig. 9.47.

We known that the emf induced at any instant t is

$$e=-\frac{d\Phi}{dt}$$

$$=-\frac{d}{dt}(4t^2+5t+12)\times10^{-3}$$

At $t=3$

$$e=-10^{-3}\times\Big[\,8t+5\,\Big]_{t=3}=-29 \text{ millivolt.}$$

As time t increases the flux through the loop increases. The $-$ve sign above shows that the direction of induced emf is such as to decrease the flux through the loop. It is possible when current passes along anticlockwise direction in the coil and hence from left to right through the resistor R.

Example 4. *A Cu-rod moves with a constant velocity of 200 cm/sec parallel to a long straight wire carrying a current of 10 amperes. Calculate the induced emf in the rod, if the ends of the rod from the wire are at distances 8 cm and 32 cm.*

Consider a small element of length dx of the rod at a distance x from the wire. The magnetic induction at this element due to the current i in wire is

$$B=\frac{\mu_0\, i}{2\pi x}.$$

As element dx moves a distance v per sec, hence the rate of change of flux will be $Bvdx$. According to the Faraday's law of induction,

emf induced in this element

$$d\mathcal{E}=Bvdx.$$

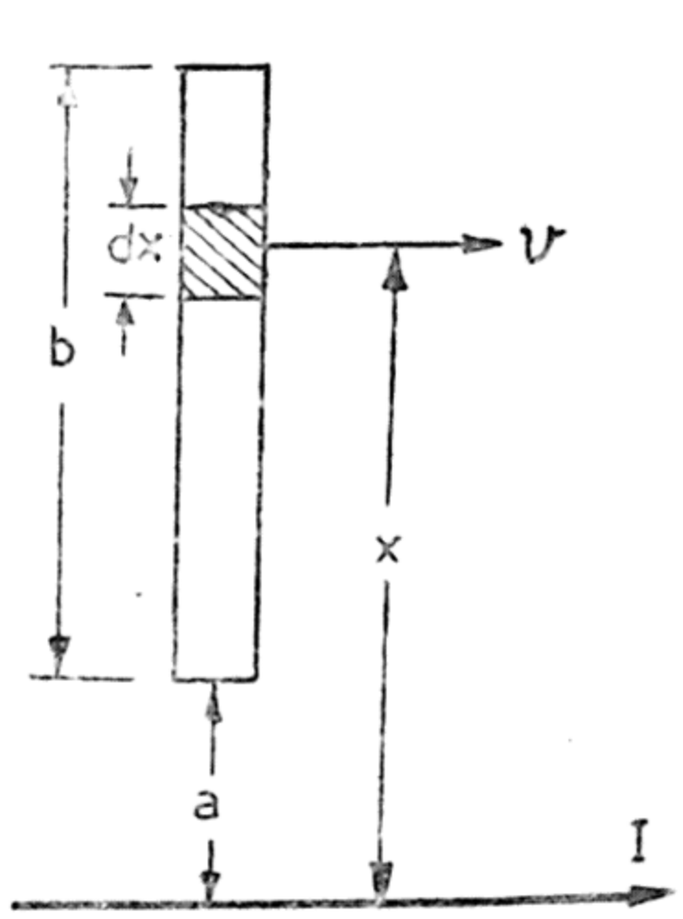


Fig. 9.48.

$$\therefore \text{ Total emf } \mathcal{E}=\int_a^b Bvdx$$

$$=\int_a^b \frac{\mu_0}{2\pi}\,iv\,\frac{dx}{x}=\frac{\mu_0}{2\pi}\,iv\log_e\frac{b}{a}$$

$$=2\times10^{-7}\times10\times2\times\log_e 4=5.55\times10^{-6}\text{ volts.}$$

Example 5. *A stiff wire bent into a semicircle of radius R is rotated at a frequency f in a uniform magnetic field* **B**. *Calculate the amplitude and frequencey of the induced voltage. If this circuit has negligible resistance and the internal resistance of the ammeter is 1000Ω, calculate the amplitude of the induced current.*

Let the magnetic field **B** be perpendicular tc the plane of paper.

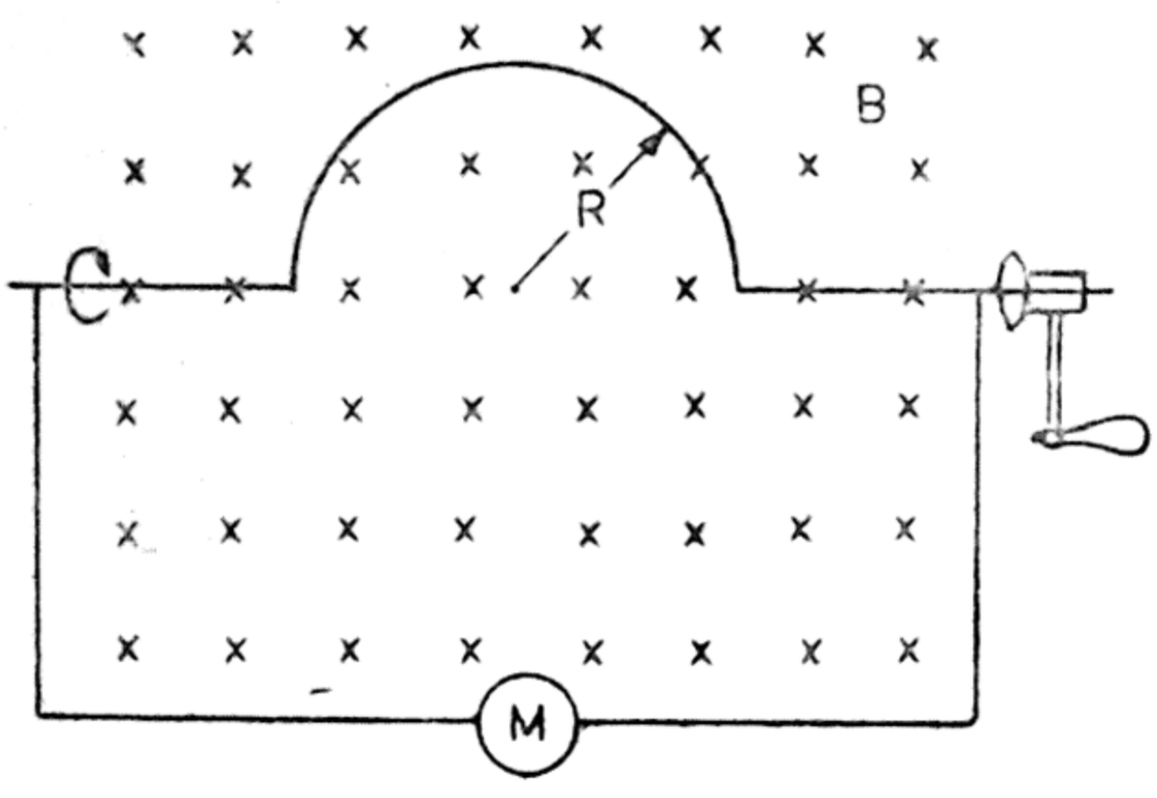


Fig. 9.49.

If at any instant t, the plane of the rotating semicircle makes an angle θ with the direction of **B**, then flux

$$\Phi = \mathbf{B}.\mathbf{A} = B(\tfrac{1}{2}\pi R^2)\cos\theta$$
$$= \tfrac{1}{2}\pi R^2\ B\cos\omega t$$
$$= \tfrac{1}{2}\pi R^2\ B\cos 2\pi f\ t.$$

As induced emf

$$\mathcal{E} = -d\Phi/dt = \pi^2 R^2\ Bf \sin 2\pi f\ t,$$

$\therefore$ Amplitude of the induced emf $= \pi^2 R^2 Bf$.

As induced current $= \mathcal{E}/R = [\pi^2 R^2 Bf \sin 2\pi ft]/1000$.

$\therefore$ Amplitude of the induced current $= \pi^2 R^2 Bf \times 10^{-3}$ amp.

Example 6. *A rectangular loop of wire is placed in a uniform magnetic field* **B** *acting normally to the plane of the loop. If we attempt to put it out of the field with velocity* **v**, *calculate the power required for this purpose.*

If l is the width of the loop and a length x is with in the field region. The flux linked with the portion of the loop which is with in the field is

$$\Phi = \mathbf{B}.\mathbf{A} = BA = Blx$$

Due to the motion of the loop, the change in flux in time dt will be $d\Phi = Bldx$. Hence the induced emf

$$\mathcal{E} = -d\Phi/dt$$
$$= -Bldx/dt = Blv.$$

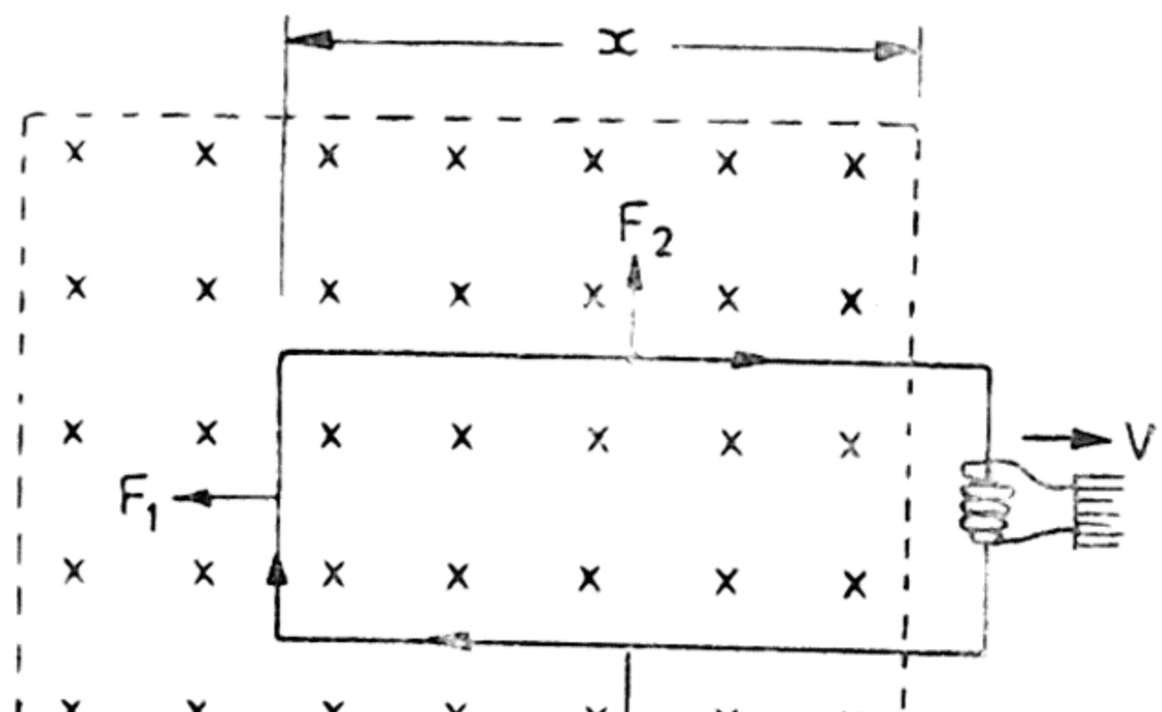


Fig. 9.50.

This induced emf sets up a current i in the loop. If R is the resistance of the loop, the current

$$i = \mathcal{E}/R = Blv/R.$$

From Lenz's law this current must be clockwise. It produces a force on each arm of the loop. Forces $\mathbf{F}_3$ and $\mathbf{F}_2$ are equal and opposite and having same line of action. Hence the net force is $\mathbf{F}_1$, where

$$\mathbf{F}_1 = i\mathbf{l} \times \mathbf{B} \quad \text{or} \quad F_1 = ilB = B^2 l^2 v/R.$$

$\therefore$ Work done for the displacement dx against this force

$$= F_1\, dx = (B^2l^2v/R)\, dx$$

or the rate of doing work $= \dfrac{dW}{dt} = \dfrac{B^2l^2v}{R}\ \dfrac{dx}{dt} = \dfrac{B^2l^2v^2}{R}$.

It is the power required for the motion.

Example 7. *A square wire of length l, mass m and resistance R slides down without friction over two parallel conducting rails of negligible resistance. The rails are connected to each other at the bottom by a resistanceless rail parallel to the wire, so that a closed loop is formed. The plane of the rails makes an angle θ with the horizontal and a uniform vertical field of magnetic induction* **B** *exists throughout the region.*

(a) Show that the wire acquires a steady state velocity of magnitude

$$v = (mg\ R/B^2l^2)\ \tan\theta\ \sec\theta.$$

(a) Show the validity of the principle of conservation of energy.

When the wire slides down, the area of the loop decreases and thus the induced emf and the induced current is set up in this circuit. The induced emf

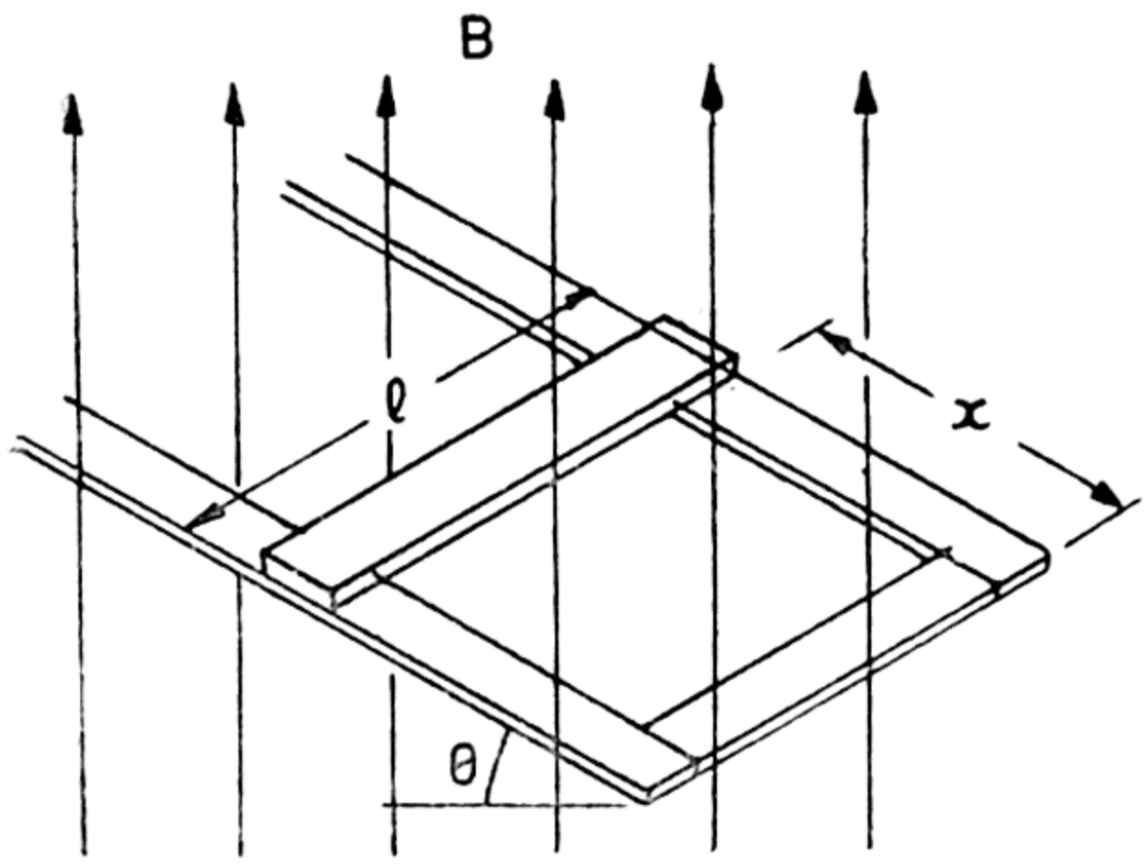


Fig. 9.51.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(\mathbf{B}\,.\,\mathbf{A})$$

$$= -\frac{d}{dt}\,(B\,.\,lx\,.\,\cos\theta)$$

$$= Bl\cos\theta\,(dx/dt).$$

$\therefore$ The induced current

$$i = \frac{\mathcal{E}}{R} = \frac{Bl\cos\theta}{R}\ \frac{dx}{dt}$$

$$= \frac{Bl\cos\theta}{R}\,v.$$

The direction of current is according to the Fleming's right hand rule and is such as to increase the flux.

The force experienced by the square wire is

$$\mathbf{F} = i\mathbf{l}\times\mathbf{B} \quad \text{or} \quad F = ilB.$$

The component of force along the rails $= F_{\parallel} = Bil\cos\theta$.

At the steady state this force is balanced by the gravitational force along the rails, hence

$$B(Bvl\cos\theta/R)\ l\cos\theta = mg\sin\theta,$$

or $$v = mgR\sin\theta/B^2l^2\cos^2\theta = (mg\ R/B^2l^2)\tan\theta\sec\theta.$$

For the second part of question, the rate of work done on the wire

$$P_1=\mathbf{F}\ .\ \mathbf{v}=Bilv\cos\theta=B^2l^2v^2\cos^2\theta/R.$$

The rate at which the Joule heat is produced in the resistor

$$P_2=i^2R=(B^2l^2\cos^2\theta\ v^2/R^2)\ R=B^2l^2v^2\cos^2\theta/R.$$

Hence, we get $P_1=P_2$ or the *principle of conservation of energy is valid for this problem.*

Example 8. *A railway line 1.2 m wide runs along the magnetic meridian. The vertical component of the Earth's magnetic field is 0.5 oersted. Calculate the emf in volts that will exist between the rails when a train runs on the line at a speed of 60 km/hr.*

The induced emf $\mathcal{E}$ due to the rate of change of flux is given by

$$\mathcal{E}=-\frac{d\Phi}{dt}=-\frac{d}{dt}(\mathbf{B}\ .\ \mathbf{A})=Blv\cos\theta.$$

In the present problem, rails are in magnetic meridian and magnetic field B is the vertical component of the earth's magnetic field, thus θ between **B** and **A** is zero and

$$\mathcal{E}=B_{\mathrm{V}}lv.$$

Substituting the numerical values,

$$B_{\mathrm{V}}=0.5 \text{ oersted}=0.5\times10^{-4} \text{ weber/meter}^2$$

length l = distance between the rails = 1.2 m

and velocity of the train $v=60$ km/hr

$$=60\times1000/60\times60=16.66 \text{ m/sec},$$

we get

$$\mathcal{E}=0.5\times10^{-4}\times1.2\times16.66$$

$$=10^{-3} \text{ volt.}$$

Example 9. *A ballistic galvanometer of 2.2×10^{-9} ampere per cm current sensitivity and 10 sec periodic time is connected in a circuit of total resistance 500 ohms. Find the deflection, in cm, produced in the galvanometer if the magnetic flux linked with the circuit changes by 1000 emu.*

The charge q due to a change in magnetic flux, through the galvanometer is given as

$$q=\frac{\text{Change in magnetic flux } \phi}{\text{Resistance } R}$$

$$=\frac{1000 \text{ emu}}{500 \text{ ohm}}$$

$$=1000\times10^{-8} \text{ weber/500 ohm}$$

$$=2\times10^{-8} \text{ coulomb.}$$

The charge-sensitivity (charge per unit deflection, *i.e.*, q/θ) is related with the current sensitivity as

$$\text{Charge sensitivity } \frac{q}{\theta} = \frac{T}{2\pi} \times \text{current sensitivity } (S_i)$$

$$\therefore \qquad q = \theta T S_i / 2\pi \quad \text{or} \quad \theta = 2\pi q / T S_i.$$

Substituting numerical values, we get

$$\theta = 2 \times 3.14 \times 2 \times 10^{-8} / 10 \times 2.2 \times 10^{-9}$$
$$= 5.7 \text{ cm.}$$

Example 10. *A fan blade of length 2a rotates with frequency f cycles per second perpendicular to a magnetic field* **B.** *Find the* **p.d.** *between the centre and end of the blade.*

Let us imagine the dotted circuit along which blade slides. As the contact at point D slides along, the area A of the loop ODC increases, we thus have

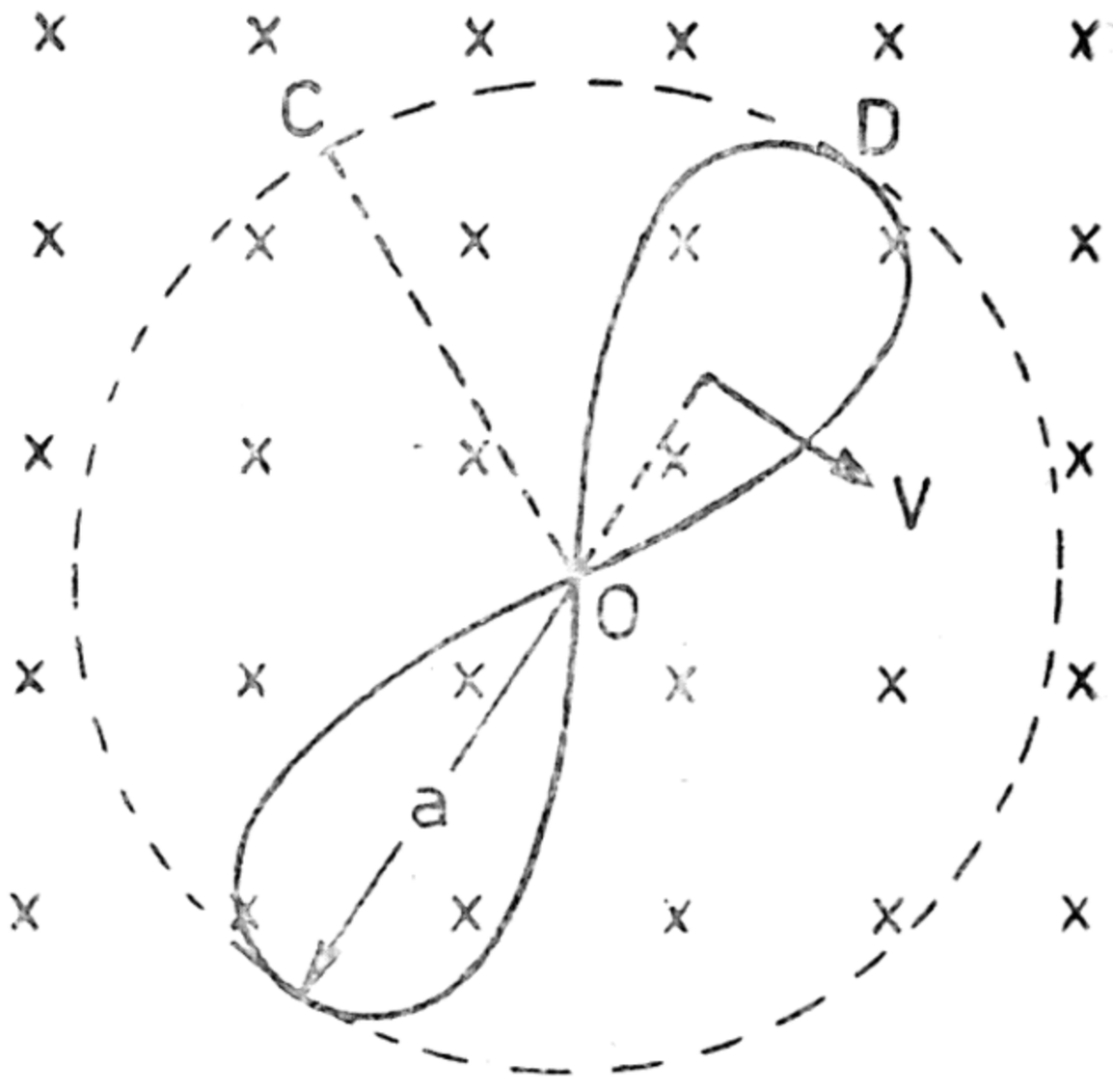


Fig. 9.52.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(BA)$$

$$= -B\frac{dA}{dt}.$$

In one revolution (time = time period = $1/f$) blade sweeps out an area of πa^2.

$$\therefore \quad \mathcal{E} = -B\frac{\pi a^2}{1/f} = -\pi B a^2 f.$$

This is the induced emf in the loop and is also the *pd* between O and D.

Same result can be obtained as :

$$V_D - V_0 = -\int_O^D E_r dr = -\int_0^a vB dr = -\int_0^a r\omega B dr$$

$$= -\int_0^a 2\pi r f B dr = -\pi f B a^2.$$

Example 11. *Find an expression for the induced emf produced between the ends of a coil of radius r having N turns and placed in a uniform magnetic field* $B = B_0 \sin \omega t$ *making an angle* θ *with the normal to the plane of the coil.*

The induced emf produced between the open ends of the coil of one turn is given by

$$\mathcal{E} = -\frac{\partial}{\partial t}\int \mathbf{B} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\left[(B_0 \sin \omega t)\, \pi r^2 \cos \theta \right]$$

$$= -\pi r^2 \omega B_0 \cos \theta \cos \omega t.$$

Here the integral is evaluated by taking surface as within the periphery of the coil. For a coil of N turns, the net result is roughly equivalent to the notion that each turn is separately linked by the magnetic flux. In each turn the induced voltage is $\mathcal{E}$, These voltages are in series, hence the total emf across the ends of a coil of N turns is N times greater and given by

$$\mathcal{E}_{total} = N\mathcal{E} = -N\pi r^2 \omega B_0 \cos\theta \cos\omega t.$$

Example 12. *In a certain belatron the maximum magnetic field was 4000 gauss, operating at 50 cycles/sec with a stable orbit diameter of 60 inches. Calculate the average energy gained per revolution and the final energy of the electrons.*

If B is the magnetic field at the end of the accelerating cycle at the orbit of radius R, the final energy

$$E = BeRc$$
$$= 4000 \times 10^{-4} \times 1.602 \times 10^{-19} \times 30 \times 2.54 \times 10^{-2} \times 3 \times 10^8,$$
$$= 91.4 \text{ MeV}.$$

Total No. of revolutions

$$N = \frac{C}{4\omega R} = \frac{3 \times 10^8}{4 \times 2 \times 3.14 \times 50 \times 30 \times 2.54 \times 10^{-2}}$$
$$= 3.13 \times 10^5.$$

$\therefore$ Average energy gained per revolution

$$= 91.4 \times 10^6 / 3.13 \times 10^5 = 291.7 \text{ eV}.$$

Example 13. *Calculate the self inductance of the thin wire of circular cross section and bent into a curve whose radius of curvature is very large compared with the radius of cross section of the wire.*

The magnetic field at any point P on a surface rimmed by the wire may be calculated on the assumption that the current is concentrated along the axis of the wire. Hence if M is the mutual

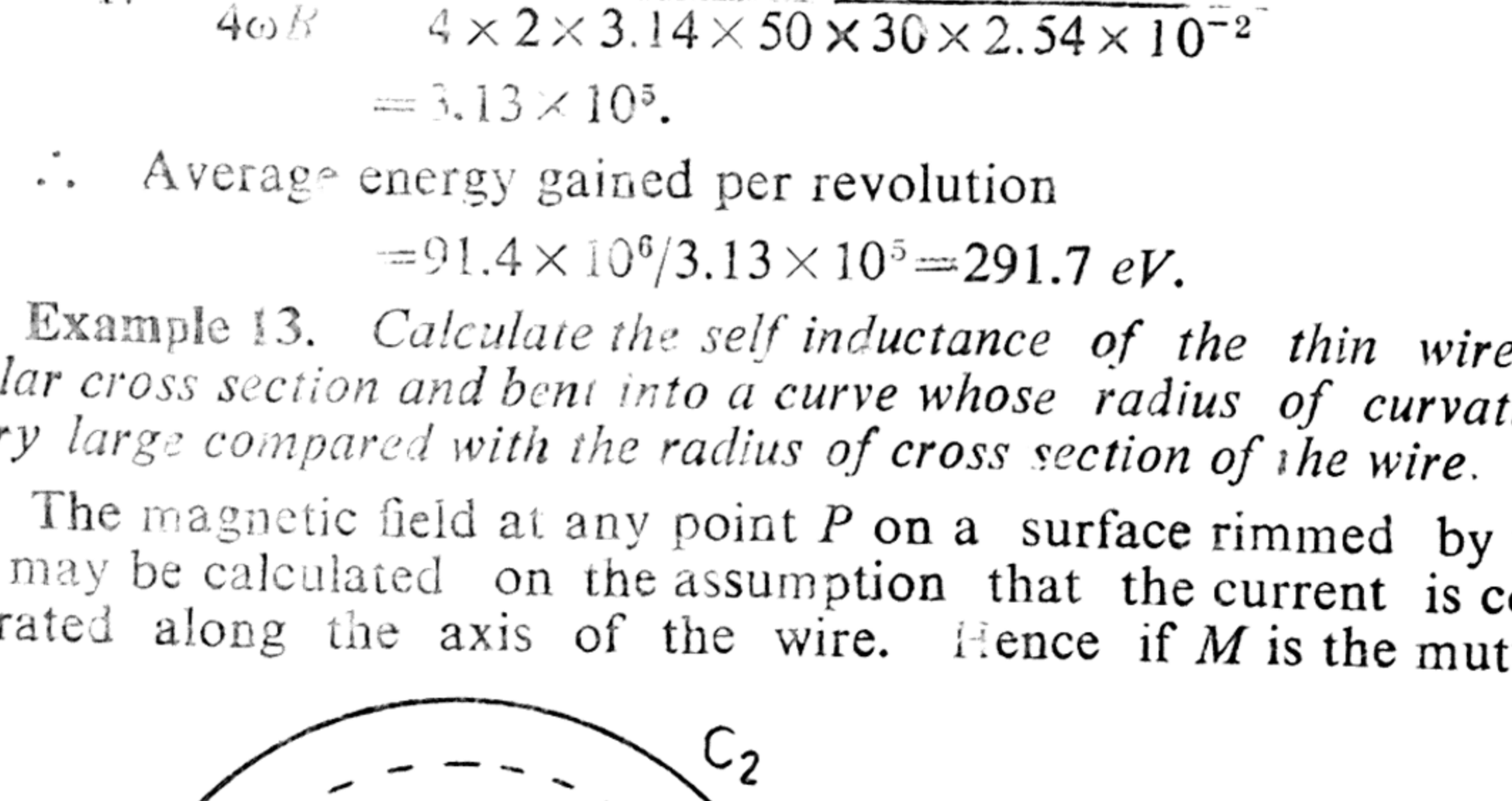


Fig. 9.53.

inductance between the circuits C and C_1 (the line of centres of the cross-section and the inner boundary of the wire), the flux through C_1 when unit current passes through the cross-section of the wire is approximately M.

The additional flux through C_1 and the filaments through an element ds of a cross sectional area is $l \int_r^R Bdr$,

where l is the length of the wire and R the radius of its cross section. Here B is the magnetic field at point P. Due to the current unity, distributed uniformly over its cross section, it is given by

$$B = \mu_0 r/2\pi R^2.$$

$$\therefore \quad \text{Total flux } \Phi = M + l\int_r^R \frac{\mu_0 r}{2\pi R^2} dr = M + \frac{\mu_0}{4\pi}\, l\, \frac{R^2 - r^2}{R^2}.$$

The self induction of the wire $L = \int\int \Phi di = \int\int \Phi (ds/\pi R^2)$

$$= \int\int \left(M + \frac{\mu_0}{4\pi}\, l - \frac{\mu_0}{4\pi}\, l\, \frac{r^2}{R^2} \right) \frac{ds}{\pi R^2}$$

$$= M + \frac{\mu_0}{4\pi}\, l - \frac{\mu_0 l}{4\pi^2 R^4} \int\int r^2 ds.$$

As $\quad \int\int r^2\, ds = \int_0^R \int_0^{2\pi} r^2 . dr . r d\theta = \int_0^R 2\pi r^3\, dr = \frac{\pi R^4}{2},$

$$\therefore \quad L = M + \frac{\mu_0}{4\pi}\, l - \frac{\mu_0 l}{4\pi^2 R^4}\, \frac{\pi R^4}{2} = M + \frac{\mu_0}{8\pi}\, l.$$

Example 14. *An endless solenoid (or toroid) has a square cross section of sides b. If the inner radius is a and N turns are uniformly distributed evenly over the circumference, calculate the self inductance of it.*

Half the toroid is shown in Fig. 9.54. We know that the magntic field B inside a toroid, if current i flows through it, is

$$B = \mu_0\, Ni/2\pi r.$$

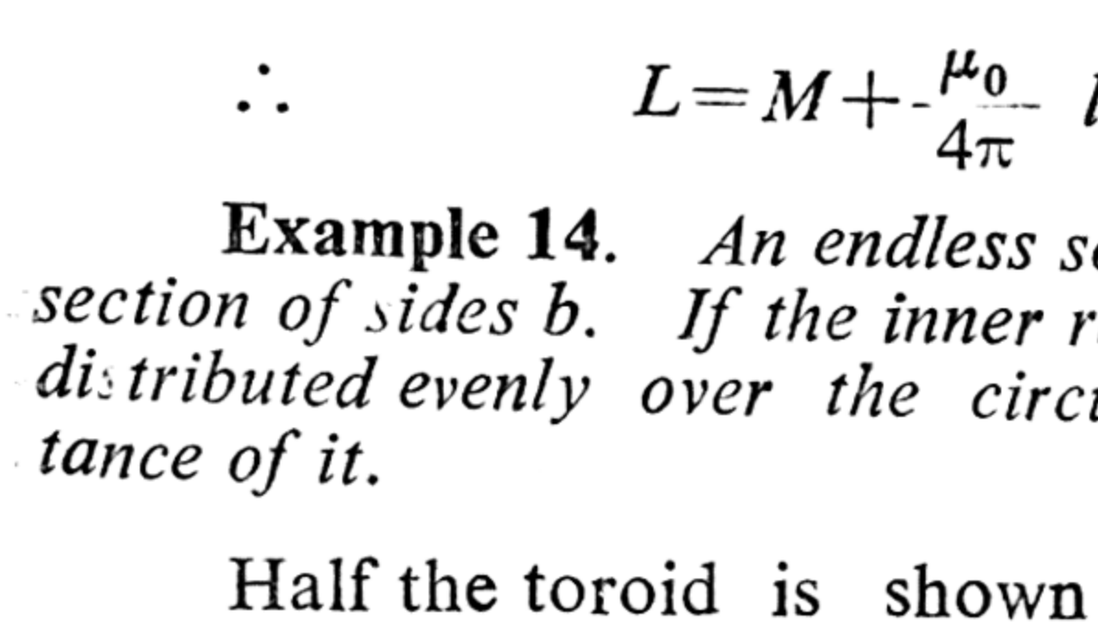


Fig. 9.54.

The fiux through the rectangular area (strip of width dr) is

$$d\Phi = B(bdr) = \frac{\mu_0 Nib}{2\pi}\, \frac{dr}{r}$$

$\therefore$ Total flux

$$\Phi = \frac{\mu_0 Nib}{2\pi} \left[\log_e r \right]_a^{a+b}$$

$$= \frac{\mu_0\, Nib}{2\pi}\, \log_e\, \frac{a+b}{a}.$$

Total flux linked with N turns $= N\Phi = \frac{\mu_0 N^2 i b}{2\pi} \log_e \frac{a+b}{a}$.

But from the definition of self inductance flux $= Li$

$\therefore$ Self inductance L of the toroid $= \frac{\mu_0 N^2 b}{2\pi} \log_e \left(1 + \frac{b}{a}\right)$.

Example 15. *Calculate the mutual inductance between two plane coaxial coils placed at a distance d apart.*

Let us consider two coils P and Q of radii a and b and total number of turns n_1 and n_2 respectively. If the centres are at a distance d apart, where $d \gg a, b$. The magnetic field at the centre of Q due to a current i in coil P is

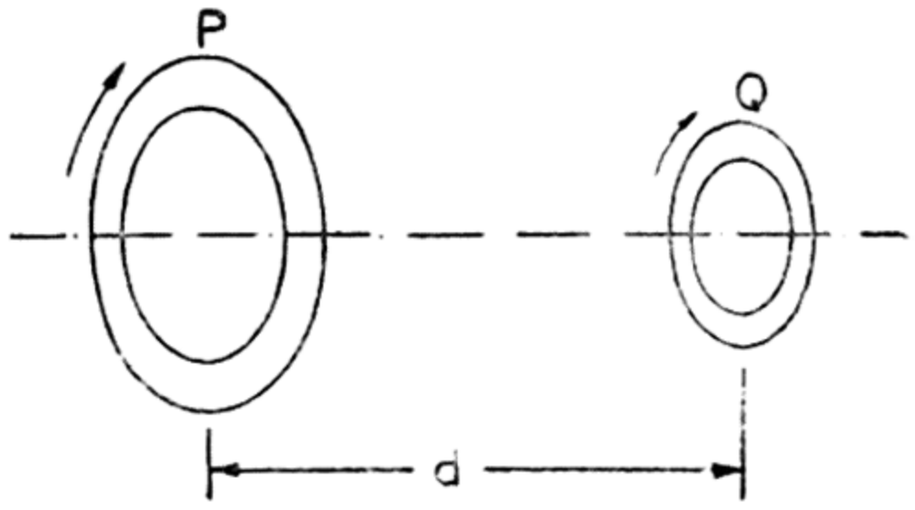


Fig. 9.55.

$$B = \mu_0 a^2 n_1 i / 2d^3.$$

As $d \gg a, b$, hence the field can be assumed uniform.

$\therefore$ Total flux through coil $Q = n_2 B \cdot \pi b^2$

$$= \mu_0 \pi a^2 b^2 n_1 n_2 i / 2d^3.$$

Thus the mutual inductance $M = \Phi / i = \mu_0 \pi a^2 b^2 n_1 n_2 / 2d^3$.

Example 16. *A straight soleniod has 50 turns per cm in the primary and 200 turns in the secondary. The area of cross-section of the solenoid is 4 cm². Calculate the mutual inductance.*

The magnetic field at any point inside the straight solenoid of primary with n_1 turns per unit length carrying a current i, is given by the relation

$$B = \mu_0 n_1 i_1.$$

The magnetic flux through the secondary of n_2 turns each of area A is given as

$$\phi = \mu_0 n_1 i_1 n_2 A.$$

If M is the mutual inductance of the solenoid, then

$$\phi = M i_1$$

$\therefore$ $$M = \mu_0 n_1 n_2 A.$$

In the present problem, $\mu = 4\pi \times 10^{-7}$ henry/*m*.

$$n_1 = 50 \text{ turns/cm} = 5000 \text{ turns/m}.$$

$$n_2 = 200 \text{ turns}$$

$$A = 4 \text{ cm}^2 = 4 \times 10^{-4} \text{ m}^2.$$

$\therefore$ $$M = 4\pi \times 10^{-7} \times 5000 \times 200 \times 4 \times 10^{-4}$$

$$= 5.024 \times 10^{-4} \text{ henry}.$$

Example 17. *Calculate the force between two coaxial solenoids carrying constant currents, one of which extends a part of it within the other.*

To neglect end effects, let us assume that the solenoids are long and thin. If the diameters are assumed nearly equal, the currents flow in the same direction, n_1 and n_2 are the numbers of turns per meter on the two solenoids, l is the length of the 1 in the 2 and S their cross sectional area, then

The mutual inductance $M=(n_1 l)\Phi_{21}/i_2=(n_2 l)\Phi_{12}/i_1$

As $\Phi_{21}=(\mu_0 n_2 i_2)\ S$ and $\Phi_{12}=(\mu_0 n_1 i_1)S$,

hence $M=\mu_0 n_1 n_2 Sl$ and $\partial M/\partial l=\mu_0 n_1 n_2 S$.

$\therefore$ $F=i_1 i_2\ (\partial M/\partial l)=\mu_0 n_1 n_2 i_1 i_2 S$.

The force is axial and is attractive, since M increases with l.

Example 18. *What will happen to the above problem, if the fluxes are kept constant instead of currents.*

We know that the magnetic energy is given by

$$\frac{dW_m}{dt}=\frac{1}{\mu_0}\int \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\,d\tau$$

or $$dW_m=\frac{1}{\mu_0}\ [B_1(dB_1)\ \tau_1+B_2(dB_2)\ \tau_2+B_1(dB_2)\ Sl+B_2(dB_1)\ Sl + B_1 B_2\ Sdl]$$

$$=\frac{1}{\mu_0}\ [(B_1 l_1 S+B_2 Sl)\ dB_1+(B_2 l_2 S+B_1 Sl)\ dB_2+B_1 B_2\ Sdl] \quad ...(i)$$

Since the total flux linkage for solenoid 1 is a constant, *i.e.*,

$$B_1 n_1 S+B_2 n_1 (l/l_1)\ S=\text{const.}$$

or $$dB_1+dB_2\ \frac{l}{l_1}+B_2\ \frac{1}{l_1}\ dl=0. \quad ...(ii)$$

Similarly for the second solenoid we have

$$dB_2+dB_1\ \frac{l}{l_2}+B_1\ \frac{1}{l_2}\ dl=0. \quad ...(iii)$$

Solving these Eqs. we get

$$dB_1=(lB_1-l_2 B_2)(l_1 l_2-l^2)^{-1}\ dl \quad ...(iv)$$

and $$dB_2=(lB_2-l_1 B_1)(l_1 l_2-l^2)^{-1}\ dl. \quad ...(v)$$

Substituting the values of dB_1 and dB_2 in Eq. (*i*) we get

$$dW_m=\frac{1}{\mu_0}B_1 B_2\ S\ dl,$$

and the axial force $F=\partial W_m/\partial l=B_1 B_2 S/\mu_0$.

Example 19. *A current i is passing in opposite directions in the inner and outer coaxial cylinders. Inner is solid of radius a and the outer hollow with inner and outer radii b and c respectively. Calculate the magnetic energy per unit length in the different regions.*

In the region of the inner cylinder, let us apply circuital law to a path of radius ρ. Thus we get

$$2\pi\rho B = \mu_0 i(\pi\rho^2/\pi a^2) \text{ or } B = \mu_0 i\rho/2\pi a^2.$$

$\therefore$ Magnetic energy per unit length inside the inner cylinder

$$W_1 = \frac{1}{2\mu_0}\int_0^a \left(\frac{\mu_0 i\rho}{2\pi a^2}\right)^2 2\pi\rho d\rho = \frac{\mu_0 i^2}{16\pi}.$$

In the region between two cylinders $B = \mu_0 i/2\pi\rho$

and $$W_2 = \frac{1}{2\mu_0}\int_a^b \left(\frac{\mu_0 i}{2\pi\rho}\right)^2 . 2\pi\rho d\rho = \frac{\mu_0 i^2}{4\pi}\log_e \frac{b}{a},$$

In the region of the outer cylinder, the current with in a circular path of radius ρ is that in the centre conductor i minus the current lying in the outer conductor between the radii b and ρ.

$$\therefore \quad B = \frac{\mu_0}{2\pi\rho}\left[i - i\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)\right] = \frac{\mu_0 i}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2}$$

and $$W_3 = \frac{1}{2\mu_0}\int_b^c \left(\frac{\mu_0 i}{2\pi\rho}\right)^2\left(\frac{c^2-\rho^2}{c^2-b^2}\right)^2 2\pi\rho d\rho$$

$$= \frac{\mu_0 i^2}{4\pi}\left[\frac{c^4}{(c^2-b^2)^2}\log_e \frac{c}{b} - \frac{3c^2-b^2}{4(c^2-b^2)}\right].$$

In the fourth region, outside the cylinders, $B=0$ and $W_4=0$, the self inductance per unit length

$$L = \frac{2(W_1 + W_2 + W_3)}{i^2}$$

$$= \mu_0\left\{\frac{1}{8\pi} + \frac{1}{2\pi}\log_e\frac{b}{a} + \frac{1}{2\pi}\left[\frac{c^4}{(c^2-b^2)^2}\log_e\frac{c}{b} - \frac{3c^2-b^2}{4(c^2-b^2)}\right]\right\}$$

The second term is most important and is the inductance associated in annular region between the conductors.

Example 20. *Explain the reason of designing transformers with laminated insulated slabs of metals instead of a solid metal. Prove it mathematically.*

Whenever a conductor is placed in the varying magnetic field, induced currents are set up. These result in heating and energy loss. The losses for laminated slabs are smaller than the single solid metal. It is the reason of designing transformers with laminated cores.

Let us consider a block of sides $a \times a$ and thickness t. We shall find the eddy current losses in this case and will compare with the losses in a block divided into four insulated parts of sides $\frac{1}{2}a \times \frac{1}{2}a$. To solve this problem, consider the block broken into a set of thin square paths. If $2x$ be the length of the side, then the total length $l = 8x$. The area enclosed is $(2x)^2$.

If **B** is directed into the face of the block and dB/dt $(=k)$ is uniform over the block. The induced emf $\mathcal{E}$ around the path $=(dB/dt)\,(2x)^2$. The average induced electric field **E** along the

$$\text{path} = \frac{\mathcal{E}}{l} = \frac{4kx^2}{8x} = \frac{1}{2}\,kx.$$

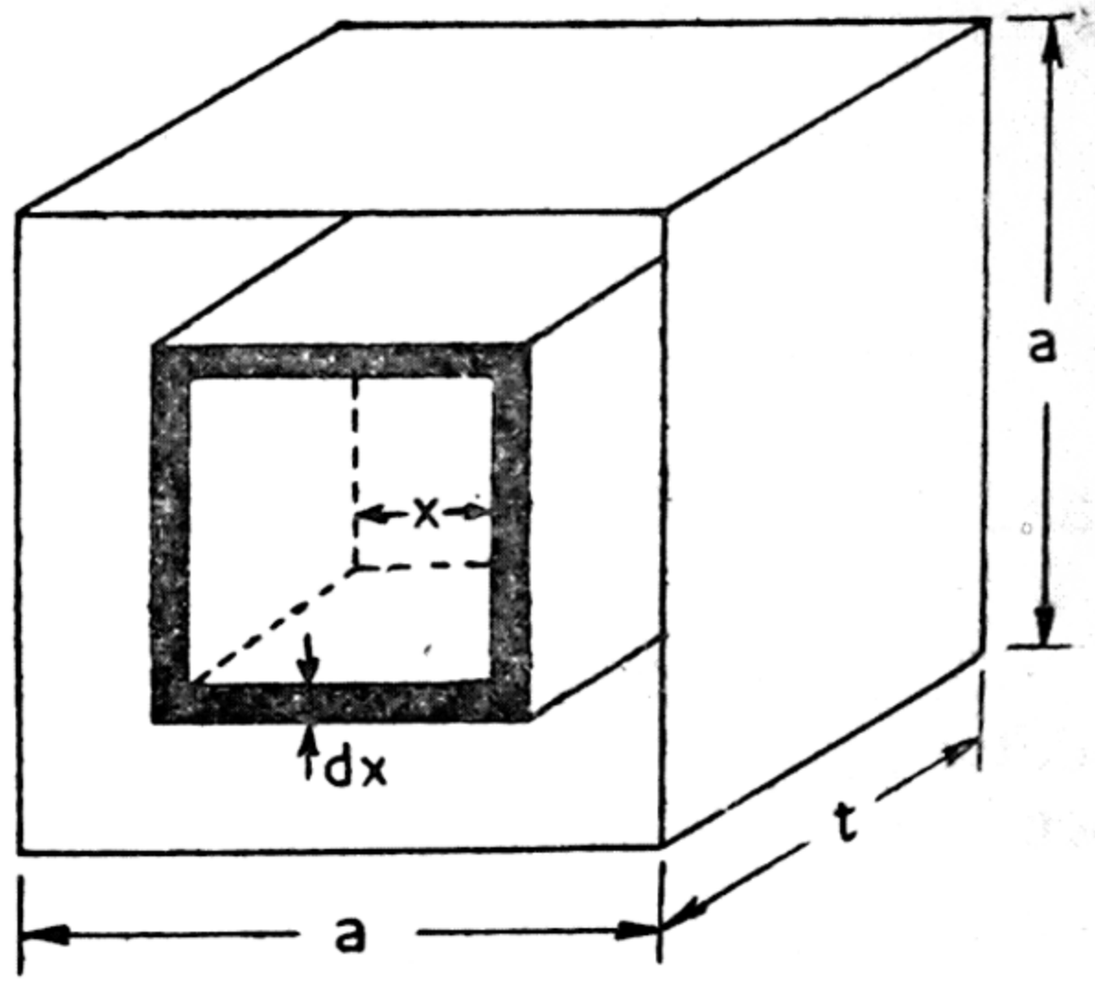


Fig. 9.56.

$\therefore$ Power dissipation due to this current

$$dP = Vi = Eli = EldAj = \sigma E^2 d\tau$$

$$\text{or} \quad P = \int \sigma E^2 d\tau$$

$$= \sigma \int_0^a \left(\frac{1}{2}\,kx\right)^2 t\, 8x dx$$

$$= \sigma t k^2 a^4/2.$$

If the block was divided into quarters, the power dissipation

$$P = 4\,\frac{\sigma t k^2}{2}\left(\frac{a}{2}\right)^4 = \frac{1}{4}\,\frac{\sigma t k^2}{2}\,a^4.$$

It is one fourth the energy dissipation in the solid block. *Hence the transformers with the laminated cores are best than the solid blocks under eddy current effects.*

Example 21. *In a circuit, a resistor of 10 mega ohms, a capacitor of 0.2 μf and a battery of emf 20 volts are connected in series. Calculate the (a) rate of growth of charge, (b) energy stored by the capacitor, (c) heat appeared in the resistor and (d) the energy delivered by the battery after 2 seconds.*

(*a*) Growth of charge in the circuit is given by the relation

$$q = q_0\,(1 - e^{-t/CR}).$$

$\therefore$ Rate of growth of charge $\dfrac{dq}{dt} = \dfrac{q_0}{CR}\; e^{-t/CR}$.

But the maximum value $q_0 = \mathcal{E}C$ coulomb.

$\therefore$ $$\frac{dq}{dt} = \frac{\mathcal{E}C}{CR}\; e^{-t/CR} = \frac{\mathcal{E}}{R}\; e^{-t/CR}$$

$$= \frac{20}{10 \times 10^6} \text{ exp. } [-2/0.2 \times 10^{-6} \times 10 \times 10^6]$$

$$= 7.4 \times 10^{-7}\ C/\text{sec.}$$

(*b*) Energy stored by the capacitor at any instant t

$$= \frac{q^2}{2C} = \frac{q_0^2}{2C}\left(1 - e^{-t/CR}\right)^2$$

∴ Rate at which the energy is being stored by the capacitor

$$\frac{dU}{dt}=\frac{q_0^2}{2C}\left[\,2(1-e^{-t/RC})\,e^{-t/RC}\,\frac{1}{RC}\,\right]$$

$$=\frac{q_0^2}{RC^2}\,e^{-t/RC}\left(\,1-e^{-t/RC}\,\right)=\frac{\mathcal{E}^2}{R}e^{-t/RC}\left(\,1-e^{-t/RC}\,\right)$$

$$=\frac{20^2}{10\times10^6}\left(\,e^{-1}\,1-e^{-1}\,\right)=9.3\times10^{-6}\text{ watt.}$$

(*c*) Heat energy dH produced in the resistor of resistance R when a current I passes through it for time dt is $I^2\,Rdt$.

∴ Rate of heat produced $dH/dt=I^2\,R=(dq/dt)^2\,R$

$$=5.5\times10^{-6}\text{ watt.}$$

(*d*) Energy delivered by the battery $dU=\mathcal{E}I\,dt$.

∴ $dU/dt=\mathcal{E}\;I=\mathcal{E}(dq/dt)=14.8\times10^{-6}$ watt.

Example 22. *A resistor R and a 2μF capacitor in series are connected at a 200 volts direct supply. Across the capacitor is a neon bulb that strikes at 120 volts. Calculate the value of R to make the bulb strike 5 sec after the switch has been closed.*

In the case of a charging of a condenser connected with a resistance R in series, the charge at any instant t is given by

$$q=q_0\,(1-e^{-t/CR}).$$

As $q=CV$, hence the potential difference at this instant t will be

$$V=V_0\,(1-e^{-t/CR}).$$

In our problem, $V=120$ volts, $V_0=200$ volts, $t=5$ sec and $C=2\times10^{-6}\,f$.

∴ $$120=200\,[1-\text{exp.}\,(-5/2\times10^{-6}\,R)]$$

$$e^{2.5\times10^6/R}=2.5 \quad\text{or}\quad 2.5\times10^6/R\,\log_e 2.5$$

∴ $$R=\frac{2.5\times10^6}{2.3026\times0.4}=2.73\times10^6\,\Omega=2.73\text{ megohms.}$$

Example 23. *A coil of resistance 20 Ω and inductance 0.5 henry is switched to direct current 200 volt supply. Calculate the rate of increase of current (i) at the instant of closing the switch and (ii) at $t=L/R$ sec after the switch is closed. Also find the value of the steady current in the circuit.*

During the growth in R-L circuit, the current at any instant is given by

$$i=i_0\,(1-e^{-Rt/L}).$$

$$\therefore \qquad \frac{di}{dt}=i_0 \ \frac{R}{L} \ e^{-Rt/L} = \frac{\mathcal{E}}{L} \ e^{-Rt/L}.$$

Hence at $t=0$, $\frac{di}{dt}=\frac{\mathcal{E}}{L}=\frac{200}{0.5}=400$ amp/sec.

at $t=\frac{L}{R}$, $\frac{di}{dt}=\frac{\mathcal{E}}{L}\cdot\frac{1}{e}=\frac{200}{0.5\times 2.718}$

$=147.2$ amp/sec.

The steady value of current in the circuit

$$i_0=\mathcal{E}/R=200/20=10 \text{ amp.}$$

Example 24. *A 5 henry inductor is placed in series with a 10 Ω resistor, an emf of 5 volts being suddenly applied to the combination. Using these values prove the principle of conservation of energy, for time equal to the time constant.*

At any instant t, current in L-R circuit is given as

$$i=i_0 \ (1-e^{-Rt/L}) = \frac{\mathcal{E}}{L} \ (1-e^{-Rt/L}).$$

At the time t=time constant L/R, we get

$$i=\frac{\mathcal{E}}{R}\left(1-\frac{1}{e}\right)=\frac{5}{10}\left(1-\frac{1}{2.718}\right)=0.316 \text{ amp.}$$

$\therefore$ The rate at which the energy is delivered by the battery is

$$P_B=\mathcal{E}i=5\times 0.316=1.58 \text{ watts}$$

At this time some energy will appear as Joule heat in the resistor, given by

$$P_J=i^2R=(0.316)^2\times 10=0.998 \text{ watt.}$$

Part of the energy is stored in magnetic field, linked with the inductor. It is given by the relation

$$P_m=Li \ (di/dt).$$

As $\frac{di}{dt}=\frac{\mathcal{E}}{L} \ e^{-Rt/L}=\frac{\mathcal{E}}{L}\cdot\frac{1}{e}$, therefore

$$P_m=Li\cdot\frac{\mathcal{E}}{L}\cdot\frac{1}{e}=\frac{\mathcal{E}i}{e}=\frac{5\times 0.316}{2.718}=0.581 \text{ watt.}$$

$\therefore \quad P_J+P_m=0.998+0.581=1.58=P_B.$

It is same as required by the principle of conservation of energy.

Example 25. *If a battery, of emf 100 volts, is connected in series with an inductance of 10 mH, a capacitor of 0.05 μF and a resistance of 100 Ω, find (i) the frequency of the oscillatory current, (ii) the logarithmic decrement and (iii) the final capacitor charge.*

$$\frac{R^2}{4L^2}=\frac{100^2}{4\times(10^{-2})^2}=2.5\times10^7$$

and $$\frac{1}{LC}=\frac{1}{10^{-2}\times0.05\times10^{-6}}=2\times10^9.$$

As in this problem $R^2/4L^2<1/LC$, *hence the charging of the capacitor is oscillatory.* The frequency of oscillation

$$f=\frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}$$

$$=\frac{1}{2\times3.14}\sqrt{(2\times10^9-2.5\times10^7)}$$

$$=7076.6 \text{ cycles/sec.}$$

$$\text{The logarithmic decrement}=\frac{RT}{2L}=\frac{100}{2\times10^{-2}\times\overline{7076.6}}$$

$$=0.707.$$

The final capacitor charge $q_0=\mathcal{E}C=100\times0.05\times10^{-6}=5\mu C.$

Example 26. *In any leakage experiment a capacitor of capacitance 1μf was first charged to a suitable potential and discharged at once, the throw was 10 cm. The capacity was again charged and was short-circuited with a resistor R for 20 sec and the throw was 7.5 cm when was discharged through a ballistic galvanometer. Calculate the resistance of the resistor.*

In the first part of the experiment, the throw d_0 was due to a full charge q_0, thus

$$q_0=\mathcal{E}C=K\theta=Kd_0.$$

If the charge q_0 reduces to q due to the leakage through resistance R, then

$$q=q_0e^{-t/RC}=Kd \quad \text{or} \quad Kd_0e^{-t/RC}=Kd$$

or

$$R=\frac{t}{C\log_e(d_0/d)}=\frac{t}{2.3026\,C\log_{10}(d_0/d)}$$

$$=\frac{20}{2.3026\times10^{-6}\times\log_{10}(10/7.5)}$$

$$=68.5\times10^6 \text{ ohm}=68.5 \text{ megohm.}$$

Oral Questions

1. A Cu-ring lies on a table with a hole through its centre. If a bar magnet with its south pole in the lower side is dropped vertically through the hole, describe the current in the ring.

2. A current carrying solenoid is moved toward a conducting loop as shown in Fig. 9.57. As viewed from the solenoid, what will be the direction of the induced current in the loop?

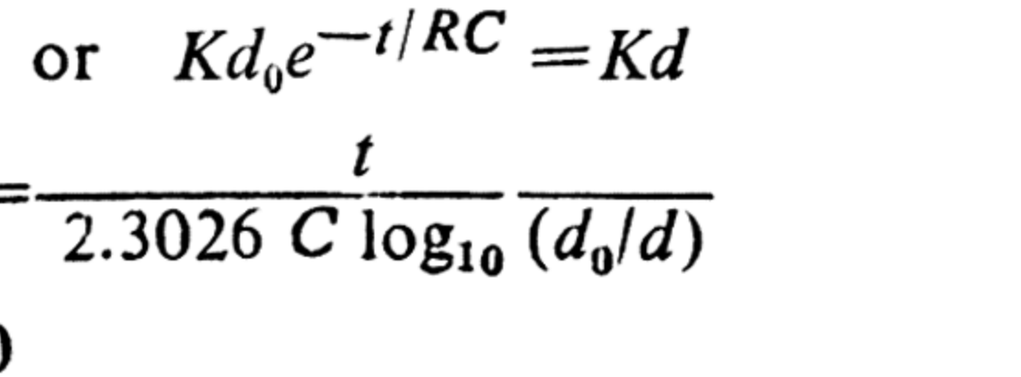


Fig. 9.57.

3. Why is the induced emf much larger when the switch is opened in a circuit consisting inductance and battery than when it is suddenly closed.

4. Describe the motion of a bar magnet which is dropped lengthwise (vertically down) in a very long *Cu*-pipe.

5. A loop is placed vertically in the gap between pole pieces of an electromagnet. What is the direction of induced current in it if pulled out vertically ?

6. The two ends of the coil of a ballistic galvanometer are generally connected together to stop swinging at once. Explain the reason.

7. A *Cu*-ring and a wooden ring of the same dimensions are placed in the same region of varying magnetic field. Compare the induced emf's.

8. Why do you use, first quarter of a cycle of alternating magnetic field in the betatron ?

9. Explain the force experienced while pulling in or out the metallic sheet placed in a magnetic field.

10. Explain how the dynamo can be used as a motor ?

11. How the motor is used as a brake in tram-cars ?

12. In a *L-R* circuit, can the induced emf across the coil of self inductance L ever be larger than the battery emf ?

Problems

1. The magnetic field inside a certain long solenoid is given by $B=0.20 \sin (300\,t)$ tesla. A coil of 100 turns of 2.0 cm^2 area is placed inside the solenoid. Find the induced emf in it when its axis makes angles 30° and 90° with the solenoid axis. [*—1.04 cos 300 t, 0*]

2. How many volts are generated in a wire 12 cm long which cuts directly across a flux whose intensity is 1.4 tesla, if it moves at a speed of 2 meters/sec ? [*0.336 volts*]

3. What is the maximum electromotive force induced in a coil of 4000 turns and average radius 12 cm rotating at 30 revolutions per sec in the earth's magnetic field of intensity 0.5×10^{-4} web/m^2. [*0.27 weber*]

4. A closed coil of 50 turns of 300 cm^2 area and of resistance 40 Ω is held at right angles to a uniform field of 2×10^{-2} tesla. Find the charge induced in the coil, when turned through an angle 30° about the axis $\perp$ to the field. [10^{-5} *coulomb*]

5. If a rectangular loop of area 400 cm^2, $R=2\Omega$ is rotating with angular velocity $\omega=10$ rad sec^{-1}, (*a*) about the *x*-axis, (*b*) about *z*-axis, or (*c*) about an edge parallel to the *y*-axis, What is the maximum induced emf in each case, if the magnetic field $B=0.5$ wb/cm^2 is along *x*-axis. [*0, 0.2 V, 0.2 V*]

6. The flux density at all points within a circular portion equals to 0.5 wb/m^2 and is decreasing at the rate of 0.1 wb/m^2-sec. If a ring of radius 10 cm and resistance 2Ω is placed in this field in such a way that the flux is $\perp$ to the plane of the ring. Calculate the (*a*) magnitude and direction of the induced electric field at any point of the ring, (*b*) induced emf, and (*c*) current in the ring. [*Clockwise 0.005 V/m, 3.14 mV, 1.57 mA*]

7. Two conducting loops face each other a distance d apart. An observer sights along their common axis. If a clockwise current is suddenly established in the larger loop, find the direction of the induced current and of the force acting on the smaller loop.

8. A uniform magnetic field of induction B fills a cylindrical volume of radius R. A metal rod of length l is placed as it is parallel to the upper and lower plane surfaces of the cylinder. If the induction is changing at the rate dB/dt, show that the emf that is produced between the ends of the rod is,

$$\mathcal{E}=\frac{dB}{dt}\cdot\frac{l}{2}\sqrt{\left[4R^2-\frac{l^2}{4}\right]}.$$

9. Find the induced electric field **E** both outside and inside the solenoid of n loops per unit length and carrying a varying current $i=i_0 \sin 2\pi ft$. $[(\mu_0\pi a^2 n i_0 f \cos 2\pi ft)/r,\ \mu_0\pi n i_0 fr \cos 2\pi ft]$

10. When a copper disc of radius 10 cm makes 20 rotations per sec with its plane perpendicular to a uniform magnetic field an induced emf 3.14 milli volts is generated between the centre and the edge of the disc. Calculate the intensity of the magnetic field. $[5\times 10^{-3}\ weber/m]$

11. A circular coil of radius 7 cm and number of turns 100 is rotated about a diameter perpendicular to a magnetic field of strength 100 gauss with 3600 rotations per minute. Find the maximum emf developed. *(1 volt)*

12. In a betatron the electron completes 2×10^5 rounds around the loop before ejected to produce X-ray. If during this time, rate of change of flux is 400 V, find the energy and speed of ejected electron. $[80\ MeV]$

13. A circular disc of metal with conductivity σ, thickness t, rotates about its axis with angular velocity ω between the pole faces of a magnet which produces a uniform magnetic field over a small square area of size a^2 at the average distance r from the axis, If **B** is $\perp$ to the disc, calculate the torque acting on the disc. $[\frac{1}{3}B^2a^2r^2\omega\sigma t]$

14. Calculate the mutual inductance of the coplanar concentric rings consisting of n_1 and n_2 turns respectively. The radius of the inner ring is r_1 and that of outer ring is r_2 ($r_1 \ll r_2$). $[\mu_0\pi r_1^2 n_1 n_2/2r_2]$

15. Two coils having 50 and 60 turns form respectively, the primary and secondary of a transformer. In the primary coil, direct current is passed which reaches a maximum value of 5 amp in 0.2 sec. A magnetic flux 3×10^{-2} weber is produced in the core passing through both the coils. Calculate the mutual inductance and the emf induced in the secondary coil.

16. A long solenoid is 1.0 m long and 3.0 cm in mean diameter. It has five layers of winding of 850 turns each and carries a current of 5 amp. What is the magnetic induction B at its centre? What is the magnetic flux for a cross section of the solenoid through its centre. $[1.88\times 10^{-5}\ web]$

17 An inductor L is connected with a battery of emf $\mathcal{E}$ and a resistance R in series. Show that the current after a time t is $x(1-x)\ \mathcal{E}/R$ and that if contact be then broken the current after a further time t is $x(1-x)\ \mathcal{E}/R$, where $L \log_e x + Rt=0$.

18. If $C=1\ \mu f$ and $R=1\ M\Omega$ in a CR circuit, find the time required by e_c to attain 50% of its maximum value. What will happen if the switch in the circuit also has a small resistance of 10 Ω ?

19. The time constant of a certain inductive coil was found to be 2.5×10^{-3} sec. With a resistance of 80 ohms added in series, a new time constant of 0.5×10^{-3} sec was obtained. Find the inductance and resistance of the coil.
[5.0×10^{-2} henry ; 20 ohms)

20. A coil of resistance R and inductancc L is connected to a battery of emf $\mathcal{E}$ volts. In what time the current rises to 99% of its final value ? When will the *pd*, across inductance equal to that across resistance ?
[4.6 L/R, 0.69 L/R]

21. If a capacitor of 20 μf be charged on a *pd* of 15 *kV* and then suddenly short-circuited by a coil of negligible resistance and of impedance 2 *mH*, calculate the frequency and maximum amplitude of the resultant current oscillations.
[795 cycles/sec, 1500 amp.]

22. The capacitor of an oscillator circuit of low resistance is enclosed in a container. When the container is evacuated, frequency of the circuit is 100 kilo-cycles and when it is filled with CO_2 the frequency changes by 50 cycle. Calculate the dielectric constant of the gas. *[1.001]*

23. A condenser of capacity 0.1 μf and resistance 10 megohms is charged to a certain potential and then insulated. Find the time the potential will take to fall to half its original value. *[0.693 sec]*

24. A potential difference of 1 volt (dc) is applied to a coil of resistance 1 ohm and self inductance 1 henry. What will the current be in the circuit after 1/19 second ?

25. A condenser of capacity 0.25 μf if charged upto a certain voltage and then discharged through a circuit of inducta ıce 0.1 henry and resistance 100 ohms. Show that the discharge is oscillatory. Also calculate the frequency of the discharge,
[1004 c/sec]

26. In an oscillating circuit L=0.20 henry, C=1.2 *nf*. Find the maximum value of R so that the circuit may oscillate.
[2.58×10^4 ohms]

27. A LCR circuit is critically damped with L=0.2 henry and R=100 ohms. What is the value of C ?
[80 μf]

28. Examine if the discharge of a condenser of capacity 2 μf in a circuit containing an inductance L=0.15 H and resistance R=15 Ω shall be oscillatory.

10

Maxwell's Equations and Electromagnetic Waves

10.1. MAXWELL'S EQUATIONS

The basic laws of electricity and magnetism, which we have discussed in the preceding chapters can be summarized in differential form by the four equations.

Gauss's law (Electrostatics) $\nabla . \mathbf{E} = \rho/\varepsilon_0$, ...(1)

Gauss's law (magnetism) $\nabla . \mathbf{B} = 0$, ...(2)

Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, ...(3)

Faraday's law of Induction $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. ...(4)

These equations are written for fields in vacuum, in the presence of electric charge of density ρ, and electric current of density $\mathbf{j}$. These are in macroscopic form and in S.I. units. In the presence of simple dielectrics, ε_0 is replaced by ε in Eq. (1). If simple ferromagnetic materials are present, μ_0 is replaced by μ in the Eq. (3).

The first equation is equivalent to Coulomb's law and describes the charge and the electric field. The second equation describes the magnetic field and shows that it is impossible to create an isolated magnetic pole. The third equation describes the magnetic field due to current density and shows that the current in a conductor sets up a magnetic field near it. The fourth equation describes the electric field of a changing magnetic field. It shows that a bar magnet moving with respect to a closed loop of wire will set up a current in the loop.

Let us recall that all but Faradays's law were derived from steady-state observations. Maxwell while studying the electromagnetic laws, mentioned above, noticed that there was some thing strange about Eq. (3). As the divergence of a curl is always zero, hence Eq. (3) becomes

$$\text{div curl } \mathbf{B} = \text{div } \mu_0 \mathbf{j} = \mu_0 \text{ div } \mathbf{j} = 0. \quad ...(5)$$

Equation of continuity $\text{div } \mathbf{j} + \partial\rho/\partial t = 0$ shows that $\partial\rho/\partial t$ must thus be zero, *i.e.*, the total flux of current out of any closed surface

is zero. But this certainly cannot in general be zero because we know that the charges can be moved from one place to another.

Maxwell proposed that Eq. (3) was not correct for moving charges and suggested that for changing charge distributions

$$\text{curl } \mathbf{B}=\mu_0 \mathbf{j}+?. \qquad \ldots(6)$$

To know this unknown, Maxwell suggested that similar to the electric field due to a changing magnetic field (Faraday's law of induction) there would be a magnetic field due to the changing electric field, *i e.*,

$$\text{curl } \mathbf{B}=\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}=\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}. \qquad \ldots(7)$$

Here constant term $1/c^2$ or $\mu_0\varepsilon_0$ corresponds to the system of units in *S I.* units. Using this term as missing term, we get

$$\text{curl } \mathbf{B}=\mu_0 \mathbf{j}+\mu_0\varepsilon_0\, \partial\mathbf{E}/\partial t. \qquad \ldots(8)$$

To check this equation, take divergence of both sides

$$\begin{aligned}\text{div (curl } \mathbf{B})&=\text{div } (\mu_0 \mathbf{j})+\text{div}[\mu_0\varepsilon_0\, \partial\mathbf{E}/\partial t]\\ &=\mu_0 \text{ div } \mathbf{j}+\mu_0 \text{ div } (\partial\mathbf{D}/\partial t]\\ &=\mu_0(\text{div } \mathbf{j}+\partial\rho/\partial t). \qquad \text{(as div } \mathbf{D}=\rho)\end{aligned}$$

As div (curl **B**) is zero, hence right hand side is zero, which is also zero by virtue of the continuity condition.

Maxwell called the added term in Eq. (8) the **displacement current density.**

A physical situation which combines electric fields and currents in a straight forward way is shown in Fig. 10.1. A current i enters the positive plate and leaves the negative plate of a parallel plate condenser. This current can not continue long and will become zero when the capacitor becomes fully charged. If q is the charge on the plates at any instant t and A the plate area, then the electric field in the gap

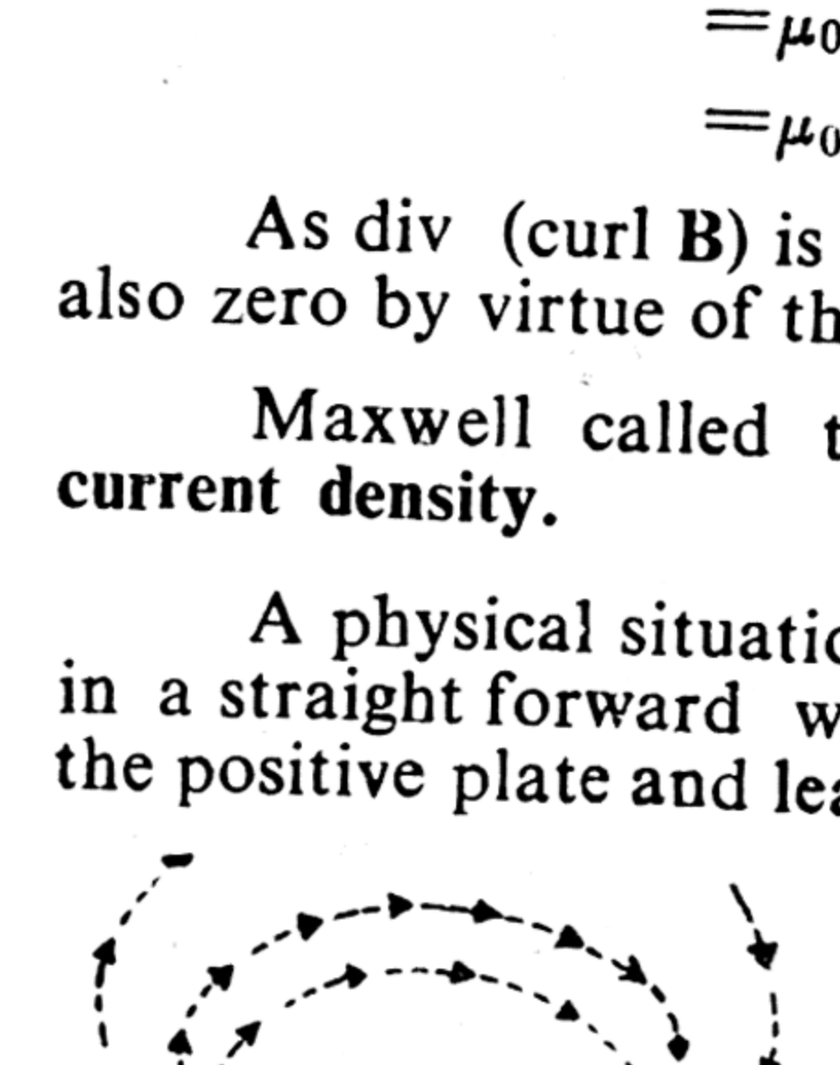


Fig. 10.1.

$$E=q/\varepsilon_0 A.$$

$$\therefore \quad \frac{dE}{dt}=\frac{1}{\varepsilon_0 A}\frac{dq}{dt}=\frac{1}{\varepsilon_0 A}\,i'$$

or
$$i'=\varepsilon_0 A\,\frac{dE}{dt}. \qquad \ldots(9)$$

Hence the current density in the gap of the plates

$$=i'/A=\varepsilon_0 dE/dt.$$

or
$$\mathbf{j}d=\varepsilon_0 d\mathbf{E}/dt.$$

Thus the variation of E with time produces an additional current of current density $\mathbf{j}_d$ in space, called *displacement current density*. In the case of a parallel plate air condenser, the displacement current in the space is such as shown by dotted arrows, while conduction current i in the connecting wire and plates is such as shown by continuous arrows in the Fig. 10.1. Thus Eq. (8) becomes

$$\text{curl } \mathbf{B}=\mu_0 \mathbf{j}+\mu_0 \mathbf{j}_d=\mu_0(\mathbf{j}+\mathbf{j}_d). \quad ...(10)$$

The set of equations thus becomes

$$\nabla \cdot \mathbf{E}=\rho/\epsilon_0, \quad ...(11)$$

$$\nabla \cdot \mathbf{B}=0, \quad ...(12)$$

$$\nabla \times \mathbf{E}=-\partial \mathbf{B}/\partial t, \quad ...(13)$$

$$\nabla \times \mathbf{B}=\mu_0[\mathbf{j}+\epsilon_0 \partial \mathbf{E}/\partial t]. \quad ...(14)$$

These equations were first stated and used in coherent way by Maxwell, and are thus known as *Maxwell's equations*. These are the basis of all electromagnetic phenomena. They are also valid *for nonhomogenous, nonlinear and even nonisotropic media.*

The lack of symmetry in these equations with respect to **B** and **E** is entirely due to the presence of electric charge and conduction current In empty or free space ρ and **j** are zero and Maxwell's equations become $\nabla \cdot \mathbf{E}=0$, $\nabla \cdot \mathbf{B}=0$, $\nabla \times \mathbf{B}=\mu_0\epsilon_0 \partial \mathbf{E}/\partial t$ and

$$\nabla \times \mathbf{E}=-\partial \mathbf{B}/\partial t. \quad ...(15)$$

The last two equations of relations (15) are also of similar type except the constants.

10.2 MAXWELL'S EQUATIONS IN INTEGRAL FORM

In place of above equations, which are in differential form, let us write them in integral form. The integral form of Maxwell's equations is easier to interpret physically and is useful in driving the boundary conditions that the field vectors must satisfy.

(1) Maxwell's first equation $\nabla \cdot \mathbf{E}=\rho/\epsilon_0$ may also be written as

$$\int_V (\nabla \cdot \mathbf{E})dV=\int_V (\rho/\epsilon_0)dV, \quad ...(16)$$

where V is any arbitrary volume.

Applying Gauss's divergence theorem

$$\oint_S \mathbf{F}.d\mathbf{S}=\int_V \text{div } \mathbf{F}dV. \quad ...(17)$$

to the left hand side of Eq. (16), wet get

$$\oint_S \mathbf{E}.d\mathbf{S}=\frac{1}{\epsilon_0}\int_V \rho dV, \quad ...(18)$$

i.e., the outward flux of **E** *throughout any closed surface S is* $(1/\epsilon_0)$ *times the net charge inside the volume enclosed in S.*

(2) The Eq. (12) may be written as

$$\int_V (\nabla \cdot \mathbf{B}) dV = 0$$

where V is any arbitrary volume. On applying Gauss's divergence theorem, (Eq. 17), we get

$$\oint_S \mathbf{B}.d\mathbf{S} = 0. \qquad \ldots(19)$$

i e., the outward flux of magnetic induction vector **B** *throughout any closed surface S is zero.*

(3) The Eq. (13) may be written as

$$\oint_S (\nabla \times \mathbf{E}).d\mathbf{S} = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}, \qquad \ldots(20)$$

where S is any surface bounded by a curve C.

Applying Stoke's theorem

$$\oint_C \mathbf{F}.\mathbf{dl} = \oint_S (\text{curl } \mathbf{F}).d\mathbf{S}$$

to the left hand side of Eq. (20), we get

$$\oint_C \mathbf{E}.d\mathbf{l} = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \oint_S \mathbf{B}.d\mathbf{S}. \qquad \ldots(21)$$

Here we have taken out $\partial/\partial t$ from the integral as area and time are independent variables. This equation may be stated as : *the induced e.m. f. around curve C is equal to minus the rate of change of magnetic flux through the surface enclosed by the curve C.*

(4) Integrating Eq. (14) over on area S bounded by a curve C and then using the Stoke's theorem, we get

$$\int_S (\nabla \times \mathbf{B}).d\mathbf{S} = \mu_0 \int_S \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{S}$$

or

$$\oint_C \mathbf{B}\, d\mathbf{l} = \mu_0 \int_S \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{S}, \qquad \ldots(22)$$

i.e., the magnetomotive force around C is equal to the sum of the free current and the displacement current linking C.

The first two equations show that the net flux of lines of **E** or **B** is zero throughout any arbitrary volume of no source. They show that lines of **E** and **B** are continuous. The third equation shows that the electric field due to the variation of **B** is not only a statement

of Faraday's law but a property of space independent of the presence of a conducting loop of wire. The fourth equation shows the new concept of magnetic field generation by displacement current.

10 3. ELECTROMAGNETIC WAVES

Before proceeding towards mathematics, let us see the qualitative features. For this purpose, consider an electric dipole antenna attached to an alternating voltage generator. When the upper end of the antenna is +vely charged, the electric field close to the antenna will be in −ve y-direction as shown in Fig. 10.2. After a half cycle the oscillator will have reversed the charges on the ends and the electric field at any point P will be in the +ve y-direction. If the charge on one end oscillates sinusoidally as

$$q = q_0 \sin 2\pi f t, \qquad \ldots(23)$$

then the electric field at point P will be given as

$$E_y = -E_0 \sin 2\pi f t. \qquad \ldots(24)$$

The −ve sign occurs, as electric field is in downward when upper end is +ve.

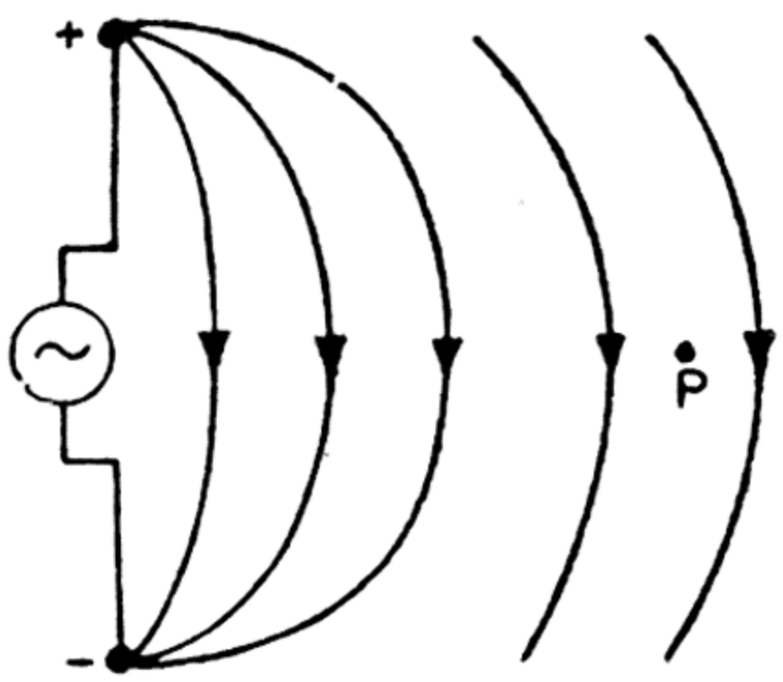


Fig. 10.2.

If the point P is at some large distance x from the antenna, one can expect the field to take a time t_0 $(=x/v)$ to reach this point. Here v is the velocity with which the field travels from the antenna to point P. The field at point P will be that existed at the antenna at time $t-t_0$, not that at time t. Thus the field at time t at point P is given by

$$E_y = -E_0 \sin 2\pi f (t-t_0) = -E_0 \sin [2\pi f (t-x/v)]. \qquad \ldots(25)$$

At large distance E_0 will vary very slowly with x and is constant approximately. Hence at any particular instant, the field along the x-axis appears as sinusoidal wave. As successive portions of the electric field (or wave) move through point P, the field will oscillate sinusoidally up and down in x-y plane, thus giving rise to the alternating electric field **E** associated with an electromagnetic wave.

The oscillating charges constitute a current, which generates a magnetic field **B**. By applying the right hand rule, we conclude that the magnetic field must circle the line joining two ends of the antenna and hence ⊥ to the plane of Fig. 10.2. Since the current in the antenna will be oscillatory, hence the magnetic field should also be oscillatory. Thus the magnetic wave will move in x-z plane. Electric and magnetic fields are shown together in a three dimensional figure (Fig. 10 3).

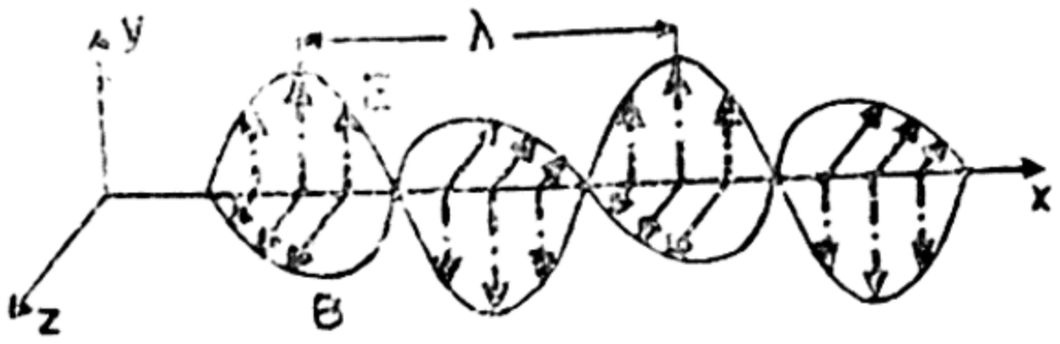


Fig. 10.3.

Wave Equation. Let us consider only the case of plane waves with the direction of propagation along the x-axis, where the electric and magnetic fields are directed transverse to the direction of travel.

Consider a rectangle of infinitesimal size in xy plane (Fig. 10.4 *a*). As the electromagnetic wave passes over it, the magnetic flux Φ_B through it will change. It will give rise to induced electric fields around the rectangle, which are the electric component of the travelling wave. As the wave is moving to the right through the rectangle, the flux Φ_B is decreasing with time. If the rectangle is assumed to be a conducting loop, a counter clockwise induced current would appear in it. This current would produce a magnetic field within the rectangle, that would oppose the decrease in Φ_B. Since there is no conducting loop, but the net induced electric field **E** acts counter clockwise around the rectangle (**E** at right edge is greater than at the left edge), hence *the electric field configuration is as induced by the changing magnetic field.*

As **E** is directed in the $+y$ direction, hence $\mathbf{E}.d\mathbf{l}$ will be zero over the ends of length $\triangle x$. If we go around the rectangle counter clockwise, we get from Maxwell's Eq. (21)

$$\int_2 E_y dl - \int_1 E_y dl = -\frac{\partial B_z}{\partial t} l \triangle x.$$

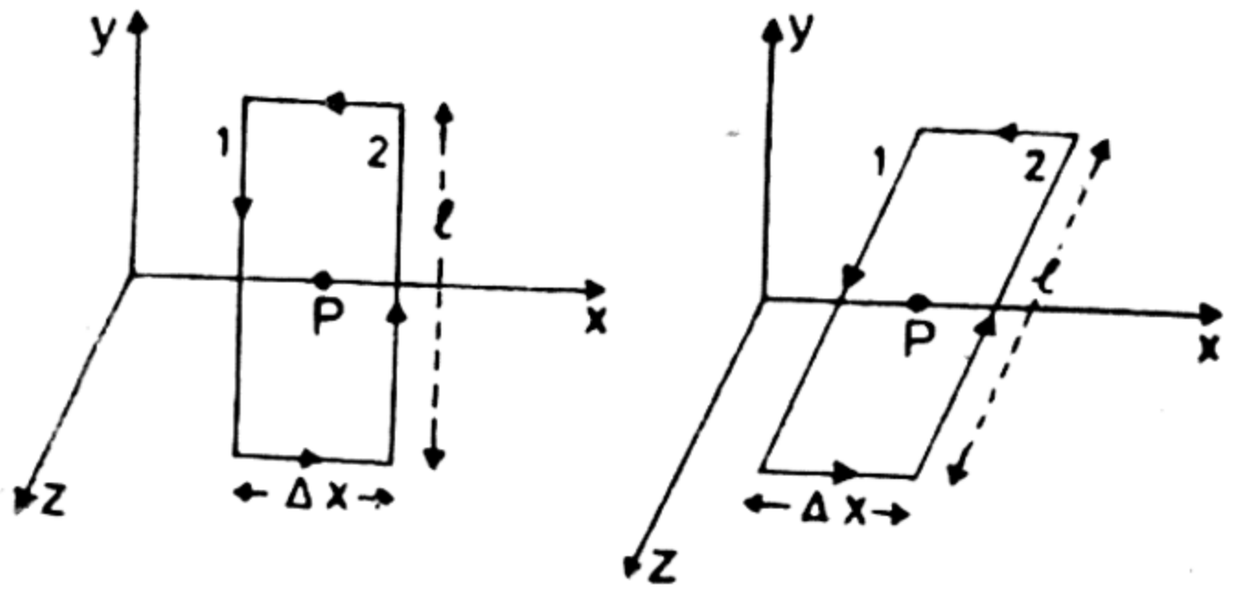
(As $dS = l \triangle x$ is along z-direction). Assuming E_y as E_{y1} and E_{y2} along paths 1 and 2 respectively, we have

$$(E_{y2} - E_{y1}) l = -l \triangle x (\partial B_z / \partial t).$$

Since $\triangle x$ is such a small distance, we can assume E_y to change linearly with x in this region. Thus we can write

$$E_{y_2} = E_{y_1} + (\partial E_y / \partial x) \triangle x$$

or $E_{y_2} - E_{y1} = (\partial E_y / \partial x) \triangle x.$



(*a*) (*b*)

Fig. 10.4.

$$\therefore \left(\frac{\partial E_y}{\partial x}\right) \triangle x l = -l \triangle x \left(\frac{\partial B_z}{\partial t}\right) \text{ or } \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \quad \ldots(26)$$

In evaluating $\partial E_y / \partial x$, it is assumed that t is constant, because Fig. 10.3 is an *instantaneous snap shot.* In evaluating $\partial B_z / \partial t$, it is assumed that x is constant because the time rate of change of E at a particular place is required. The minus sign in this equation is appropriate, as E_y is increasing with x and B_z is decreasing with time t.

Now consider a rectangle in xz plane (Fig. 10.4 *b*) for which flux Φ_E is decreasing with time as the wave moves through it. This change will induce a magnetic field at points around the periphery of the rectangle. This induced field is the magnetic component of

the electromagnetic wave. As clear from Eq. (22), the closed line integral of **B** around the rectangle, if we proceed in counter clockwise direction, is negative, while it was positive for **E**. *Thus we see that the line of induced fields E and B circulate in opposite directions, even though fluxes* Φ_B *and* Φ_E *respectively are changing with time in the same way.*

Using similar operations for Eq. (22), here too the **B**.*dl* will be zero over the ends of the closed path in xz plane, Fig. 10.4 (b). Hence Eq. (22) with no conduction current becomes

$$\int_1 B_z\, dl - \int_2 B_z\, dl = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int E_y l \triangle x.$$

Assuming Bz as B_{z_1} and B_{z_2} along paths 1 and 2 respectively, we have

$$(B_{z_1} - B_{z_2})\, l = \mu_0 \varepsilon_0\, l \triangle x (\partial E_y / \partial t).$$

If Bz changes with x linearly with in this small distance $\triangle x$, then we have

$$B_{z_2} = B_{z_1} + (\partial B_z / \partial x) \triangle x.$$

$$\therefore \quad \left(\frac{\partial B_z}{\partial x} \right) \triangle x l = -\mu_0 \epsilon_0 l \triangle x \frac{\partial E_y}{\partial t} \text{ or } \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}. \quad \ldots(27)$$

Differentiating Eq. (26) with respect to x and Eq. (27) with respect to t and subtracting, we get

$$\partial^2 E_y / \partial x^2 = \mu_0 \varepsilon_0 (\partial^2 E_y / \partial t^2). \quad \ldots(28)$$

Similarly we can get an equation involving B_z only after differentiating Eqs. (26) and (27) in opposite order and then subtracting, as

$$\partial^2 B_z / \partial x^2 = \mu_0 \varepsilon_0\, (\partial^2 B_z / \partial t^2). \quad \ldots(29)$$

Comparison with the equation for a transverse wave on a string shows immediately that Eqs. (28) and (29) represent the differential equations that imply wave propagation. The velocity of propagation along x-direction is given by

$$v = 1/(\mu_0 \varepsilon_0)^{1/2}. \quad \ldots(30)$$

Substitution of numerical values $\mu_0 = 4\pi \times 10^{-7}$ N-sec$^2/C^2$ and $\epsilon_0 = 8.85 \times 10^{-12}$ $C^2/N - m^3$, gives $v = 2.998 \times 10^8$ m/sec, which is the measured velocity of light c. This shows that light may be electromagnetic radiation.

To show the relationship between the magnitude of the electric and magnetic vectors in the wave, let us differentiate Eq. (25) with respect to t. This gives

$$\partial E_y / \partial t = -E_0 2\pi f \cos [2\pi f (t - x/v)]. \quad \ldots(31)$$

Substituting this value in Eq. (27), we get

$$\partial B_z / \partial x = \mu_0 \epsilon_0 E_0 2\pi f \cos [2\pi f\, (t - x/v)].$$

On integrating with respect to x, we get

$$B_z = -\mu_0\varepsilon_0 E_0 v \sin [2\pi f(t-x/v)] = \mu_0\varepsilon_0 v E_y$$

or $$B_z = (\mu_0\varepsilon_0)^{1/2} E_y = E_y/c. \qquad ...(32)$$

This shows that E_y and B_z are in phase.

If the vacuum is replaced by matter, all the equations given above must be modified by using μ and ε for μ_0 and ε_0. In this case

$1/\sqrt{\mu_0\varepsilon_0} \to 1/\sqrt{\mu\varepsilon}$ *i.e.*, $c \to v$ and the equation $c=\lambda f$ becomes $v=\lambda_m f_m$, where the subscript m refers to the quantities measured in medium.

Unpolarized plane waves are made up of a superposition of polarized plane waves, with the random distribution of **E** and **B** in the yz-plane. The electric and magnetic vectors of each component of these plane waves are perpendicular to each other.

Transverse Nature of Plane Waves. We now show that for a plane wave travelling in the x-direction the electric and magnetic fields vectors are limited in the y-z-plane only. Maxwell's first equation tells us that the net flux of **E** is zero through any arbitrary volume consisting zero charge distribution. Since, **E** does not vary with y or z, hence the net flux through the faces $dx\,dz$ and $dx\,dy$ is zero. Therefore we get

$$\oint \mathbf{E}\,d\mathbf{S} = \int_{xy} \mathbf{E}\,d\mathbf{S} + \int_{yz} \mathbf{E}.d\mathbf{S} + \int_{xz} \mathbf{E}.d\,,\ -$$

or $$\int_{yz} \mathbf{E}.d\mathbf{S} = 0. \qquad ...(33)$$

It is only possible, if E_∞ does not vary with x. Similarly we may prove that Bx does not vary with x. These results show that *the wave has no field component in the direction of travel*, or the **E** and **B** are transverse to the direction of propagation.

10.4. THE GENERAL SOLUTION OF MAXWELL'S EQUATIONS

Maxwell equations may be converted to differential equations giving the fields in terms of ρ and **j**. The vector potential **A** is introduced in the same way as in the case of a static magnetic field. Since divergence of a curl of vector is always zero, hence Maxwell's equation div **B**=0 shows that we may define **B** as

$$\mathbf{B} = \text{curl } \mathbf{A}. \qquad ...(34)$$

Maxwell's equation curl $\mathbf{E} = -\partial\mathbf{B}/\partial t$ may be expressed in terms of **A**, as

$$\text{curl } \mathbf{E} = -\frac{\partial}{\partial t}(\text{curl } \mathbf{A}) = -\text{curl } \frac{\partial \mathbf{A}}{\partial t}$$

or $$\text{curl } (\mathbf{E} + \partial\mathbf{A}/\partial t) = 0. \qquad ...(35)$$

Here we have changed the order of differentiation as time ans space coordinates are independent variables. This equation showd that $\mathbf{E}+\partial\mathbf{A}/\partial t$ is a vector whose curl is zero. Therefore it may be put in the form of the gradient of a scalar function, as

$$\mathbf{E}+\partial\mathbf{A}/\partial t=-\text{grad}\,\phi \quad \text{or} \quad \mathbf{E}=-\text{grad}\,\phi-\partial\mathbf{A}/\partial t. \qquad ...(36)$$

We have thus solved two of Maxwell's equations. Now we return to the two remaining Maxwell equations which will give us relations between the potentials and the sources ρ and $\mathbf{j}$.

Putting $\mathbf{E}=-\nabla\phi-\partial\mathbf{A}/\partial t$ into $\nabla.\mathbf{E}=\rho/\varepsilon_0$, we get

$$\nabla.\left(-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}\right)=\frac{\rho}{\varepsilon_0}$$

or
$$\nabla^2\phi+\frac{\partial}{\partial t}\,(\text{div}\,\mathbf{A})=-\frac{\rho}{\varepsilon_0} \qquad ...(37)$$

Similarly equation $\nabla\times\mathbf{B}=\mu_0\,[\mathbf{j}+\varepsilon_0\,(\partial\mathbf{E}/\partial t)]$ becomes

$$\nabla\times(\nabla\times\mathbf{A})=\mu_0\left[\mathbf{j}+\varepsilon_0\,\frac{\partial}{\partial t}\left(-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}\right)\right]$$

or
$$\nabla\,(\nabla.\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{j}-\mu_0\varepsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right)-\mu_0\varepsilon_0\,\frac{\partial^2\mathbf{A}}{\partial^2 t}$$

or
$$\nabla^2\mathbf{A}-\mu_0\varepsilon_0\,\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\,\mathbf{j}+\text{grad}\left[\text{div}\,\mathbf{A}+\mu_0\varepsilon_0\frac{\partial\phi}{\partial t}\right] \qquad ...(38)$$

The equations (37) and (38) are mixed, each equation consists both ϕ and $\mathbf{A}$. Lorentz found a more general condition for making the equations separable, namely $\text{div}\,\mathbf{A}+\mu_0\epsilon_0\,\partial\phi/\partial t=0$. Above equations then become

$$\nabla^2\phi-\mu_0\varepsilon_0\,\frac{\partial^2\phi}{\partial t^2}=-\rho/\varepsilon_0 \qquad ...(39)$$

and
$$\nabla^2\mathbf{A}-\mu_0\varepsilon_0\,\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\,\mathbf{j}. \qquad ...(40)$$

To find solution of these equations let us consider an equation

$$\nabla^2\psi-\frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2}=-f(x,\,y,\,z,\,t). \qquad ...(41)$$

The theory of differential equations gives its solution as

$$\psi\,(x,\,y,\,z,\,t)=\frac{1}{4\pi}\int_V\frac{f(x',\,y',\,z',\,t\pm r/c)}{r}\,dV. \qquad ...(42)$$

where $dV=dx'dy'dz'$ and $r=|\,\mathbf{r}-\mathbf{r}'\,|$. This equation shows that for time argument with negative sign, ψ at the point of x, y, z at time t depends on the value of f at other point x', y', z' at an earlier time $t-r/c$. This time lag r/c is the time taken by a signal to travel

from x', y', z' to x, y, z. The solution of (47) is called the *retarded solution.* Corresponding to the plus sign, the value of ψ at time t is determined by the values of f at other points of space at some later time $t+r/c$ and the solution is called the *advanced solution.* Using equation (40), the components of vector potential **A** may be obtained by substituting corresponding components of μ_0 **j** for the value of f. To find ϕ, f must be replaced by ρ/ε_0.

Generation of Electromagnetic Waves. The simplest radiators of electrostatic waves are the *Hertz Oscillator* and *current carrying loop.* The former consists of the two metal spheres joined by a conductor. If equal and opposite charges are given to these spheres, charge will exchange between the spheres periodically and an oscillator current will flow in the conductor joining these spheres. The resistance of the conductor is kept small otherwise damping will effect these oscillations. This system will work as an electric dipole with a moment p varying with time for the distances much greater then the distance between the spheres. If the dipole moment of the linear oscillator varies as $p=p_0\, e^{i\omega t}$ and the magnetic moment of the current loop varies as $M=M_0 \cos \omega t$, then for $M_0 = cp_0$, the electric and magnetic fields of a dipole are equal to the fields of a current carrying loop. The only difference is of the directions. For a dipole the electric field is directed along the meridians and for a loop it is perpendicular to the meridianal planes. The orientation of the magnetic induction differs in a similar manner.

10.5. POYNTING VECTOR

The propagation of electric and magnetic fields through space implies the energy transmission. The rate of energy transmission across a unit area placed $\perp$ to the direction of propagation of energy is represented by **S**, the *poynting veetor.* Let us consider a elementary volume $dx\, dy\, dz$. If the electromagnetic energy within this volume be U, then the rate of change of this energy is

$$\partial U/\partial t = \int \mathbf{S}.d\mathbf{A}. \qquad \ldots(43)$$

Here we have used $d\mathbf{A}$ for surface area instead of $d\mathbf{S}$ to avoid confusion with **S**. The integral of S is taken over the surface surrounding the volume. The negative sign on the right hand side of the equation shows that the energy is entering the volume.

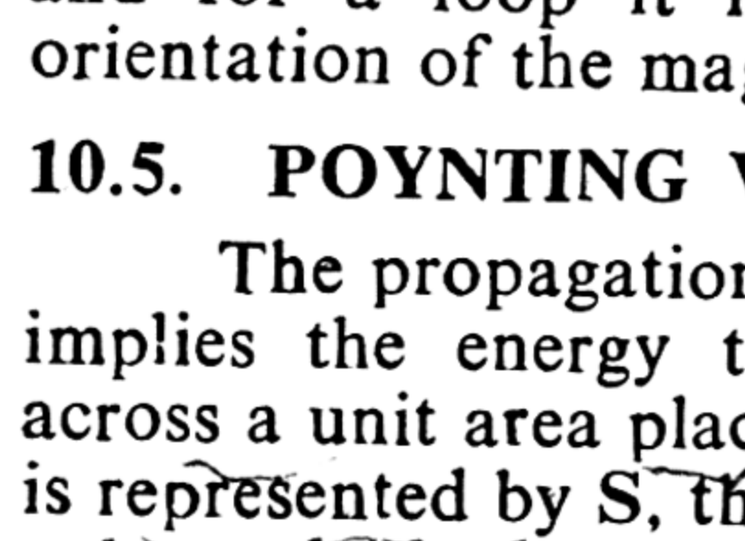


Fig. 10.5.

Let us consider a parallelopiped with sides dx, dy, and dz thus of volume $dx\, dy\, dz$. If electromagnetic wave is passing in the x-direction, the energy flow must be in the x-direction into and out of the volume through the $dy\, dz$ faces only. In this case

$$\int \mathbf{S}.d\mathbf{A} = \int S_x\, dA_x = (Sx\, dy\, dz)_{x+dx} - (S_x dx dz)_x$$
$$= [(Sx)_{x+dx} - (Sx)_x]\, dy\, dz = (\partial S_x/\partial x)\, dx.dy.dz$$

Since the energy flow is due to E_y and B_z, hence we have the relation for energy in the volume as

$$U=[\tfrac{1}{2}\varepsilon_0 E_y^2+\tfrac{1}{2}(\mu_0)^{-1} B_z^2]\, dxdydz$$

$$\therefore \quad \frac{\partial U}{\partial t}=\left[\varepsilon_0 E_y \frac{\partial E_y}{\partial t}+\mu_0^{-1} B_z \frac{\partial B_z}{\partial t}\right] dxdydz.$$

Using relations (26) and (27), we have

$$\frac{\partial U}{\partial t}=-\frac{1}{\mu_0}\left[E_y \frac{\partial Bz}{\partial x}+B_z \frac{\partial E_y}{\partial x}\right] dxdydz=-\frac{1}{\mu_0}\frac{\partial}{\partial x}(E_y B_z)\, dxdydz$$

$$\therefore \quad \frac{1}{\mu_0}\frac{\partial}{\partial x}(E_y B_z)\, dxdydz=\frac{\partial S_x}{\partial x} dxdydz$$

or

$$\frac{1}{\mu_0}\frac{\partial}{\partial x}(E_y B_z)==\frac{\partial Sx}{\partial x}.$$

Integration with respect to x gives

$$Sx=E_y Bz/\mu_0.$$

For the general case it can be written as

$$\mathbf{S}=(\mathbf{E}\times\mathbf{B})/\mu_0 \qquad \text{joules/sec}\cdot m^2=\text{watt}/m^2. \qquad \ldots(44)$$

This shows that the direction of energy flow is perpendicular to the plane containing **E** and **B**. In this equation E and B are the instantaneous values. In the case of sinusoidally varying fields, we have

$$\mathbf{S}_{av}=\frac{1}{\mu_0}\mathbf{E}_{eff}\times\mathbf{B}_{eff}=\frac{1}{\mu_0}\frac{\mathbf{E}_0}{\sqrt{2}}\times\frac{\mathbf{B}_0}{\sqrt{2}}$$

$$=(\mathbf{E}_0\times\mathbf{B}_0)/2\,\mu_0. \qquad \ldots(45)$$

Law of Conservation of Energy for the Electromagnetic field. By the law of conservation of energy the Joule heat is evolved at the expense of the energy of the fields, since there are no other sources of energy. We know an expression for the Joule heat evolved in unit volume in unit time as

$$Q/V=\mathbf{J}.\mathbf{E}=\lambda E^2=J^2/\lambda.$$

Hence for a volume V enclosed by a surface A, we have

$$Q=\int_V \mathbf{J}.\mathbf{E}\, dV.$$

Substituting the value of **j** from Eq. (14), we get

$$Q=\int_V\left[\frac{1}{\mu_0}\,\text{curl}\,\mathbf{B}-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right].\mathbf{E}\, dV.$$

Using the vector identity $\text{div}\,(\mathbf{A}\times\mathbf{B})=\mathbf{B}\,.\,\text{curl}\,\mathbf{A}-\mathbf{A}\,.\,\text{curl}\,\mathbf{B}$ and Eq. (13) for curl **E**, we get

$$Q=-\frac{1}{\mu_0}\int_V \text{div}\,(\mathbf{E}\times\mathbf{B})\,dV-\int_V\left[\,\varepsilon_0\mathbf{E}.\,\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu_0}\mathbf{B}.\,\frac{\partial\mathbf{B}}{\partial t}\right]dV \quad ...(46)$$

We know the self-evident equalities

$$\mathbf{E}\,.\,\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{2}\,\frac{\partial}{\partial t}\,(\mathbf{E}\,.\,\mathbf{E})=\frac{1}{2}\,\frac{\partial}{\partial t}\,(E^2)$$

and $$\mathbf{B}\,.\,\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{2}\,\frac{\partial}{\partial t}\,(\mathbf{B}\,.\,\mathbf{B})=\frac{1}{2}\,\frac{\partial}{\partial t}\,(B^2)$$

$$\therefore\int_V\left[\varepsilon_0\,\mathbf{E}.\,\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu_0}\,\mathbf{B}.\,\frac{\partial\mathbf{B}}{\partial t}\right]dV$$

$$=\frac{1}{2}\,\frac{\partial}{\partial t}\int_V\left(\varepsilon_0E^2+\frac{1}{\mu_0}\,B^2\right)dV=\frac{dU}{dt}\,.$$

Here we have taken out dt as V is independent of time. Using Gauss's theorem, the first term of right hand of Eq. (46) becomes

$$\frac{1}{\mu_0}\int_V\text{div}\;(\mathbf{E}\times\mathbf{B})dV=\frac{1}{\mu_0}\int_A\,(\mathbf{E}\times\mathbf{B}).d\mathbf{A}=\int_A\,\mathbf{S}.d\mathbf{A}$$

Hence Eq. (46) reduces to

$$Q=-\int_A\,\mathbf{S}.d\mathbf{A}=\frac{dU}{dt}\,. \quad ...(47)$$

i.e., Joule Heat=flow of energy through the surface A enclosing the volume minus the energy flow of electromagnetic field within the volume V.

10.6. REFLECTION AND REFRACTION OF ELECTRO-MAGNETIC WAVES

Consider a plane surface dividing a space into two regions, 1 and 2, with different dielectric constants. If a plane wave with wave vector **k** and frequency ω is incident on this surface from the side of region 1. part of this wave will reflect into region 1 and part will refract and transmit into region 2. If the refracted and reflected waves have wave vectors **k′** and **k″** respectively, then the electric field for the three waves are

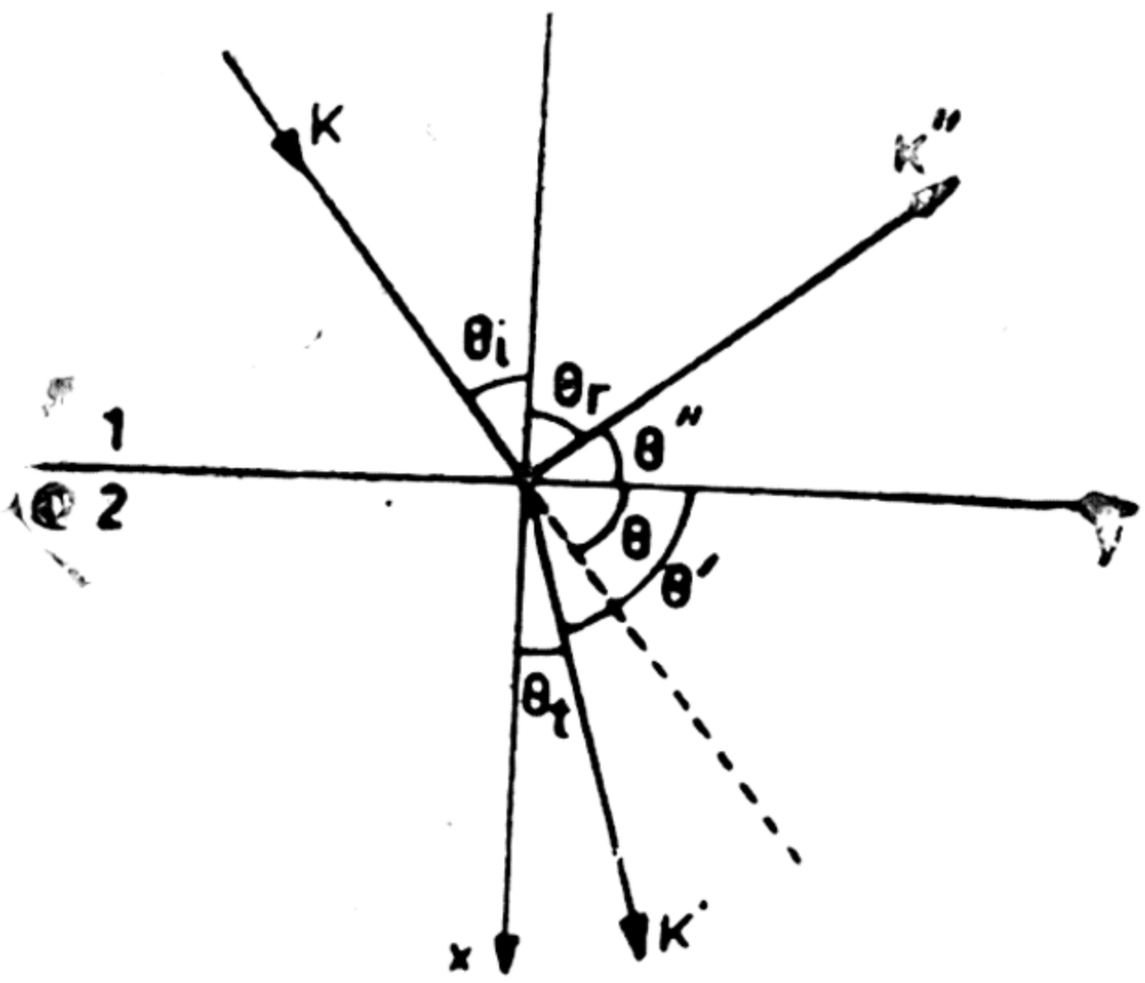


Fig. 10.6.

$$\mathbf{E}\,(\mathbf{r},\,t)=E_0\,e^{i(\omega t-\mathbf{k}\,.\,\mathbf{r})}$$

(Incident wave)

$$\mathbf{E}'\,(\mathbf{r},\,t)=E_0'e^{i(\omega' t-\mathbf{k}\,.\mathbf{r})}$$

(Refracted wave)

$$\mathbf{E}''\,(\mathbf{r},\,t)=E_0''\,e^{i(\omega'' t-\mathbf{k}''.\mathbf{r})}\,.$$

(Reflected wave)

To prove that the frequency of the wave does not change under reflection and refraction, let us use the boundary condition : *the tangential components of the electric field vector remain continuous at the boundary of two media.*

$$E_{ot}\, e^{i(\omega t-\mathbf{k}.\mathbf{r})}+E''_{ot}\, e^{i(\omega'' t-\mathbf{k}''.\mathbf{r})}=E'_{ot}\, e^{i(\omega' t-\mathbf{k}'.\mathbf{r})}$$

$$\text{or}\quad E_{ot}\, e^{-i\mathbf{k}.\mathbf{r}}\, e^{i\omega t}+E''_{ot}e^{-i\,\mathbf{k}''.\mathbf{r}}\, e^{i\omega'' t}=E'_{ot}\, e^{-i\mathbf{k}'.\mathbf{r}}\, e^{i\omega' t} \quad \ldots(48)$$

Differentiating both sides of the equation with respect to t, we have

$$i\omega E_{ot}\, e^{-i\mathbf{k}.\mathbf{r}}\, e^{i\omega t}+i\omega'' E''_{ot}e^{-i\mathbf{k}''.\mathbf{r}}\, e^{i\omega'' t}=i\omega' E'_{ot}\, e^{-i\mathbf{k}'.\mathbf{r}}\, e^{i\omega' t} \quad \ldots(49)$$

Substituting the value of E''_{ot} from Eq. (48) in Eq. (49) and solving, we have

$$iE_{ot}\, e^{-i.\mathbf{k}.\mathbf{r}}\,(\omega-\omega')\, e^{i\omega t}=iE'_{ot}\, e^{-i\mathbf{k}'.\mathbf{r}}\,(\omega'-\omega'')\, e^{i\omega' t} \quad \ldots(50)$$

This equation is satisfied for all values of t, which is possible only if

$$\omega=\omega''.$$

Similarly, by eliminating E'_{ot} from Eq. (48) and (49) and with the same argument, we have

$$\omega=\omega'$$

$$\therefore \quad \omega=\omega'=\omega''. \quad \ldots(51)$$

To prove that the incident, reflected and refracted rays lie in the same plane, let us write the boundary condition (48) as

$$ae^{-i\mathbf{k}.\mathbf{r}}+b\,.e^{-i\mathbf{k}''.\mathbf{r}}=c\,e^{-i\mathbf{k}'.\mathbf{r}} \quad \ldots(52)$$

where a, b and c are independent of $\mathbf{r}$. Using the vector relation

$$\mathbf{r}.\nabla\, e^{-i\mathbf{k}.\mathbf{r}}=\left(x\frac{\partial}{\partial x}+y\,\frac{\partial}{dy}+z\,\frac{\partial}{\partial z}\right)\, e^{-i\mathbf{k}.\mathbf{r}},$$

$$=-i(\mathbf{k}.\mathbf{r})\;\; e^{-i\mathbf{k}\,\mathbf{r}}$$

Eq. (52) now may be written as

$$-i\,a\,(\mathbf{k}.\mathbf{r})e^{-i\mathbf{k}.\mathbf{r}}\;-ib\;(\mathbf{k}''.\mathbf{r})\;e^{-i\mathbf{k}''.\mathbf{r}}=-ic\;(\mathbf{k}'.\mathbf{r})\;e^{-i\mathbf{k}'.\mathbf{r}} \quad \ldots(53)$$

Substituting the value of $c\,e^{-i\mathbf{k}'.\mathbf{r}}$ from eq. (52) in eq. (53), we have

$$i\,a\,[(\mathbf{k}\,.\,\mathbf{r})-(\mathbf{k}'\,.\,\mathbf{r})]\,e^{-i\mathbf{k}\,.\,\mathbf{r}}=i\,b\;[\mathbf{k}'\,.\,\mathbf{r})-(\mathbf{k}''\,.\,\mathbf{r})]e^{-i\mathbf{k}''\,.\,\mathbf{r}}$$

This is valid for all vectors **r** lying in the boundary plane, which is possible only if

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k}'' \cdot \mathbf{r}.$$

Similarly, by eliminating $b\,e^{-i\mathbf{k}'' \cdot \mathbf{r}}$ from equations (52) and (53) and with the same argument, we have

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k}' \cdot \mathbf{r}.$$

$$\therefore \quad \mathbf{k} \cdot \mathbf{r} = \mathbf{k}' \cdot \mathbf{r} = \mathbf{k}'' \cdot \mathbf{r}. \qquad \ldots(54)$$

It follows that **k**, **k**′ and **k**″ are coplanar.

To find the relationship between the Angles. Let us take the xy plane as the plane of incident, reflected and refracted rays. The y-axis lies along the boundary and the x-axis is perpendicular to it. If we choose the origin at some point on the negative y-axis, the direction of **r** will be along the positive y-axis. Therefore Eq. (54) may be written as

$$kr \cos\theta = k'r \cos\theta' = k''r \cos\theta''. \qquad \ldots(55)$$

The wave numbers k, k' and k'' may be written in terms of velocities of the incident, refracted and reflected waves respectively as

$$k = \omega/v,\ k' = \omega'/v' = \omega/v',\ k'' = \omega''/v'' = \omega/v''.$$

Since the incident and reflected waves are propagated in the same medium, we may write

$$v = v'' \text{ and } k = k''.$$

This alongwith Eq. (55) gives

$$\cos\theta = \cos\theta'' \quad \text{or} \quad \theta = \theta''$$

Hence $$\theta_i = \theta_r. \qquad \ldots(56)$$

i.e. The angle of incident is equal to the angle of reflection.

Eq. (55) may also be written as

$$(1/v)\cos\theta = (1/v')\cos\theta'.$$

Since $\cos\theta = \sin\theta_i$ and $\cos\theta' = \sin\theta_t$, hence we may write this relation as

$$\sin\theta_i/\sin\theta_t = v/v'. \qquad \ldots(57)$$

If n and n' be the refractive indices of the first and second media respectively, then we have

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v}{v'} = \frac{n'}{n} = n_{12}, \qquad \ldots(58)$$

i.e, the ratio of the sine of the angle of incident to the sine of the angle of refraction is equal to the ratio of the refractive index of the second medium with respect to the first. This is called *Snell's law.*

To find Fresnel's Equations. Let us consider the case of normal incidence of waves on the boundary. Let the x-axis be along the direction of incidence wave, y-axis along the direction of **E** for the incident wave and the y-and z-axes in the boundary plane. For the incident wave, we thus have

$$E_y = E_o\, e^{i(\omega t - \mathbf{k}.\mathbf{r})} = E_o\, e^{i(\omega t - kx)}, \quad E_x = E_z = 0.$$

and $$B_z = (\varepsilon_1\mu_1)^{1/2}\, E_y = (\varepsilon_1\mu_1)^{1/2}\, E_o\, e^{i(\omega t - kx)}, \quad B_x = B_y = 0.$$

For the refracted waves these vectors may be written as

$$E'_y = E'_o\, e^{i(\omega t - \mathbf{k}'.\mathbf{r})} = E'_o\, e^{i(\omega t - k'x)}, \quad E'_x = E'_z = 0$$

and $$B'_z = (\varepsilon_2\mu_2)^{1/2}\, E_o'\, e^{i(\omega t - k'x)}, \quad B'_x = B'_y = 0.$$

The expressions for the vectors **E** and **B** in the reflected wave may be obtained by taking into account the two facts :

(1) The reflected wave travels in the negative x-direction.

(2) As **E**, **B** and **k** are mutually perpendicular vectors, hence for the reflected wave either **E** or **B** is reversed. Therefore the field vectors for the reflected wave (for which **B** is reversed) are

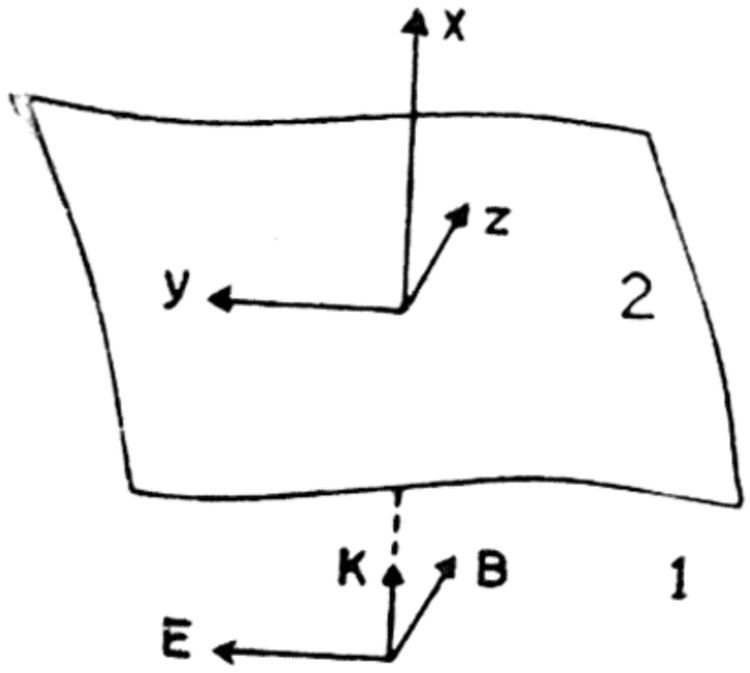


Fig. 10.7.

$$E''_y = E''_o e^{i(\omega t - \mathbf{k}''.\mathbf{r})} = E''_o\, e^{i(\omega t + k''x)},$$

$$E''_x = E''_z = 0$$

and $$B''_z = (\varepsilon_1\mu_1)^{1/2} E''_o\, e^{i(\omega t + k''x)},$$

$$B''x = B''_y = 0.$$

Using the boundary conditions : *the tangential components of* **E** *and* **B** *are continuous*, at the point on the boundary (say the origin), we get

$$E_o + E''_o = E'_o \qquad \ldots(59)$$

and $$(\varepsilon_1\mu_1)^{1/2}\, E_o - (\varepsilon_1\mu_1)^{1/2}\, E''_o = (\varepsilon_2\mu_2)^{1/2}\, E'_o. \qquad \ldots(60)$$

Solving these equations, we get

$$E''_o = \frac{(\varepsilon_1\mu_1)^{1/2} - (\varepsilon_2\mu_2)^{1/2}}{(\epsilon_1\mu_1)^{1/2} + (\varepsilon_2\mu_2)^{1/2}}\, E_o \qquad \text{(Reflected wave)}$$

and $$E'_o = \frac{2(\varepsilon_1\mu_1)^{1/2}}{(\varepsilon_1\mu_1)^{1/2} + (\varepsilon_2\mu_2)^{1/2}}\, E_o. \qquad \text{(Refracted wave)}$$

For dielectrics, $\mu_1 = \mu_2 = \mu_0$. If $(\varepsilon_2/\varepsilon_1)^{1/2} = n_{12}$, then above relations become

$$E''_o = \frac{1 - n_{12}}{1 + n_{12}}\, E_o \text{ and } E'_o = \frac{2}{1 + n_{12}}\, E_o. \qquad \ldots(61)$$

This pair of equations is known as *Fresnel's equations* for plane polarized waves, herewith the plane of polarization normal to the plane of incidence.

The intensity of the wave is described by the absolute magnitude of the Poynting vector.

$$S=\frac{E_y B_z}{\mu}=\frac{E_y}{\mu}(\varepsilon\mu)^{1/2}E_y=\left(\frac{\epsilon}{\mu}\right)^{1/2}E_y^2. \qquad ...(62)$$

Since **E** and **B** vary harmonically, the mean values of these vectors

$$<E^2>=E_o^2<\cos\omega t>=\tfrac{1}{2}E_o^2 \text{ and } <B^2>=\tfrac{1}{2}B_o^2. \qquad ...(63)$$

$$\therefore \qquad S_{av}=\tfrac{1}{2}(\varepsilon/\mu)^{1/2}E_o^2. \qquad ...(64)$$

The mean intensities per period for the incident, refracted and reflected waves are

$$S_o=\tfrac{1}{2}\left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2}E_o^2 \qquad ...(65)$$

$$S_o'=\tfrac{1}{2}\left(\frac{\varepsilon_2}{\mu_2}\right)^{1/2}E'^2_o=\tfrac{1}{2}\left(\frac{\epsilon_2}{\mu_2}\right)^{1/2}\frac{4}{(1+n_{12})^2}E_o^2 \qquad ...(66)$$

$$S_o''=\tfrac{1}{2}\left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2}E''^2_o=\tfrac{1}{2}\left(\frac{\epsilon_1}{\mu_1}\right)\left(\frac{1-n_{12}}{1+n_{12}}\right)^2E_o^2. \qquad ...(67)$$

These equations are also called *Fresnel's formulae.*

$$\textit{Coefficient of refraction}=\frac{S'_o}{S_o}=\frac{4n_{12}}{(1+n_{12})^2} \qquad ...(68)$$

$$\textit{Coefficient of reflection}=\frac{S''_o}{S_o}=\left(\frac{1-n_{12}}{1+n_{12}}\right)^2. \qquad ...(69)$$

Relations (65) to (67) show that

$$S_o=S'_o+S''_o, \qquad ...(70)$$

i.e., the *conservation of energy* is satisfied in the process of reflection and refraction at a boundary separating dielectrics.

10.7. ELECTROMAGNETIC CAVITIES

In the oscillating mass-spring system the *K.E.* is stored in the moving mass and the *P.E.* is in the spring, *thus the two forms of energy appear in separate parts of the system.* These are not separated in a mechanical oscillating systems, *e.g.*, acoustic cavity resonator (*organ pipe*), but are present throughout the volume of the cavity. Such a cavity is called the oscillating system with distributed elements, while the spring-mass system is called the system with *lumped elements.*

A simple L—C circuit is an example of lumped elements, as the two kinds of energy are stored in different places (L and C) in this case. In modern practice, the *electromagnetic cavity resonator* replaces the lumped parameter LC resonant circuit. It is a distributed electromagnetic oscillator. A metallic enclosure acts as an electromagnetic resonator, when a system of oscillating electric and magnetic fields such as a magnetron is set up in it. The oscillations can be sustained within the enclosure with a very small expenditure of power, the power, required to compensate the power loss in the cavity walls. The resonant frequencies are far higher than the acoustic frequencies in the cavities of the same size, because of this fact velocity of *em*-waves in free space is much greater than the velocity of sound in air. In the fundamental mode (or lowest frequency) the angular frequency of oscillations for the cavity of length l is given by

$$\omega = 2\pi\nu = 2\pi v/\lambda = 2\pi v/2l = \pi v/l. \qquad \ldots(71)$$

Electromagnetic cavities are used as the resonant circuit in high frequency tubes, (*e.g.*, klystron), for bandpass filters, for wavemeters and for a number of other applications.

Consider a close cylinder of radius a and length l, whose walls are made of copper or some other good conductor. If it is connected to a source of electromagnetic radiation through a small hole in its side wall, the electric and magnetic lines of force are set up in the cavity. These vary with time. The electric lines are the horizontal lines originating on positive charges at one end of the cylinder and terminating on negative charges at the other end. At any point in the cavity the energy stored per unit volume in the electric field is $\frac{1}{2}\varepsilon_0 E^2$. The magnetic lines form circles about the cylinder axis. It is maximum when the electric field is zero. At any point in the cavity the energy stored in the magnetic field per unit volume is $B^2/2\mu_0$. Thus the energy is shuttled back and forth between the electric and magnetic fields as in the LC circuit. These fields no longer occupy completely separate regions of space hence it is the case of distributed elements.

It is easy to prove that the angular frequency of oscillation for the electromagnetic cavity in the fundamental mode is given by

$$\omega = 1.19\ c/a. \qquad (\textit{cavity of radius } a) \qquad \ldots(72)$$

The structure of the interior fields depends on the geometry or shape of the cavity.

10.8. TRANSMISSION LINES

Transmission line is a device to transmit high frequency electromagnetic energy from a given source (*generator*) to the load (*antenna*). It consists of more uniform and parallel conductors. The cross-sections of several common forms of transmission lines are shown in

Fig. 10.8 [(*a*) shows the two wire line, (*b*) the shielded two wire line and (*c*) the coaxial line].

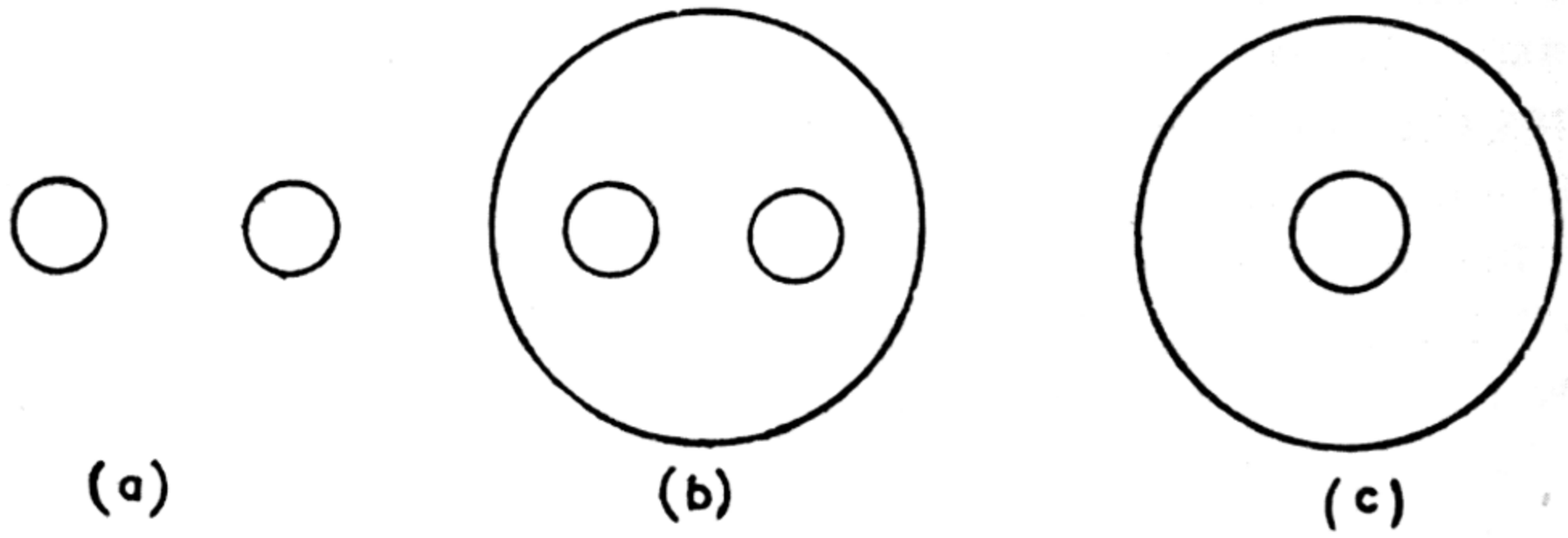


Fig. 10.8.

Consider a coaxial cable. If a potential difference is applied suddenly between the central and the outer conductors of the cable, it does not appear instantaneously all along the line but propagate, with a finite speed. For resistanceless line this speed is exactly equal to c, the velocity of light. At a distance l along the line, the potential difference rises suddenly at a time given by $t=l/c$ from zero to the emf applied. At any instant $t(=t_1)$, the signal has not yet reached points where $x>ct_1$. If a sinusoidal output of an electromagnetic oscillator is applied the sinusoidal wave will be propagated. If the oscillator frequency is 50 cycles/sec, the wavelength of the wave propagated will be

$$\lambda=3\times10^8/50=6\times10^6 \text{ meters.}$$

It is very high. Frequencies in the radio or the microwave range are much higher and the wavelengths are smaller.

We deal with lumped elements if the wavelength is much larger than the dimensions of the system and with distributed elements otherwise. Therefore a transmission line 60 meters long will be a lumped system for electromagnetic radiation at 50 cycles/sec ($\lambda=6\times10^6$ m) but a distributed system at 10^8 cycles/sec ($\lambda=3$m). In a lumped system the analysis is carried out in terms of parameters L, C and R. In a distributed system the analysis is carried out in terms of the fields.

The distributed parameters of a transmission line are :

(1) *Distributed resistance.* The wires of a transmission line are of some conducting material, hence have resistance. It is distributed uniformly along their entire lengths.

(2) *Distributed inductance.* The magnetic field developed about the wires of transmission line opposes the change in line current. Hence the inductance, which is distributed uniformly along the entire length of the wire, is present in series with line resistance.

(3) *Distributed conductance.* The dielectric between the wires is not perfect, a conduction current will flow between the wires. It is measured in terms of shunt conductance.

(4) *Distributed capacitance.* The two parallel conducting lines are separated by a dielectric, hence form a capacitor. It shunts the line.

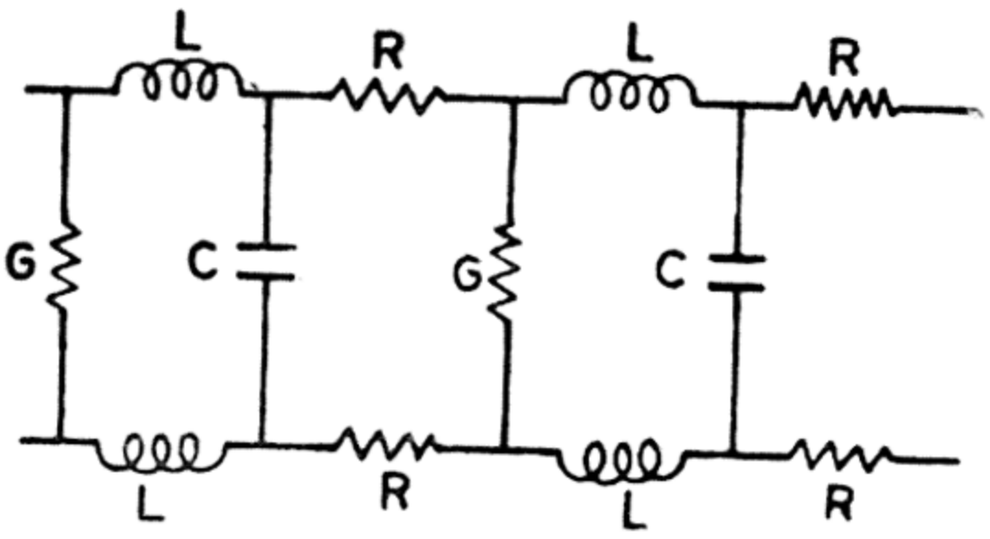


Fig. 10.9.

Therefore a transmission line is equivalent to a circuit with resistances and inductances in series and a conductance and capacitances in parallel (Fig. 10.9).

In the coaxial cable, the current moves along the inner conductor in one direction and returns along the outer conductor in opposite direction. The space between the inner and outer conductors is filled with a dielectric and an electric field acts between these conductors. The lines of force of the magnetic field are concentric circles with their centres on the axis of the cable. Since there is no loss of energy during transmission along it, hence div **E** $=0$. The direction of motion of the electromagnetic energy remains the same if the current in the cable is reversed.

At high frequencies, the transmission line begins to radiate electromagnetic energy and acts as an aerial. At very high frequencies large losses of energy occur in the insulation of conductors. Therefore transmission lines are not suitable for transmitting energy at high frequencies ($\lambda \sim$ centimeter).

10.9. WAVE GUIDES

At the high frequencies ($\lambda < 10$ cm) the losses in a transmission line become quite large. It is convenient to use hollow conducting tube for the propagation of electromagnetic energy at such frequencies. The wave is confined to move in the direction of the tube. Such a wave is called a *quided wave* and the hollow conducting tube as *wave guide*. In the wave guide the wave is altered from a plane wave, as is required in order that the electric and magnetic boundary conditions at the surface of the conductor be satisfied. There are many different modes (patterns) of electromagnetic waves that can satisfy tne boundary conditions and propagate energy down a guide. There are two most commonly used types : *rectangular and circular.* The geometry and size of a guide allows only one mode of transmission for a limited range of frequencies

We know that *electric field must always be perpendicular to the surface of a conductor. As the magnetic field is always perpendicular to the electric field in e.m. waves, hence must be parallel to the surface of the conductor.* Therefore the lines of **B** form closed loops in planes parallel to the broad face of the rectangular guide. The electric field

configuration is similar to that in a polarized plane wave. Lines of **B** and **E** for wave guide of rectangular cross section are shown in Fig. 10.10, (*a*) is showing a side view of the lines of **E** and (*b*) is showing the top view of the lines of **B**.

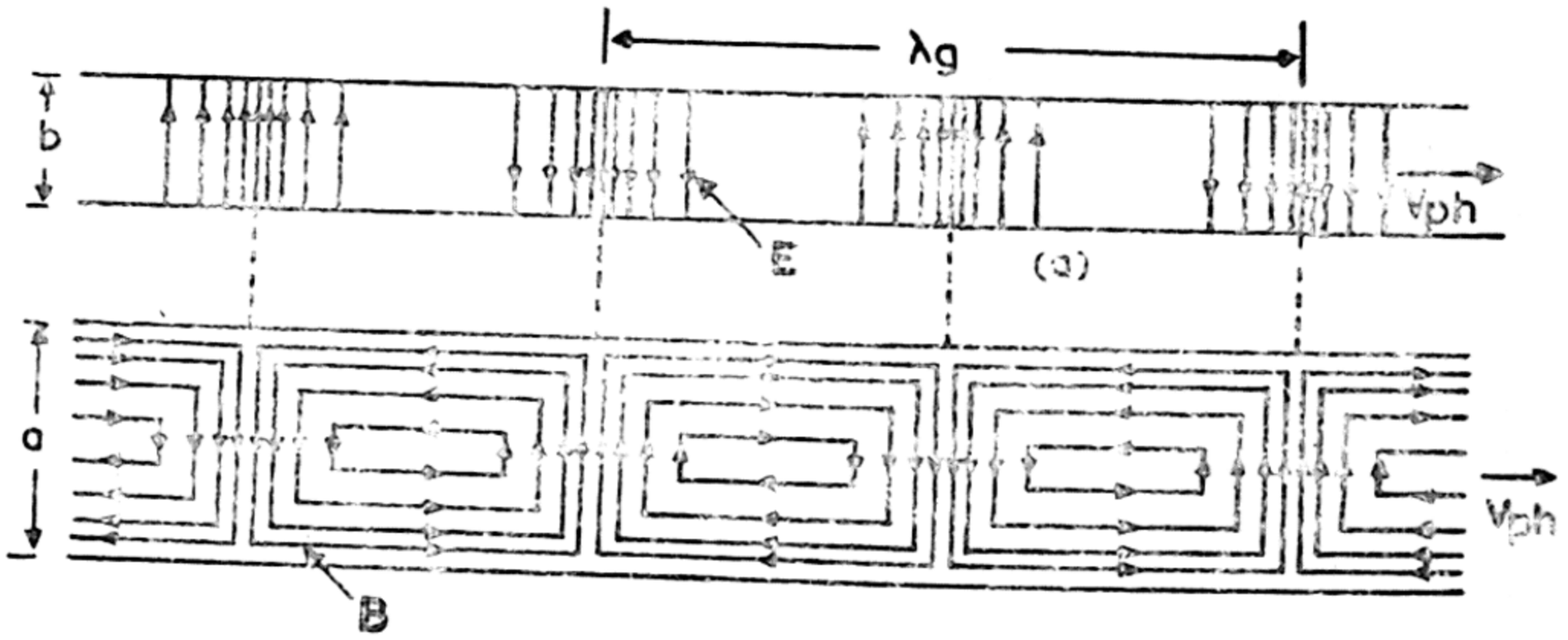


Fig. 10.10.

Maxwell's equations, subject to the boundary conditions at the conductor surface, can determine the configuration of electromagnetic field inside the guide. Consider a rectangular guide along the x-axis. The transmission is confined to a finite range between two xy planes and two xz planes. The expresssions obtained for the transverse components in terms of transverse derivatives of the longitudinal components may be divided into two sets—the transverse electric (TE mode) with $Ex=0$ and the transverse magnetic waves (TM mode) with $B_x=0$.

For a given frequency ω of electromagnetic waves to be transmitted, we select a guide whose dimensions are such that ω is larger than the *cut off* frequency ω_0 for the *dominant mode*. The cut off frequency of a waveguide depends on its dimensions and the mode of transmission. In a wave guide, we consider two kinds of velocity. The first is the *phase velocity* v_{ph}, the velocity with which the wave patterns along the guide travel. It is obtained by measuring the distance between peaks in the wave (λ_g). The other is the velocity with which energy or information carrying signals travel along the guide. This is called the *group velocity* v_{gr}.

From Maxwell's equations it can be shown that

cut off frequency $$f_c=\frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

and the *cut off wavelength* $$\lambda_c=\frac{c}{f_c}=\frac{2}{\sqrt{(m/a)^2+(n/b)^2}}\ . \qquad \ldots(73)$$

Here m and n are the integers which specify the mode of transmission. The first is connected with the distance a and the second with b.

The phase velocity of propagation along the waveguide axis is c at the critical frequency, but at higher frequencies

$$v_{ph} = \frac{\omega}{\sqrt{\omega^2 \varepsilon_o \mu_o - (m\pi/a)^2 - (n\pi/b)^2}} \quad \ldots(74)$$

Hence $$\lambda_g = \frac{v_{ph}}{f} = \frac{2\pi}{\sqrt{\omega^2 \varepsilon_o \mu_o - (m\pi/a)^2 - (n\pi/b^2)}} \quad \ldots(75)$$

These relations are identically the same for TE mode as those for TM mode. Mode of transmission is represented by subscripts m and n, where m is always taken across wide dimension of the guide and n is taken along narrow dimension. In the TM mode, the field components are zero for $m=n=0$, hence the lowest cut off frequency will occur for $m=n=1$. This particular wave is called TM_{11} wave. In the TE-mode, the field components do not vanish if either m or n (but not both) is zero. Thus TE_{10} and TE_{01} waves are possible. Since $a > b$, hence the TE_{10} wave has a lower cut off frequency than the TE_{01} wave.

$$\lambda_g = \frac{2\pi}{\sqrt{\omega^2/c^2 - (\pi/a)^2}} = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} \quad \ldots(76)$$

and $$v_{ph} = \frac{c}{\sqrt{1-(\lambda/2a)^2}} \quad \ldots(77)$$

At first glance $v_{ph} > c$ shows a direct contradiction to relativity theory. Actually the relativisitic limitation on velocities ($v \leqslant c$) refers to mass or energy transformation, hence we must compare v_{gr} which is always smaller than or equal to c as in general

$$v_{ph}\, v_{gr} = c^2. \quad \ldots(78)$$

Above relations also show that *the energy cannot be transmitted down the guide for* $\lambda \geqslant 2a$.

10.10. ANTENNA

An electromagnetic transmission line or a waveguide may be terminated in many ways and the energy may escape into the space beyond. The electromagnetic energy radiated into the space is of the form of a travelling electromagnetic wave in free space. The transmission and reception of electromagnetic waves used for radio communication are accomplished by radiators and collectors exposed in space. *These radiators and collectors are respectively named as transmitting and receiving antennas*. An antenna is a device composed of a system of one or more linear conductors, the dimensions of which vary from a fraction to several wave lengths. It may be assumed as a circuit with distributed constants. The lumped inductance and lumped capacitance of an antenna act as an oscillatory circuit.

The potential difference between the two conductors terminated at the end of a coaxial cable (Fig. 10.11) varies sinusoidally as the wave reaches them. Its effect is equivalent to that of the electric dipole and the antenna may thus be treated to be equivalent to an oscillating electric dipole. As the potential difference varies sinusoidally, hence the dipole moment is also sinusoidal.

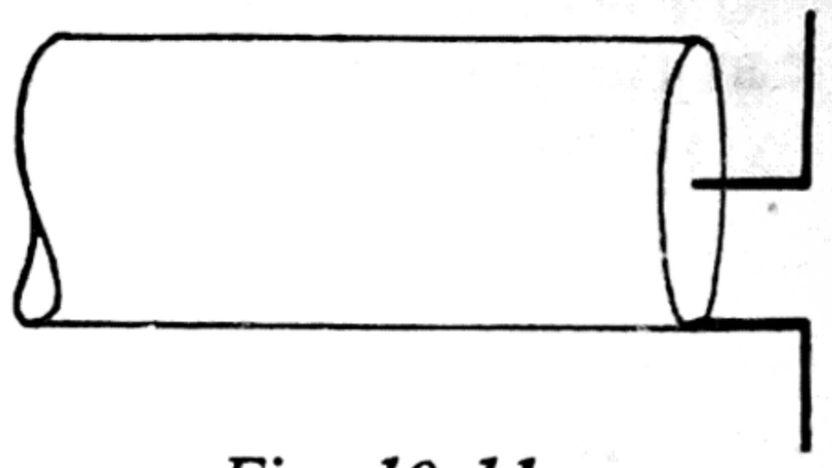


Fig. 10.11.

The velocities of the charges, the electric dipole moment, the equivalent current i and the lines of **E** are shown in Fig. 10.12. The four views, one eighth of a cycle apart, shown in this figure show how the electric lines of force break away from the dipole and form

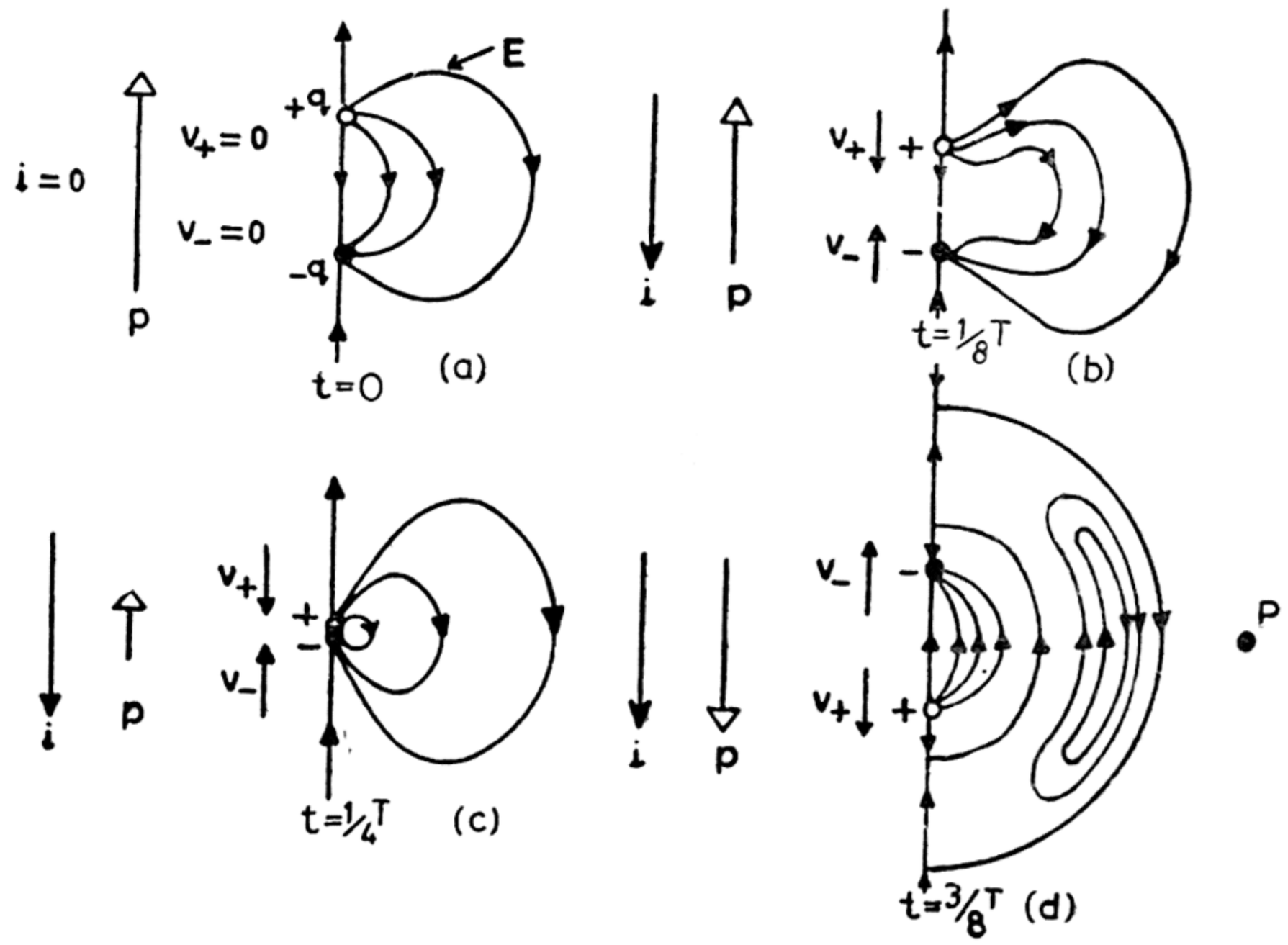


Fig. 10.12.

closed loops. These closed loops travel in free space with speed c. The oscillating charges constitute oscillatory currents, which generate a magnetic field B. The lines of B also form closed loops and move away from the dipole with speed c.

10.11. RADIO WAVE PROPAGATION

Radio waves mean that band of electromagnetic energy which covers the frequency range from a few *Kc/s* to a few *KMc/s*. Propagation of radio waves from the transmitting to the receiving antenna may take place in one or more of the following ways :

(1) **Ground wave or surface wave propagation.** This is of practical importance at low frequency broadcasting and can exist when the transmitting and receiving antennas are close to the earth surface. The curvature of the earth gives an advantage to the surface wave for long distance. Surface waves can bend round the curvature of the earth just as the electromagnetic waves in a wire can bend round its curvature.

Surface wave does not mean that most of the wave energy is confined to the earth's surface. Actually a greater part of energy is in the air above the surface. But in this particular form of propagation, the wave is guided along the earth's surface and around its bends, and curvatures in a manner analogus to that in a guided wire wave.

(2) **Space wave propagation.** In this type of propagation the wave energy travels from transmitting to receiving antenna either straight through air (or space) or being reflected at ground. The space wave propagation is useful mainly for very high frequencies (>30 *Mc/s*). It is limited by earth's curvature.

(3) **Sky wave propagation.** In this type of propagation the wave is reflected back to earth from ionized regions in the upper atmosphere (ionosphere). This region starts about 80 kilometers above the earth surface. The sky waves are of greater importance for very long distance radio communication. These waves are of medium and high frequencies (short waves) and are in the frequency range of 2 to 30 *Mc*/sec.

Since the propagation of space and surface waves are limited by the curvature of the earth, hence are useless for large distances communication. Sky wave propagation is the only way for propagation over long distances (>1000 km). By multiple reflections of the waves with the ionosphere and earth surface, it is possible for a wave of appropriate frequency to travel to any distance round the earth. The waves are also reflected from different parts of the ionized regions and the received signal is the vector sum of all components. This ionized region is produced in the upper atmosphere chiefly by ultra violet radiation from the sun. At such a hight the pressure of the gas is very low and the ultra violet radiation

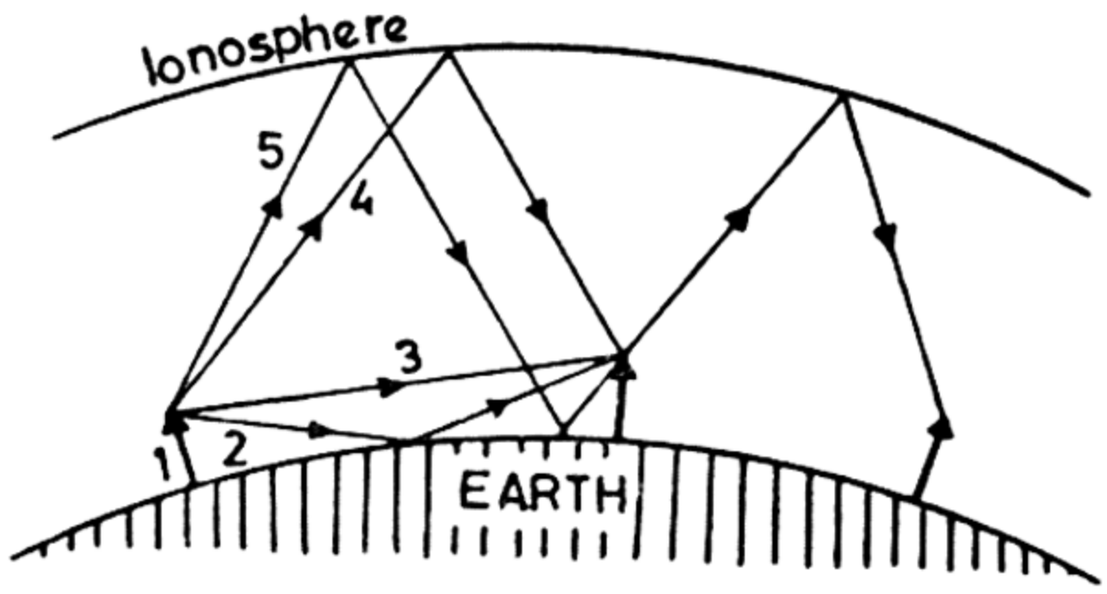


Fig. 10.13. Surface waves (1), Space waves (2, 3), Sky waves (4, 5).

from the sun is very intense to produce ionization. The gases at different altitudes differ in their tendency to become ionized, thus different layers of ions of different densities are formed at different heights.

Exercises

Example 1. *Calculate the retarded potentials for an oscillating electric dipole.*

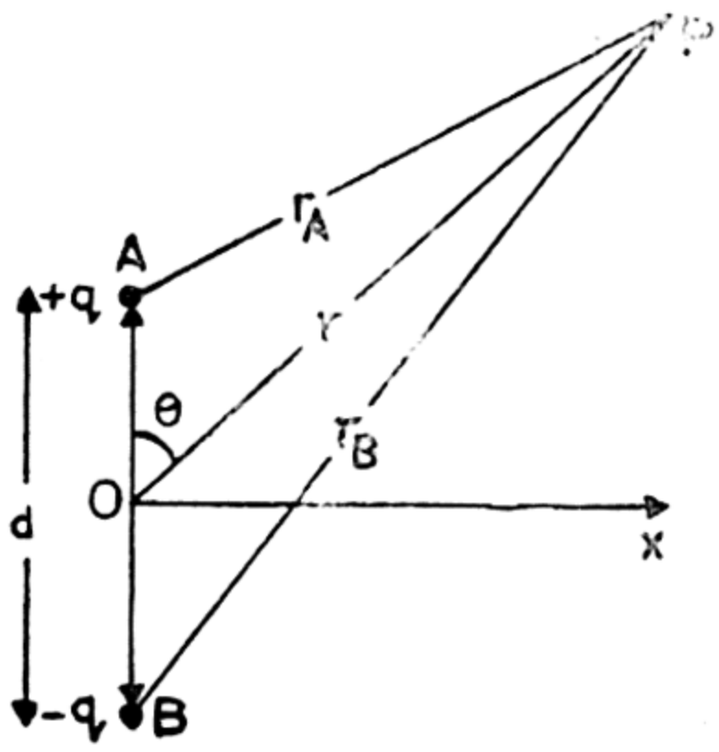


Fig. 10.14.

An electric dipole is a system of two equal and opposite point charges placed at a small distance. Its dipole moment

$$\mathbf{p}=q\mathbf{d}=q_0\, e^{j\omega t}\mathbf{d}=\mathbf{p}_0\, e^{j\omega t},$$

where $\mathbf{p}_0=q_0\mathbf{d}$ is positive in the direction from $-q$ to $+q$ charge. As charge q is oscillating periodically, hence $\mathbf{p}$ is also oscillating. The upward current i through the wire connecting the both charges is thus given as

$$i=dq/dt=j\omega q_0\, e^{j\omega t}=i_0\, e^{j\omega t}.$$

We know that the potential at a distance r from a point charge Q is given by the relation $V=Q/4\pi\epsilon_0 r$. In this case the potential is due to oscillating charges. If the wave propagates in vacuum the potential at an instant t at point P will be due to the wave transmits at the time $t-r/c$. As potential is a scalar quantity, hence at point P at any instant t due to both the charges

$$\text{Potential } V=\frac{q_0 e^{j\omega\,(t-r_B/c)}}{4\pi\epsilon_0 r_B}-\frac{q_0 e^{j\omega(t-r_A/c)}}{4\pi\epsilon_0 r_A}.$$

Similarly the vector potential due to oscillatory current i is

$$\mathbf{A}=\frac{\mu_0 i}{4\pi r}\, e^{j\omega(t-r/c)}\mathbf{d}.$$

Example 2. *An electric field in free space has the components* $Ex=E_y=0$ and $Ez=E_0\cos(2\pi x/\lambda)\cos\omega t$. Find the magnetic field **B**.

We know the relation between fields **E** and **B** in free space, as

$$\text{curl } \mathbf{E}=-\partial\mathbf{B}/\partial t.$$

In the given problem

$$Ex=E_y=0 \text{ and } Ez=Eo\cos\left(\frac{2\pi x}{\lambda}\right)\cos\omega t$$

$$\therefore \quad \text{curl } \mathbf{E}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & Ez \end{vmatrix}=\mathbf{i}\,\frac{\partial Ez}{\partial y}-\mathbf{j}\,\frac{\partial Ez}{\partial x}.$$

$$=\mathbf{j}\left(\frac{2\pi}{\lambda}\right)Eo\sin\left(\frac{2\pi x}{\lambda}\right)\cos\omega t.$$

$$\therefore \quad \frac{\partial \mathbf{B}}{\partial t} = -\mathbf{j}\left(\frac{2\pi}{\lambda}\right) E_o \sin\left(\frac{2\pi x}{\lambda}\right) \cos \omega t$$

or $$\mathbf{B} = -\mathbf{j}\left(\frac{2\pi}{\lambda\omega}\right) E_o \sin\left(\frac{2\pi x}{\lambda}\right) \sin \omega t$$

$$= -\mathbf{j}\left(\frac{1}{c}\right) E_o \sin\left(\frac{2\pi x}{\lambda}\right) \sin \omega t.$$

Example 3. *A laser beam has a power of 25 gigawatts and diameter of 1 mm. Calculate the peak values of* **E** *and* **B** .

As the instantaneous value of the poynting vector at a given point in space is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{or} \quad Sx = \frac{1}{\mu_0} E_y B_z = \frac{1}{\mu_0 c} E_y^2.$$

For a plane sinusoidal wave, the average value of S is

$$S_{av} = \frac{1}{T} \int_0^T \frac{1}{\mu_0 c} \left\{ E_0 \sin\left[2\pi f \left(t - \frac{x}{v} \right) \right] \right\}^2 dt = \frac{E_0^2}{2\mu_0 c}$$

or $$E_0 = [2\mu_0 c \ S]^{1/2} = \left[2 \times 4\pi \times 10^{-7} \times 3 \times 10^8 \times \frac{25 \times 10^9}{\pi \times 10^{-6}} \right]^{1/2}$$

$$= 2.45 \times 10^9 \text{ volt/meter.}$$

Hence the peak value of B

$$B_0 = E_0/c = 2.45 \times 10^9 / 3 \times 10^8 = 8.16 \text{ teslas.}$$

Example 4. *An observer is at a distance 1 meter from a point source whose power output is 10^3 watts. Calculate the electric and magnetic fields as observed by the observer.*

As the power P_0 is radiated in all directions. Hence at any distance r from this radiation centre the average value of Poynting vector

$$\bar{S} = P_0/4\pi r^2 = E_0^2/2\mu_0 c$$

or $$E_0 = \frac{1}{r} \sqrt{\left(\frac{\mu_0 c P_0}{2\pi} \right)} = \frac{1}{1.0} \sqrt{\left(\frac{4\pi \times 10^{-7} \times 3 \times 10^8 \times 10^3}{2\pi} \right)}$$

$$= 244.95 \text{ volt/meter.}$$

The relation $E_0 = cB_0$ gives

$$B_0 = E_0/c = 244.95/3 \times 10^8 = 8.16 \times 10^{-7} \text{ weber/m}^2.$$

Example 5. *Calculate the net average energy flow per sec through an imaginary cylinder in the vacuum, if placed along the energy propagation.*

Let us consider an imaginary cylinder of length L and of cross-sectional area A at a distance x_0 from the source. If the amplitude of the electromagnetic wave is constant, the average energy at one end is same as at other and hence net outflow is zero.

If the source emits energy, such that amplitude decreases linearly with time, then the value of **E** at any instant t at the source is

$$E=E_0(1-at)e^{j\omega t}.$$

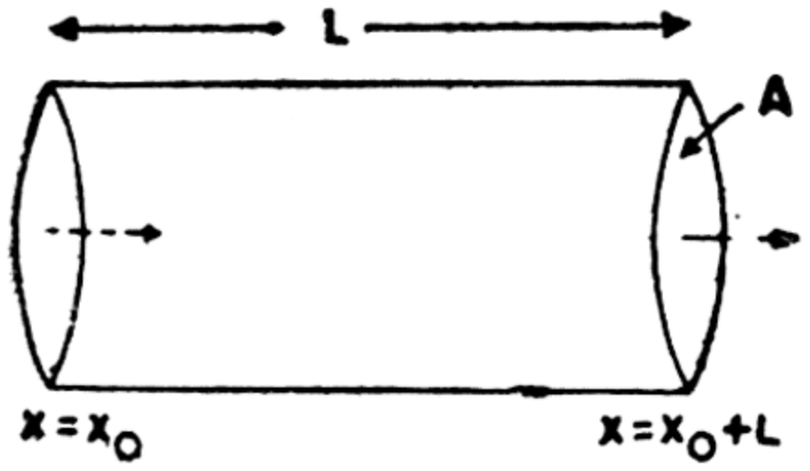


Fig. 10.15.

Its value at a distance x from the source will be same as that at the source at a previous time $t-x/c$.

$$\therefore \quad E=E_0[1-a(t-x/c)]e^{j\omega(t-x/c)}$$

$$=E'_0e^{j\omega(t-x/c)}.$$

The net outward energy flow is obtained by evaluating the surface integral of $\mathbf{S}=(\mathbf{E}\times\mathbf{B})/\mu_0$ over the two ends of the cylinder. Since the flow of energy is along x-direction hence $\mathbf{E}\times\mathbf{B}$ and **A** are parallel.

We have thus

$$\frac{1}{\mu_0}\int_S(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{A}=\frac{1}{2\mu_0 c}\left\{\left[E_0'^2A\right]_{x=x_0+L}-\left[E_0'^2A\right]_{x=x_0}\right\}$$

$$=\frac{A}{2\mu_0 c}\left\{E_0^2\left[1-a\left(t-\frac{x_0+L}{c}\right)\right]^2-E_0^2\left[1-a\left(t-\frac{x_0}{c}\right)\right]^2\right\}$$

$$=\frac{AE_0^2}{2\mu_0 c^2}\left\{2a\left(1-at\right)L+\frac{a^2}{c}\left(2x_0+L\right)L\right\}$$

$$=\epsilon_0 AE_0^2La\left\{\left(1-at\right)+\frac{a}{c}\left(x_0+\frac{L}{2}\right)\right\}.$$

Oral Questions.

1. Why is the quantity $\epsilon_0\partial E/\partial t$ referred to as a displacement current ?
2. Discuss the symmetries that appear between two groups of two equations of Maxwell's.
3. An electromagnetic wave consists of two distinct types of electric and magnetic field vibrations. Where the vibration is in time or in space ?

4. Find the frequencies of electromagnetic waves having the wave length (*a*) 1 meter, (*b*) 0.1 *A*, in free space.

5. An electromagnetic wave travels along the *x*-axis with its electric vector in the *y*-direction. What effect does the wave have on a loop of wire placed (*a*) in *xy*-plane, and (*b*) in *xz*-plane.

6. A wave, with *E* plane polarised normal to the plane of incidence on a perfect conductor at some non zero angle with the normal, find the positions of nodes and antinodes.

7. What do you mean by ionosphere ?

Problems

1. If the *a* field is purely electric and the *b* field is purely magnetic, then prove that under such limitations

$$\nabla \cdot (\mathbf{E}_a \times \mathbf{B}_b) = 0.$$

2. An *ac* source is connected across the parallel plate capacitor of circular plates each of area *A*. Show that the magnetic field *B*, induced by the resulting displacement current, at any point between the plates is given by

$$B = (\mu_0 q_0 r \omega / 2A) \cos \omega t.$$

where *r* is the distance from the axis of symmetry and charge $q = q_0 \sin \omega t$.

3. Calculate the peak values of *E* and *B* for sunlight striking the earth with an intensity of 2.0 cal/cm²—min.

[$E_0 = 1020$ volts/m, $B_0 = 3.4 \times 10^{-6}$ web/m².]

4. Prove that for an electromagnetic waves, the electric and magnetic energy densities are in phase and are equal.

5. A 30 megahertz plane electromagnetic wave propagates in free space and its peak value of electric field intensity is 10 *mV/m*. Calculate the peak value of voltage reduced in a 1 m² 10 turn receiving loop oriented with its plane making an angle 30° with the *E*-direction and containing the normal to a wave front.

[54.4 milivolts]

11

Magnetic Properties of Matter

11.1. MAGNETIC INDUCTION AND MAGNETIC MATERIALS

We know that the electric lines of force change when a dielectric is placed between the parallel plates of a capacitor, this was explained by the simple atomic theory leading to the idea of electric dipoles aligned in the dielectric. Let us now see the effect of placing a piece of iron in a magnetic field. For this an experimental arrangement, known as *Rowland Ring*, is used. This consists of a toroidal winding with magnetic material, say iron, inside it in the form of a ring. This is equivalent to a toroidal solenoid supported on a core of magnetic material. Secondary few turns are connected to a ballistic galvanometer G, as shown in Fig. 11.1 (*a*).

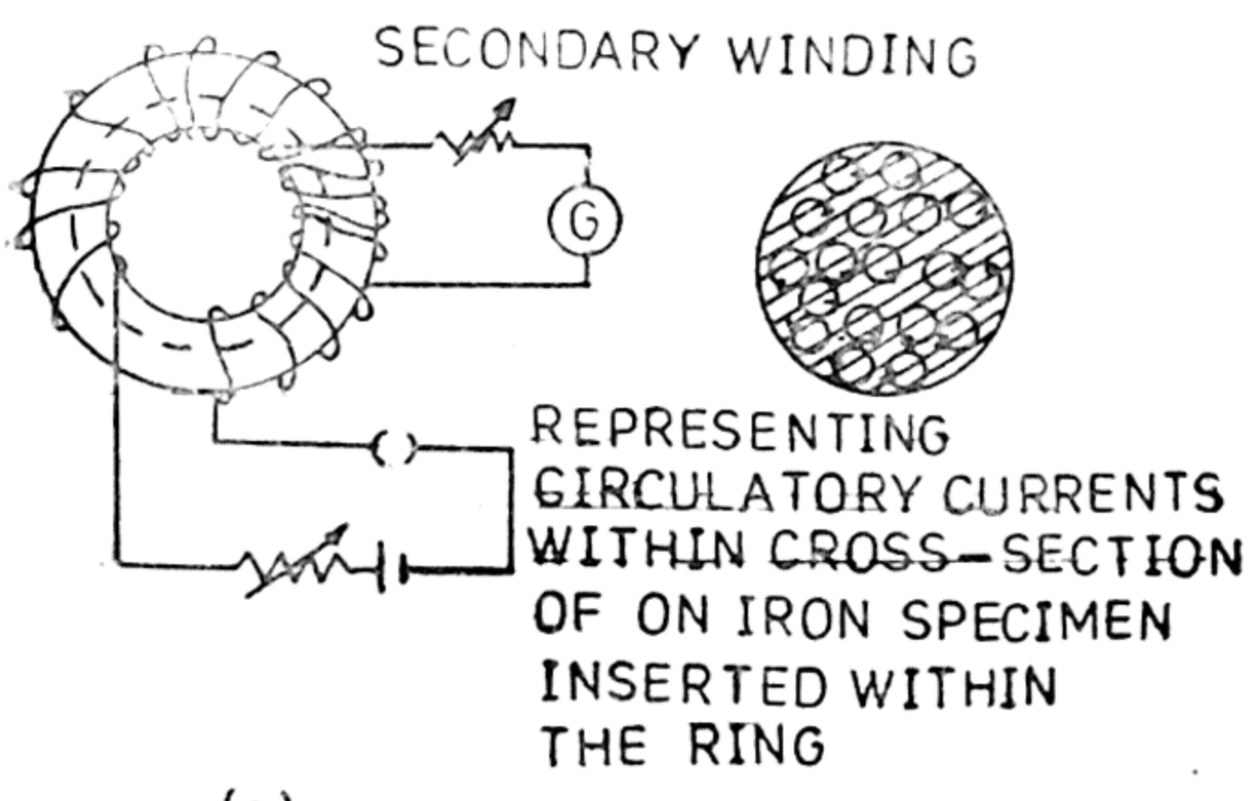


Fig. 11.1.

With the space inside the toroid empty, the change in flux is observed with the help of ballistic galvanometer by turning off the current suddenly. The experiment is repeated with matter filling the entire toroid, the deflection is found for the same current. Experiments show that the magnetic induction **B** is modified by the presence of matter in the toroid. This experiment can be repeated with the core of different materials. Few substances, such as O_2, air, *Pt*, *Al*, *Cr*, *Mn*, etc., show a very small increase in flux in the interior of the toroidal solenoid and are said to be *paramagnetic*. Few other substances, such as H_2, H_2O, *Bi*, *Cu*, *Zn*, *Sb*, *Au*, etc., show a very small decrease in flux, and are said to be *diamagnetic*. There are other substances like *Fe*, *Co*, etc., through which the flux changes to a larger value and are known as *ferromagnetic*.

A circular cross section of the Rowland ring, when the external field is applied, exhibits magnetic effects. This effect can be represented by series of minute current loops, where the all current

directions are the same [Fig. 11.1. (*b*)]. The resultant current is equivalent to the sum of these mesh currents and equal to the current round the boundary of the ring and is known as the surface current.

11.2. THE MAGNETIC DIPOLE CONCEPT OF ATOM

According to the atomic theory, atom consists of electron rotating along closed orbital paths round the nucleus. The nucleus is a central heavy core consisting of neutrons and protons. Let us consider a simple model of one electron of charge e moving with speed v in a circular path of radius r round a nucleus with a proton charge. At any instant, the electron and the proton would appear as an electric dipole of dipole moment p_E, but the average value of p_E over a complete round is zero. Hence *net electric field due to an atom at a certain distance is zero.*

Magnetic Moment of an Electron. The electron rotating with velocity v in a circular path of radius r is equivalent to a current

$$i = e \times \text{frequency } \nu = e(\omega/2\pi) = ev/2\pi r.$$

We have dicussed in art. 8.12 (*B*) that the current carrying loop is equivalent to a magnetic dipole. Hence the magnetic dipole moment of circulating electron is

$$m_l = \text{circulating current} \times \text{area of loop} = (ev/2\pi r)\ \pi r^2 = \tfrac{1}{2}evr. \quad \ldots(1)$$

This is known as the orbital magnetic moment of an electron. In ordinary lump of matter there must be as many electrons going round one way as the other and m_l can be assumed as distributed evenly over all directions in space. Hence the *net orbital magnetic moment will be zero in the absence of an external magnetic field.*

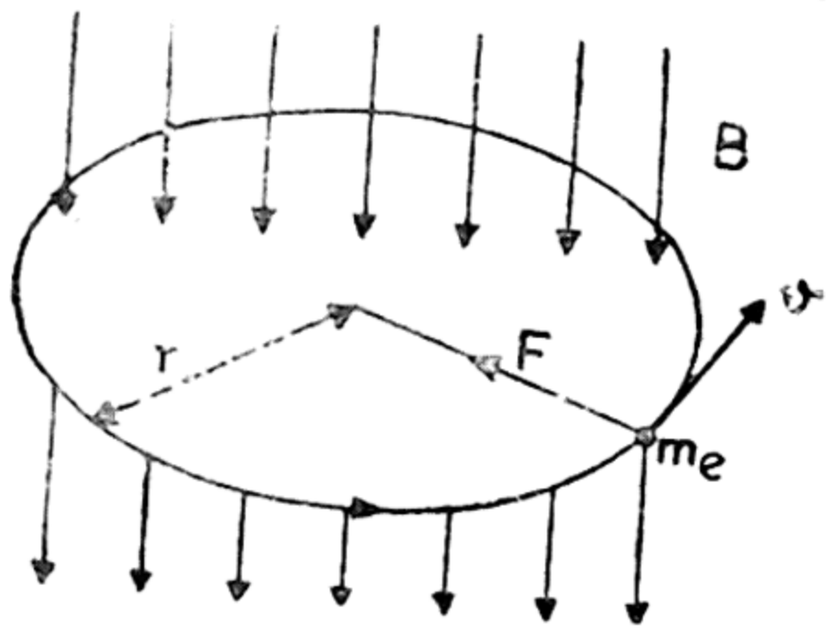


Fig. 11.2.

Let us now see the effect of an external magnetic field of flux density **B.** If the electron of mass me is rotating along anticlockwise direction and the magnetic field **B** is applied $\perp$ to the plane of the orbit in downward direction. When the field **B** changes, the flux through the circular orbit will change and an induced emf $\mathcal{E}$ will thus set up, such that

$$\mathcal{E} = -d\Phi/dt = -\pi r^2\, dB/dt.$$

But $$\mathcal{E} = -\oint \mathbf{E}.d\mathbf{l} = -2\pi r E, \text{ hence}$$

$$E = \tfrac{1}{2} r\ (dB/dt).$$

This electric field will accelerate the electron, hence

$$me\ \frac{dv}{dt} = eE = \frac{er}{2}\ \frac{dB}{dt} \quad \text{or} \quad dv = \frac{er}{2me}\ dB.$$

As e, r and m_e are fixed, the net change in v in the process of bringing the field upto the final value B_0 is

$$\triangle v = erB_0/2m_e.$$

$$\therefore \quad \triangle\omega = \triangle v/r = eB_0/2m_e. \qquad \ldots(2)$$

This is called the *Larmor angular velocity* and is written as ω_L.

The increase in velocity means the increase in the upward magnetic moment $\mathbf{m}_l$. The negatively charged electrons are thus decelerated and the magnetic moment is decreased. Therefore from Eq. (1), the change in magnetic moment

$$\triangle \mathbf{m}_l = -\frac{er}{2}\left[\frac{er\mathbf{B}_0}{2m_e}\right] = -\frac{e^2r^2}{4m_e}\mathbf{B}_0. \qquad \ldots(3)$$

The revolving electron also possesses angular momentum L. Its magnitude is given by

$$L = m_e vr.$$

The magnetic moment of the orbital electron is thus given by

$$m_l = eL/2m_e \quad amp.\ m^2. \qquad \ldots(4)$$

The ratio $m_l/L = e/2m_e$ is called the *orbital gyro-magnetic ratio* or the *orbital magneto-mechanical ratio for the electron.*

From the quantum idea, the orbital angular momentum of the electron is expressed as $L = nh/2\pi$, where n is known as *quantum number.*

$$\therefore \quad m_l = neh/4\pi m_e = nM_B, \qquad \ldots(5)$$

where $M_B = eh/4\pi m_e = Bohr\ magneton.$

Above relation shows that the orbital magnetic moment of the electron is an integer multiple of Bohr magnetron.

Orbital angular momentum **L**, a vector of magnitude $mevr$, is in the upward direction $\perp$ to the plane containing vector **r** and **v**.

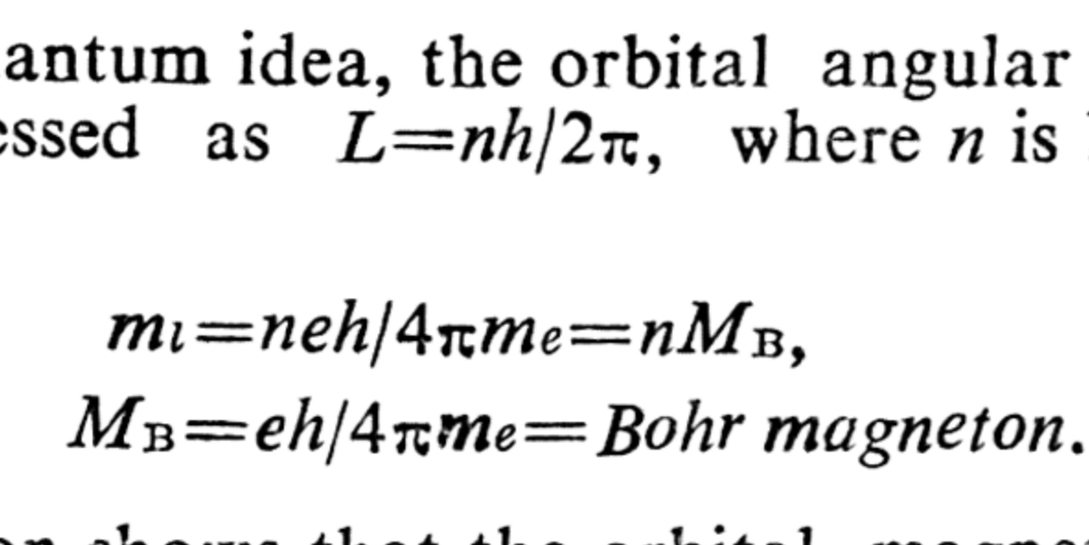


Fig. 11.3.

As the electron is in anticlock-wise motion, hence the current will be in clockwise direction. Thus the magnetic moment $\mathbf{m}_l$ will be in downward direction and is given by

$$\mathbf{m}_l = -(e/2m_e)\,\mathbf{L}. \qquad \ldots(6)$$

Spin Magnetic Moment of an Electron. The electrons not only have orbital motion, but rotate about their own axes, thus possess angular momentum which is independent of orbital motion.

This is called *intrinsic spin of the electron.* The magnetic moment associated with this intrinsic angular momentum (or spin angular momentum) is referred to as the *spin magnetic moment.* As the electron has spin $\frac{1}{2}$, or angular momentum $\frac{1}{2}(h/2\pi)$, hence the spin magnetic moment of electron is $eh/4\pi me$. It shows that

$$\mathbf{m}_s = -(e/me)\mathbf{s}. \qquad \ldots(7)$$

It has purely quantum mechanical explanation and thus can not be discussed here.

The magnitude of the spin magnetic moment ($\mathbf{m}_s$) is always the same, however it alignes with the applied magnetic field of flux density **B**. This orientation of $\mathbf{m}_s$ along **B** is due to the torque $\tau = \mathbf{m}_s \times \mathbf{B}$ experienced by $\mathbf{m}_s$ when placed in **B**.

In any atom, several electrons are usually present. Each has angular momentum and magnetic moment corresponding to both orbital and spin motions. As discussed earlier, the direction of the magnetic moment is opposite to that of angular momentum. The ratio of these two is $e/2m_e$ for the orbital motion and e/m_e for the spin motion. Therefore for the atom as a whole, we may write

$$\mathbf{m} = -g(e/2m_e)\,\mathbf{J}, \qquad \ldots(8)$$

where 'g' (*gyromagnetic ratio*) is a factor which is characteristic of the state of the atom. It is called the '*Lande g-factor*'. It is unity for pure orbital motion and two for a pure spin motion and lies some where in between these two values for the combined motion. Here **J** is the total angular momentum of the atom.

Magnetic Moment of a Proton. In the nucleus, there are neutrons and protons. Many nuclei possess magnetic moments, typically less then atomic moments by a factor of about 10^3 (due to the inverse dependence of m on mass). The proton, like an electron, may move around in some kind of orbit and have an intrinsic spin. The proton is positively charged, the magnetic moment **m** is therefore parallel to the angular momentum **J**. In analogy with Eq. (8), the magnetic moment of a proton

$$\mathbf{m} = g\,(e/2m_p)\,\mathbf{J}, \qquad \ldots(9)$$

where m_p is the mass of the proton and g the nuclear g-factor. By analogy with the Bohr magneton M_B for atomic electrons we define a nuclear magneton M_n as

$$M_n = eh/4\pi m_p = M_B\,(m_e/m_p). \qquad \ldots(10)$$

Thus we see that the nuclear magneton is 1836 times smaller than the Bohr magneton.

If the spinning proton behaved like a uniformly charged classic sphere, then $g_s=1$ and its magnetic dipole moment should be one half nuclear magneton. If the proton behaved like a particle following Dirac's relativistic quantum mechanics, then $g_s=2$ and its magnetic dipole moment should be one nuclear magneton. Actually the spin magnetic dipole moment of the free proton is 2.763 nuclear magnetons.

Magnetic Moment of a Neutron. Neutron should not have magnetic dipole moment on account of its zero charge. Surprisingly the neutron is associated with a fairly large magnetic dipole moment. (−1.913 nuclear magnetons). Actually the neutron has inner constitution such that its spin angular momentum $s=\frac{1}{2}$. The sign of spin magnetic moment for neutron is negative and therefore stimulates the rotation of negative charge in the spin direction.

An atom in a magnetic field. We know that the electrons, protons and neutrons possess magnetic moments. Naturally most of the atoms may have net magnetic moments. If such an atom is placed in a magnetic field, its energy depends on the orientation of this magnetic moment relative to the magnetic field (*the external field*). According to quantum mechanics, the magnetic moment can point only in certain permitted directions. The restriction is called *space quantization*. Classical theory has no explanation for this quantization. Since the magnetic moment has a discrete set of possible orientations, the energy of the atomic level splits into a finite set of energies. In the absence of a magnetic field, all directions in space are equivalent and all the states have the same energy.

In an external magnetic field **B**, the interaction energy of the atom associated with each permitted orientation is given by

$$U=-\mathbf{m}.\mathbf{B}. \qquad \ldots(11)$$

Since the contribution of neutron and proton to the magnetic moment is negligible in comparison to the magnetic moment of the electrons. The latter may be written as

$$\mathbf{m}=-\frac{e}{2m_e}(g_s\mathbf{S}+g_l\mathbf{L}), \qquad \ldots(12)$$

where **S** is the total spin and **L** the total orbital angular momenta of the electrons.

Since $g_s=2$ and $g_l=1$, therefore we have

$$U=\frac{e}{2m_e}[2\,\mathbf{S}.\mathbf{B}+\mathbf{L}.\mathbf{B}] \qquad \ldots(13)$$

The energy shift of a state depends on whether or not **S.B** and **L.B** are constants of the motion. If they are not, the energy shift must be deduced as an average value of U. Two limiting cases are of special interest :

(*a*) *The strong field limit* in which the interaction energy of the atom with the external field greatly exceeds the *internal spin-orbit interaction* which is proportional to **S.L.**

(*b*) *The weak-field limit* in which the interaction with the external field is much less than the spin-orbit interaction.

Let us choose the *z*-axis to lie along the direction of the external field **B.** If **B** is sufficiently great, then Eq. (13) may be written as

$$U=\frac{eB}{2m_e}(2S_z+L_z). \qquad ...(14)$$

This interaction will change the pattern of spectral lines. Its influence is called the *Paschen-Back* effect.

In a weak field, the interaction energy is not sufficient to break down the $L-S$ coupling and the total angular momentum **J** couples with the field **B.** We thus have

$$\mathbf{m}=-g_j\frac{e}{2m_e}\mathbf{J}. \qquad ...(15)$$

and

$$U=-m_zB=g_j\frac{e}{2m_e}J_zB. \qquad ...(16)$$

The influence of this weak interaction is called *Zeeman effect.*

Since J_z is a constant of the motion, given by $J_z=n_jh/2\pi$.

For a given j, the energy splits a level into $2j+1$ equally spaced levels. Since the extreme values of J_z are $jh/2\pi$ and $-jh/2\pi$. Therefore the energy gap between these levels is

$$(\triangle U)_{max}=2g_j\frac{e}{2m_e}\left(\frac{jh}{2\pi}\right)B, \qquad ...(17)$$

where the factor g_j may be written as

$$g_j=\frac{3J^2+S^2-L^2}{2J^2}. \qquad ...(18)$$

In the quantum mechanical form, *i.e.*, in terms of the quantum numbers j, s and l, we get

$$g_j=\frac{3j(j+1)+s(s+1)-l(l+1)}{2j(j+1)}. \qquad ...(19)$$

11.3. MAGNETIZATION OF MATTER

A material body is consisting of large number of atoms and thus large number of electrons. Each electron produces orbital and spin magnetic moments and can be assumed as a dipole. In the absence of any external magnetic field, the dipoles of individual atoms are

randomly orientated due to thermal agitation and the magnetic moments thus cancel. In some materials the cancellation is complete and net magnetic moment is zero.

When we apply an external magnetic field of the flux density B, two processes may occur :

(1) All atoms which have non zero magnetic moment are aligned with the applied field due to the torque τ. The increase in magnetic moment is along B.

(2) If the atom has a zero magnetic moment, the applied magnetic field distorts the electron orbit and thus induces magnetic moment. The induced magnetic moment is given by Eq. (3).

In materials, referred to as *diamagnetic*, such as *Cu, Bi, Zn, Ag, Pb* and *Hg*, the individual atoms do not have an intrinsic magnetic moment. When an external magnetic field is applied the second process occurs. The induced magnetic moment is thus set-up in the direction opposite to **B**. In this case the magnetic flux density in the interior of the body will be less than that of the external field B.

In materials, referred to as *paramagnetic*, such as. *Al, Pt, Cr, Mn*...etc., the constituent atoms have intrinsic magnetic moments, and the cancellation of magnetic moment vectors is not complete. When an external magnetic field is applied, both of the above processes occur and the resultant magnetic moment is always in the direction of magnetic field B as the first effect predominates over the second.

Thus we see that the material body can be assumed as a volume distribution of magnetic dipoles, if it is acted on by an external magnetic field. If m is the average magnetic dipole moment per atom or molecule and there are N such effective dipoles per unit volume, the *magnetization* is defined as

$$\mathbf{M}=N\mathbf{m}.$$

It is also known as *magnetic dipole polarization per unit volume.* If the dipoles are not aligned parallel to each other, then **M** is the net magnetic moment per unit volume. It is measured in amperes meter and corresponds to the polarization **P** in the dielectric.

If the magnetic moment of i^{th} atom is $\mathbf{m}_i$, we may define the magnetization vector **M** as

$$\mathbf{M}=\lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\sum_i \mathbf{m}_i, \qquad \ldots(20)$$

where $\Delta\tau$ the volume element of the material is made very small from the macroscopic point of view, but it contains a statistically large number of atoms. The quantity **M** thus becomes a vector point function. In the unmagnetized matter, M is zero as a result of random orientation of $\mathbf{m}_i$. The action of applied magnetic field changes the orientation of $\mathbf{m}_i$. The degree of magnetization depends on this applied field as well as on the material. From this

point of view, all known materials fall into three groups discussed earlier.

Any material placed in a magnetic field due to external currents is thus in a state of magnetization in which an additional magnetic field results from the orbital motion of the electrons around the nucleus of the atoms of the material and also the spin of the electrons and nuclei about their own axes with the help of *Ampere's model*. Consider a cylindrical rod of 1 meter length which has a uniform magnetization parallel to its axis. We know that such a uniform magnetization means a uniform density of circulatory currents everywhere inside the material. Assuming the cylinder as made up of large number of cells of very small area and of length 1 meter. As the atomic current is passing through each cell, hence the current in one cell is equal and opposite to that in the adjoining cells. The net current is thus on the surface of the rod, Fig. 11.4. Due to this reason a *uniformly magnetized rod is said to be equivalent to a long solenoid carrying an electric current.*

Fig. 11.4.

If λe be the net current on the surface of rod of area of cross section S, and magnetization M. Then the magnetic moment of the current λe round the periphery will be equal to the sum of the magnetic moments in the cells, *i e.*,

$$m = \text{circulatory current} \times \text{area of the loop}$$

$$= \lambda e \times S = MS$$

or $$\lambda e = M. \qquad \ldots(21)$$

If inside the material, the magnetization M is constant, so all its derivatives are zero. The magnetization is inhomogenous specially due to the discontinuity at the boundary. Fig 11.5 (*a*) shows, for simplicity, two little blocks in x and y plane. If M_z is the component of M along z axis on block 1, then it will be $M_z + \Delta M_z$ on the block 2. Block 1 will produce a current I_1 and block 2 will produce a current I_2 in the common surface of the blocks. Therefore the net current

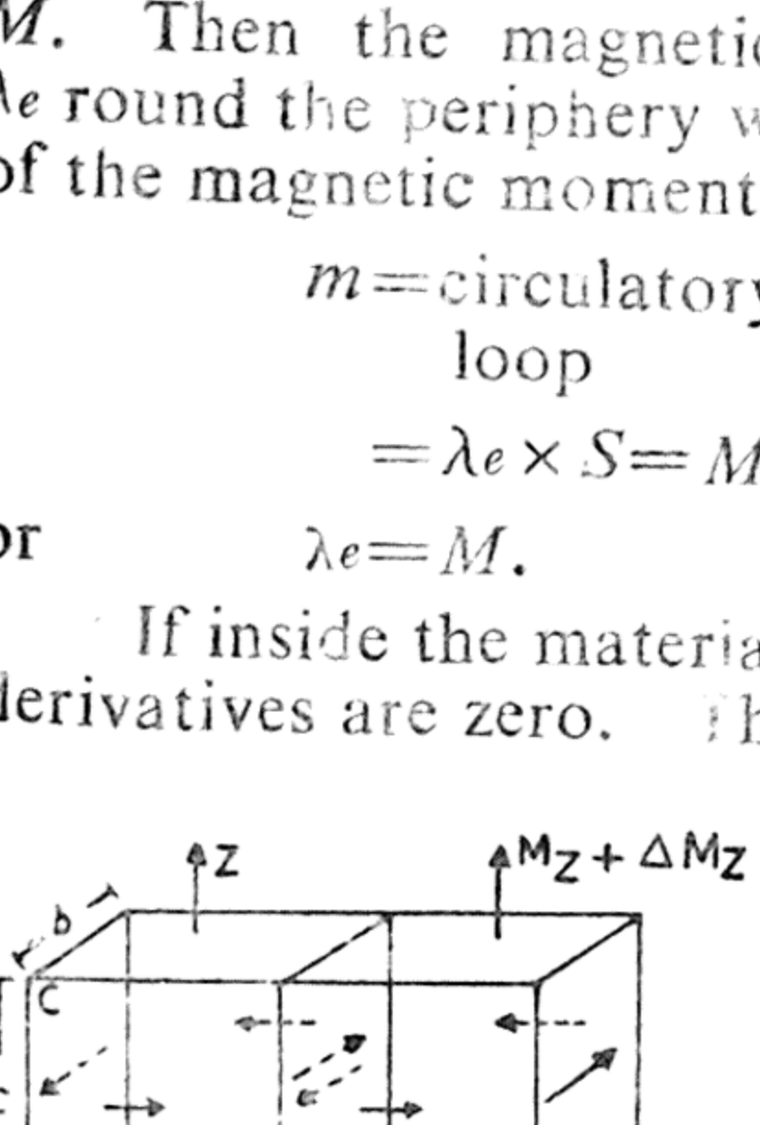


Fig. 11.5

$$I_y = I_1 - I_2 = M_z c - (M_z + \Delta M_z) c$$

$$= - \Delta M_z c.$$

As $\Delta M_z = \partial M_z / \partial x \; a$, hence

$$I_y = - [\partial M_z / \partial x] \; ac,$$

Therefore, y-component of current density $j_y = - \partial M_z / \partial x$. There will be an other term in j_y, can be obtained from

the variation of M_z with z. For this consider two little blocks one on top of the other, Fig. 11.5 (*b*). Here the currents are assumed in $y-z$ plane. Similar to the above relation, the net current may be expressed as

$$I_y = I_2 - I_1 = (M_x + \triangle M_x)a - M_x a = \triangle M_x a$$

As $\triangle Mx = \frac{\partial Mx}{\partial z}\ c$, hence $I_y = \frac{\partial Mx}{\partial z}\ ac$.

$\therefore$ y-component of current density $j_y = \partial Mx/\partial z$.

Hence the total current density in y-direction

$$jy = \frac{\partial Mx}{\partial z} - \frac{\partial Mz}{\partial x} = (\nabla \times \mathbf{M})_y$$

or $$\mathbf{j}e = \nabla \times \mathbf{M}. \qquad \ldots(22)$$

This shows the net effective molecular current due to incomplete concellation of adjacent current loops in the interior of the magnetization body. Thus one can calculate **B** by replacing the magnetized material by its equivalent currents, λe and $\mathbf{j}e$.

11.4. AMPERE'S MODEL FOR THE EQUIVALENT CURRENTS

We know from article 8.15 that the vector potential A due to a current loop of magnetic dipole moment **m** is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\ \frac{\mathbf{m} \times \mathbf{r}}{r^3} = -\frac{\mu_0}{4\pi}\ \mathbf{m} \times \text{grad}\left(\frac{1}{r}\right). \qquad \ldots(23)$$

Here r, the distance of the point from the centre of the loop, is large compared to the size of the loop. The contribution of the vector potential from the magnetic polarization of a material body will be obtained by integrating the contribution from each elementary dipole element $Md\tau'$. Hence

$$\mathbf{A}\ (x, y, z) = -\frac{\mu_0}{4\pi}\int_{\tau'} \mathbf{M}\ (x', y', z') \times \text{grad}'\left(\frac{1}{r}\right) d\tau', \qquad \ldots(24)$$

where τ' is the volume of the material body. Using vector identities for scalar function ϕ and vector function **F** as

$$\text{curl}\ (\phi \mathbf{F}) = \phi\ \text{curl}\ \mathbf{F} + \text{grad}\ \phi \times \mathbf{F} \qquad \ldots(25)$$

and $$\int \tau\ \text{curl}\ \mathbf{F} d\tau = \oint s\ \mathbf{n} \times \mathbf{F}\ dS, \qquad \ldots(26)$$

the integral (24) may be transformed as

$$\mathbf{A} = \frac{\mu_0}{4\pi}\left[\int_{\tau'}\left[\frac{1}{r} \nabla' \times \mathbf{M}\right] d\tau' + \int_{s'}\left(\frac{\mathbf{M}}{r} \times \hat{n}\right) dS'\right] \qquad \ldots(27)$$

where S' is the surface bounding the volume τ'.

If we compare this equation with the expression for A due to distribution of currents, we get

$$\nabla' \times \mathbf{M} = \text{Equivalent volume current density } \mathbf{j}e.$$

and $$\mathbf{M} \times \hat{n} = \text{Equivalent surface current density } \vec{\lambda e} \text{ or } \mathbf{j}e_s$$

$$\therefore \quad \mathbf{A} = \frac{\mu_0}{4\pi}\left[\int_{\tau'} \frac{\mathbf{j}e}{r}\, d\tau' + \int_s \frac{\vec{\lambda e}}{r}\, dS'\right]. \qquad \ldots(28)$$

If the magnetization $\mathbf{M}$ is constant throughout the body, $\nabla' \times \mathbf{M}$ is zero and the first term is zero. The second term is zero when $\mathbf{M}$ is perpendicular to surface, *i.e.*,

$$\mathbf{M} \times \hat{n} = 0.$$

The equation (28) gives us an expression for the vector potential $\mathbf{A}$ due to an element of magnetized matter in terms of the equivalent current densities $\mathbf{j}e$ and $\vec{\lambda e}$. We are actually interested in $\mathbf{B}$, not $\mathbf{A}$. To calculate $\mathbf{B}$, we take the curl as

$$\mathbf{B} = \text{curl } \mathbf{A} = \nabla \times \mathbf{A}$$

$$= \frac{\mu_0}{4\pi}\left[\int_{\tau'} \nabla \times (\mathbf{j}e/r)\, d\tau' + \int_{s'} \nabla \times (\vec{\lambda e}/r)\, dS'\right]. \qquad \ldots(29)$$

Here the differential operators in the curl act on the unprimed coordinates. Using the vector identity (25), we get

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\int_{\tau'} \frac{1}{r}(\nabla \times \mathbf{j}e)\, d\tau' + \int_{\tau'} \nabla\left(\frac{1}{r}\right) \times \mathbf{j}e\, d\tau'\right.$$

$$\left. + \int_{s'} \frac{1}{r}(\nabla \times \vec{\lambda e})\, dS' + \int_{s'} \nabla\left(\frac{1}{r}\right) \times \vec{\lambda e}\, dS'\right]$$

Since the curl will operate with respect to unprimed coordinates (x, y, z) while the current densities $\mathbf{j}e$ and $\vec{\lambda e}$ depend on primed coordinates (x', y', z'). Thus the first and third terms will not contribute to $\mathbf{B}$. Therefore

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\int_{\tau'} \nabla\left(\frac{1}{r}\right) \times \mathbf{j}e\, d\tau' + \int_{s'} \nabla\left(\frac{1}{r}\right) \times \vec{\lambda e}\, dS'\right]. \qquad \ldots(30)$$

Equations (28) and (30) show that the vector potential $\mathbf{A}$ and the magnetic induction vector $\mathbf{B}$ due to the effect of a piece of magnetized matter can be computed with the help of surface current density $\vec{\lambda e}$ and a volume current density $\mathbf{j}e$. These surface and volume current densities are also known as bound currents. These are associated with molecular or atomic magnetic moments including the intrinsic magnetic moment of particles with spin.

In the presence of magnetized matter, the magnetic field is not only due to the conventional or true or free currents which can be measured in laboratory directly, but the bound currents also contribute effectively.

11.5. MAGNETIC FIELD INTENSITY H

When dealing with dielectric substances in an electric field, the electric displacement **D** is introduced in order to eliminate the effect of electric polarization **P**. A similar procedure is used for the magnetic case.

We know that for steady currents and nonmagnetic materials

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \qquad \ldots(31)$$

Consider a magnetic material in which a free current density $\mathbf{j}_f$ exists. This current gives rise to a magnetic polarization of the material. The magnetic field from the magnetic polarization may be evaluated from equivalent volume polarization current density $\mathbf{j}_e$. This current is also known as *bound current* and is associated with molecular or atomic magnetic moments. Thus the total magnetic field **B** is due to the free currents as well as the bound or equivalent current. Hence

$$\text{curl } \mathbf{B} = \mu_0 (\mathbf{j}_e + \mathbf{j}_f). \qquad \ldots(32)$$

Since the equivalent volume polarization current $\mathbf{j}_e$ is equal to the curl of the magnetic dipole polarization **M**, hence

$$\text{curl } \mathbf{B} = \mu_0 \text{ curl } \mathbf{M} + \mu_0 \mathbf{j}_f$$

or
$$\text{curl } (\mathbf{B}/\mu_0 - \mathbf{M}) = \mathbf{j}_f. \qquad \ldots(33)$$

The vector $(\mathbf{B}/\mu_0 - \mathbf{M})$ is replaced by a new vector quantity **H** called *the magnetic field intensity* or *magnetic field strength*. Thus we have

$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}. \qquad \ldots(34)$$

This vector eliminates the necessity of dealing directly with the polarization as its curl directly gives the free current density. Thus we see that the magnetic field intensity **H** is related to the free current in the way **B** is related to the total current $(\mathbf{j}_{free} + \mathbf{j}_{bound})$.

We know that the divergence of a magnetic field **B** in a free space is always zero. It is also true if the magnetized matter is present. This is because **B** = curl **A** and the divergence of a curl is always zero.

As div **B** is always zero, hence Eq. (34) gives div **H** = − div **M**. Thus **H** does not necessarily have divergence. As Eq. (33) can be written as curl **H** = $\mathbf{j}_f$, hence on integrating over a surface S we have

$$\int (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int \mathbf{j}_f \cdot d\mathbf{S}.$$

Using Stoke's theorem on the left hand side, we get

$$\int_c \mathbf{H} \cdot d\mathbf{l} = \int_s \mathbf{j}_f \cdot d\mathbf{S} = i_f, \qquad \ldots(35)$$

where C is the curve bounding the surface S and i_f is the current of free charges or free current linking the curve C. The left hand term is called the *magnetomotance*, in analogus with the electromotive force in electric fields. The above equation is the more general form of *Ampere's circuital law*, as it can be used to calculate **H** even in the presence of magnetic material. The relation div $\mathbf{B}=0$ shows that the average macroscopic field inside matter is **B** *not* **H**, or **B** is the *fundamental magnetic field vector*. Eq. (35) shows that the unit of H is ampere/meter.

H at some point inside the magnetized material is the vector sum of $\mathbf{B}/\mu_0$ and $-\mathbf{M}$. The lines of **H** are as it should be for if magnetic poles really were the source of magnetization, instead of electric currents. We therefore conclude our results as :

(*i*) $\mathbf{H}=\mathbf{B}/\mu_0-\mathbf{M}$ in magnetic materials, while $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}$ in dielectrics Thus **H** is the vector sum of **B** and **M**, while **D** is that of **E** and **P**.

(*ii*) The line integral of **H** is the free or true current and does not include the magnetizing current. Similarly the surface integral of **D** gives free charges only not polarization charges.

(*iii*) **H** can be directly calculated by Ampere's circuital law (Eq. 35), while **D** can be directly calculated by Gauss's theorem $\int \mathbf{D}.d\mathbf{S}=q$.

Thus we see that **H** plays the same part in magnetic materials as does the vector **D** in dielectrics. From this point of view, it should be more correct to call **H**, the magnetic induction vector. But this name belongs historically to **B**, although **B** plays a role in the magnetic materials analogus to that of **E** in dielectrics. It follows that the analogus of permittivity ϵ in the electric field theory is $1/\mu$ in the magnetic field theory. It is impossible now to change the terminology, but we must remember the true significance of the magnetic field vectors **H** and **B**.

11.6. MAGNETIC SUSCEPTIBILITY AND PERMEABILITY

We know that both diamagnetic and paramagnetic substances develop a magnetic moment depending on the applied field. Leaving aside the saturation value of **M** at very low temperatures in fairly strong magnetic field the relation between **M** and **H** is linear. The ratio of magnetization **M** to magnetic field intensity **H** is a constant in general and is called the *magnetic susceptibility*. It is a measure of how susceptible the material is to polarization

by an applied magnetic field. It is the characteristic of the material and is denoted by χm.

$$\mathbf{M}=\chi m\mathbf{H}. \qquad \ldots(36)$$

If the magnetic moment of unit mass of a material is $\mathbf{M}_{mass}$, then

$$\mathbf{M}_{mass}=\chi_m, {}_{mass}\,\mathbf{H} \text{ or } \chi_m, {}_{mass}=\chi_m/\rho \qquad \ldots(37)$$

If the magnetic moment of one molecule is $\mathbf{M}_{mol}$, then

$$\mathbf{M}_{mol}=\chi_m, {}_{mol}\,\mathbf{H} \text{ or } \chi_m, {}_{mol}=\chi m\, A/\rho. \qquad \ldots(38)$$

where ρ and A are the density and molecular weight of the material respectively. Here χm, χm, $_{mass}$, and χ_m, $_{mol}$ are known as volume, mass and molar susceptibilities respectively.

χm is positive for paramagnetic and negative for diamagnetic materials. For these both cases it is less than one. Magnetic susceptibility of a paramagnetic material decreases with increasing temperature as $\chi_m=C/T$, where the constant C is known as *Curie constant*. χ_m is independent of temperature for diamagnetic substances.

Combining Eqs. (34) and (36), we get

$$\begin{aligned}\mathbf{B}&=\mu_0\,(\mathbf{H}+\chi m\mathbf{H})=\mu_0\,(1+\chi m)\,\mathbf{H}\\&=\mu\mathbf{H}, \qquad \ldots(39)\end{aligned}$$

where $\mu=\mu_0\,(1+\chi_m)$ is called the *magnetic permeability* of the material. The dimensionless quantity $\mu/\mu_0=km$ is called the *relative permeability* of the material km is unity for vacuum. Thus the permeability of vacuum is μ_0. For diamagnetic materials k_m is slightly less than unity and for paramagnetic materials it is slightly greater than unity. Thus for the former **M** and **H** are antiparallel and for the latter these are parallel.

Table 1. The magnetic susceptibilities of paramagnetic and diamagnetic materials at approx, room temperature in SI units

Paramagnetic Materials $\chi_{O,\ mass}\ 10^{-8}\ (m^3/kg)$		*Diamagnetic Materials* $\chi m,\ m_{ass}\times 10^{-8}\ (m^3/kg)$	
Aluminum (Al)	0.82	Bismuth (Bi)	−1.7
Iron Aluminum alloy	38.2	Cadmium (Cd)	−0.25
Calcium (Ca)	1.4	Copper (Cu)	−0.11
Chromium (Cr)	4.5	Diamond	−0.62
Cuprous oxide (CuO)	1.5	Germanium (Ge)	−0.15
Ferric oxide (Fe_2O_3)	26.00	Gold (Au)	−0.19
Magnesium (Mg)	0.69	Helium (He)	−0.59
Manganese (Mn)	1	Lead Pb)	−0.18
Platinum (Pt)	1.65	Silver/Sodium	−0.25
Tantalum (Ta)	1.1	Zin (Zn)	−0.20

11.7. BOUNDARY CONDITIONS FOR B AND H

Gauss's law for magnetism, similar to the law in electrostatics, tells us that the algebraic sum of the number of magnetic flux lines coming out of the surface of a volume is always zero, or

$$\Phi_B=\int_S \mathbf{B}.d\mathbf{S}=0.$$

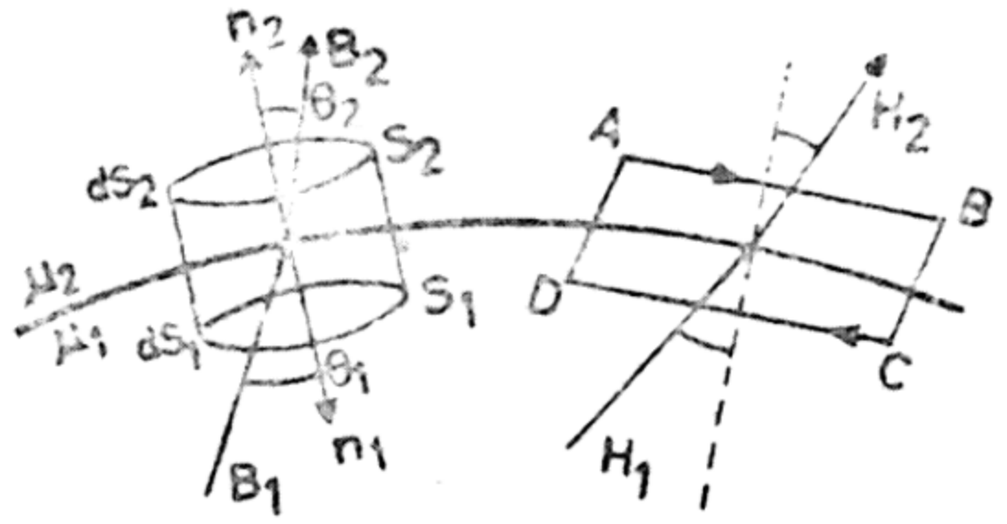


Fig. 11.6.

The magnetic field generally changes as it goes from one material to another. To find the continuity of **B** or **H** across the interface of two magnetic media, let us proceed as for **E** and **D**.

To find the boundary conditions on the normal components of **B**, let us assume a short cylindrical volume (Fig. 11.6) whose top and bottom faces are parallel and infinitely close to the interface. Since the total net flux through the cylindrical surface is zero, hence

$$\Phi=\int_{S_1}\mathbf{B}_1.d\mathbf{S}_1+\int_{S_2}\mathbf{B}_2.d\mathbf{S}_2+\int_C \mathbf{B}.d\mathbf{S}=0.$$

The height of the cylinder is assumed to be so small that the magnetic flux through the curved surface is negligible.

$$\therefore \qquad \int_{S_1}\mathbf{B}_1.d\mathbf{S}_1+\int_{S_2}\mathbf{B}_2.d\mathbf{S}_2=0.$$

As $d\mathbf{S}_1=\mathbf{n}_1 dS$, $d\mathbf{S}_2=\mathbf{n}_2 dS=-\mathbf{n}_1 dS$.

$$\therefore \quad \mathbf{B}_1.\mathbf{n}_1=\mathbf{B}_2.\mathbf{n}_1 \quad \text{or} \quad B_{1n}=B_{2n}. \qquad \ldots(40)$$

i.e., the normal component of **B** *is continuous across any interface.*

The corresponding boundary condition on the normal component of **H** may be obtained by using relation $\mathbf{B}=\mu\mathbf{H}$.

$$\therefore \qquad \mu_1 H_{1n}=\mu_2 H_{2n}. \qquad \ldots(41)$$

To find the boundary conditions on the tangential components of **H**, let us assume a closed rectangular path $ABCD$ with its long sides AB and CD parallel to the interface and close to it. If no true surface current is enclosed by this loop, then

$$\oint \mathbf{H}.d\mathbf{l}=\int_A^B \mathbf{H}_2.d\mathbf{l}_2+\int_B^C \mathbf{H}.d\mathbf{l}+\int_C^D \mathbf{H}_1.d\mathbf{l}_1+\int_D^A \mathbf{H}.d\mathbf{l}=0.$$

For $AB=CD=\Delta\mathbf{l}$, $\int d\mathbf{l}_2=\Delta\mathbf{l}$ and $\int d\mathbf{l}_1=-\Delta\mathbf{l}$. As the sides BC and DA are negligibly small, hence their contributions will be zero.

$$\therefore \qquad \mathbf{H}_1.\Delta\mathbf{l}=\mathbf{H}_2.\Delta\mathbf{l} \qquad \text{or} \qquad H_{1t}=H_{2t}. \qquad \ldots(42)$$

The corresponding boundary conditions for the tangential components of **B** are

$$B_{1t}/\mu_1 = B_{2t}/\mu_2. \qquad \ldots(43)$$

Equations (40) and (42) are identical with those for **D** and **E** found for dielectric boundaries except that—(*a*) the normal component of **B** is strictly continuous across the boundary while that of **D** is continuous only if there is no surface charge, (*b*) the tangential component of **E** is strictly continuous, while that of **H** is continuous across the boundary only if there is no surface current. *These differences confirm the existence of true electric charge and the absence of true magnetic charge.*

If the field **B** makes angles θ_1 and θ_2 with the normal on the surface of media 1 and 2 respectively, then we have

$$B_1 \cos\theta_1 = B_2 \cos\theta_2 \text{ and } B_1 \sin\theta_1/\mu_1 = B_2 \sin\theta_2/\mu_2.$$

$$\therefore \quad \tan\theta_2 = (\mu_2/\mu_1)\tan\theta_1. \qquad \ldots(44)$$

Thus the lines of **B** or of **H** are farthest away from the normal to the interface in the medium of larger permeability. *It is similar to the refraction of electric lines of force.*

11.8. THE FIELD OF A PERMANENT MAGNET

Let us discuss the materials with permanent magnetic polarization The permanent magnets can be made from alloys and compounds of the ferromagnetic substances. In the case of permanent magnets the alignment of dipoles results from internal forces within the matter. **M**, **H** and **B** all have non-vanishing values inside the magnet even though there is no true current. Let us take following three important cases only.

(*a*) **Bar Magnet.** To find **B** and **H** fields, that the self magnetization produces both inside and outside the matter, the use is made of equation $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$. From Gauss's law for **B**, we know

$$\int_S \mathbf{B}.d\mathbf{S} = 0 \quad \text{or} \quad \mu_0 \int_S \mathbf{H}.d\mathbf{S} + \mu_0 \int_S \mathbf{M}.d\mathbf{S} = 0,$$

$$\therefore \quad \int_S \mathbf{H}.d\mathbf{S} = -\int_S \mathbf{M}.d\mathbf{S}. \qquad \ldots(45)$$

If A is the area of cross section of the magnet and **M** its magnetization for the uniform magnetization, we have

$$-\int_S \mathbf{M}.d\mathbf{S} = MA = \int_S \mathbf{H}.d\mathbf{S} \qquad \ldots(46)$$

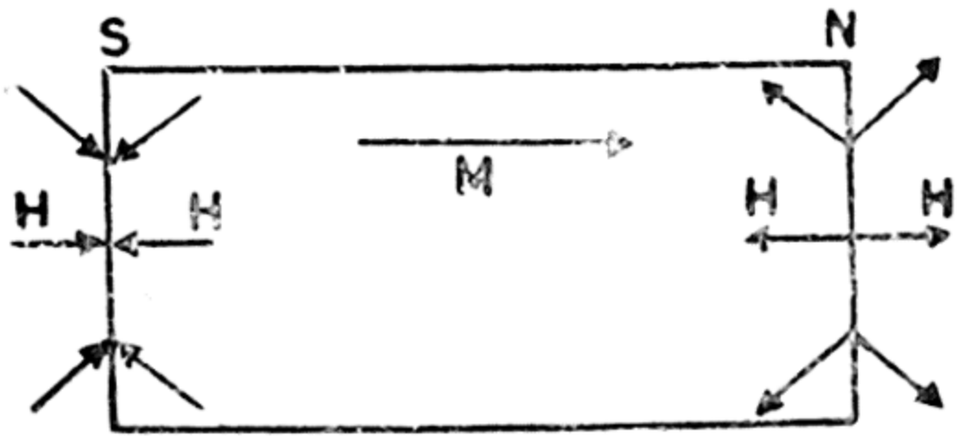


Fig. 11.7.

The negative sign results from the fact that the lines of **M** are directed inward at the gaussian surface.

As there are no real currents at the surface, hence $\oint \mathbf{H}.d\mathbf{l} = 0$. Therefore the lines of **H** are uniformly distributed. This leads directly to the result that Coulomb's law is obeyed, *i.e.*, the effects of **H** originating from a small end element

fall off as $1/r^2$. By analogy with the electrostatic case (*i.e.*, $-\int \mathbf{P}_s.d\mathbf{S}=q_p$, the polarization charge), we can define the effects of the end discontinuity by

$$\int_S \mathbf{M}.d\mathbf{S}=q_m \ amp/m. \qquad ...(46)$$

where q_m is called the strength of the magnetic pole or magnetic charge.

Gauss's law then becomes

$$\int_S \mathbf{H}.d\mathbf{S}=MA=q_m \ amp/m. \qquad ...(47)$$

The end of the magnet from which lines of **H** emerge is called the *north pole* and the end on which lines of **H** converge is called the *south pole*. The values of **B** and **H** at points outside and inside the magnet are as shown in Fig. 11.8 (*a*) and (*b*) respectively. Thus we see that **M** and **H** are in opposite directions inside the magnet. At a point far away from the end of the magnet, so that the side of the end region is small compared with the distance *r*, the pole can be taken as a point. Now applying Gauss's law for **H** over a Gaussian sphere of radius *r*, we get

$$\int \mathbf{H}\,d\mathbf{S}=q_m \quad \text{or} \quad 4\pi r^2 H=q_m$$

$$\text{or} \qquad B=\mu_0 H=\frac{\mu_0}{4\pi}\frac{q_m}{r^2}. \qquad ...(48)$$

The magnetic field at any point outside the bar magnet can be calculated by using the above equation for both the ends.

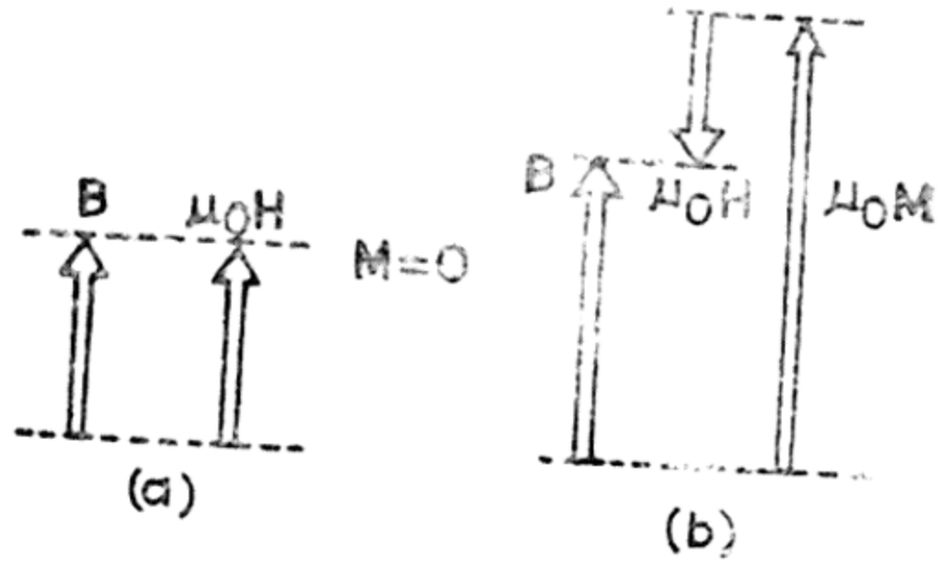


Fig. 11.8.

(*b*) **Cylindrical or Disc shaped Magnet.** Let us consider a cylindrical magnet of length **L** and radius *a*, magnetized uniformly with a magnetization $\mathbf{M}_0$ in the positive *z* direction. In the interior the volume density of polarization charge $j_m=\nabla\times\mathbf{M}_o$. Since $\mathbf{M}_0$ is constant, hence $\nabla\times\mathbf{M}_0=0$ and all the interior currents effectively cancel. On the two ends we have an equivalent magnetic charge

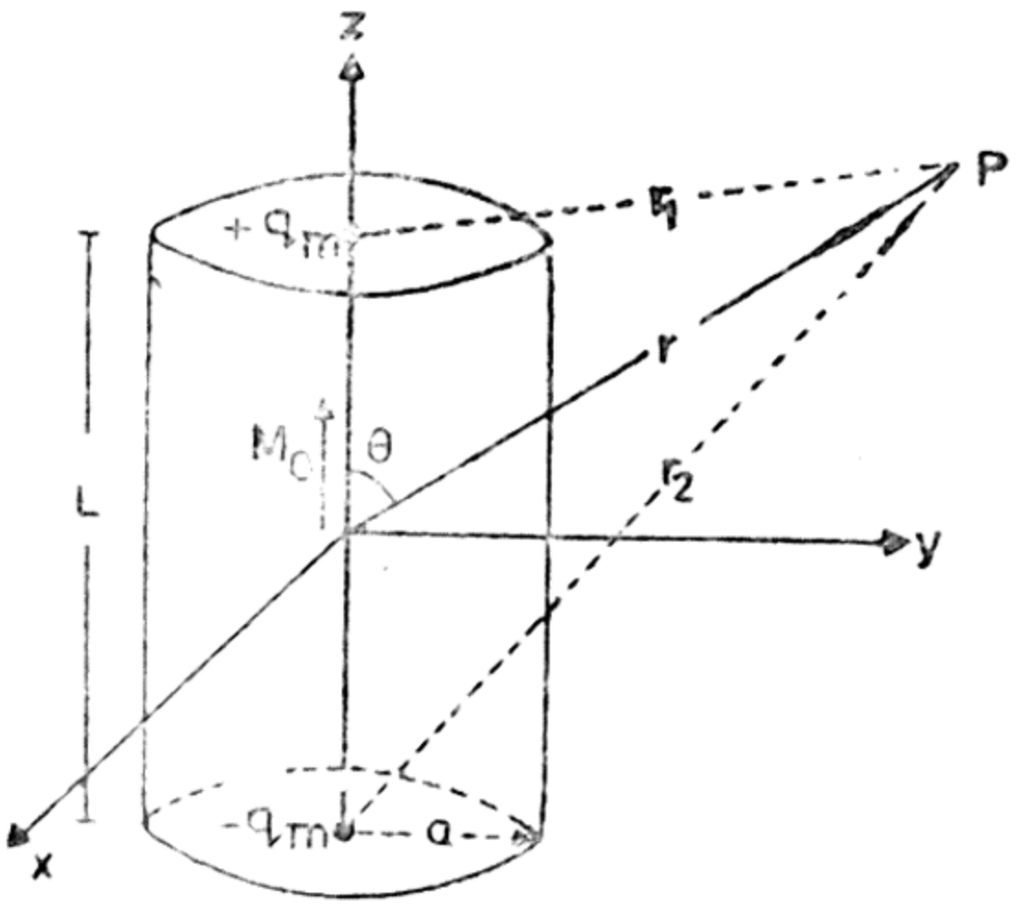


Fig. 11.9.

$$q_m=-\int \mathbf{M}_0.d\mathbf{S}=\pi a^2 M_0.$$

Thus the charges on the upper and lower surfaces are $\pi a^2 M_0$ and $-\pi a^2 M_0$ respectively. Since M_0 is parallel to the sides, hence there will be no surface charges on the sides.

For the greater distances ($r \gg a$) one can assume the charges on the end surfaces as at the centres of each side face. The magnetic scalar potential due to these point magnetic charges is thus given by

$$\phi_m=\frac{1}{4\pi}\,qm\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=\frac{\pi a^2 M_0}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right). \quad \ldots(49)$$

Since $H=B/\mu_0=-\nabla\phi$, hence

$$H=-\frac{\pi a^2 M_0}{4\pi}\nabla\left(\frac{1}{r_1}-\frac{1}{r_2}\right). \quad \ldots(50)$$

For $r \gg L$, we may write Eqs. (49) and (50) as

$$\phi m=\frac{\pi a^2 M_0}{4\pi}\,\frac{L\cos\theta}{r^2}=\frac{pm\cos\theta}{4\pi r^2} \quad \ldots(51)$$

and $$H=-\frac{pm}{4\pi}\,\nabla\left(\frac{\cos\theta}{r^2}\right), \quad \ldots(52)$$

where pm is the total effective dipole moment of the bar magnet.

(c) Magnetized sphere. Let us examine the magnetic lines of forces due to a magnetic sphere placed in a uniform magnetic field H_0. Field perturbation due to the magnetized sphere in a uniform field is found to be same as in the case of a dielectric sphere placed in a uniform electric field. On account of this similarity, one can assume demagnetization factor for the sphere as equal to the depolarization factor for a dielectric sphere. Therefore H inside the sphere is

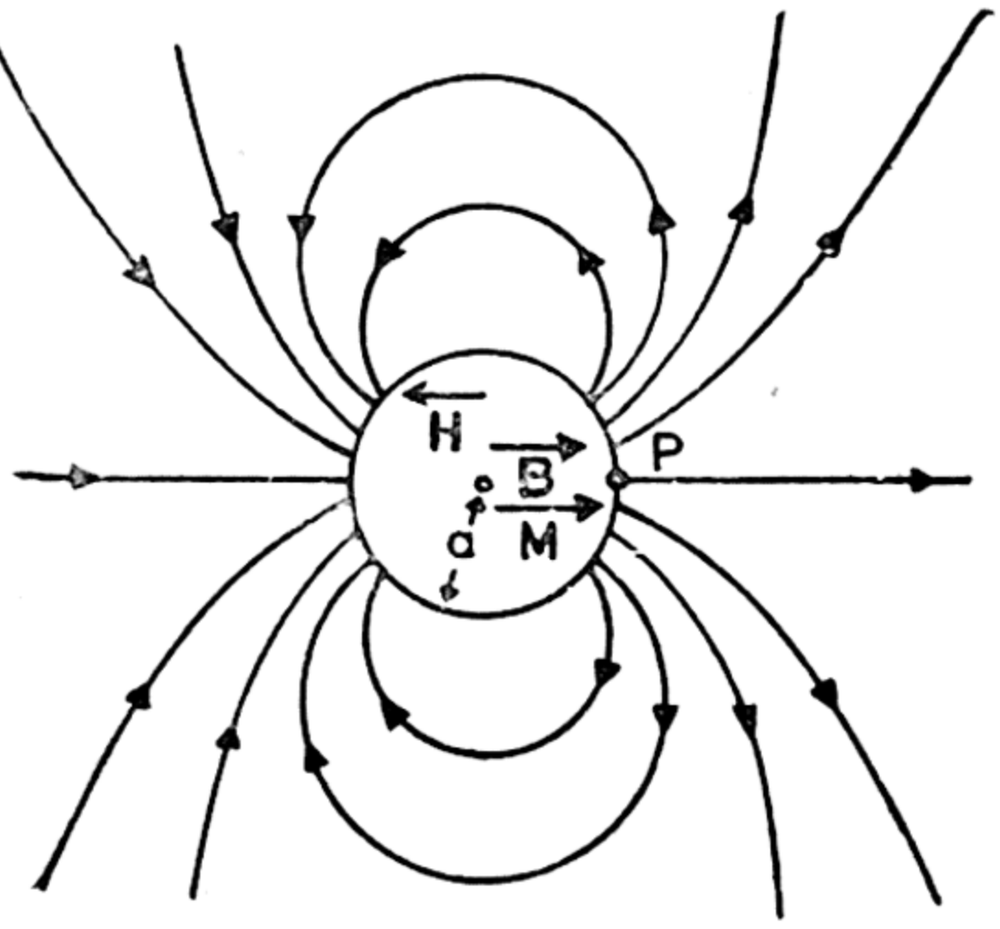


Fig. 11.10.

$$H=H_0-\tfrac{1}{3}M. \quad \ldots(53)$$

$$\therefore \quad B=\mu_0(H+M)=\mu_0[(H_0-\tfrac{1}{3}M)+M]=\mu_0[H_0+\tfrac{2}{3}M)]. \quad \ldots(54)$$

In the absence of an external field H_0 above relation reduces to $B=\frac{2}{3}\mu_0 M$, and the field inside the sphere is given by the relation

$$H=-\tfrac{1}{3}\,M. \quad \ldots(55)$$

Hence **H** is uniform and opposite to the magnetization **M**. Just as in the dielectric case the perturbation on the external field due to the uniformly magnetized field will be exactly the field of a dipole. To find out the magnitude of the dipole moment, boundary conditions are matched at the surface of the sphere at the point on the sphere along its axis of magnetization (point P). **B** is normal to the surface at this point. Since $B_{n1}=B_{n2}$, hence **B** inside and just outside the surface must have the same value.

$$\tfrac{2}{3}\mu_0 M=\frac{\mu_0}{4\pi}\,\frac{2pm}{a^3}$$

or $$pm=\tfrac{4}{3}\pi a^3 M. \quad \ldots(56)$$

It is just its magnetization times its volume. The magnetic field **B** at any point outside the sphere may be obtained by taking the vector sum of the two components Br and B_θ.

11.9. MAGNET AND EQUIVALENT SOLENOID

Consider a coil of N-turns, area **A** and carrying a current i amperes placed in a magnetic field of flux density **B**. If θ is the angle between **B** and **A**, then the torque acting on the coil is given by

$$\tau = Ni\,\mathbf{A}\times\mathbf{B} \qquad \text{or} \qquad \tau = NiAB\sin\theta.$$

Let us compare this with the expression for the torque on an electric dipole of dipole moment $\mathbf{p}_E$ placed in an electric field **E**, *i.e.*,

$$\tau = p_E E\sin\theta.$$

This comparison suggests that NiA can be taken as the *magnetic dipole moment* pm, also named as *electromagnetic moment*.

$$\tau = pmB\sin\theta. \qquad \ldots(57)$$

The unit of pm is ampere meter^2. It may therefore be defined as the maximum torque which can act on the body when it is immersed in a magnetic field of uniform flux density of unit strength, provided the torque is wholly due to the interaction between the body and the **B**.

A solenoid when immersed in a uniform electric field experiences a torque. Consider a solenoid of length $2l$ and of N turns per unit length (Fig. 11.11). It may be divided up into a large number of sections, each of length Δx. The torque acting on the sole-noid due to the interaction with **B** is

$$\tau = \Sigma(n\Delta x)AiB\sin\theta$$
$$= AiB\sin\theta\,\Sigma(n\Delta x) = NAiB\sin\theta.$$

On comparison, we get the dipole moment of the solenoid as NAi.

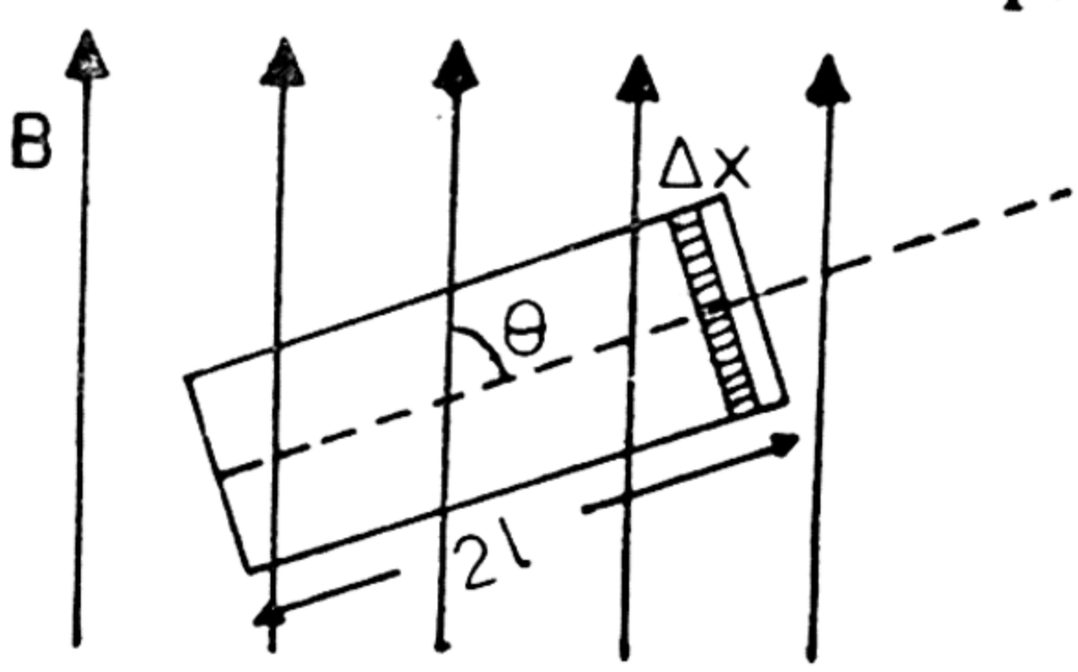


Fig. 11.11.

We know that the magnetic field at a distance d (along the axis) from the centre of a solenoid of length $2l$, radius r and of total turns N is given as

$$B = \frac{\mu_0}{2}\,\frac{N}{2l}\,i\,(\cos\theta_2 - \cos\theta_1)$$
$$= \frac{\mu_0 Ni}{4l}\left[\frac{d-l}{[r^2+(d-l)^2]^{1/2}} - \frac{d+l}{[r^2+(d+l)^2]^{1/2}}\right]$$
$$= \frac{\mu_0 Ni}{4l}\left\{\left[1-\frac{r^2}{2(d-l)^2}\right] - \left[1-\frac{r^2}{2(d+l)^2}\right]\right\}$$

$$B=\frac{\mu_0 Ni}{4l}\ \frac{2r^2ld}{(d^2-l^2)^2}=\frac{\mu_0}{4\pi}\ \frac{2pmd}{(d^2-l^2)^2},$$

where $pm=\pi r^2 Ni$, the magnetic moment of the equivalent dipole. If the solenoid is short compared with the distance d, then we have

$$B=\frac{\mu_0}{4\pi}\ \frac{2pm}{d^3}. \qquad ...(58)$$

At any point on the perpendicular bisector of the solenoid, we can get the relation

$$B=\frac{\mu_0}{4\pi}\ \frac{pm}{d^3}. \qquad ...(59)$$

Relations (58) and (59) show that pm is a vector quantity and its direction is the same as the direction of B it generates.

To obtain the expression for the force on a magnetic pole, we modify the torque equation by replacing pm by $qm\ (2l)$, to obtain

$$\tau=qm(2l)B \sin \theta.$$

This is exactly analogous to the case of an electric dipole in an electric field, where $F=qE$. By analogy, the force* on a magnetic dipole is given by the relation

$$F=qmB. \qquad ...(60)$$

*In few standard books the reader will find the force equation as

$$F=\frac{1}{4\pi\mu_0}\ \frac{qm\ qm'}{r^2}, \qquad ...(i)$$

similar to the Coulomb's law in the electrostatics, *i.e.*, $F=q_1q_2/4\pi\epsilon_0 r^2$, where ϵ_0 has been replaced by μ_0 and the electrostatic charges by magnetic charges.

The above relation will be obtained, if we define the magnetic dipole moment of the current loop by

$$pm=\mu_0 iAN. \qquad (ii)$$

In this way the unit of pm will be web. m.

The torque Eq. (57) will be

$$\tau=\frac{1}{\mu_0} pmB \sin \theta=pmH \sin \theta, \qquad ...(iii)$$

and the force Eq. (60) will be

$$F=\frac{1}{\mu_0}\ qmB=qmH. \qquad ...(iv)$$

There is only one law of magnetostatics, and that is Biot and Savart's law. The Coulomb's law of force between magnetic charges (or poles) cannot be considered a fundamental law as magnetic monopoles do not exist. On account of this reason, the author does not agree with the relations (*i*) to (*iv*).

For a north pole q_m is positive and the force is parallel to **B**, and for a south pole the force is antiparallel to **B**.

Thus the force between the magnetic poles of pole strengths (or magnetic charges) q_m and q_m' separated by a distance r in a vacuum is given by

$$F=\frac{\mu_0}{4\pi}\frac{q_m q_m'}{r^2}, \qquad ...(61)$$

where μ_0 is the permeability of vacuum. Eq. (61) is known as Coulomb's law for magnetostatics.

Let us now consider two following simple cases :

(*a*) **Bar Magnet.** To compare the solenoid with the bar magnet, let us draw magnetic lines of induction for a bar magnet and for a short solenoid [Fig. 11.12 (*a*) and (*b*)]. Lines of **E** for an electric dipole are also shown in Fig. 11.12 (*c*). The comparison shows that the magnetic field B for a bar magnet is of same type as that due to a small solenoid. At large enough distances the fields in these two cases vary like those for a dipole. Hence the flux density

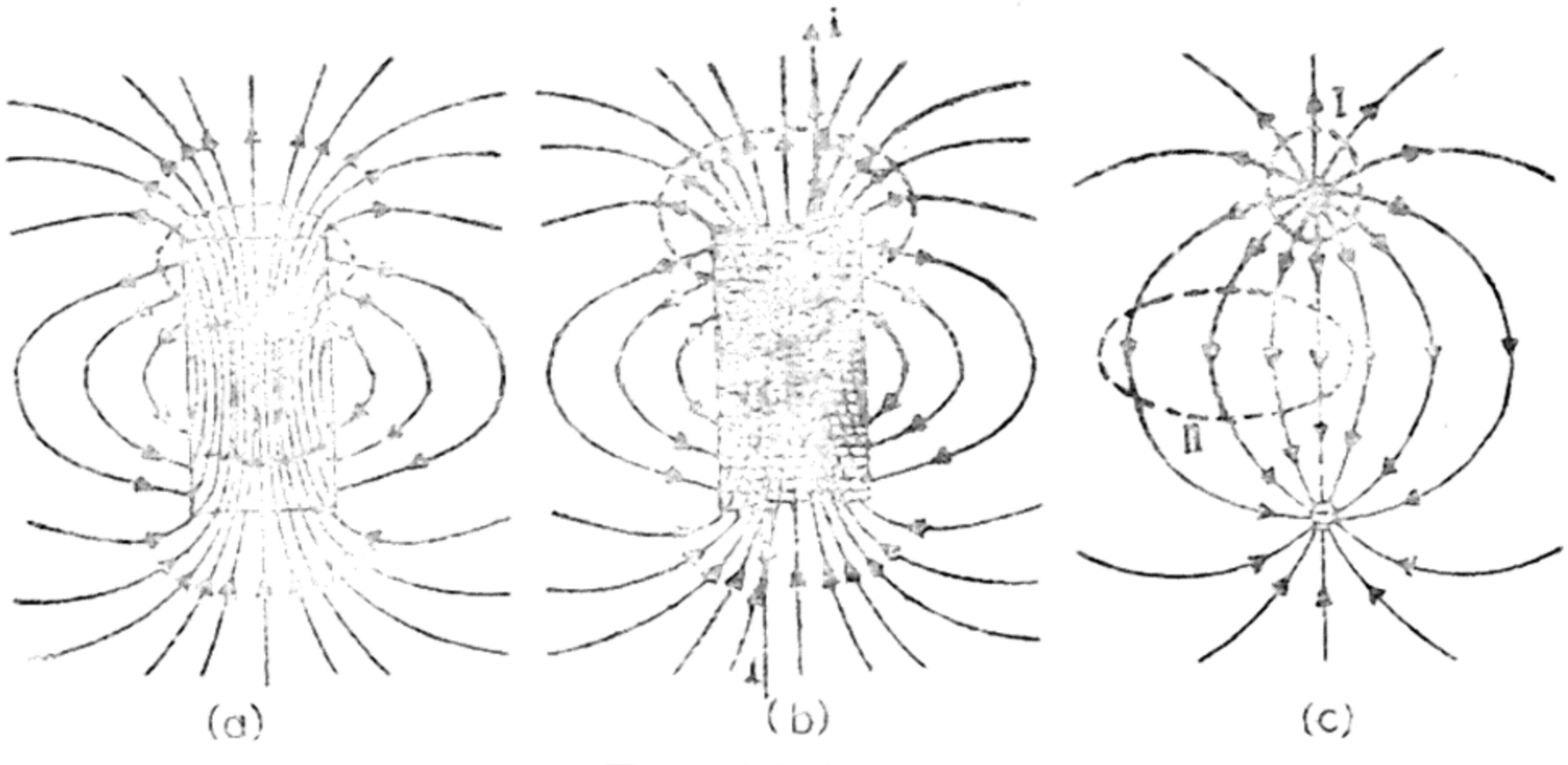


Fig. 11.12.

on the axis and on the perpendicular bisector of a bar magnet may also be obtained by using equations (58) and (59) respectively. In this case $2l$ is slightly shorter than the actual length of the magnet. This similarity in the lines of forces show that the torque acting on a bar magnet of density B may be given by Eq. (57). Hence a freely suspended magnet comes at rest when the magnet (pm) is parallel to the field B.

(*b*) **Magnetized sphere.** We know that the magnetized body is equivalent to a solenoid and its effect can be stimulated by the amperian surface current density λ_e, where

$$\lambda_e = \text{Magnetization } M. \qquad ...(62)$$

Fig. 11.13 shows a few of the turns of wire wound around the surface in which current i is producing magnetic field equivalent to the field produced

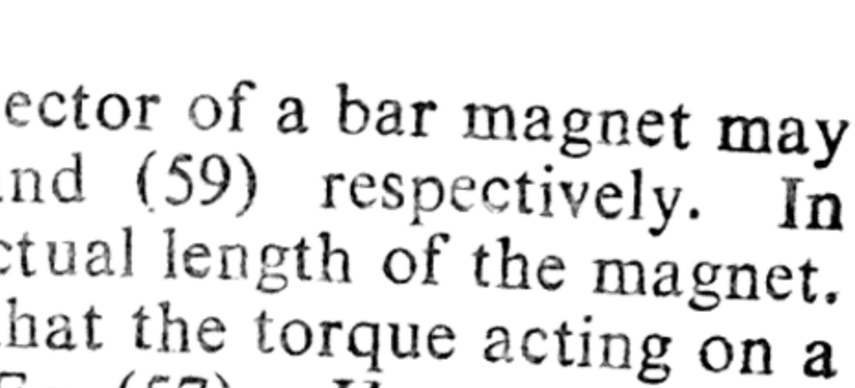


Fig. 11.13.

by M. The magnetic field at the centre of the sphere due to one turn at distance x from the centre.

$$dB=\frac{\mu_0 i}{2}\frac{y^2}{(x^2+y^2)^{3/2}}.$$

If dx is the thickness of one turn, then the current $i=dx\times$ current density j. As $x=R\cos\theta$, $dx=-R\sin\theta\, d\theta$, and $y=R\sin\theta$, hence

$$dB=\frac{\mu_0 j}{2}\frac{R^3\sin^3\theta\, d\theta}{R^3}=\frac{\mu_0 j}{2}\sin^3\theta d\theta.$$

or
$$B=\frac{\mu_0 j}{2}\int_0^{\pi}\sin^3\theta\, d\theta=\tfrac{2}{3}\mu_0 j.$$

Since $j=\lambda_e=M$, hence

$$B=\frac{2}{3}\mu_0 M=\frac{\mu_0}{4\pi}\frac{2pm}{R^3}, \qquad ...(63)$$

where $pm=\frac{4}{3}\pi R^3 M$, the total magnetic moment of the sphere.

The Eq. (63) is same as due to a magnetic dipole. Hence the field at any point outside the sphere can be obtained by simple relations for electric or magnetic dipoles

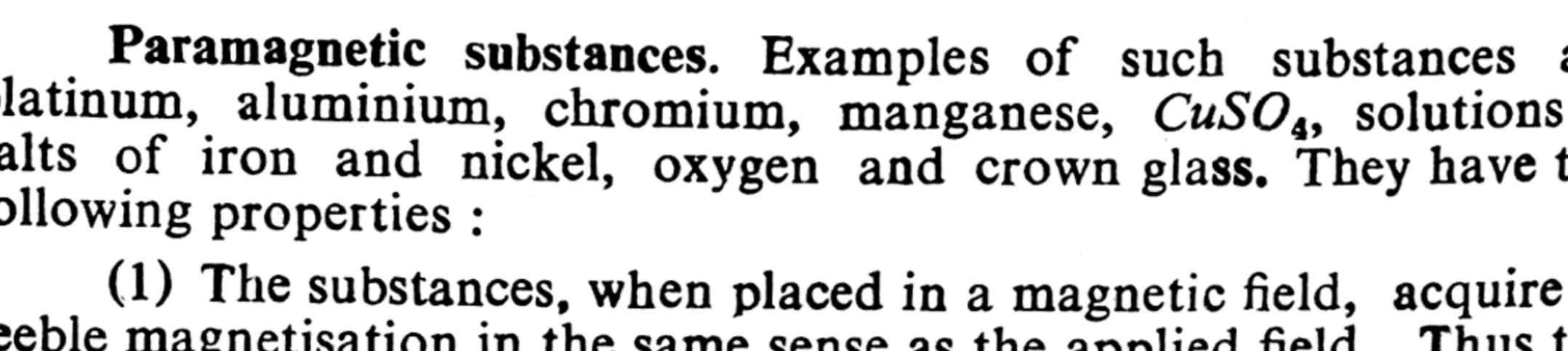
$$B_r=\frac{\mu_0}{4\pi}\frac{2\mathrm{p}m\cos\theta}{r^3} \text{ and } B_\theta=\frac{\mu_0}{4\pi}\frac{pm\sin\theta}{r^3}. \qquad ...(4)$$

11.10. PROPERTIES OF MAGNETIC MATERIALS

As discussed earlier, all substances, whether solid, liquid or gaseous may be classified into three categories, in terms of their susceptibility : (*i*) *paramagnetic*, (*ii*) *diamagnetic* and (*iii*) *ferromagnetic*.

Paramagnetic substances. Examples of such substances are platinum, aluminium, chromium, manganese, $CuSO_4$, solutions of salts of iron and nickel, oxygen and crown glass. They have the following properties :

(1) The substances, when placed in a magnetic field, acquire a feeble magnetisation in the same sense as the applied field. Thus the magnetic induction inside the substance is slightly greater than outside to it, Fig. 11.14 (*a*).



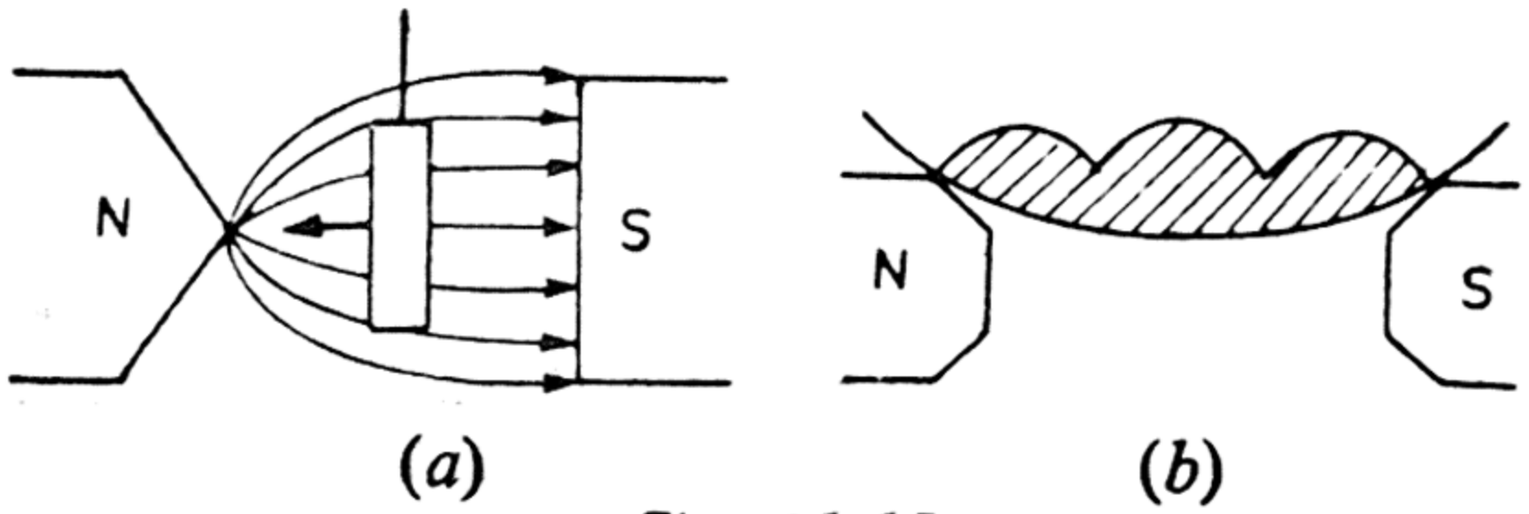
(*a*) (*b*)

Fig. 11.14.

(2) In a uniform magnetic field, these substances rotate until their longest axes are parallel to the field, Fig. 11.14 (*b*).

(3) These substances are attracted towards regions of stronger magnetic field when placed in a non-uniform magnetic field. Fig. 11.15 (*a*) shows a strong electromagnet in which one of the pole pieces is sharply pointed while the other is flat. Magnetic field is much stronger near the pointed pole than near the flat pole and the field is as diverging from pointed pole. If a small piece of test material is suspended in this region, a slight force can be observed by the slight displacement of the material, int he direction of arrow.

If a paramagnetic liquid is placed in a watch glass resting on the poles of a powerful electromagnet, the liquid is found to move so that the general depth is at points of greatest magnetic field Fig. 11.15 (*b*).



(*a*) (*b*)

Fig. 11.15.

(4) If a paramagnetic liquid is filled in a narrow U-tube and one limb is placed in between the pole pieces of an electromagnet such that the level of the liquid is in line with the field, then the liquid will rise in the limb as the field is switched on (Fig. 11.16).

Similarly when a paramagnetic gas is allowed to ascend between the poles of an electromagnet, the gas spreads along the field.

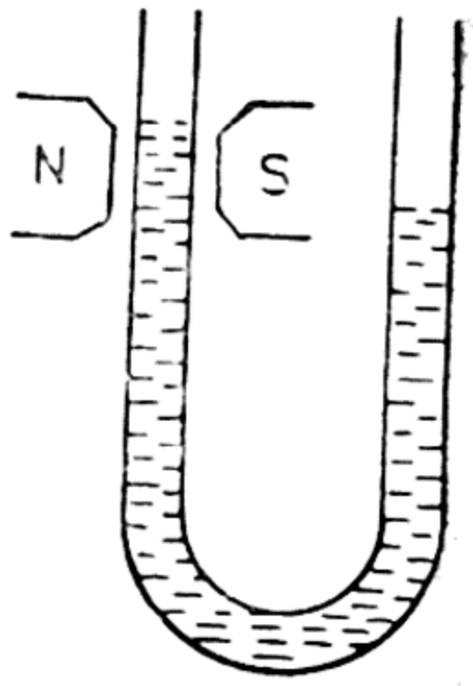


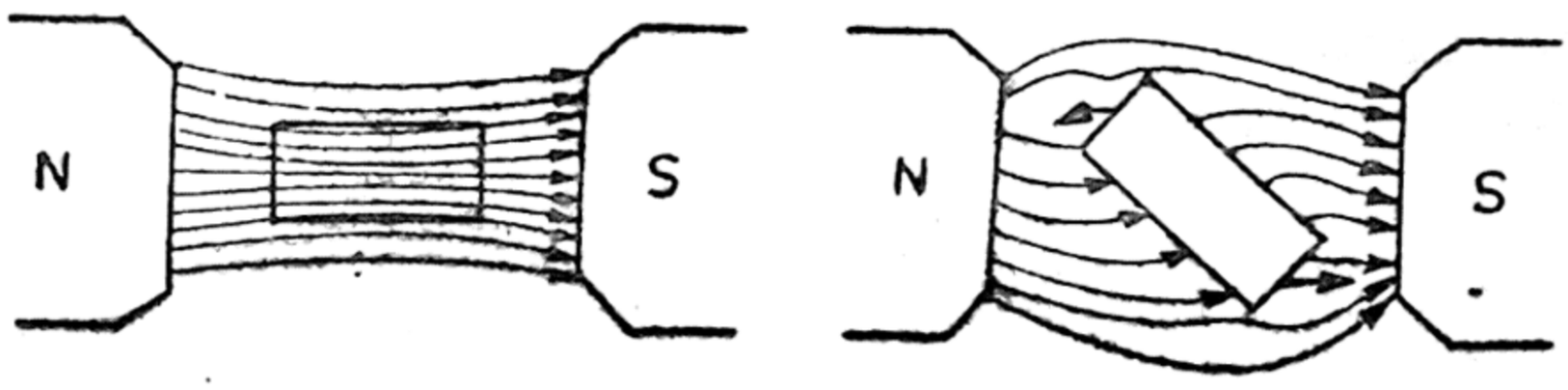
Fig. 11.16.

(5) The relative permeability K is slightly greater than one. K=1.000022 for aluminium, 1.00036 for platinum and 1.00047 for chromium. The most paramagnetic element is manganese for which K=1.00013.

(6) At a given temperature the susceptibility χ does not change with the magnetising field. However it varies inversely as the absolute temperature. As temperature increases, χ decreases. At some high temperature, χ becomes negative and the substance becomes diamagnetic.

Diamagnetic substances. Examples of such substances are bismuth, antimony, gold, quartz, water, alcohol, hydrogen. They have following properties.

(1) These substances, when placed in a magnetic field, acquire feeble magnetisation in a direction opposite to that of the applied field. Thus the lines of induction inside the substance is smaller than that outside to it, Fig. 11.17 (*a*).



(*a*) (*b*)

Fig. 11.17.

(2) In a uniform magnetic field, these substances rotate until their longest axes are normal to the field, Fig. 11.17 (*b*).

(3) In a non-uniform field, these substances move from stronger to weaker parts of the field. It is represented by the Fig. 11.18.

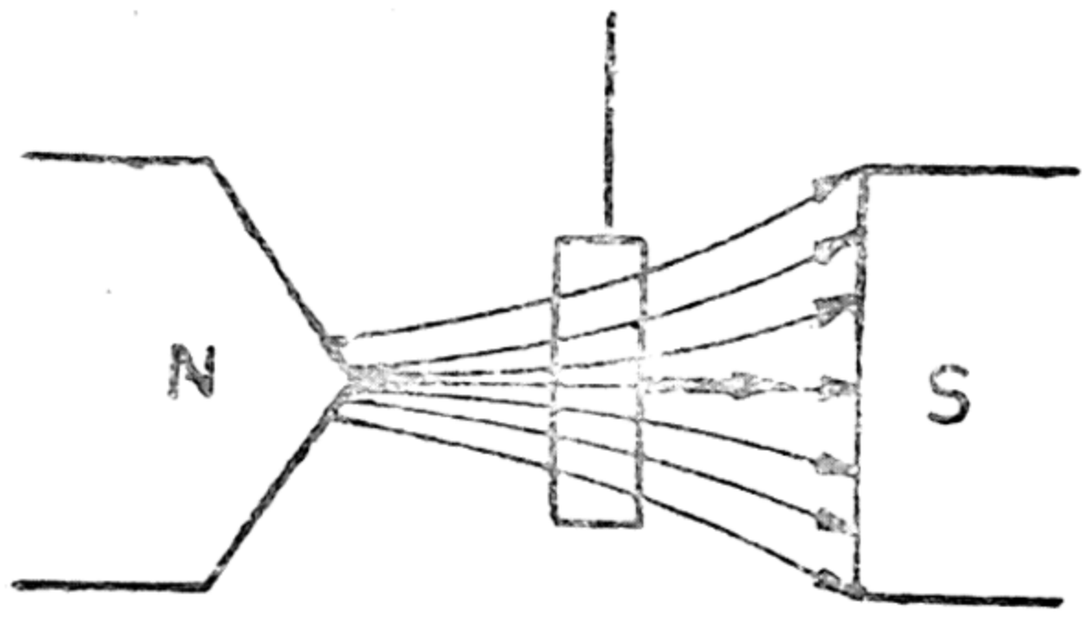


Fig. 11.18.

If a diamagnetic liquid is placed in a watch glass resting on the pole of a powerful electromagnet, the liquid is found to accumulate on the sides, where the field is weaker.

(4) If a diamagnetic liquid is filled in a narrow *U*-tube and one limb is placed in between the pole pieces of an electromagnet, the level of the liquid depresses when the field is switched on.

Similarly, when a diamagnetic gas is allowed to ascend between the poles of an electromagnet, the gas spreads across the field.

(5) The relative permeability K is slightly less than 1.

(6) The susceptibility χ of such substances is always negative. It is constant and does not vary with field or the temperature. There are few exceptions, *e.g.*, bismuth at low temperatures.

Ferromagnetic Substances. Examples of such substances are iron, nickel, steel, cobalt and their alloys. These substances resemble to a higher degree with paramagnetic substances as regard their behaviour. They have following additional properties :

(1) These substances are strongly magnetised by even a weak magnetic field. The relative permeability is very large and is of the order of hundreds and thousands. The susceptibility is positive and very large.

(2) The intensity of magnetization **M** is proportional to the magnetising field intensity **H** for its smaller values, increases rapidly

for larger values and attains a constant value for very large values of **H**, Fig. 11.19.

(3) Susceptibility χ thus remains constant for very small values of **H**, increases for larger values of **H** and then decreases for very large values of **H**.

(4) The magnetic induction **B** varies with **H** in a similar manner as **M** does, except that **B** does not attain a constant value for very large values of **H**.

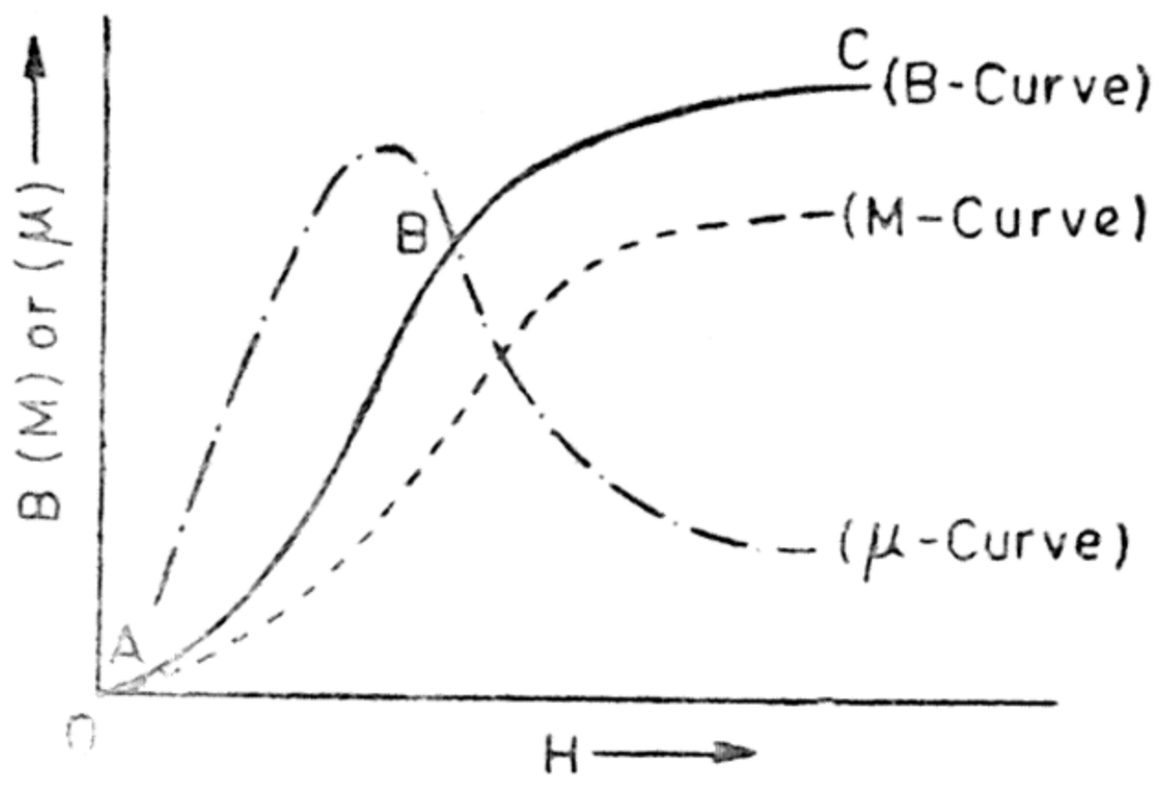


Fig. 11.19.

(5) Permeability μ also varies as χ, except at very high magnetic fields where μ decreases slowly in comparison to χ.

(6) The susceptibility χ decreases steadily with the rise of temperature. Above a certain temperature, known as Curie temperature, the ferromagnetic substances become paramagnetic. It is 1000°C for iron, 770°C for steel, 360°C for nickel and 1150°C for cobalt.

11.11. PARAMAGNETISM

We know that there are three mechanisms or properties of atoms that give rise to magnetic dipole moment.

1. The electrons moving around the nucleus in the orbits act as small current loops and contribute magnetic moments

$$m = A.i = \pi r^2 (ef) = \pi r^2 ev/2\pi r = evr/2.$$

2. The spinning electron has an intrinsic magnetic dipole moment of value $eh/4\pi me$.

3. The nucleus contributes to the total magnetic moment. It is due to the motion of charge within the nucleus.

The magnitude of nuclear moments is about 10^{-3} that of electronic moments or the spin magnetic moments as the latter two are of the same order.

Most of the circulatory electrons do not contribute to the magnetic moment, as they pair off in such a way that they produce equal and opposite moments that cancel out. In atoms and ions, the electrons also try to pair up with their opposite spins. Thus we have spin magnetism either due to the odd numbered unpaired electron or due to incomplete cancellation of spin magnetic moment. All the rare earth and transition elements have a permanent magnetic moments as their inner electron shells are not completely filled.

Paramagnetism is mainly due to the intrinsic or spin magnetic dipole moment of the electron. If there are N atoms per unit volume, each having a magnetic dipole m, then magnetization

$$\mathbf{M}=\sum_{i=1}^{N}\mathbf{m}_i. \qquad \ldots(65)$$

This is equal to Nm when all the dipoles are aligned parallel.

Langevin's Theory. Langevin gave a theory for a paramagnetic gas. He assumed that the interaction between neighbouring dipoles was negligible. The dipoles were assumed of same magnetic moment m but pointed in any direction. In an applied magnetic field of flux density **B**, the magnetic energy $W=-\mathbf{m}.\mathbf{B}=-mB\cos\theta$, where θ is the angle which the dipole makes with the field **B**. From Boltzmann statistics, the relative probability of finding a dipole making an angle θ with the B direction is $e^{-W/kT}$. Here k is the Boltzmann constant and T the absolute temperature of the gas which is responsible for the random nature of the dipoles.

The number of atoms having an energy in the range W and $W+dW$

$$dN=c\,e^{-W/kT}\,dW.$$

One can also write it as the number of atoms whose dipole lies between θ and $\theta+d\theta$. We therefore have

$$dN=c\,e^{mB\cos\theta/kT}\,mB\sin\theta\,d\theta$$

$$=c'\,e^{mB\cos\theta/kT}\sin\theta\,d\theta.$$

As total number of atoms per unit volume is N, hence the constant $c'=cmB$ can be obtained as

$$N=c'\int_0^{\pi} e^{\alpha\cos\theta}\sin\theta d\theta=\frac{2c'}{\alpha}\sinh\alpha$$

or $$c'=N\alpha/(2\sinh\alpha). \qquad (\text{where } \alpha=mB/kT)$$

$\therefore$ $$dN=\frac{N\alpha}{2\sinh\alpha}\,e^{\alpha\cos\theta}\sin\theta\,d\theta. \qquad \ldots(66)$$

The component of each dipole moment parallel to **B** is $m\cos\theta$, hence the total magnetic moment per unit volume

$$M=\int_0^{\pi} m\cos\theta\,dN=\frac{mN\alpha}{2\sinh\alpha}\int_0^{\pi} e^{\alpha\cos\theta}\sin\theta\cos\theta\,d\theta,$$

$$M=\frac{m\,N\,\alpha}{2\sinh\alpha}\left[\frac{e^{\alpha\cos\theta}}{\alpha^2}-\frac{\cos\theta\;\;e^{\alpha\cos\theta}}{\alpha}\right]_0^{\pi}$$

$$=Nm\;(\coth\alpha-1/\alpha). \qquad ...(67)$$

Since Nm is the saturation value of the magnetic moment, say M_0, hence

$$M=M_0\,(\coth\alpha-1/\alpha). \qquad ...(68)$$

This bracket term $(\coth\alpha-1/\alpha)$ is known is *Langevin function.* It is plotted as a function of α, in Fig. 11.20. This curve indicates that for large values of α the function tends to unity, *i.e.*, $M\to M_0$. It is the case when all the atomic dipoles are parallel to **B**. For small values of α, the curve is linear, as for smaller values of α

$$\coth\alpha-1/\alpha=\alpha/3=mB/3kT$$

For small values of α and thus for higher values of T

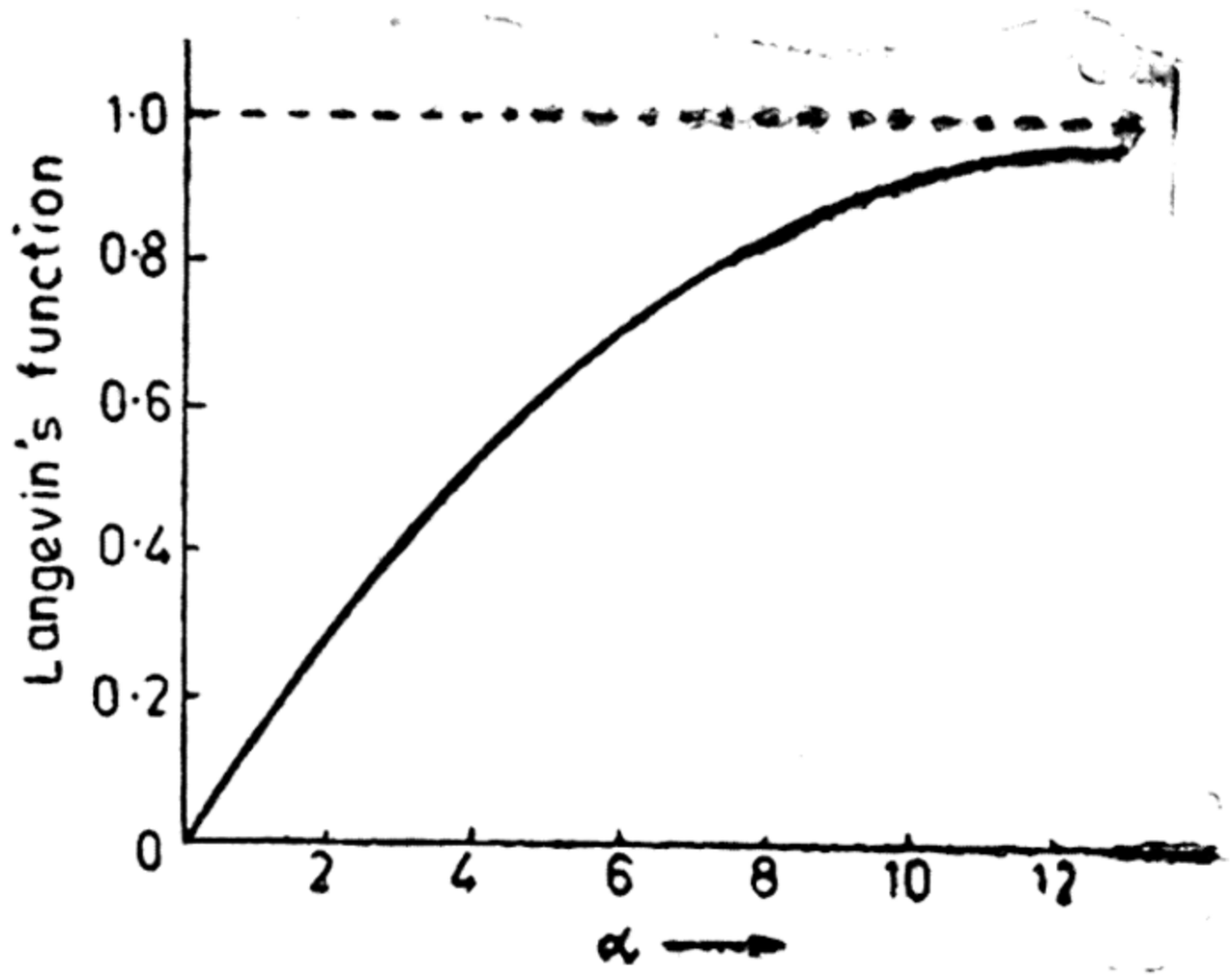


Fig. 11.20.

$$\text{Susceptibility } \chi_m=\frac{M}{H}=\frac{M_0(\coth\alpha-1/\alpha)}{H}=\frac{m^2N\mu_0H}{3kTH}$$

$$=\frac{\mu_0M_0^2}{3NkT}=\frac{\text{const}}{T}\;. \qquad ...(69)$$

Here the constant is called the *Curie constant* and this relation is known as *Curie law.* Curie law is well verified experimentally and is very reasonable as at higher temperatures the thermal agitation opposes the dipole alignments along the **B**-direction. It breaks down at low temperatures. The temperature below which this law ceases can reach as low as $1.3°K$, for some paramagnetic substances.

Langevin's theory applies strictly only to gases. In a gas or liquid the atoms are making collisions with one another continuously, while in a solid the atoms are undergoing thermal oscillations. Under these conditions the various magnetic dipoles can interchange magnetic energy with the thermal energy of their environment. The thermal energy of the system tries to produce a completely random orientation of the magnetic dipoles, but the orientations along or near the applied field **B** have the lower magnetic energy and are thus favoured. For solids NkT in equation (69) is the total kinetic energy of the atoms. It may be regarded as a measure of the resistivity against the turning of the atoms (molecules) in the direction of **B**. The mutual action of the solid molecules affects the orientation of the molecules in the same way as their kinetic energy. Thus the effect of taking

this action into account is equivalent to an increase of *K.E.* by a term $\angle$. Hence in the case of solids

$$\chi_m = \mu_0 M_0^2/3(NkT + \Delta). \qquad \ldots(70)$$

If $\Delta = Nk\Omega$, where Ω is a constant independent of temperature, then the above relation becomes

$$\chi_m = \mu_0 M_0^2/3Nk(T + \Omega). \qquad \ldots(71)$$

11.12. DIAMAGNETISM

In electromagnetism we know Lenz's law, which states that when the flux through an electrical circuit is changed, an induced current is set up in the circuit in such direction as to oppose the change of flux. The magnetic moment associated with this induced current is called the diamagnetic moment. It is in the direction opposite to the applied field, hence the susceptibility is negative.

Langevin's theory of diamagnetism. In diamagnetic substances, the electrons associated with different atoms rotate in orbits oriented so that the net magnetic dipole moment is zero. According to *Langevin*, the net magnetic dipole moment is not zero, when a magnetic field of flux density **B** is applied, but is negative for these substances.

To calculate net magnetic moment in the presence of magnetic field, let us consider a very simple model consisting of an electron orbit whose plane is $\perp$ to the **B**-direction. According to the Lenz's law there is an emf induced in the electron orbit, given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(B\pi R^2 \right) = -\pi R^2 \frac{dB}{dt}, \qquad \ldots(72)$$

where the radius R of the orbit is assumed constant. It is the case when the magnetic field **B** is such that it makes very small changes in the circular motion.

The emf produced by changing **B** results in an electric field **E** which acts tangentially to the direction of the electronic motion. This field $E(=\mathcal{E}/2\pi R)$ will accelerate the electrons. Hence by Newton's law

$$F = me\frac{dv}{dt} = eE = \frac{e\mathcal{E}}{2\pi R} \qquad \ldots(73)$$

Hence the combination of Eqs. (72) and (73) gives

$$\frac{dv}{dt} = -\frac{e}{2\pi Rme}\frac{d\Phi}{dt} = -\frac{eR\,dB}{2m_e\,dt}. \qquad \ldots(74)$$

The total change in velocity, during the time the **B** is being increased from 0 to Bm, is thus given by

$$\Delta v = -eRBm/2me.$$

$\therefore$ Change in angular velocity

$$\triangle\omega = \triangle v/R = -eBm/2me. \qquad \ldots(75)$$

This is called *Larmor frequency*, also written as ω_L. Negative sign shows that ω_L is $-$ve (clock-wise) when B is positive (outside).

The magnetic moment will be induced due to the Larmor frequency of an electron and thus the induced current. We know that the magnetic moment of a circular current is $m = iA$, where i is the current and A is the area of the loop. Since the induced current

$$i = e(\triangle\omega/2\pi) = e\omega_L/2\pi = -e^2\ Bm/4\pi me.$$

$\therefore$ Induced magnetic moment $m = iA = i(\pi R^2) = -e^2R^2Bm/4m_e.$

In diamagnetic materials, the velocity of one electron increases while that of second decreases by the Larmar frequency in the presence of applied magnetic field. Thus we have a net magnetic moment. Therefore, the net induced magnetic moment of any atom may be obtained by adding m's for all the electrons in an atom, as

$$m' = -\frac{e^2\ Bm}{4\ me}\Sigma R_i^2. \qquad \ldots(76)$$

For a substance containing N-atoms per unit volume, the total magnetic moment per unit volume, *i e.*, the induced magnetization

$$M = m'N = -\frac{e^2NBm}{4me}\Sigma R_i^2 = -\frac{e^2\mu_0NH_m}{4me}R_i^2. \qquad \ldots(77)$$

Hence the susceptibility of the diamagnetic substance is given by

$$\chi_m = \frac{M}{Hm} = -\frac{\mu_0e^2N}{4me}\Sigma R_i^2. \qquad \ldots(78)$$

In the derivation of this result, B_m is assumed to be perpendicular to the planes in which electrons circulate. When the orbit is inclined, so that a normal to the orbit makes an angle θ_i with the field Bm, the above relation can be written as

$$\chi_m = -\frac{\mu_0e^2N}{4me}\Sigma R_i^2\ \cos^2\theta_i. \qquad \ldots(79)$$

Substitution of known values gives χ_m of the orders of $\sim 10^{-8}$. This relation indicates that the diamagnetic susceptibility is independent of the temperature. Small variation in χ_m with temperature T is due to changing of lattice constants with temperature.

Diamagnetism is present in all types of matter but its effect is masked generally by stronger effects of paramagnetic or ferromagnetic behaviour that also occurs simultaneously in the meterial. Diamagnetism is prominent in materials which consist atoms with closed electron shells, as in these paramagnetic effects cancel out.

As paramagnetic susceptibilities decrease with temperature, hence all materials become diamagnetic at high enough temperatures.

Some metals are diamagnetic, while others are paramagnetic. In the metals, χ_m is due to the ion cores and the conduction electrons. There is also a paramagnetic contribution from the electron spins. In some cases this paramagnetic contribution is more than required to cancel the diamagnetic term, hence *it is difficult to predict whether a given metal will be diamagnetic or paramagnetic.*

11.13. FERROMAGNETISM

In iron like elements and their alloys. known as *ferromagnetic substances*, the magnetization is not usually proportional to the applied magn.tic field, as in the previously studied materials. The susceptibility is several thousand for these materials. Such a high magnetization may be due to magnetic moments of electron spin. In ferromagnetic substances the effective field acting on each spin magnetic dipole is the vector sum of the applied field plus a *strong interaction* field arising from all the neighbouring dipoles. This interaction field is so strong that thermal vibrations even at room temperature can not destroy the alignment. These interactions, also known as *exchange coupling*, cannot be explained in terms of classical physics. They can be explained only by quantum mechanics as they exist due to the wave nature of the electrons.

Weiss, in 1907, gave a quantum mechanical theory of ferromagnetism, with the following postulates :

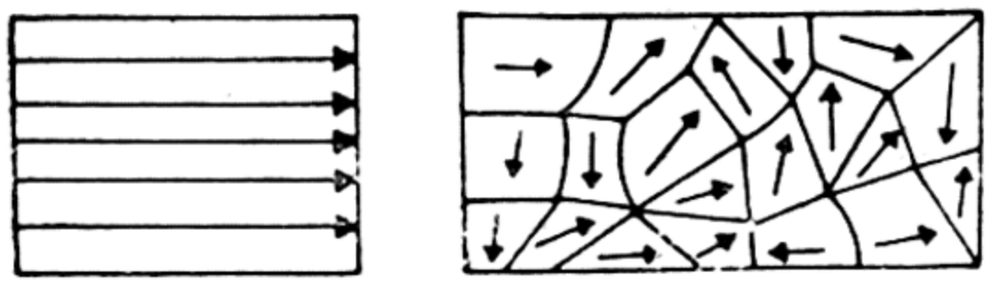
(*i*) The internal field acting on a dipole is not the external field H_0, but given by

$$B_i = \mu_0(H_0 + \beta M), \qquad \ldots(80)$$

where β is the internal field constant corresponding to the exchange coupling.

(*ii*) A ferromagnetic substance consists of large number of domains, called *ferromagnetic domains.*

Ferromagnetic Domains. A domain is a region composed of a number of elementary magnets, each pointing with its axis in the same direction even in the absence of any external field, Fig 11.21 (*a*). The size of the domain is now known to vary from about 10^{-6} cm^3 to 10^{-2} cm^3. In an unmagnetized ferromagnetic specimen the domains are oriented at random, so that their resultant magnetic moment is zero, Fig. 11.21 (*b*).



(*a*) (*b*)

Fig. 11.21.

Fig. 11.22 (*a*) shows a ferromagnetic crystal as being made up of small cubes in each of which the domain vector is ‖ to one of

the directions of preferred magnetisation. This picture is wrong as it violates the fundamental theorem of magnetism. The *normal component of magnetic field is to be conserved.* Fig. 11.22 (*b*) is favourable as **B** is continuous across every domain. Using this concept the domain structure of a single crystal is illustrated in Fig. 11.22 (*c*)

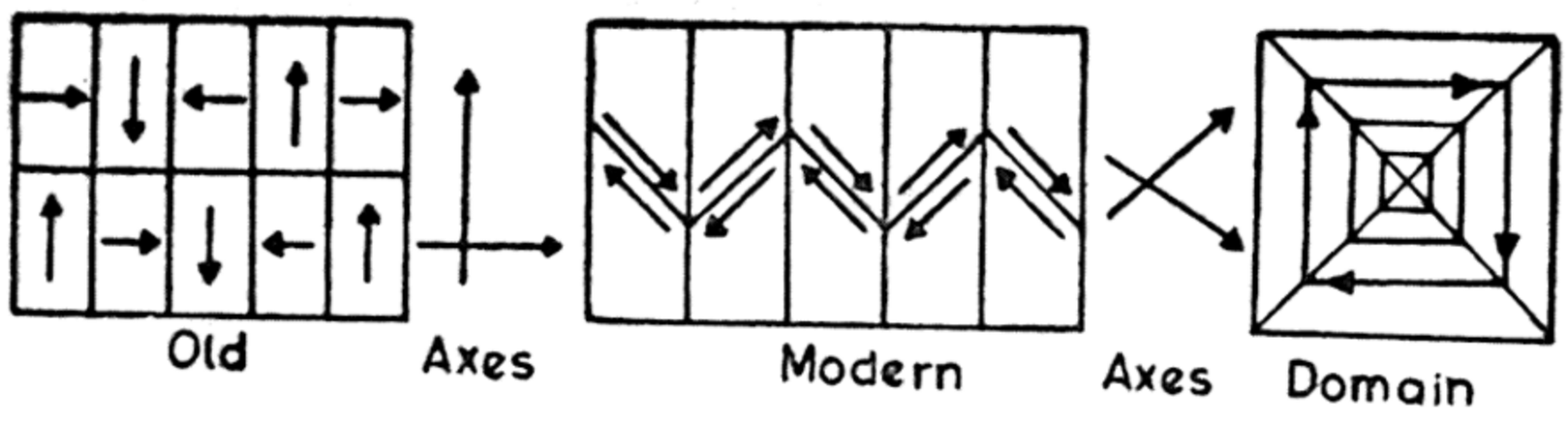


Fig. 11.22.

When the specimen is placed in a magnetic field the resultant magnetization may increase in two different ways : by an increase in the volume of the domains which are favourably oriented with respect to the field at the expense of unfavourably oriented domains, Fig. 11.23 (*a*) or by the rotation of the domain magnetization

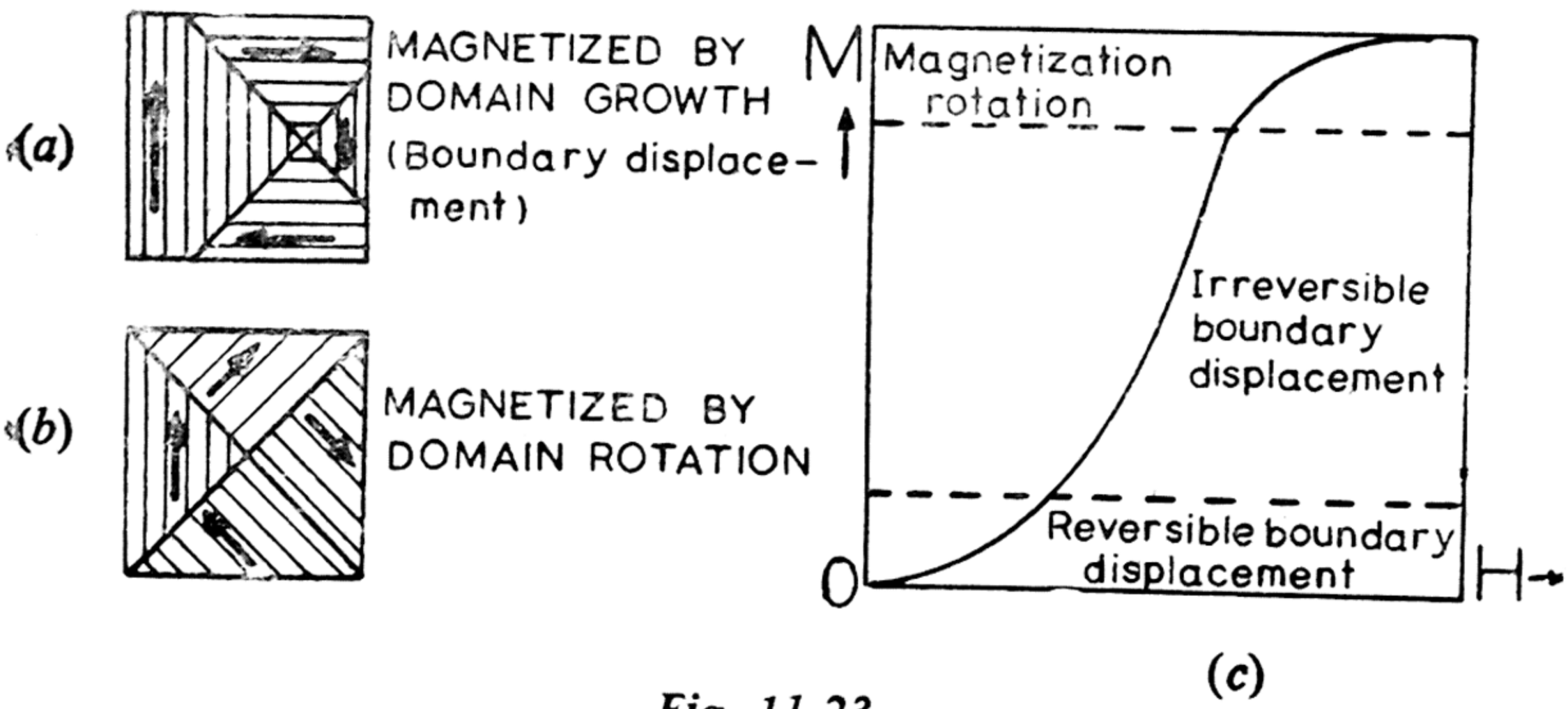


Fig. 11.23.

toward the field direction, Fig. 11.23 (*b*). In weak fields the magnetization usually changes by the former method and in strong fields by the latter method. A magnetization curve for a ferromagnetic material is shown in Fig. 11.23 (*c*). It shows the regions in which each process is dominant. In small fields the change is reversible, *i.e.*, *boundaries return to their original positions when the field is removed.* If the magnetic field is further increased, irreversible boundary displacements predominate and the magnetization increases more rapidly. When the applied field is such that the knee of the curve is reached, the domain vectors will be so oriented that they are all parallel to the direction of those domains, whose volumes increased in the early stages The further increase of the applied field increases the magnetization by the rotation of the domain vectors in the direction of the applied field.

Curie-Weiss Law. Weiss theory of the ferromagnetism can not be discussed here. The equation for the magnetization in a ferromagnetic material may be obtained by replacing B by B_i, the internal field in a ferromagnetic substance, in the equation for the magnetic moment of paramagnetic substances. Thus we get

$$M=\frac{Nm^2B_i}{3kT}=\frac{Nm^2\mu_0\,(H_0+\beta M)}{3kT}. \qquad ...(81)$$

As the magnetic moment of an orbital electron $m=eh/4\pi m_e$.

$$\therefore \qquad M=N\left(\frac{eh}{4\pi m_e}\right)^2 \mu_0\ \frac{(H_0+\beta M)}{3kT}. \qquad ...(82)$$

Hence $$M=\frac{CH_0}{T-\theta} \quad \text{or} \quad \text{Susceptibility } \chi_m=\frac{C}{T-\theta}, \qquad ...(83)$$

where constant $$C=N\left(\frac{eh}{4\pi m_e}\right)^2\frac{\mu_0}{3k} \text{ and } \theta=\beta C. \qquad ...(84)$$

This law is known as the *Curie-Weiss law*, C the *Curie constant* and θ the *Curie temperature*. For $T>\theta$, the ferromagnetic material behaves as the paramagnetic material. For $T<\theta$, this law is not applicable. In this case a finite value of the magnetization exists even when H the applied field $H_0=0$. This magnetization is called the spontaneous magnetization. It accounts for the residual magnetization in the absence of external field.

11.14. HYSTERESIS

The real distinguishing characteristic of a ferromagnetic material is not that it can be strongly magnetized but that the intensity of magnetization M is not directly proprortional to the magnetizing field H. If a gradually increasing magnetic field H is applied to an unmagnetized piece of iron, its magnetization increases non-linearly, until it reaches a maximum. If M is plotted against H, a curve like OA is obtained, this curve is known as the *magnetization curve*. At this stage, all the dipoles are aligned and M has reached to maximum or *saturated value*. It is often more convenient to plot B than M, the difference is often small except for the factor μ_0, as $B=\mu_0\,(H+M)$ and M is usually much greater than H. If a magnetic field H is now reduced, the M (or B) does not return along magnetization curve but follows path AB. At $H=0$, M (or B) does not come to its zero value, but its value is still near the saturated value. The value of M at this point (*i e*, OB) is known as *remanence*, *remanent magnetization* or *retentivity*. The value of B at this point is known as the *residual induction*. The retentivity gives the state of permanent magnetization of the specimen.

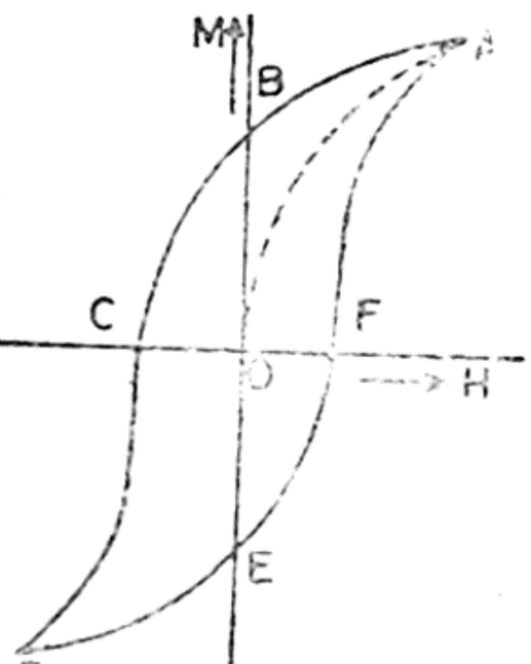


Fig 11.24.

On applying a reverse field the value of **M** falls and finally becomes zero. The abscissa *OC* represents the reversed magnetic field needed to demagnetize the specimen. This is known as *coercivity* (or coercive field) of the material. If the graph is in between **B** and **H**, then it must be noted that the numerical value of **H** to make **B**=0 is not the coercivity, as $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})$, hence **B**=0 for **M**=−**H** and the magnetization **M** is left positive.

If the reverse field is further increased, a reverse magnetization is set up which quickly reaches the saturation value. This is shown as *CD*. If **H** is now taken back from its negative saturation value to its original positive saturation value, a similar curve *DEFA* will be traced. The whole graph *ABCDEFA* thus forms a closed loop, usually known as *hysteresis loop*. The whole process described above and the property of the iron characterized by it are called *hysteresis*. *Hysteresis is the name for the lag of magnetization behind the magnetizing field.*

Demagnetization. It is clear from the hysteresis loop that the intensity of magnetization **M** does not reduce to zero on removing the magnetizing field **H**. **M** is zero when the magnetizing field **H** is equal to the coercitive force. At these points the magnetic induction is not zero, and the specimen is not demagnetized. To demagnetize a substance, it is subjected to several cycles of magnetization, each time with decreasing magnetizing field and finally the field is reduced to zero. In this way the size of the hysteresis curve goes on decreasing and the area finally reduces to zero. Demagnetization is best obtained by placing the specimen in an alternating field of continuously diminishing amplitude. It is also obtained by heating, *ferromagnetic materials becoming practically non-magnetic at sufficiently high temperatures.*

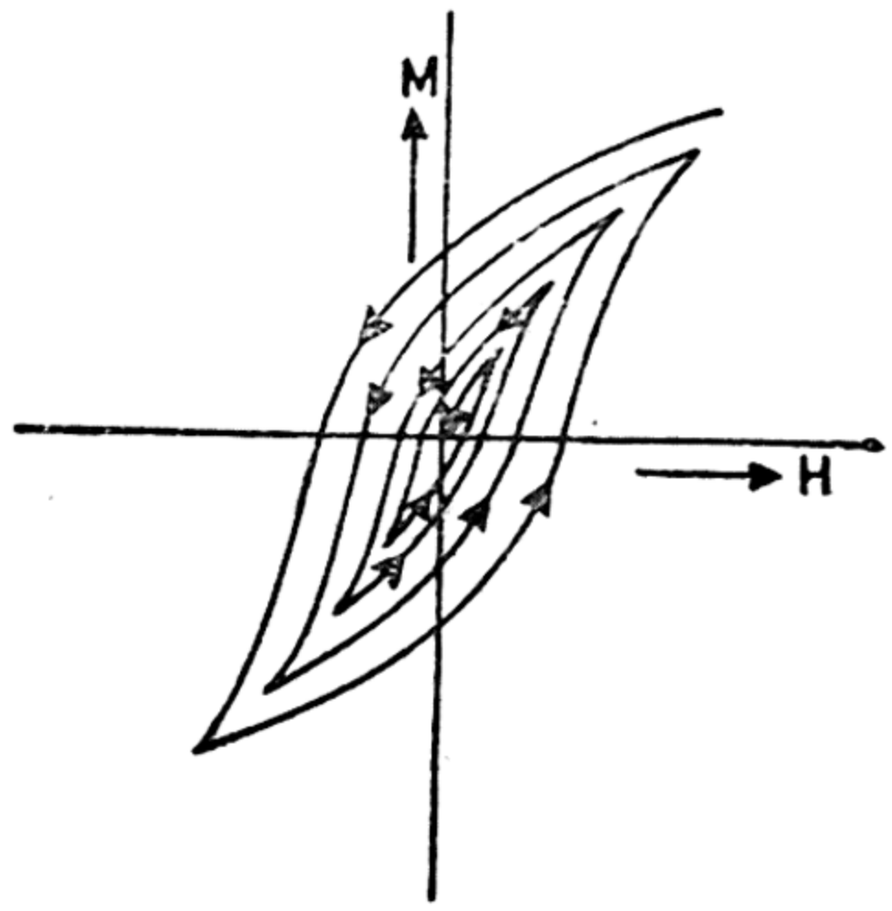


Fig. 11 25.

Ballistic Method for B.H. Curve. The *B-H* curve of a specimen can be obtained by a ballistic galvanometer method. In this method the specimen is taken in the form of a ring, known as *anchor ring*.

A magnetizing coil is wound closely round the specimen ring, forming the primary *P* of an endless solenoid. The coil is connected

through a reversing key K and a two way key K_1 to a circuit containing a battery, an ammeter A, a rheostat R_1 and a resistance R'. The resistance R' can be removed from the circuit by pressing the key K'. Over a small part of primary of the solenoid, a secondary coil S of few turns is wounded, which is connected in series with a ballistic galvanometer (*B.G.*), variable resistance R_2, a key K_2 and the secondary coil S' of the standard solenoid having primary P'. The two way key K_1 may connect either, the primary of the ring solenoid, or primary of the standard solenoid. To damp the motion of ballistic galvanometer, a tapping key K_3 is connected across it. The whole arrangement is shown in Fig. 11.26.

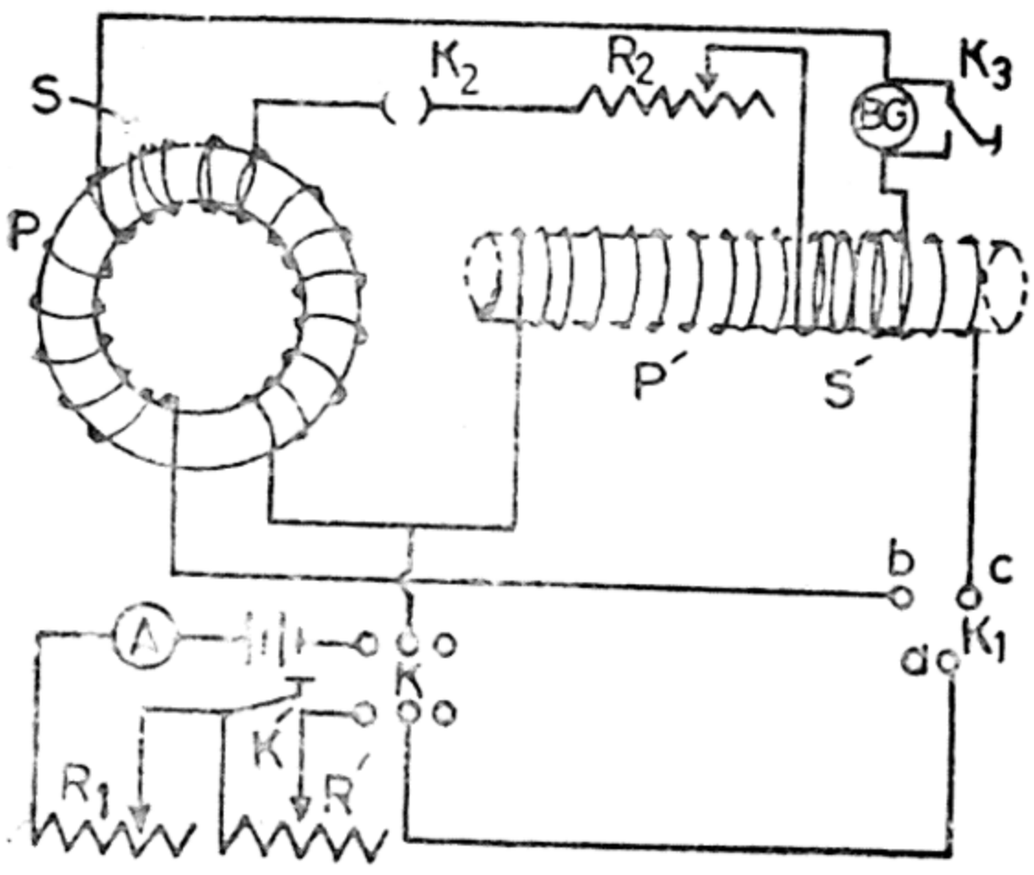


Fig. 11.26.

If the primary of the ring solenoid consists N_1 turns and the mean circumference of the ring is l, then the magnetic field produced in the ring due to the current i in the primary is given by

$$H=N_1 i/l. \qquad \ldots(85)$$

This magnetizing field produces a magnetic flux which passes through the secondary of the ring solenoid. If B is the flux density in the ring material, N the number of turns in the secondary and A the area of cross section of the ring then the total flux linked with the secondary

$$\Phi=NBA.$$

This is the change of flux in the secondary, it sets up an induced emf in the secondary circuit. If R is the total resistance of the secondary circuit, then the charge passing through the ballistic galvanometer

$$q=NBA/R.$$

If θ is the first throw of the ballistic galvanometer coil, then

$$q=NBA/R=K\theta\,(1+\lambda/2). \qquad \ldots(86)$$

where K is the ballistic constant and λ the logarithmic decrement of the ballistic galvanometer.

To eliminate K and λ, the known current i' is passed through the primary of the standard solenoid by closing the key K_1 to the right. If n is the number of turns per unit length in the primary, then the flux density in the standard solenoid will be $\mu_0 n i'$. The corresponding flux linked with the secondary of N' turns will thus be $\mu_0 n i' (N'A')$, where A' is the area of cross-section of the secondary of the standard solenoid. This change in the magnetic flux sends

a charge through the ballistic galvanometer which deflects the coil. If θ' is the first throw in the galvanometer coil, then

$$\text{charge} = \mu_0 n N' i' A'/R = K\theta'(1+\lambda/2). \quad \ldots(87)$$

Dividing Eq. (86) by Eq. (87), we get

$$\frac{NBA}{\mu_0 n N' i' A'} = \frac{\theta}{\theta'} \quad \text{or} \quad B = \frac{\mu_0 n N' i' A'}{NA} \frac{\theta}{\theta'}. \quad \ldots(88)$$

This is the expression for magnetic induction in the specimen corresponding to the magnetic intensity H, given by Eq. (85).

For the full cycle of magnetization, the following procedure is to be followed : The key K_1 is first closed to the left by inserting a plug between *ab* gap and the resistances R_1 and R' are decreased until on closing the commutator K the galvanometer gives full scale deflection from the zero The current corresponding to this deflection is measured by an ammeter and is used as maximum current in the main experiment.

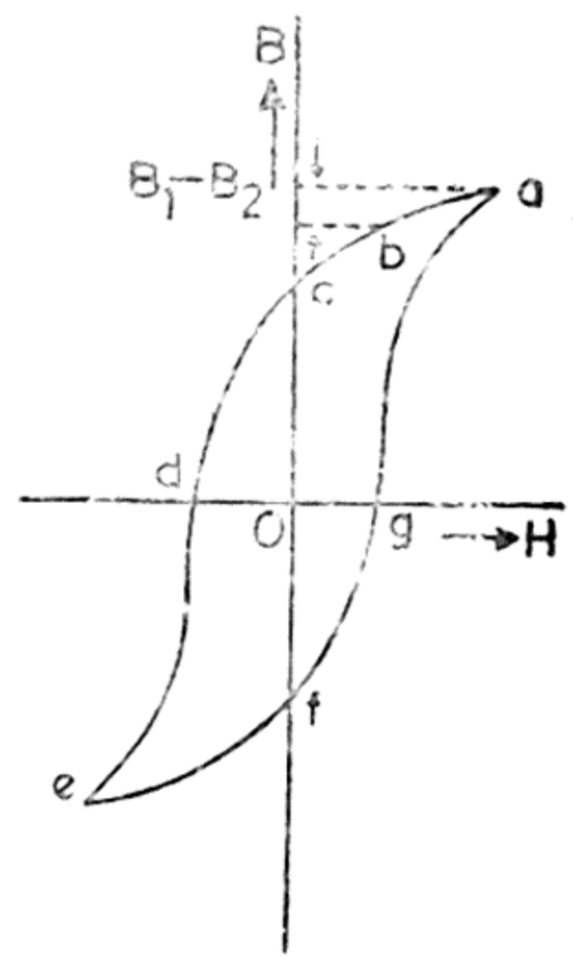


Fig. 11.27.

The residual magnetism in the specimen is reduced to zero as :—The galvanometer circuit is first broken and the resistances R_1 and R' are reduced to the minimum. The current passing through the primary of the ring solenoid is then reversed many times by means of the commutator K, the resistances R_1 and R' are increased gradually simultaneously until the current becomes very small.

The galvanometer is again put in the circuit by closing key K_2. The key K' is closed and resistance R_1 is given a value corresponding to the maximum current. The galvanometer first throw is thus noted. The values of B and H, *i e*., B_1 and H_1 are thus calculated. The corresponding point on the *B-H* curve is *a*. The galvanometer circuit is again broken and the specimen is again demagnetized by reversing rapidly the commutator K as described before. The ballistic galvanometer is again put in the circuit. Now R' is given a small value and K' is opened. This decreases the current and thus the magnetizing field H. This decrease in H produces a decrease in magnetic induction B. Corresponding to this decrease $B_1 - B_2$, a throw θ is observed. The value of magnetising field H_2 is calculated by noting ammeter reading. The corresponding point on the graph (*B H* curve) is denoted by the point *b* This process is repeated by gradually increasing R', until current and hence H becomes zero. The graph corresponding to these readings is *ac* (Fig 11.27). As after each measurement the specimen is returned to the state *a* by the reversal of maximum current, hence *point a works as the reference point.*

The *cde* of the graph is plotted by repeating the process in many steps with the reverse current. The point *e* is corresponding to the zero value of R'. The part *efga* can be drawn by symmetry or by repeating the experiment using *e* as the reference point and leaving the commutator on the left, if it was on the right previously.

In an unmagnetized specimen, the directions of magnetization in different domains are different so that net magnetization is zero. When this specimen is placed in an external magnetic field, these domains are rotated so that the magnetization is aligned with the field direction. The work is thus done and the energy is taken by the specimen under this process. When the field is removed the domain boundaries do not mave compietely back to their original positions and the specimen retains some magnetism, Thus the energy is not recovered during demagnetization and there is always a definite loss of energy for each hysteresis loop. *This energy loss is known as hysteresis loss.*

Let m be the magnetic moment of any one domain which makes an angle θ with the direction of magnetizing field **H**. The components of m along and $\perp$ to **H** are $m \cos\theta$ and $m \sin\theta$ respectively. The summation of all such terms for all the domains in volume of the specimen will be

$$\sum_{N} m \cos\theta \text{ and } \sum_{N} m \sin\theta,$$

where summation is taken over all the N domains, which are in unit volume. As the magnetisation is along **H**, hence from the definition of **M**, the intensity of magnetisation, we have

$$\sum m \cos\theta = M \text{ and } \sum m \sin\theta = 0$$

$$\therefore \quad dM = d[\sum m \cos\theta] = -\sum m \sin\theta \, d\theta.$$

As the restoring couple acting on the domain of moment m, when it is inclined at an angle θ to the **H**−direction is $m B \sin\theta$ $(=\mu_0 mH \sin\theta)$. Hence the work done when it moves through a small angle $d\theta$ is

$$dW' = \mu_0 mH \sin\theta \, d\theta.$$

∴ Work done per unit volume of the material

$$= \Sigma \mu_0 mH \sin\theta \, d\theta. \qquad \ldots(89)$$

If the intensity of magnetization M is increased to $M+dM$ by increasing H to $H+dH$, then the work done per unit volume as M increases (or the dipoles are rotated more along H, *i.e.*, θ decreases) is given as

$$dW = \Sigma \mu_0 m (H+dH) \sin\theta \, d\theta. \qquad \ldots(90)$$

As dH and $d\theta$ both are very small, hence by assuming H as constant, we get

$$dW = -\Sigma\mu_0 mH \sin\theta\, d\theta = -\mu_0 H\, \Sigma m \sin\theta\, d\theta = \mu_0 H dM.$$

∴ Work done per unit volume for a complete cycle of magnetization

$$W = \mu_0 \oint H dM. \qquad \ldots(91)$$

As $B = \mu_0 (H + M)$, hence for constant magnetizing field $dB = \mu_0 dM$. Therefore,

$$W = \oint H dB. \qquad \ldots(92)$$

Let us consider $M - H$ curve (or $B - H$ curve). At point near Q on the curve (Fig. 11.28) PQ represents H and PR represents a small change in M. *i.e.*, dM (or dB). Hence the area $PQSR$ will represent H.dM (or H dB), the work on the material per unit volume. Thus the work done on the material per unit volume during the path ab will be the sum of all such areas, *i.e.*, area $OabJO$. When the material is taken from b to c, or the magnetising field is decreased to zero, the energy equal to the area $bJcb$ is restored back per unit volume of the material. Thus the horizontally shaded areas represent the work done by the material per unit volume, whilst the areas which are also vertically shaded represent the work done on the material per unit volume. Hence the excess of work done on the material over that done by it, per unit volume, is represented by the area enclosed within the hysteresis loop $abcdefa$.

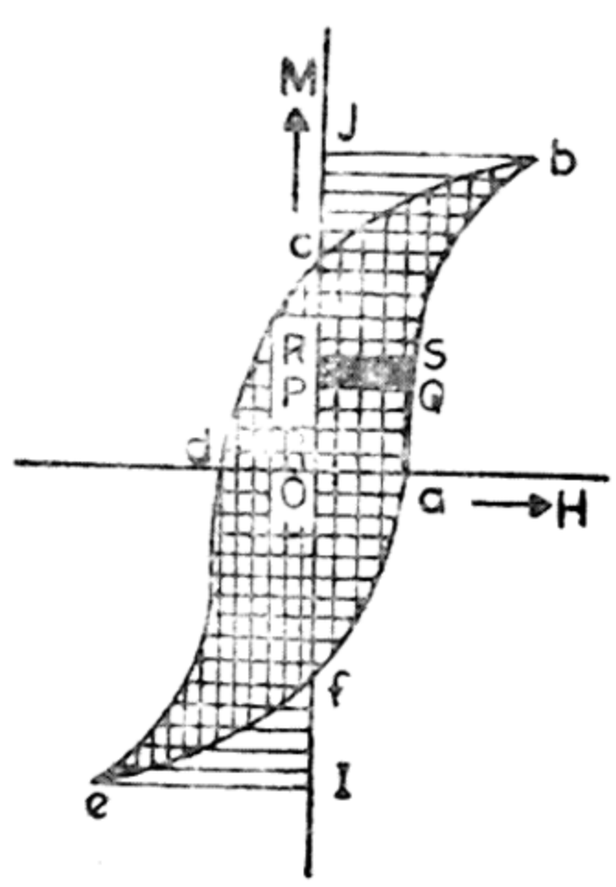


Fig. 11.28.

∴ *Work done per unit volume per cycle* $= \mu_0 \times$ *Area of the* $M - H$ *loop* $=$ *Area of* $B - H$ *loop.*

A rough idea of the magnitude of hysteresis losses may be obtained from an empirical formula of *Steinmetz.*

Work done or the Energy loss per unit volume per cycle

$$W = \eta\, (B_{max})^{K}, \qquad \ldots(93)$$

where B_{max} is the maximum magnetic induction in the loop, η, the *Steinmetz coefficient*, is a constant varying from material to material. The exponent K varies between 1.4 and 1.8 for the various ferromagnetic materials, most accepted value is 1.6.

11.15. SOFT AND HARD MAGNETIC MATERIALS

The magnetic properties of a ferromagnetic substance can be obtained from the size and shape of the hysteresis loop. For a few materials the hysteresis curves are shown in Fig. 11.29. Curves (*a*), (*b*), (*c*) and (*d*) are respectively for the specimen of soft iron, steel, cobalt and nickel materials. The study of these curves gives the following informations :

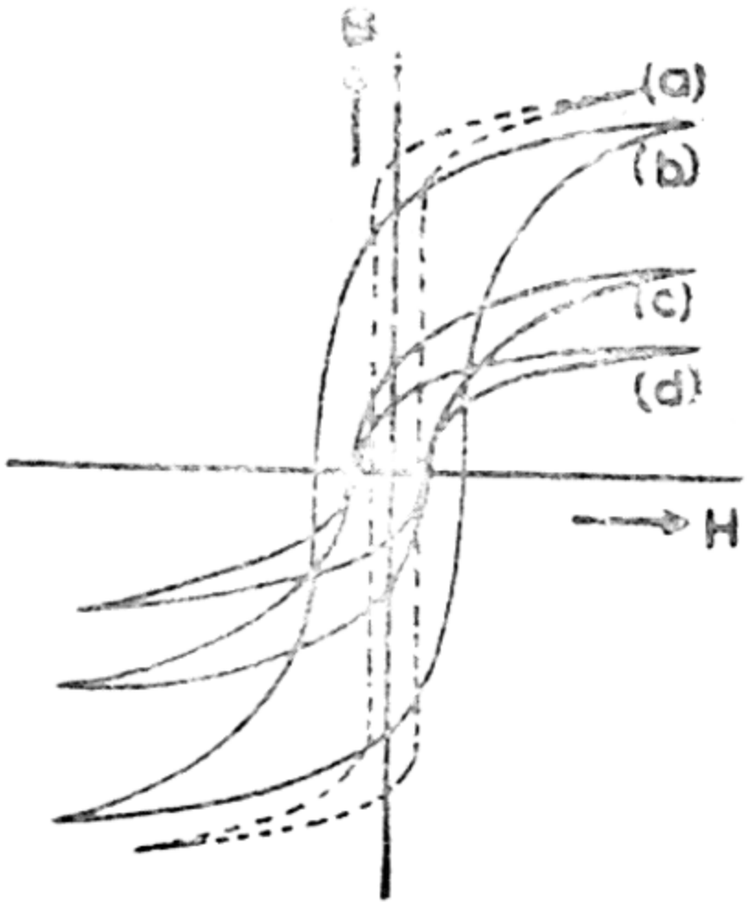


Fig. 11.29.

(*i*) *Susceptibility.* The susceptibility, the intensity of magnetisation per unit magnetizing field, (*i.e.*, M/H) is greater for soft materials than for hard materials.

(*ii*) *Permeability.* The permeability, the magnetic induction per unit magnetizing field (*i.e.*, B/H) is greater for soft materials than for hard materials.

(*iii*) *Retentivity.* When a magnetic specimen is first magnetized and then the magnetizing field is reduced to zero, the specimen retains intensity of magnetisation (or the magnetic induction), this is known as the *retentivity* (or *the remanent induction*). It is greater for soft materials than for hard materials.

(*iv*) *Coercivity.* To demagnetize the magnetic specimen completely a—ve field is required. The value of reverse field H required to reduce the intensity of magnetisation to zero is known as the *coercivity*. It is less for soft materials than for hard materials.

(*v*) *Hysteresis Loss.* The area of the hystersis loop and hence hysteresis loss per unit volume per cycle is less for soft materials than for hard materials.

Choice of Magnetic Materials. The choice of a magnetic materials for different uses can be decided from the hyteresis curve of a specimen of the material.

(1) *Permanent Magnets.* The materials for a permanent magnet should have : (*a*) high retentivity (so that the magnet is strong) and (*b*) high coercivity (so that the magnetizing is not wiped out by stray magnetic fields, mechanical ill-treatment or by temperature change). As the material in this case is never put to cyclic changes of magnetization, hence hysteresis is immaterial. From the view point of these facts, *steel is more suitable for the construction of permanent magnets than soft iron.* The fact that the retentivity of iron is little greater than that of steel is outweighted by the much smaller value of its coercivity. Recently a number of alloys, having large values of coercivity have been developed for the purpose of construction of permanent magnets. The very suitable alloy of highest coercivity is named as *vicalloy* (vanadium, iron and cobalt).

(2) *Electromagnets.* The materials for the construction of electromagnets should have (*a*) high initial permeability, (*b*) low hysteresis loss and (*c*) maximum magnetic induction **B** with comparatively small value of magnetizing field. From the view point of these facts *the soft iron is an ideal material for this purpose.*

(3) *Cores of transformers and Chokes, Armatures of Dynamos and Motors; and for telephone Diaphragms.* As the magnetic material used in these cases is subjected to cyclic changes. Thus the essential requirements for the selection of the material are : (*a*) high initial permeability to obtain large flux density **B** for low values of magnetizing field **H**, (*b*) low hysteresis loss to prevent the breakdown of insulation of the windings as less dissipation of energy produces a small heating effect, and (*c*) high specific resistance to reduce eddy current losses.

Soft iron is better than steel for these purposes. Soft iron has initial permeability about 250. The permeability is greatly increased by alloying it with 4% silicon. It is very useful and is known as *transformer steel.* There are some other alloys of iron and nickel, known as *permalloys. Mumetal* (*Ni* 76%, *Cu* 5%, *Fe* 17.5%, *Cr* 1.5%) has an initial relative permeability between 10,000 and 100,000. *Radiometal* (*Ni* 48%, *Fe* 48.5%, *Cu* 3%, *Mn* 0.5%) has an initial relative permeability between 2,000 to 15,000.

11.16. FERRITES

According to the Heiserberg Theory of ferromagnetism, the change from parrallel to antiparallel spin alignment of neighbouring atoms is associated with the change in electrostatic energy. If this energy change favours parallel alignment and of sufficient magnitude, the material composed of these atoms is called a *ferromagnet.* If the

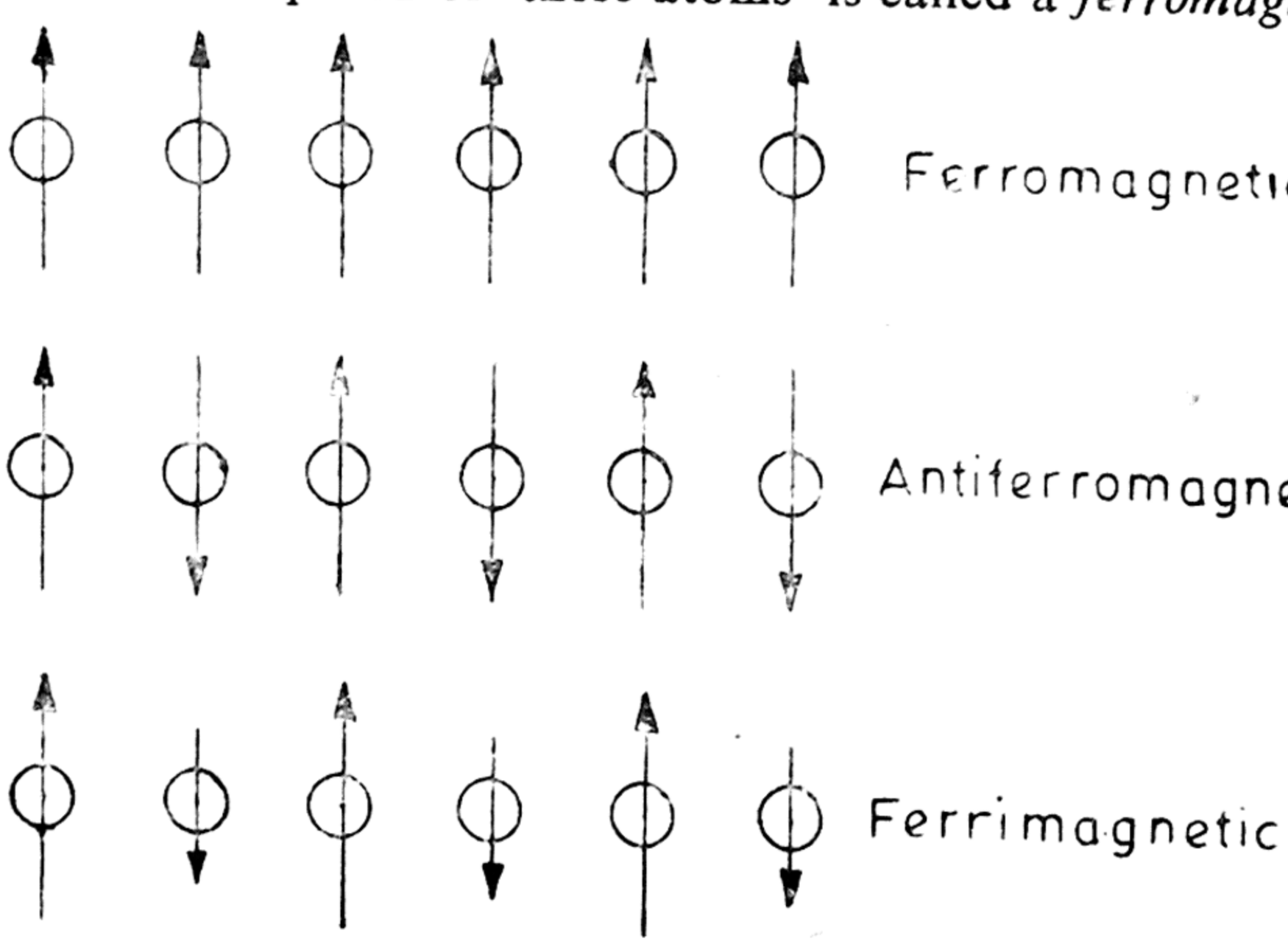
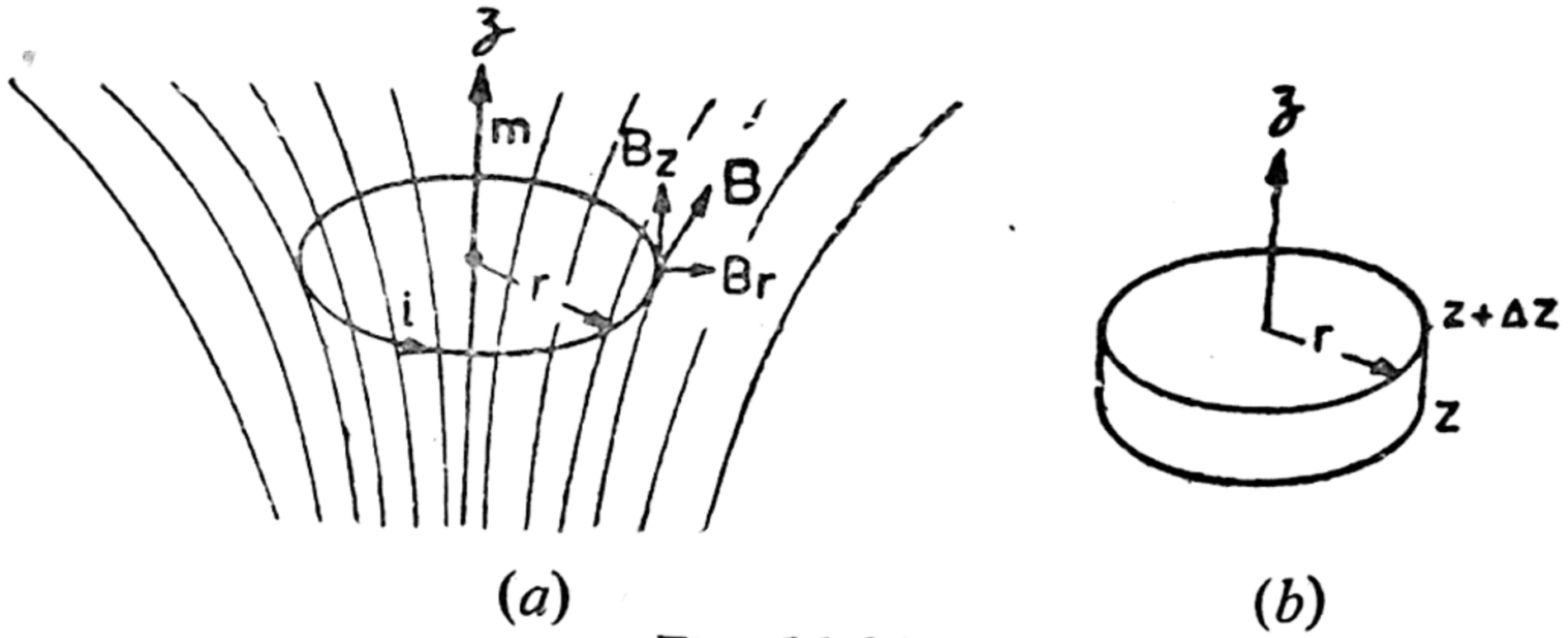


Fig. 11.30.

energy change favours antiparallel alignment and the spins are alternating from atom to atom (ordered spin structure) with zero net magnetic moment, the material is called an *antiferromagnet.* In most general ordered spin structure the net magnetic moment in one of these (up or down) directions is not zero, such a material is called a *ferrimag net* or a *ferrite.* The Schematic representation of atomic spins in ordered spin structures is shown in Fig. 11.30. The example of a ferrite is the mineral magnetite (Fe_3O_4). The ferrites are of considerable technical importance because : (*a*) They have relatively large saturation magnetization, (*b*) They are poor conductors of electricity, (*c*) They can be used for high frequency applications where eddy current losses are significant, and (*d*) The resistivities fall in the range from 1 to 10^4 ohm-meter.

11.17. MAGNETIC DIPOLE IN A NON-UNIFORM MAGNETIC FIELD

Consider a small circular current loop of radius *r* carrying a current *i* and placed in a non-uniform magnetic field **B.** Let us assume for simplicity that the field is symmetric about the z-axis and decreases as we proceed in the *z*-direction. In this problem, we want to find the force on the current loop due to the external field. Since the net force on the current loop due to its own field is zero, we therefore ignore its own field and consider **B** as the total field under consideration.



(*a*) (*b*)

Fig. 11.31.

At every point on the current loop [*Fig. 11.31.* (*a*)] the magnetic field *B* has two components, *Br* in the plane of loop directed radially outwards, and B_z in the *z* direction. On account of the symmetry of loop about the *z*-axis. The force due to the B_z component is zero on the current loop and the net force is due to the radial component *Br* only, which is given as

$$\mathbf{F} = -\mathbf{k}\, lBr\, \Sigma dl = -\mathbf{k}\, 2\pi r\, i\, Br. \qquad \ldots(94)$$

The negative sign shows that the direction of the force is along the negative *z*-direction.

The value of *Br* can be easily evaluated by using Gauss law. *i.e.*, div **B**=0 or the net amount of flux of magnetic field out of any

volume is every where zero. Consider a small cylinder of radius r and hight $\triangle z$. The outward flux from the curved side surface is $2\pi r \triangle z\, Br$ and the net outward flux from the end surfaces is $\pi r^2 [-B_z(z)+Bz(z+\triangle z)]$ Therefore

$$\oint_S B \,.\, dS = 2\pi r \triangle z B_r + \pi r^2[-B_z(z)+B_z(z+\triangle z)]=0.$$

or
$$B_r = -\tfrac{1}{2} r\,[-B_z(z)+B_z(z+\triangle z)]/\triangle z.$$

Since $B_z\,(z)$ and $B_z\ (z+\triangle z)$ are the magnetic fields at z and $z+\triangle z$ respectively. To the first approximation the bracket term may thus be written as $(\partial B_z/\partial z)\triangle z$

$$\therefore \quad Br = -\frac{r}{2}\ \frac{\partial B_z}{\partial z}\,. \qquad \ldots(95)$$

Since B_z decreases as z increases, Br is therefore positive. Substituting the value of Br in equation (94), we get

$$\mathbf{F} = -\mathbf{k}\ 2\pi\ r\ i\left(-\frac{r}{2}\ \frac{\partial B_z}{\partial z}\right) = \mathbf{k}\pi r^2 i\ \frac{\partial B_z}{\partial z}$$

$$= \mathbf{k}\ m\ \partial B_z/\partial z. \qquad \ldots(96)$$

Here $m=\pi r^2 i$ is the magnetic moment of the current loop.

In the above derivation, we have been taken the special case of a circular loop. Since the force is found to depend on the dipole moment 'm' of the current loop not on its shape and the magnetic dipole moment **m** is along the z-direction in the above case, hence the equation for the general case may be written as

$$\mathbf{F}=(\mathbf{m}\,.\,\nabla)\,\mathbf{B}. \qquad \ldots(97)$$

Thus we see that the force acts in the direction of increasing magnetic field, when the dipole moment is parallel to **B**, in the direction of decreasing magnetic field when the dipole moment is antiparallel to **B**. If the external field **B** is uniform this force is zero.

11.18. MEASUREMENT OF SUSCEPTIBILITY

If a sample of the given material is placed in an inhomogeneous magnetic field, it will tend to move to the region of small magnetic intensity if it is diamagnetic and to the region of higher magnetic intensity if it is paramagnetic. Thus a force **F** acts on the sample placed in a non uniform magnetic field. The work is done by this force which is equal to the change in potential energy caused by the displacement of the sample. The susceptibility of the given material is measured by measuring the force on the substance in an homogeneous magnetic field. If instead of a permanent dipole, we have a specimen of volume V and of susceptibility χ. The magnetic moment of the specimen will be

$$\mathbf{m}=\mathbf{M}V=\chi\mathbf{H}V.$$

$$\therefore \qquad \mathbf{F}=\mu_0(\mathbf{m}.\nabla)\mathbf{H}=\chi\mu_0 V(\mathbf{H}.\nabla)\mathbf{H}.$$

and $$\triangle F_z = \tfrac{1}{2}\chi\ \mu_0\ V\left(\frac{\partial}{\partial z} H^2\right). \qquad \ldots(98)$$

If the specimen has a susceptibility χ_1 and is immersed in a medium with susceptibility χ_2, the force

$$\triangle F_z = \tfrac{1}{2}\ (\chi_1 - \chi_2)\mu_0 V \frac{\partial}{\partial z}(H^2). \qquad \ldots(99)$$

If the specimen is made into a long cylinder of uniform cross section A, the vertical force in the z-direction on an element $A\triangle z$ is

$$\triangle F_z = \tfrac{1}{2}\ (\chi_1 - \chi_2)\ \mu_0\ A \frac{\partial}{\partial z}\ (H^2)\ \triangle z$$

$$\therefore \quad F_z = \tfrac{1}{2}\ (\chi_1 - \chi_2)\ \mu_0 A \int \frac{\partial H^2}{\partial z} dz$$

$$= \tfrac{1}{2}\ A\mu_0\ (\chi_1 - \chi_2)\ (H_1^2 - H_2^2), \qquad \ldots(100)$$

where H_1 is the field at the lower end of the specimen and H_2 that at the upper end. Above relation can be employed for determining the susceptibility of a dia-or paramagnetic material (solid or liquid) as given below :

(*a*) **Susceptibility of solids (Gouy'method).** This method was first employed by Gouy. In this method the specimen is taken in the form of a thin rod and suspended from one arm of a sensitive microbalance in the magnetic field between the wedge shaped pole pieces of an electromagnet (Fig. 11.32). The pole pieces are sufficiently

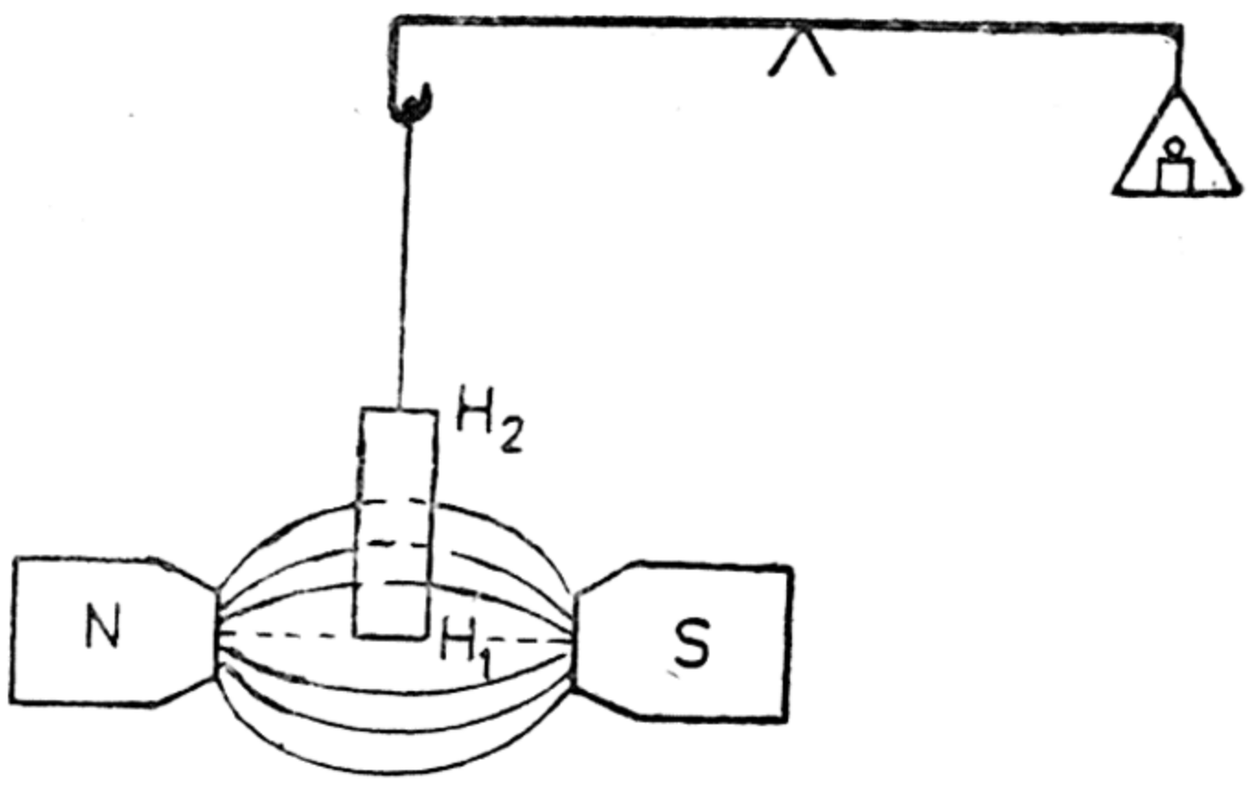


Fig. 11.32.

wide apart and the magnetic field is uniform horizontally in the central part of the field where the specimen is suspended. Since the vertical dimension of the pole pieces is small, the magnetic field will fall off rapidly in the vertical direction. Due to this change in magnetic field along the specimen rod, the specimen will experience a force along its length (*i.e.*, vertical direction), upward or downward depending on whether the specimen is dia- or paramagnetic respectively.

In this experiment, the specimen is first balanced by putting weights on the other arm of the balance. The magnetic field is switched on and the balance is again made by adjusting the weights. The weights m added or removed in the second case give us the force experienced by the specimen due to the magnetic field. As the magnetic field falls off rapidly in the z-direction, neglecting H_2 and comparing the forces we get

$$mg = \tfrac{1}{2}\,\mu_0 A\,(\chi_s - \chi_a) H_1^2. \qquad \ldots(101)$$

The magnetic field H_1 can be measured by the fluxmeter. Thus by putting the susceptibility of air χ_a and other known quantities in the above equation, the χ_s of the given specimen is calculated.

(*b*) **Susceptibility of liquids (Quincke method).** Quincke devised a method for determining susceptibilities of acqueous solutions, many liquids and liquified gases. This method is particularly useful in investigating the variation in the susceptibility of acqueous solution of iron salt with concentration. *The susceptibility, in such solutions is found to be proportional to the iron content of the solution.*

A carefully clean glass tube of about 2 cm in diameter is drawn out in an air gas flame so that part of it is in the form of a long narrow capillary of uniform bore. This thin part is bent to the form of U-tube. The U-tube is placed so that its plane is vertical and its narrow limb passes between the poles of an electromagenet. The tube is partially filled, so that the surface of the liquid in the narrow limb is near the centre of the field. When the field is switched on, the liquid meniscus rises or falls.

If the surface tension effects are negligible and Z is the vertical distance between the levels of the menisci when the field is applied, then

Change in hydrostatic pressure = Pressure due to magnetic field.

In our present problem, field H_2 at the end B is zero and H_1 at the end A is the field applied H, $\chi_1 = \chi_s$ the susceptibility of solution and $\chi_2 = \chi_a$ that of air. Hence we have

$$g(\rho - \sigma)Z = \tfrac{1}{2}\,\mu_0(\chi_s - \chi_a)H^2. \qquad \ldots(102)$$

Here ρ is the density of the solution and σ that of air.

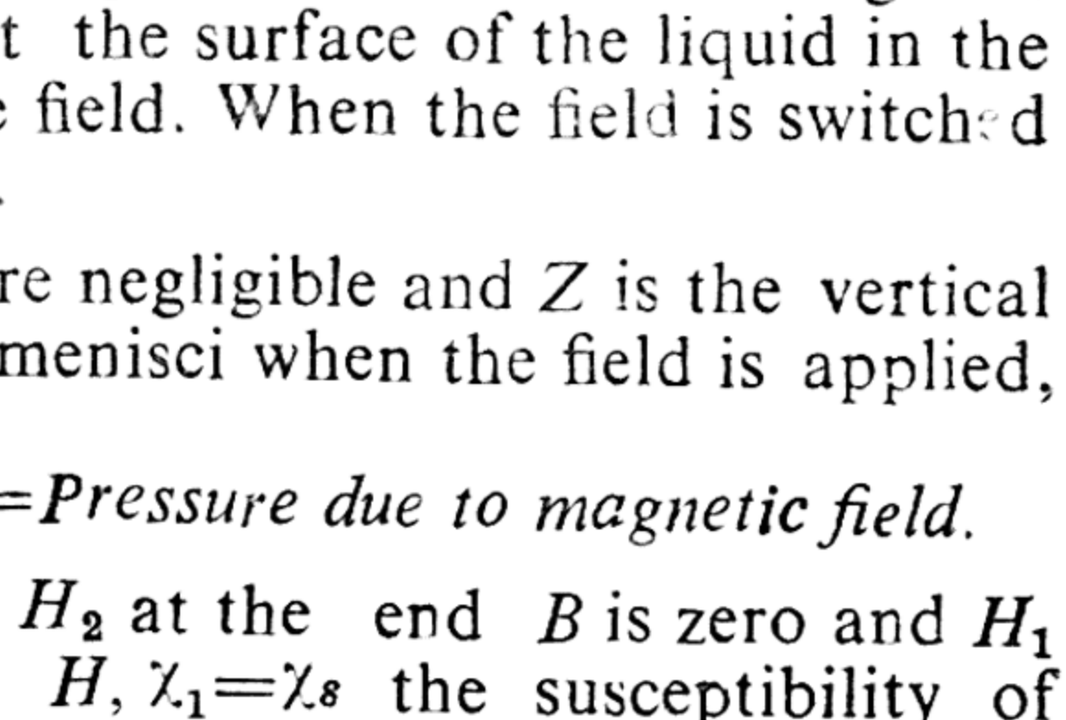


Fig. 11.33.

Let α be the cross sectional area of the narrow limb and β that of the wide limb, z the observed rise of the meniscus in the narrow tube and z_0 the fall in the wide tube. As total volume remains constant, hence

$$\beta z_0 = \alpha z.$$

$$\therefore \qquad Z = z + z_0 = z\,(1 + \alpha/\beta).$$

Hence $\frac{1}{2}\mu_0 (\chi_s - \chi_a) H^2 = g (\rho - \sigma) z (1 + \alpha/\beta)$. ...(103)

With the help of this equation susceptibility of any solution can be calculated. Unknown field in the narrow gap can be calculated as a reverse process, if the susceptibility is known.

(c) Curie Balance Method. This method was first used by Curie in 1895 for the measurement of susceptibility of weakly magnetic substances. The specimen is placed in a glass tube which is suspended from a light arm carrying a balance weight and a copper vane. The *Cu*-vane moves in the gap of a permanent magnet and serves to damp the oscillations of the arm by inducing eddy currents in the vane. The light arm is suspended by a silk or quartz fibre. The moving system also carries a mirror to measure deflection accurately. A permanent magnet of ring shape is mounted on the one end of an arm, which may be rotated so that the air gap of the magnet may be moved toward or away from the tube containing the specimen.

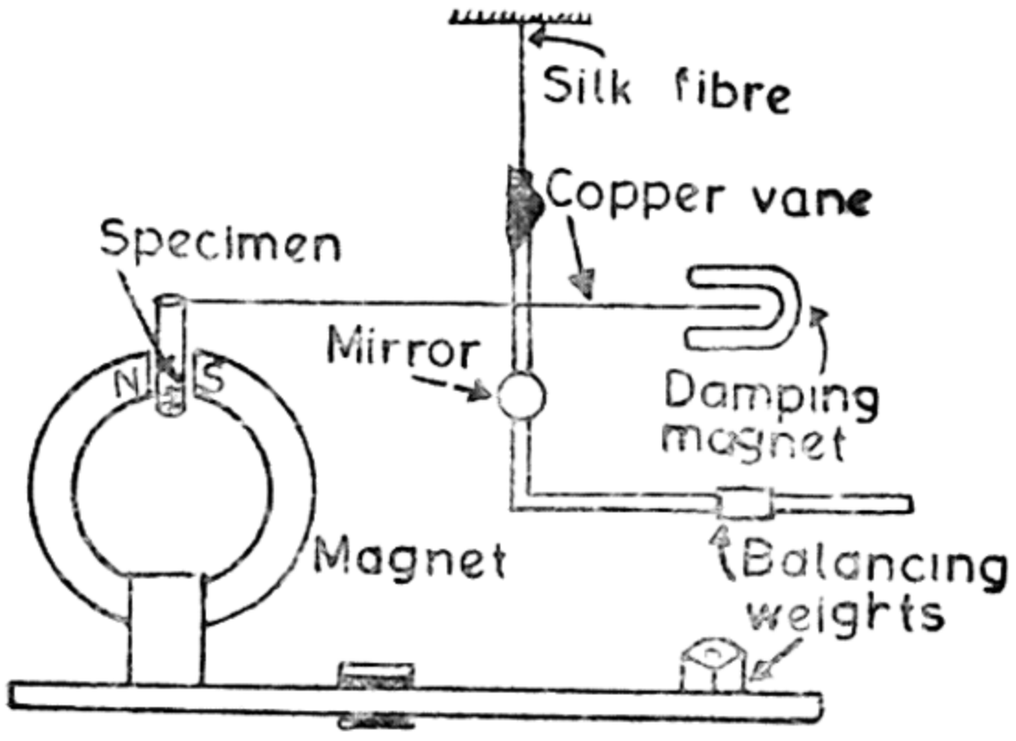


Fig. 11.34.

This method depends on measuring the force on the substance in an inhomogeneous magnetizable material of susceptibility χ and volume V is given by Eq. (98). The pole tips of an electromagnet is adjusted so that $\partial H_x^2/\partial z$ is large while $\partial H_y^2/\partial z$ and $\partial H_z^2/\partial z$ are very small, special shapes of pole tips are designed to make $\partial H_x^2/\partial z$ uniform over a large volume so that different specimens may be placed in the same position. If the specimen of susceptibility χ_1 is immersed in a medium with susceptibility χ_2, then the force

$$Fz = \tfrac{1}{2} (\chi_1 - \chi_2) \mu_0 V (\partial H^2/\partial z).$$

This is measured in terms of the rotation of the torsion head, which is proportional to the torque due to this force.

First of all, by using the glass tube alone the magnet is rotated until maximum deflection is obtained. The specimen is then placed in the tube and the maximum deflection θ_1 of the moving system is observed. The specimen is then replaced by some material whose susceptibility is known (distilled water is generally used for this purpose) and a deflection θ_2 is observed. If θ_0 was the deflection with the glass tube only, then

$$\frac{\text{Susceptibility of specimen 1}}{\text{Susceptibility of specimen 2}} = \frac{\chi_1}{\chi_2} = \frac{m_2 (\theta_1 - \theta_0)}{m_1 (\theta_2 - \theta_0)}, \quad \text{...(104)}$$

where m_1 and m_2 are the masses of the specimen and the standard material respectively.

11.19. THE STERN GERLACH EXPERIMENT

Stern and Gerlach in 1922 performed an atomic beam experiment and demonstrated that the directions of vector magnetic moment of an atom placed in an external magnetic field were quantized with respect to the field applied.

The Stern-Gerlach atomic beam experiment is illustrated in Fig. 11.35. A strap-shaped beam of silver atoms was formed by evaporating silver in a heated oven O and allowing atoms from the vapour to stream out through collimating slits S_1 and S_2. This beam was allowed to pass along and very close to the knife edge pole piece P_1 of the magnet and condensed on the glass plate P. This specially

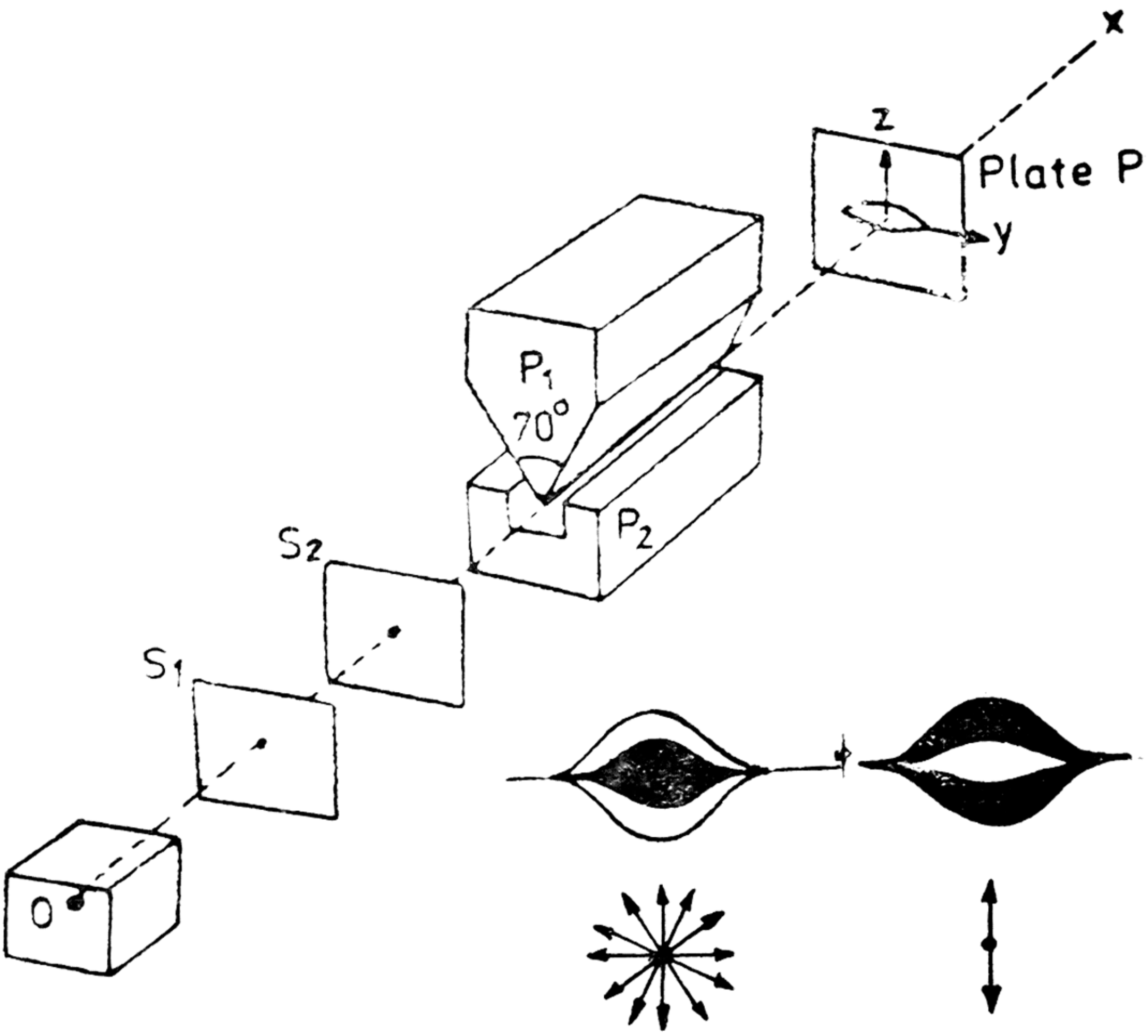


Fig. 11.35. The Stern-Gerlach Experiment.

Fig. 11.36. Pattern of Silver deposite.

shaped magnet had produced the non-homogeneous magnetic field, the field was much stronger near the edge than elsewhere. Thus the $\partial B_z/\partial z$ was large near the knife edge The silver atoms were not deflected by the Lorentz force, as they were electrically neutral.

The silver atom has a single electron outside the closed shells. The orbital angular momentum of an electron thus produces a magnetic dipole moment. On account of it, the silver atoms may be

considered as small magnets. Suppose that the beam of atoms having magnetic dipole of strength **m** travels in the direction of x-axis across a nonuniform magnetic field parallel to the z-axis. Then atoms which have a component of their magnetic moment in the direction of the field are deflected by a force $Fz = m_z \, \partial B_z/\partial z$. The amount of the deflection depends on the length of the atom's path and on its velocity. The measurement of the distribution of silver deposited on the glass plate will give the value of the m_z.

Classically, each atom will enter the field with its magnetic axis inclined at some angle ϕ to the field and executed a Larmor precession about the field. Since all the values of ϕ occur among the atoms, thus a continuous distribution of m_z values will be expected and atoms will be drawn out into a continuous band [Fig. 11.36 (*a*)]. The observed silver deposite is shown in Fig. 11.36 (*b*). It reveals only two components of m_z, one parallel to B and one antiparallel, thus gives direct proof that the directions of **m** with respect to the magnetic field are quantised. This may be explained quantum mechanically as : The silver atom is normally in a $2S_{1/2}$ state ($g=2$), thus the half the atoms have $M_J = \frac{1}{2}$ and a magnetic moment in the direction of the field of $gM_J = 1$ Bohr magneton, and the other half have $M_J = -\frac{1}{2}$ and magnetic moment —1 Bohr magneton. The silver beam is thus broken up into separate beams and forms on the target a series of distinct spots one for each possible value of M_J.

11.20. MAGNETOSTRICTION

The *magnetostriction* is the phenomenon which refers to the changes in the dimensions of a ferromagnetic material due to the magnetising field. The longitudinal and transverse effects when the specimen is magnetised in a specific direction were first observed by Joule, hence are called *Joule effects*. If the magnetic specimen exhibits a bending effect due to the magnetising field, such an effect is called *Guillemin effect*. If the specimen material is twisted when subjected to circular and longitudinal magnetising field, such an effect is called *Wiedemann effect*. There are changes in volume due to the magnetising field and thus the modulii of elasticity of a ferromagnetic material.

The reverse effect, known as *Villari effect*, is also seen as a change in magnetic induction due to (*i*) changes in the longitudinal and transverse mechanical stresses and (*ii*) the bending and twisting of ferromagnetic material when placed in a magnetic field.

The Joule effect may be demonstrated easily by placing an iron rod inside a solenoid. On passing a current through the solenoid, it is found that the rod lengthens first, then decreases and becomes shorter than in the unmagnetised condition. The change in length *dl* of the rod of length *l* is observed by pressing one end of the rod against a rigid support and by putting the

other end on a bent glass tube to the vertical limb of which is attached a mirror. As the length of the rod increases the limb of the tube rolls along and rotates the mirror. The rotation in the mirror is measured with the help of lamp and scale arrangement. The variation of dl/l with the magnetising field B is shown in Fig. 1..37 for *Fe*, *Co* and *Ni*.

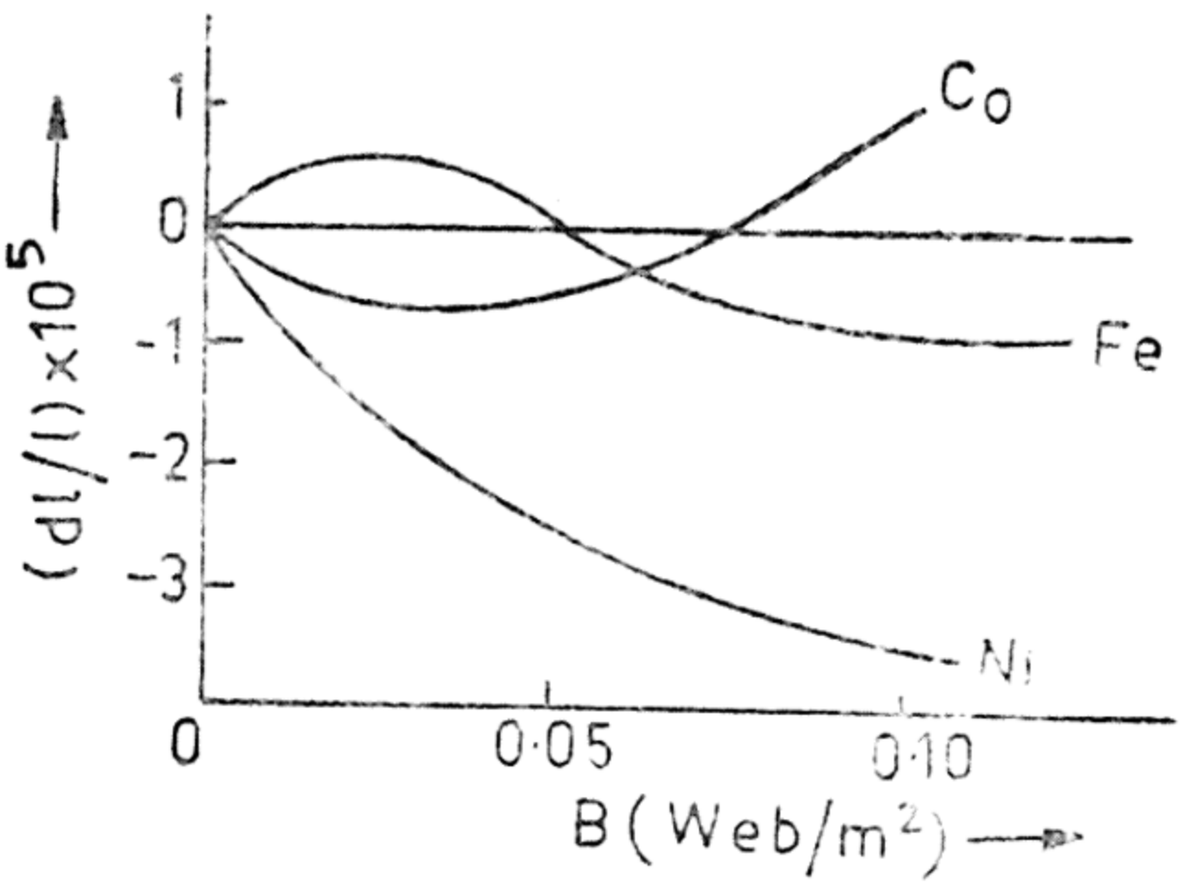


Fig. 11.37.

If a small rod of strongly magnetostriction alloy (54% *Fe*, 36% *Ni* and 10% *Cr*) is placed in a solenoid and an alternating current is allowed to pass For a particular length of the rod, it is possible to produce resonance. The experimental arrangement of this type is called *magnetostriction oscillator* and is used in echo-sounding devices.

11.21. MAGNETIC CIRCUIT

A magnetic circuit is the closed path taken by the magnetic flux set up in an electrical machine or apparatus by a magnetizing force which is a current coil or a permanent magnet. It consists chiefly of some ferromagnetic material. The magnetic circuits may be classified as : (*i*) undivided and (*ii*) divided. The type of circuit to be prepared depends mainly on the purpose and design of the electrical apparatus.

The undivided magnetic circuit carries one and the same flux across its all parts. Its simplest example is the toroid or Rowland ring in which the magnetic flux is confined to the region inside the toroidal winding.

The magnetic flux is generally different in different limbs of the divided magnetic circuit. Divided circuits may be classified as symmetrical and unsymmetrical. The former is made up of identical undivided components carrying one and the same flux in all of their parts. In the latter case these conditions are not met. The magnetic circuits of various electromagnets are the examples of divided circuits.

Let us now discuss the two simple cases :

(*a*) **Rowland Ring.** Consider a Rowland ring of ferromagnetic material magnetised by a current i in a closely fitting toroidal winding of total turns N. We know from Ampere's circuital law, that

$$\oint \mathbf{H}.d\mathbf{l} = Ni.$$

As the line integral of electric field **E** over a circuital path is the electromotive force (emf), hence by analogy, the line integral of **H** is termed as *magnetomotive force* (mmf).

$$\therefore \qquad \text{mmf} = \oint \mathbf{H}.d\mathbf{l} = Ni.$$

If A is the cross-sectional area of the ring specimen, then the magnetic flux

$$\Phi = BA = \mu HA \quad \text{or} \quad H = \Phi/\mu A.$$

$$\therefore \qquad \text{mmf} = \oint H dl = \oint \Phi dl/\mu A = \Phi \oint dl/\mu A = Ni. \qquad \ldots(104)$$

Φ is taken outside the integral as we consider it to be essentially constant at all points in the circuit. The Eq. (104) shows that it is analogous to the equation for a series current circuit (*i.e.*, ***Ohm's law***).

$$emf = Current \times Resistance = i \oint \rho \, (dl/A). \qquad \ldots(105)$$

The Eqs (104) and (105) suggest that

(*a*) The magnetomotive force ($\oint \mathbf{H}.d\mathbf{l}$) is analogous with emf ($\oint \mathbf{E}.d\mathbf{l}$).

(*b*) The magnetic flux Φ is analogous with current i in Ohm's law.

(*c*) The magnetic resistance, known as *magnetic reluctance* ($\oint dl/\mu A$) is analogous with electric resistance ($\oint \rho dl/A$), provided $1/\mu$ corresponds with resistivity ρ.

$$\therefore \qquad \text{mmf} = \text{Flux} \times \text{Reluctance},$$

$$\text{or} \qquad \text{Total flux } \Phi = \frac{\text{mmf}}{\text{Reluctance}} = \frac{Ni}{\oint dl/\mu A}. \qquad \ldots(106)$$

However, this analogy is only in the form of the expressions, as there are following points which show dissimilarity between magnetic and electrical circuits :

(*a*) In an electric circuit the electric current is a flow of electrons (or ions), but in a magnetic circuit there is no flow of such particles.

(*b*) In an electric circuit, energy is expended as long as the current flows, but in a magnetic circuit energy is expended in establishing the magnetic flux and not in maintaining it.

(*c*) At a given temperature, the resistivity ρ is independent of the current strength, while the corresponding quantity l/μ in reluctance varies with magnetic flux Φ.

If the magnetic circuit is made up of a number of components, *i.e.*, various types of ferromagnetic materials and possibly air gaps, then the total reluctance,

$$\oint \frac{dl}{\mu A} = \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3} + \ldots = \sum \frac{l}{\mu A} \quad \ldots(107)$$

This indicates that the reluctance of a magnetic circuit is the sum of the reluctances of its parts, similar to the resistance of an electric circuit which is the sum of its component resistances.

$$\therefore \qquad mmf = \Phi \times \Sigma\,(l/\mu A). \qquad \ldots(108)$$

This formula for magnetic circuit is used by engineers for calculating the number of ampere turns in the winding required to produce a desired flux density in electromagnets, motors or in dynamos.

(*b*) **Electromagnet.** Let us consider an electromagnet, the vertical portions of the iron core are wound with coils of wire carrying a current *i*. Most of the lines of the **B** field are confined to the core and the air gap. The **H** field due to the current *i* in the coils (H_C) is everywhere clockwise in the core, but not have same magnitude at all points. The poles set up a demagnetizing

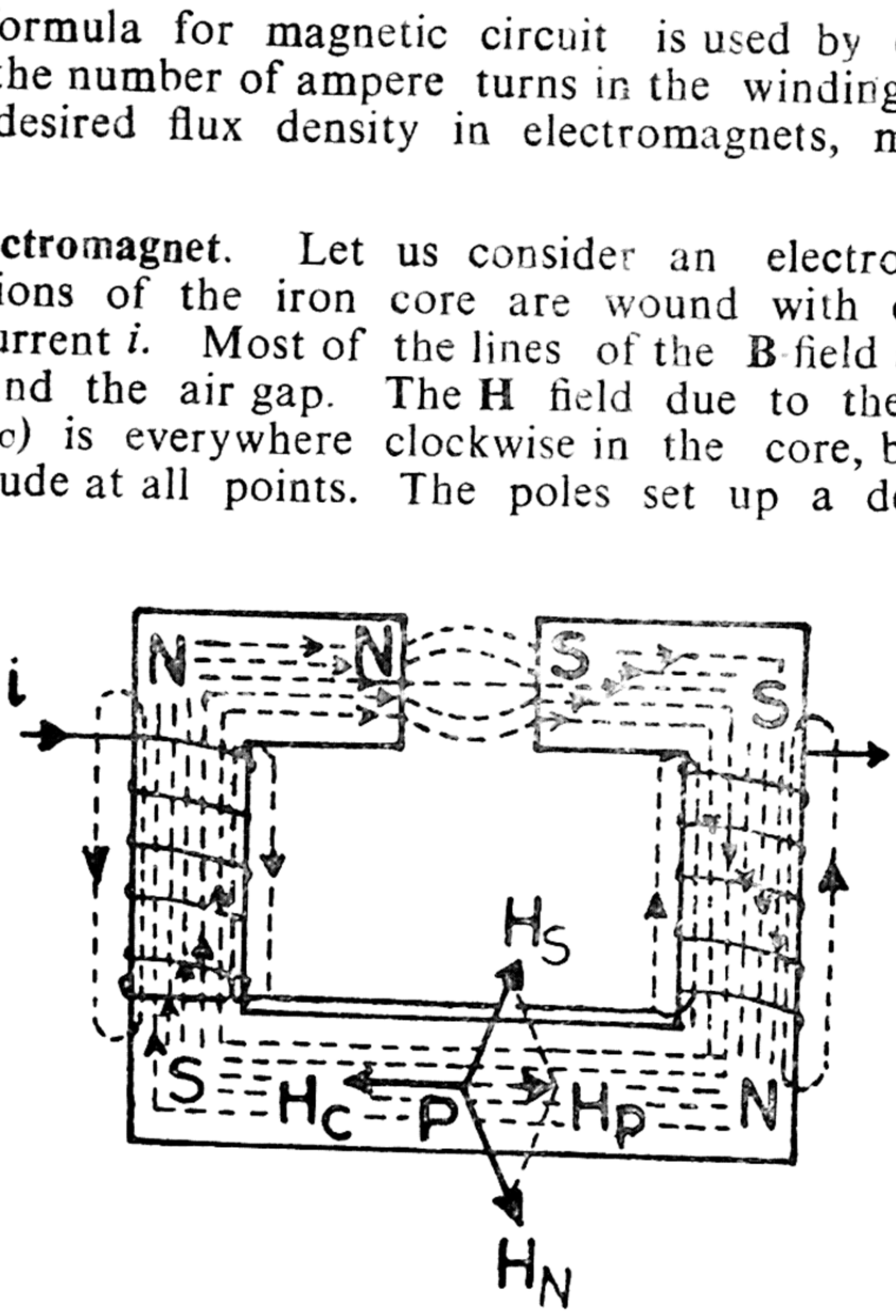


Fig. 11.38.

field H_P which at all points in the core is opposite to H_C. H_P at any point is the vector sum of the fields H_N and H_S due to north and south poles at this point.

An exact analytic solution of the problem is very tedious. For this let us neglect any leakage of flux. If the flux Φ is the same over all cross sections of the air gap. If the core is of same cross section A_1 at all points and the air gap is of larger cross-section A_2, then flux densities in the core and the gap are

$$B_1 = \Phi/A_1 \text{ and } B_2 = \Phi/A_2. \qquad \ldots(109)$$

The corresponding values of H are

$$H_1=B_1/\mu_1=\Phi/\mu_1 A_1 \text{ and } H_2=B_2/\mu_2=\Phi/\mu_2 A_2. \qquad ...(110)$$

If l_1 and l_2 are the path lengths in the core gap and the air gap respectively, then

$$\oint \mathbf{H}.d\mathbf{l}=H_1 l_1+H_2 l_2=\Phi[l_1/\mu_1 A_1+l_2/\mu_2 A_2].$$

As **H** is the vector sum of $\mathbf{H}_C$ and $\mathbf{H}_P$, hence

$$\oint \mathbf{H}.d\mathbf{l}=\oint \mathbf{H}_C.d\mathbf{l}+\oint \mathbf{H}_P.d\mathbf{l}=Ni+0,$$

where N is the total number of turns in the winding.

$$\therefore \qquad \Phi=\frac{Ni}{(l_1/\mu_1 A_1)+(l_2/\mu_2 A_2)}. \qquad ...(111)$$

Thus the magnetic flux in the air gap is equal to the mmf Ni divided by the sum of the reluctances of the iron yoke and of the air gap. This is a general relation. It shows that the reluctances in a magnetic circuit add in the same way as the conductances in an electric circuit.

11.22 MAGNETIC FIELD ENERGY

The energy is required to establish the magnetic field. If a source of supply is applied to a circuit, the current I through the circuit can be expressed as

$$\mathcal{E}_0+\mathcal{E}=IR,$$

where $\mathcal{E}_0$ is the emf of the external source, $\mathcal{E}$ the induced emf, and R the resistance of the circuit.

The work done by the source in moving the charge increment dq $(=Idt)$ through the circuit is

$$\mathcal{E}_0 dq=\mathcal{E}_0 Idt=-\mathcal{E}Idt+I^2Rdt$$

$$=Id\Phi+I^2Rdt. \qquad ...(112)$$

The last term $I^2R\,dt$ represents the energy dissipated as heat in the circuit resistance. The term $I\,d\Phi$ represents the work done against the induced emf in the circuit. This work done is positive when the flux change $d\Phi$ th ough the circuit is in the same direction as the flux produced by the current I.

When the circuit is connected to the energy source, the current rises gradually from zero to its maximum. In this process the gradual change in the flux linkage occurs. If i is the current at any instant and $d\Phi$ the increment in flux, the increase in magnetic field energy

$$dW_M=id\Phi.$$

Thus the magnetic field energy corresponding to the current I and flux Φ may be obtained by the area between the I, Φ curve and the axis of p.

$$\therefore \qquad W_M=\tfrac{1}{2}I\Phi. \qquad ...(113)$$

In the case of an electric circuit of constant inductance

$$W_M = \tfrac{1}{2}\Phi I = \tfrac{1}{2}LI^2 = \Phi^2/2L. \qquad \ldots(114)$$

In the case of a system comprising two magnetically coupled circuits

$$W_M = W_{11M} + W_{22M} \pm W_{21M}$$

$$= \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 \pm MI_1 I_2, \qquad \ldots(115)$$

where L_1 and L_2 are the self inductances of the first and sccond circuits and M the mutual inductance.

Similar to the electrostatic energy. The energy density in a magnetic field

$$W = \tfrac{1}{2}\mathbf{H}.\mathbf{B} = \tfrac{1}{2}\mu\mathbf{H}.\mathbf{H} = \tfrac{1}{2}\mu H^2 = B^2/2\mu. \qquad \ldots(116)$$

Exercises

Example 1. *Calculate the force acting on a dipole placed on the axis of the current carrying loop of wire.*

Consider a small magnet (dipole) of magnetic moment $\boldsymbol{m}$ placed at point P, at a distance x along the axis of a loop of wire of radius a, carrying a current i. The magnetic field at P due to the current is

$$B = \mu_0 i a^2/(a^2+x^2)^{3/2}.$$

It is along the axis of the coil. The magnet will therefore experience a couple. The torque due to this couple is

$$\tau = mB \sin\theta = \frac{\mu_0 i a^2 m}{2(a^2+x^2)^{3/2}} \sin\theta.$$

This couple will tend to reduce θ. If the centre of the magnet is fixed at P and the magnet is free to rotate about P, then the equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\frac{\mu_0 i a^2 m \sin\theta}{2(a^2+x^2)^{3/2}}$$

$$\text{or} \quad \frac{d^2\theta}{dt^2} + \frac{\mu_0 i a^2 m}{2I(a^2+x^2)^{3/2}} \sin\theta = 0,$$

where I is the moment of inertia of the dipole. If θ is small, then $\sin\theta = \theta$ and the above equation may be written as

$$\frac{d^2\theta}{dt^2} + n^2\theta = 0, \qquad \text{where} \quad n^2 = \frac{\mu_0 i a^2 m}{2I(a^2+x^2)^{3/2}}.$$

It is the equation of S.H.M. If i is known, the measurement of time period gives us m.

If **m** lies along the axis *OP*. In this case no couple acts on the magnet, there is a force F only which is along the axis and is of value.

$$F=m\frac{\partial B}{\partial x}=-\frac{3\mu_0 ia^2mx}{(a^2+x^2)^{5/2}}.$$

Example 2. *Two small magnets are kept at a distance r from each other, their magnetic moments* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are inclined at angles* θ_1 *and* θ_2 *with the line joining them. If r is greater than the size of each magnet, calculate : (a) the torque acting on each of them, (b) the forces experienced by each, and (c) the mutual potential energy of two magnets.*

We know that the potential energy of a magnetic dipole of magnetic moment **M** and placed in a magnetic field **B** is

$$U=-\mathbf{M}.\mathbf{B}=-MB\cos\theta.$$

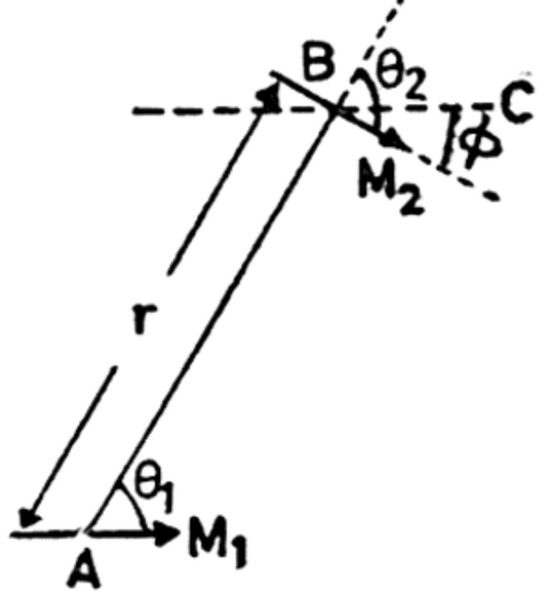


Fig. 11.40.

Magnetic field **B** at point B due to magnetic dipole at A is having two components $2\mu_0 M_1 \cos\theta_1/4\pi r^3$ along AB or **r** and $\mu_0 M_1 \sin\theta_1/4\pi r^3$ along normal to **r**. If this field **B** is assumed as the resultant of B_1 along BC and B_2 along BD, then

$$B_1\cos\theta_1+B_2=\frac{2\mu_0 M_1\cos\theta_1}{4\pi r^3}$$

and
$$B_1\sin\theta_1=-\frac{\mu_0 M_1\sin\theta_1}{4\pi r^3}.$$

$$\therefore \qquad B_1=-\frac{\mu_0 M_1}{4\pi r^3} \quad \text{and} \quad B_2=\frac{3\mu_0 M_1\cos\theta_1}{4\pi r^3}.$$

Hence in vector form, magnetic field is

$$\mathbf{B}=-\frac{\mu_0\,\mathbf{M}_1}{4\pi r^3}+\frac{3\mu_0(\mathbf{M}_1\cdot\mathbf{r})\,\mathbf{r}}{4\pi r^5}.$$

Thus the **potential energy** of the magnet M_2 in the field **B** due to magnet M_1 is given by

$$U=-\mathbf{M}_2\cdot\mathbf{B}=\frac{\mu_0\mathbf{M}_2\cdot\mathbf{M}_1}{4\pi r^3}-\frac{3\mu_0(\mathbf{M}_1\cdot\mathbf{r})(\mathbf{M}_2\cdot\mathbf{r})}{4\pi r^5}.$$

When the magnets are coplaner, this becomes

$$U=\mu_0 M_1 M_2\,[\cos\phi-3\cos\theta_1\cos\theta_2]/4\pi r^3.$$

Torque exerted by dipole M_1 on M_2 is

$$\tau_{12}=\mathbf{M}_2\times\mathbf{B}=\mathbf{M}_2\times\left[-\frac{\mu_0\,\mathbf{M}_1}{4\pi r^3}+\frac{3\mu_0(\mathbf{M}_1\cdot\mathbf{r})\,\mathbf{r}}{4\pi r^5}\right].$$

or $$\tau_{12}=-\frac{\mu_0\mathbf{M}_2\times\mathbf{M}_1}{4\pi r^3}+\frac{3\mu_0\,(\mathbf{M}_1\,.\,\mathbf{r})\,(\mathbf{M}_2\times\mathbf{r})}{4\pi r^5}.$$

In the special case of coplaner magnets, we get

$$\tau_{12}=-\frac{\mu_0 M_1 M_2\sin(\theta_2-\theta_1)}{4\pi r^3}+\frac{3\mu_0 M_1 M_2\cos\theta_1\sin\theta_2}{4\pi r^3}$$

$$=\frac{\mu_0 M_1 M_2}{4\pi r^3}[-\sin\theta_2\cos\theta_1+\cos\theta_2\sin\theta_1+3\cos\theta_1\sin\theta_2]$$

$$=\mu_0 M_1 M_2[2\cos\theta_1\sin\theta_2+\cos\theta_2\sin\theta_1]/4\pi r^3.$$

It must be $\perp$ to the plane containing $\mathbf{M}_2$ and $\mathbf{B}$.

The torque acting on M_1 by M_2 is not same as that on M_2 by M_1. The expression can be similarly obtained and is given as

$$\tau_{21}=\mu_0 M_1 M_2[2\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_1]/4\pi r^3.$$

Let us compute the force exerted by M_1 on M_2 tending to move the latter without rotation. Suppose $\mathbf{M}_2$ undergoes a displacement $\delta\mathbf{r}$. The force is given as

$$\mathbf{F}_{12}\,.\,\delta\mathbf{r}=-\delta U=-\delta[\mu_0\mathbf{M}_1.\mathbf{M}_2/4\pi r^3-3\mu_0(\mathbf{M}_1.\mathbf{r})\,(\mathbf{M}_2.\mathbf{r})/4\pi r^5]$$

$$=\frac{3\mu_0\mathbf{M}_1.\mathbf{M}_2}{4\pi r^4}\delta r+\frac{3\mu_0(\mathbf{M}_1.\delta\mathbf{r})(\mathbf{M}_2.\mathbf{r})+3\mu_0(\mathbf{M}_1.\mathbf{r})(\mathbf{M}_2.\delta\mathbf{r})}{4\pi r^5}$$

$$-\frac{15\mu_0(\mathbf{M}_1.\mathbf{r})(\mathbf{M}_2.\mathbf{r})\delta r}{4\pi r^6}$$

or $$\mathbf{F}_{12}=(\alpha\mathbf{r}+\beta\mathbf{M}_1+\gamma\mathbf{M}_2)/4\pi r^5,$$

where $$\alpha=3\mu_0[(\mathbf{M}_1.\mathbf{M}_2)-5(\mathbf{M}_1.\mathbf{r})\,(\mathbf{M}_2.\mathbf{r})r^{-2}],$$

$$\beta=3\mu_0(\mathbf{M}_2\,.\,\mathbf{r})\text{ and }\gamma=3\mu_0\,(\mathbf{M}_1\,.\,\mathbf{r}).$$

Example 3. *Two small magnets lie in the same plane and make angles θ_1 and θ_2 with the line joining their centres. Show that the line of action of the resultant force between them divides the line of centre, in the ratio*

$$(tan\,\theta_2+2\,tan\,\theta_1):(tan\,\theta_1+2\,tan\,\theta_2).$$

Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the magnetic moments of the small magnets, placed at points A and B. Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are coplaner, hence $\mathbf{F}_{12}$ and τ_{12} also lie in the plane of $\mathbf{M}_1$ and $\mathbf{M}_2$ and thus reduce to a single force. If the line of action of this force cuts AB at P, such that $\mathbf{AP}=\lambda\mathbf{r}$ and $\mathbf{BP}=-\mu\mathbf{r}$. Changing of the origin from A to P follows that the couple at P must vanish.

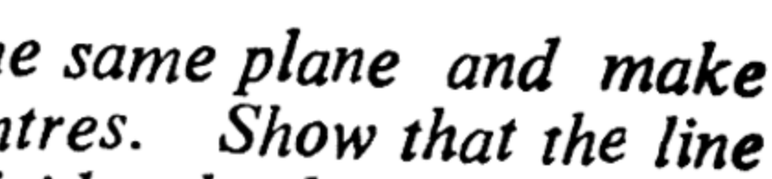


Fig. 11.41.

$$\therefore\quad\tau_{21}+\mu\mathbf{r}\times\mathbf{F}_{12}=0.$$

Similarly $\tau_{21}-\lambda\mathbf{r}\times\mathbf{F}_{21}=\tau_{21}+\lambda\mathbf{r}\times\mathbf{F}_{12}=0$

$\therefore \qquad \lambda\tau_{12}=\mu\tau_{21}.$

Using relations for torque from example 2, we get

$$\frac{\lambda}{\mu}=\frac{2\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_1}{2\sin\theta_2\cos\theta_1+\sin\theta_1\cos\theta_2}$$

$$=\frac{2\tan\theta_1+\tan\theta_2}{2\tan\theta_2+\tan\theta_1}.$$

Example 4. *A coil of 200 turns and 2 cm² area of cross-section carries a current of 10 amp. If it is kept in such a way that its magnetic moment* **M** *is parallel to the horizontal component of earth's magnetic field* **H**. *Calculate the radius of the horizontal circle around the coil at which the total horizontal field is everywhere radially directed. Given* $H=3\times10^{-5}$ *weber/meter².*

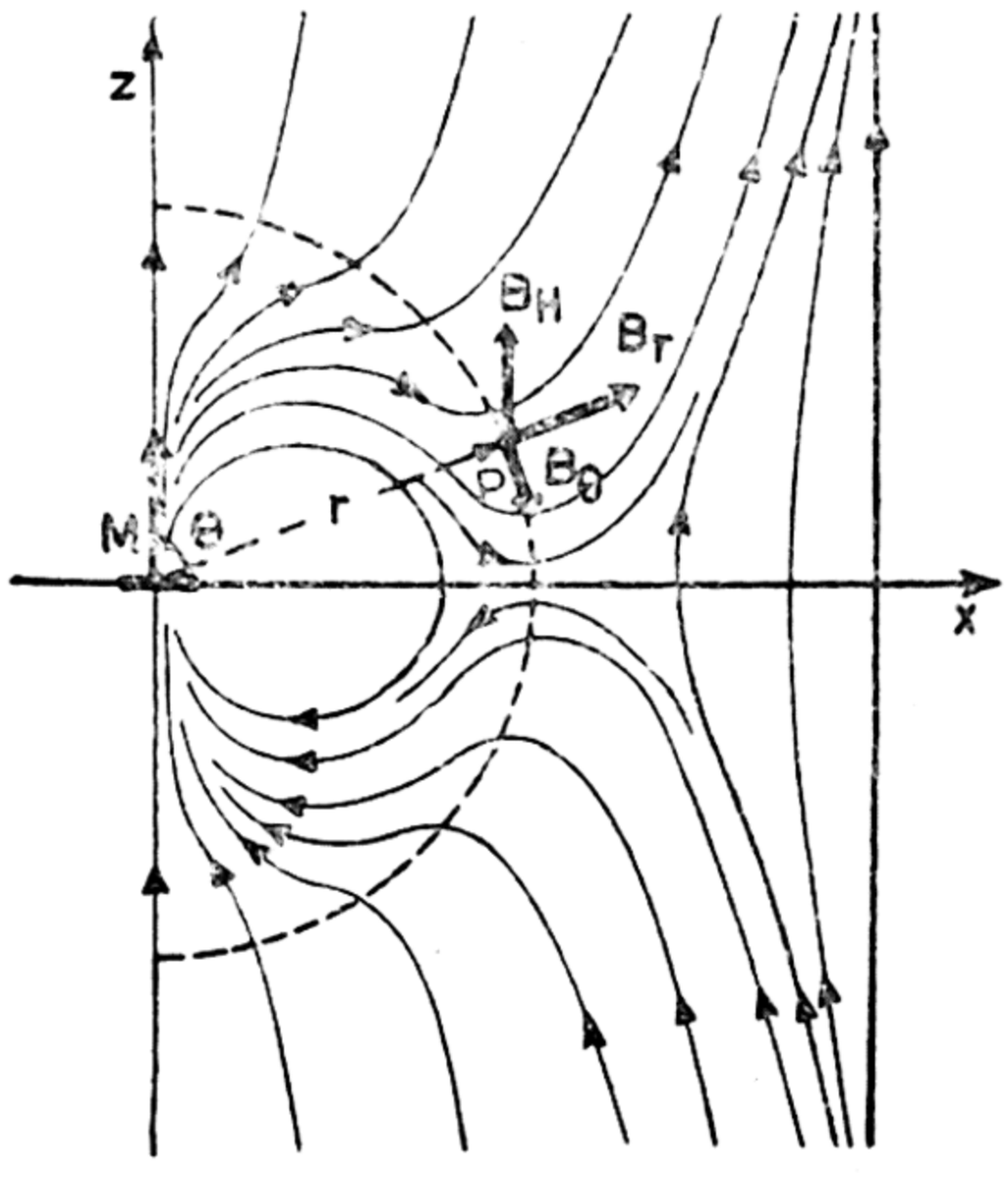


Fig. 11.42.

Let the x-z plane be the horizontal plane and let the horizontal component of earth magnetic field B_H is parallel to the +ve z-direction. Magnetic field at point P due to the magnetic moment **M** is

$$Br=2\mu_0 M\cos\theta/4\pi r^3$$

and $$B\theta=\mu_0 M\sin\theta/4\pi r^3.$$

If the magnetic field is radial at point P, then

$$B\theta=B_H\sin\theta$$

or $$\mu_0 M\sin\theta/4\pi r^3=B_H\sin\theta$$

$$\therefore \qquad r=R=[\mu_0 M/4\pi B_H]^{1/3}.$$

In our problem $M=iAn$

$$=10\times2\times10^{-4}\times200=0.4$$

$$\therefore \qquad r=[4\pi\times10^{-7}\times0.4/4\pi\times3\times10^{-5}]^{1/3}$$

$$=0.11 \text{ m}.$$

Example 5. *Prove that the radius of the orbital motion of an electron around its nucleus remains same, whether the magnetic field is applied or not in the direction* $\perp$ *to its plane of motion.*

We know that in the absence of magnetic field, the electrostatic force between the nucleus and electron is balanced by the centripetal force. Thus for an electron of mass m_e, radius r and linear velocity v, we have

Electrostatic force $F=mev^2/r=mer\omega^2$.

When a magnetic field is applied the angular velocity changes to $\omega+\omega_L$ (Here sign of ω_L will depend on the direction of B with respect to the original motion). In addition to it a new force e ($\mathbf{v}\times\mathbf{B}$) is applied to balance the new angular velocity. Thus in the presence of magnetic field

$$F=mer(\omega+\omega_L)^2+evB$$
$$=mer(\omega+\omega_L)^2+er(\omega+\omega_L)B$$
$$=mer\omega^2+mer\omega_L^2+2mer\omega\omega_L+er\omega B+er\omega_L B.$$

As $\omega_L=-eB/2me$, hence third and fourth terms cancel, leaving $F=mer\omega^2+$terms consisting ω_L and ω_L^2. As $\omega_L \ll \omega$, hence we have same value of electrostatic force for equilibrium, or the radius of the electron orbit remains same. The magnetic field only causes change in velocity.

Example 6. *A sphere of radius r carries a charge q, distributed uniformly over its surface. If it is rotating about an axis of sphere with angular velociry ω, calculate the magnetic moment of the sphere.*

The magnetic moment of a charge q of mass m and of angular momentum L is given by the Eq, $M=qL/2m$.

The angular momentum

$$L=mvr=mr^2\omega.$$

$$\therefore \quad M=qr^2\omega/2.$$

Let us consider the given sphere as consisting of narrow discs, The area of one disc$=2\pi r \sin\theta\, r d\theta$. As charge is uniformly distributed over the surface, hence surface charge density

$$\sigma=q/4\pi r^2.$$

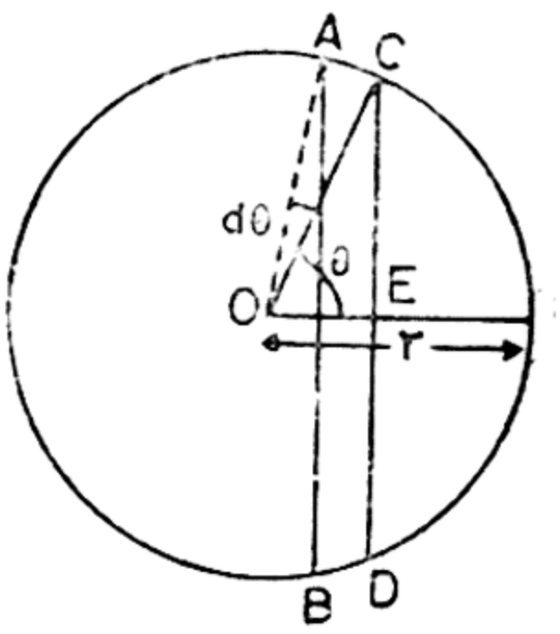


Fig. 11.43.

$\therefore$ Charge on the disc$=2\pi r^2 \sin\theta\, d\theta\sigma$

$$=\tfrac{1}{2}\, q \sin\theta\, d\theta.$$

The magnetic moment due to this disc

$$dM=(\tfrac{1}{2}\, q \sin\theta\, d\theta)\,(r\sin\theta)^2\,\omega/2$$
$$=\tfrac{1}{4}q\omega r^2 \sin^3\theta\, d\theta.$$

$\therefore$ Total magnetic moment of the whole sphere

$$=\int_0^{\pi} \tfrac{1}{4}q\omega r^2 \sin^3\theta\, d\theta$$

$$=\tfrac{1}{4}q\omega r^2.4/3=\tfrac{1}{3}q\omega r^2.$$

Example 7. *In the Rowland ring 2.0 amp current is passing through the winding of number of turns per unit length 10 turns/cm*

Magnetic induction measured is 1.0 weber/m². Calculate (a) H, (b) M. and (c) the magnetising current, both when the core is placed and when it is removed.

As H is independent of the core material and is given by

$$H=ni=(1000 \text{ turns/m})\times 2.0 \text{ amp}=2000 \text{ amp/m}.$$

M is zero where no core is present. In the presence of core

$$M=\frac{(B-\mu_0 H)}{\mu_0}=\frac{1.0-4\pi\times 10^{-7}\times 2\times 10^3}{4\pi\times 10^{-7}}$$

$$=7.94\times 10^5 \text{ amp/m}.$$

Effective magnetising current

$$im=\frac{H}{n}=\frac{7.94\times 10^5}{2000}=397 \text{ amp}.$$

The relative permeability

$$k=\frac{B}{\mu_0 H}=\frac{1}{4\pi\times 10^{-7}\times 2000}=397.7.$$

Example 9. *The magnetic field of 20 CGS units (emu or oersted) produces a flux of 2400 CGS units (emu or maxwell) in a bar of iron of cross section 0.2 cm². Calculate the permeability and susceptibility of the bar.*

$$B=\frac{\Phi}{A}=\frac{2400\times 10^{-8}}{0.2\times 10^{-4}}=1.20 \text{ weber/m}^2.$$

The permeability of the bar material

$$\mu=\frac{B}{H}=\frac{1.2}{20/4\pi\times 10^{-3}}=7.54\times 10^{-4} \text{ henry/meter}.$$

The magnetic susceptibility and the permeability of a material are related with each other as

$$\mu=\mu_0(1+\chi_m)$$

or
$$\chi_m=\mu/\mu_0-1=\frac{7.54\times 10^{-4}}{4\pi\times 10^{-7}}-1=600-1=599 \text{ mks units}.$$

Example 10. *An iron rod 20 cm long, 1 cm in diameter and of a permeability 1000 is placed inside a solenoid 1 meter long wound uniformly with 600 turns. If the current of 0.5 ampere is passed through the solenoid, find the magnetic moment of the rod.*

We know that B, H and M are related with each other as

$$B=\mu_0 (H+M).$$

Since $B=\mu H=\mu_r\mu_0 H$, where μ is the permeability of the material and μr its relative permeability, Hence

$$M=\frac{B-\mu_0 H}{\mu_0}=\frac{\mu_0\mu r H-\mu_0 H}{\mu_0}=(\mu r-1)\,H.$$

In a solenoid of n turns per unit length carrying a current i, the magnetic field intensity H is given by the relation

$$H=ni.$$

$$\therefore \quad M=(\mu r-1)ni.$$

The substitution of the numerical values gives

$$M=(1000-1)\times 600\times 0.5=299700 \text{ amp/m}.$$

As the intensity of magnetisation is defined as the magnetic moment per unit volume, therefore the

$$\text{Magnetic moment}=M\times\text{volume}=M\times\pi r^2 l$$

$$=299700\times 3.14\times(0.5\times 10^{-2})^2\times 0.20$$

$$=4.71 \text{ amp. } m^2.$$

Example 11. *Prove that the tangential component of* **H** *is discontinuous across a current sheet and its normal component is continuous.*

The surface of a closely wound solenoid constitutes a surface current sheet. The line integral of **H** around the contour or the closed loop is equal to the total current passing through the loop, passing through the surface $\triangle S$ (say). If the enclosed current has a volume density jv. then we will have

$$\oint c\mathbf{H}.d\mathbf{l}=\int \mathbf{j}_v d\mathbf{S}.$$

If the contour (or loop) is of rectangular shape of width very small h and length $\triangle l$. The ends contributions are negligible and the line integral is mainly due to long sides.

$$\therefore \quad (H_{1t}-H_{2t})\ \triangle l=jv\ \triangle S=jvh(\triangle S/h)$$

$$=j_v\ h\ \triangle l=j_s\triangle l.$$

or $$H_{1t}-H_{2t}=j_s.$$

Thus the tangential component of **H** is discontinuous across a current sheet by an amount equal to the surface current density. It is same as that of **D** at a polarization current sheet.

One can prove very well that the normal component of **H** is continuous across the sheet as the normal component of **B** is continuous and $\mathbf{H}=\mathbf{B}/\mu_0$.

Example 12. *An iron ring of mean circumference 60 cm and cross sectional area 8 sq cm is wound uniformly with a primary coil of 240 turns. A secondary coil containing 10 turns of insulated wire is*

wound over the primary and is connected to a ballistic galvanometer, the total resistance of the secondary circuit being 200 ohms. When a current of 2.0 amp. flowing in the primary is reversed, the galvanometer gives a throw of 60 scale divisions. When a condenser of capacity 2.0μF charged to potential of 25 volts is discharged through the galvanometer the throw is 40 scale divisions. Calculate (a) the magnetic field in the primary, (b) the magnetic inductance in the iron and (c) the permeability of iron.

The magnetic field intensity H in the primary is given by

$$H=n_1 i=(N_1/2\pi r)i$$

$$=(240/60\times 10^{-2})\times 2.0=800 \text{ amp. turn/meter.}$$

If μ is the permeability of iron, the magnetic induction in it is given by

$$B=\mu H=800\ \mu\ Wb/m^2.$$

When the current in the primary is reversed the charge passing in a secondary circuit of total number of turns N and having a resistance R is given by

$$q=\frac{2\ NBA}{R}=\frac{2\times 10\times 800\mu\times 8\times 10^{-4}}{200}=6.4\times 10^{-2}\ \mu \text{ coul.}$$

If K is the ballistic constant of the galvanometer, the deflection θ due to this charge is given by the relation

$$q=K\theta.$$

or $$6.4\times 10^{-2}\mu=K\times 60. \qquad ...(i)$$

When a condenser of capacity C is charged to a potential of V volts, the charge q is given by

$$q=CV=K\theta'$$

or $2.0\times 10^{-6}\times 25=K\times 40$ or $K=1.25\times 10^{-6}$ coul/scale division.

Substituting this value of K in equation (i), we get

$$6.4\times 10^{-2}\mu=1.25\times 10^{-6}\times 60$$

or $$\mu=11.72\times 10^{-4}.$$

$$\therefore \quad B=800\ \mu=800\times 11.72\times 10^{-4}=0.9375 \text{ wb./m}^2.$$

Example 13. *Calculate the hourly loss of energy in the iron core of a transformer, the hysteresis loop of which is equivalent in area to 2500 ergs/cc. Given frequency 50 cycles/sec ; density of ion 7.5 gm/cc and weight of the core 10 kg.*

The volume of the core $=M/\rho=10/7.5\times 10^{+3}=1.33\times 10^{-3}\ m^3.$

The energy loss in the iron core per cycle/cc

$$=2500 \text{ ergs/cc}=2500\times 10^{-1} \text{ joules/m}^2$$

The energy loss in the iron core per cycle
$$=2500\times10^{-1}\times1.33\times10^{-3}\text{ joule.}$$

Number of cycles in an hour
$$=60\times60\times50=180000.$$

∴ The hourly loss of energy in the iron core
$$=2500\times1.33\times10^{-4}\times180000$$
$$=5.985\times10^{4}\text{ joules.}$$

Example 14. *When 23.15 gm of $NiCl_2$ are dissolved in 100 gm water, the density of the solution is 1255 kg/m³. Show that the maximum height of a column of the solution which can be supported by magnetic force. when one surface of the column is in a uniform magnetic induction of 1 web/m² is 3.0 mm. Given that the susceptibilities of 1 kg of $NiCl_2$ and water are 0.438×10^{-6} and -0.009×10^{-6} respectively; volume susceptibility of air $=0.4\times10^{-6}$.*

(Since 23.15 gm. of $NiCl_2$ is dissolved in 100 gm. of H_2O, the mass susceptibility of the solution is given

$$(\chi_{sol})_{mass}=\chi_1\times\frac{m_1}{m_1+m_2}+\chi_2\frac{m_2}{m_1+m_2}$$

$$=0.438\times10^{-6}\times\frac{23.15}{123.15}+(-0.009\times10^{-6})\times\frac{100}{123.15}$$

$$=7.5028\times10^{-8}.$$

$$\therefore\quad(\chi_{sol})_{vol}=(\chi_{sol})_{mass}\times\rho=7.5028\times10^{-8}\times1255$$
$$=9.416\times10^{-5}.$$

The column of the solution is balanced by a magnetic force, if

$$g(\rho-\sigma)h=\tfrac{1}{2}\mu_0(\chi_{sol}-\chi_a)H^2=\tfrac{1}{2}(\chi_{sol}-\chi_a)\frac{B^2}{\mu_0}$$

or
$$h=(\chi_{sol}-\chi_a)B^2/2\ \mu_0\ g\ (\rho-\sigma).$$

Since the air density σ is negligible in comparison to ρ, the density of the solution, hence the substitution of numerical values gives

$$h=(9.416\times10^{-5}-0.4\times10^{-6})\times1^2/2\times(4\times3.14\times10^{-7})\times9.81\times1255$$
$$=3.03\times10^{-3}\text{ meter}=0.303\text{ cm.}$$

Example 15. *A toroidal winding of N turns surrounding a ferromagnetic specimen in which a narrow gap of width d has been cut. Calculate the value of magnetic field both in the gap and in the material.*

Let us consider a toroid of mean radius r and of circular cross-section with a radius $a\ (<<r)$. The tangential component of

H is continuous across the boundary separating the toroid material and the air region just out side. **B** is μ_0**H** and μ**H** outside and inside respectively. Using Ampere's circuital law for **H**, we have

$$\oint \mathbf{H}.d\mathbf{l}=2\pi r\, H=Ni \quad \text{or} \quad H=Ni/2\pi r.$$

If a small section of thickness d is removed from the toroid, the normal component of **B** must be continuous, hence **B** normal to the faces has the same value in the interior of the toroid. The field H is thus B/μ_0 in the gap and B/μ in the toroid. If we represent these fields as H_g and H_i respectively, then

$$\oint \mathbf{H}.d\mathbf{l}=H_i\,(2\pi r-d)+H_g d=Ni.$$

or $$\frac{B}{\mu}\left(\,2\pi r-d\,\right)+\frac{B}{\mu_0}d=Ni \quad \text{or} \quad B=\frac{Ni\mu\mu_0}{2\pi r\mu_0+(\mu-\mu_0)d}$$

or $$H_i=\frac{B}{\mu}=\frac{Ni\mu_0}{2\pi r\mu_0+(\mu-\mu_0 d)} \text{ and } H_g=\frac{B}{\mu_0}=\frac{Ni\mu}{2\pi r\mu_0+(\mu-\mu_0)d}$$

In our present problem the current $i=0$ and the **B** and **H** are due to permanent magnetization. Hence

$$H_i(2\pi r-d)-H_g d=0$$

or $$H_i=-H_g d/(2\pi r-d).$$

Thus the field inside the toroid must be oppositely directed to the field in the gap. As H and B are in the same direction as the **B** is continuous into the toroid, hence in the toroid B and H_i are oppositely directed.

Example 16. *A circular ring of mild steel having a mean diameter 20 cm and a cross sectional area of 2.5 sq cm is wound with 500 turns of wire Calculate (i) the reluctance of the magnetic circuit and (ii) the current necessary to produce a flux of 25000 lines. (μ for mild steel=800 in CGS units).*

The total reluctance of the magnetic circuit is given by

$$R=\sum\frac{l}{\mu A}=\frac{2\times 3.14\times 10\times 10^{-2}}{(800\times 4\times 3.14\times 10^{-7})(2.5\times 10^{-4})}$$

$$=25\times 10^5 \text{ amp·turns/web.}$$

The total magnetic flux linked with the magnetic circuit

$$\Phi=\frac{\text{m.m.f. }(Ni)}{\text{Reluctance}} \quad \text{or current } i=\frac{\Phi\times\text{Reluctance}}{N}$$

$$\therefore \quad i=(25000\times 10^{-8})\,(25\times 10^5)/500$$

$$=1.25 \text{ amp.}$$

Example 17. *Calculate the current required in the winding of the toroidal ring of mean length 50 cm and of cross sectional area*

4 cm² to produce a flux of 4×10^{-4} weber in the ring. Given : the no. of turns of wire is 200 and permeability of iron at this flux density is 65×10^{-4} wb/Amps. m. If an air gap of 0.1 cm length is cut in the ring, find the value of current required to maintain the same flux.

$$\text{Flux density } B=\frac{\Phi}{A}=\frac{4\times10^{-4}}{4\times10^{-4}}=1 \text{ wb/m}^2.$$

$$\text{Hence the reluctance } R=\frac{l}{\mu A}=\frac{0.5}{65\times10^{-4}\times4\times10^{-4}}$$

$$=1.92\times10^5 \text{ amp turns/weber}$$

$$\therefore \text{ Magnetomotive force (mmf.)}=\text{Flux}\times\text{Reluctance}$$

$$=4\times10^{-4}\times1.92\times10^5=76.9 \text{ amp turn}$$

If the ring is wound with 200 turns, the current required is

$$=76\ 9/200=0.385 \text{ amp.}$$

If the air gap 0.1 cm in length is cut in the ring, the reluctance of the air gap is

$$R=\frac{l}{\mu_0 A}=\frac{10^{-3}}{4\pi\times10^{-7}\times4\times10^{-4}}=1.99\times10^6 \text{ amp turns/weber.}$$

As 0.1 cm is negligible in comparison with the total length of the ring, hence the reluctance of the ring will be same.

$$\therefore \text{ Total reluctance of the entire circuit}=1.92\times10^5+1.99\times10^6$$

$$=21.82\times10^5 \text{ amp turns/weber}$$

Hence the number of amperes required is $4\times10^{-4}\times21.82\times10^5$ =872.8 amp turns and the corresponding current is 872.8/200 =4.36 amp.

Example 18. *Design a magnetic circuit that produces a magnetic field of 1.5 tesla in air gap 1 cm long and 48 cm² in area.*

Assuming a magnetic circuit, energized by a pair of permanent magnets. As the ends of the permanent magnet act as either sources or sinks of **H**, hence the flux of **H** over a closed surface is not necessarily equal to zero. Since there are neither free currents nor displacement currents, hence

$$\oint \mathbf{H}.d\mathbf{l}=H_iL_i+H_gL_g-H_mL_m=0,$$

where the subscripts i, g, m refers respectively to the soft iron, the air gap and the permanent magnets. L_i, L_g and Lm are the lengths in respective regions. As the reluctance of the iron is much smaller than of the air gap, hence H_iL_i is negligible and

$$HmLm\approx H_gL_g.$$

If the leakage flux is negligible, the flux Φ is same over any cross section.

$$A_iB_i=A_gB_g=A_mB_m.$$

As for an alloy Alnico, the energy product $HmBm$ is maximum at $Bm=0.80$ tesla and $Hm=3.6\times10^4$ amp turn/meter. Hence the area of cross section of the permanent magnet.

$$Am=\frac{A_gB_g}{Bm}=\frac{48\times10^{-4}\times1.5}{0.80}=9\times10^{-3}\text{ meter}^2.$$

Length of the magnetic region $Lm=H_gL_g/Hm=B_gL_g/\mu_0Hm$

$$=\frac{1.5\times10^{-2}}{4\pi\times10^{-7}\times3.6\times10^4}=0.33\text{ meter.}$$

Thus we require two permanent magnets each of about 16 cm long and of 11 cms dia. If the magnets of suitable shape are not available, smaller magnets may be used in series and parallel combinations.

Oral Questions

1. Why cannot one obtain an isolated north or south pole by breaking a magnet into two pieces ?

2. Why is diamagnetism, unlike paramagnetism, relatively temperature independent ?

3. Is the saturated magnetization for a paramagnetic substance very much different from that for a ferromagnetic substance of the same size ?

4. A hollow cylinder is uniformly magnetized in the direction parallel to its axis. How are B and H affected by the introduction of a cylinder of ferromagnetic substance inside the former ?

5. Name the corresponding quantities in magnetostatics, of the following electrostatics ; E, D, and P.

6. Name the corresponding quantities in magnetic circuit of the following quantities in electric circuit ;

Current, Current density, Conductivity, Electromotance,
Resistance, Conductance.

7. Why is the magnetic susceptibility of diamagnetic substances negative ?

8. What are the main differences in the boundary conditions for electrostatic fields, and magnetostatic fields ?

9. Why does a magnet attract small pieces of iron ?

10. How can you distinguish two identical bars, one is magnet and other is not ? Please note that you are not allowed to suspend either bar or to use any other apparatus.

11. Two iron bars attract each other, no matter in which combinations their ends are brought near each other. What do you conclude about their magnetization ?

12. Why do you use substance in the form of an anchor ring while going to draw B-H curve ?

13. How can you demagnetise magnetic substance completely ?

14. How can you explain hysteresis loss ?

15. **Which physical properties are important to note when choosing the material for electromagnets.**

Problems

1. A cylindrical bar magnet of radius a and length $2l$ is uniformly magnetised along its length with intensity I. Show that at a point on the axis a distance x from the centre of the magnet

$$\Phi_{axis} = \tfrac{1}{2} I \{[a^2+(x-l)^2]^{1/2} + [a^2+(x+l)^2]^{1/2} + 2l\}.$$

2. Two similar permanent magnets A and B, having magnetic moments M are arranged in such a way that their axes are mutually $\perp$ to each other. They are separated by a distance (x_1+x_2), large compared with their lengths. A compass needle is placed a distance x_1 from one magnet and x_2 from the other. The needle makes an angle θ with the direction of first magnet. Prove that the ratio $x_1/x_2 = (2 \cot \theta)^{-1/2}$.

3. An annealed iron ring of mean length 15 cm is wound with a toroidal coil of 100 turns. Determining the magnetic induction in the ring when the current in the winding is (*a*) 0.1 amp. and (*b*) 0.2 amp.
(*0.25 weber/m²*, *0.95 weber/m²*)

4. Two magnetic molecules lie in one plane, their axes making an angle θ and θ' with the line joining the molecules. Show that the couple on the second molecule due to first will vanish, if

$$\tan \theta + 2 \tan \theta' = 0.$$

5. Two small magnets of moment M are placed at the corners A, B of a square $ABCD$ with their axes in the directions AB aad BA respectively. If two other small magnets of moment M' free to turn about their centres at C, D take up positions of equilibrium with their axes along AC and BD. Prove that the ratio of M to M' is $2 : (\sqrt{2}-1)$.

6. A specimen of unmagnetised iron of volume 20 cc is placed in a magnetic field of 50 oersteds: If the magnetic moment produced in it is 6000 cgs units, find the flux density in it.
(*0.3818 web/m²*)

7. A magnetic field of 40 emu produces a flux of 2500 emu in an iron rod of 0.5 sq. cm. area of cross section. Calculate magnetic permeability and susceptibility of the rod material. (*1.57×10^{-4} henry/m*; *124 mks units*)

8. The susceptibility of a gm mole of helium gas is -2.4×10^{-11} (SI units). Show that this corresponds to a value for the mean square distance of each electron in the helium atom of $1.22\, a_0^2$, where a_0 is the radius of the first Bohr orbit in the hydrogen atom and having value 0.528×10^{-10} m.

9. Rayleigh found that at low values of the magnetic induction ths hysteresis loop with tips at B_0., H_0 and $-B_0$, $-H_0$ is described by the equatione

$$B = \mu H + \tfrac{1}{2} a (H_0^2 - H^2) \text{ upper half of loop}$$

and

$$B = \mu H - \tfrac{1}{2} a (H_0^2 - H^2) \text{ lower half of loop.}$$

Show that the energy loss per cycle represented by the area of the loop in each unit volume of the specimen is $(4/3) a H_0^3$.

10. Calculate a value for the loss of energy due to hysteresis when 5 kg of iron are subjected for 1 hr to cyclic magnetic changes of frequency 25 cycles/sec, if the hysteresis loop represents a loss of 2.5×10^3 erg cm^{-3} $cycle^{-1}$. The density of iron = 7.8 gm cm^{-3}.
(*1.44×10^3 joule*)

11. An iron rod of density 7.7 and specific heat 0.1 is subjected to magnetisation at 50 cycle/sec. If the area enclosed by *M-H* curve for the specimen is 5×10^4 erg find the rise in temperature per minute. *(277.6°K)*

12. The armature core of a generator, made of iron, has a hyteresis *B-H* loop of area 5×10^4 units. The core is cylindrical of length 40 cm and radius 10 cm. If the armature rotates at 100 revolutions per sec, find the rate of rise of temperature (assuming no loss due to radiation). Given the density of iron as 7.7 and specific heat 0.11.

13. An electron in an atom of hydrogen circulates in the orbit of radius 0.51×10^{-10} meter. Calculate the change in magnetic moment for this electron, if a magnetic field of induction 2.0 weber/m^2 acts at right angles to the plane of the orbit. *(3.7×10^{-23} amp/m^2)*

14. The earth has a magnetic dipole moment of 6.4×10^{21} amp. m^2. (*a*) What current would have to be set up in single turn of wire going around earth at its magnetic equator, if we wished to set up such a dipole ? *(5×10^7 amp)*

15. The dipote moment associated with an atom of iron in an iron bar is 1.8×10^{-23} amp. m^2. Assume that all the atoms in the bar of length 5 cm and of cross sectional area 1 cm^2 have their dipole moments aligned. Calculate the dipole moment of the bar and the torque required to hold this magnet at right angles to an external field of 1500gauss ? *(7.5 amp m^2, 11 newton meter)*

16. Electric charge q is uniformly distributed over the volume of a sphere of radius a. Calculate the equivalent magnetic moment if this sphere spins about its diameter with a angular velocity ω. $(q\omega\mu_0 a^2/5)$.

17. An iron ring 10 cm mean radius and 5 sq. cm. crose section has an air gap 1 cm wide. The ring is bound uniformly with a coil of 900 turns. A flux of 10^4 maxwells is required in the air gap. If K for iron is 2000, find the strength of current required. *(1.82 amp)*

12

Alternating Current

12.1. ALTERNATING CURRENT

As we have discussed in article 9.8, an induced emf in the coil of turns N, area A, rotating with an anguiar velocity ω in a uniform field B, at any instant t is given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\,[BNA\cos\omega t] = BNA\omega \sin\omega t \quad \ldots(1)$$

Thus when $\omega t = 0$, $\sin\omega t = 0$ and $\mathcal{E} = 0$

$\omega t = \pi/2$, $\sin\omega t = 1$ and $\mathcal{E} = BNA\omega$

$\omega t = \pi$, $\sin\omega t = 0$ and $\mathcal{E} = 0$

$\omega t = 3\pi/2$, $\sin\omega t = -1$ and $\mathcal{E} = -BNA\omega$

and $\omega t = 2\pi$, $\sin\omega t = 0$ and $\mathcal{E} = 0$.

Here we see that induced emf goes through a cycle, each cycle corresponds to one revolution of the coil. Instantaneous value of emf is generally represented by e and the maximum value $BNA\omega$ as $\mathcal{E}_0$. Thus relation (1) reduces to

$$e = \mathcal{E}_0 \sin \omega t. \quad \ldots(2)$$

It represents a sinusoidal alternating emf at an instant t. This will develop an instantaneous voitage v, given by

$$v = V_0 \sin \omega t. \quad \ldots(3)$$

The corresponding current i through the circuit is given by

$$i = I_0 \sin \omega t. \quad \ldots(4)$$

The time of one cycle is known as *time period* T, the number of cycles per second the *frequency* $f\,(=1/T)$, the peak value of current or voltage the *amplitude*, the angular displacement per sec the angular velocity ω.

If an alternating current is passed through an ordinary ammeter or voltmeter, it will record the mean value for the complete cycle, as the quantity to be measured varies with time. The average value of current

$$\overline{I}=\frac{\int_0^T i\,dt}{\int_0^T dt}=\frac{\int_0^{2\pi/\omega} \mathrm{I}_0 \sin\,\omega t\,dt}{\int_0^{2\pi/\omega} dt}=0. \qquad \ldots(5)$$

Similarly the average value of the emf for the whole cycle $\overline{V}=0$. Since these averages for the whole cycle are zero, hence *dc instruments will indicate zero deflection.* In each complete cycle the average value of current (or voltage) in first half cycle is equal and opposite to that in second half cycle.

In *ac*, the average value of current is defined as its average taken over half the period. Hence

$$I_{av}=\frac{\int_0^{\pi/\omega} i dt}{\int_0^{\pi/\omega} dt}=\frac{\int_0^{\pi/\omega} I_0 \sin\,\omega t\,dt}{\frac{\pi}{\omega}}=\frac{2}{\pi}I_0$$

$$=(2/\pi)\times \textit{Peak value of current.} \qquad \ldots(6)$$

Similarly $V_{av}=(2/\pi)\times$ *Peak value of voltage.*

A *dc* meter can be used in an *ac* circuit, if it is connected in the full wave rectifier circuit. The average value of the rectified *current is the same as the average current in any half cycle, i.e., $2/\pi$ times the maximum current* I_0. The meter can be calibrated accordingly.

Most *ac* meters are based upon the measurement of root mean square value. The quadrant electrometer used idiostatically gives a deflection proportional to the square of the applied voltage and the hot wire ammeter used for the measurement of current depends on the square of the current as the heat developed by a current is proportional to (current)^2. Thus the alternating current is measured by the strength of the steady current which would produce the same heating effect. The square root of the mean square value is called the *virtual value* and is the value given by *ac* instruments. As

$$\overline{I}^2=\frac{\int_0^T i^2 dt}{\int_0^T dt}=\frac{\int_0^{2\pi/\omega} I_0^2 \sin^2\,\omega t\,dt}{\int_0^{2\pi/\omega} dt}=\frac{I_0^2}{2}.$$

$$\therefore\ \textit{Virtual current}=(\overline{I^2})^{1/2}=I_{rms}=I_0/\sqrt{2}=0.707\ I_0. \qquad \ldots(7)$$

Similarly we get $V_{rms}=V_0/\sqrt{2}=0.707\ V_0$.

Thus when we speak of our household power supply as 200 volts *a*c, this means that the rms voltage is 200 volts and its voltage amplitude is

$$V_0 = \sqrt{2}\, V_{rms} = 283 \text{ volts.}$$

The *ac* instruments are calibrated in virtual volts or virtual amperes. The ratio

$$\frac{\text{rms value}}{\text{Average value}} = \frac{V_0/\sqrt{2}}{2V_0/\pi}$$

$$= \frac{\pi}{2\sqrt{2}} = 1.11 \quad \ldots(8)$$

is known as *form factor.*

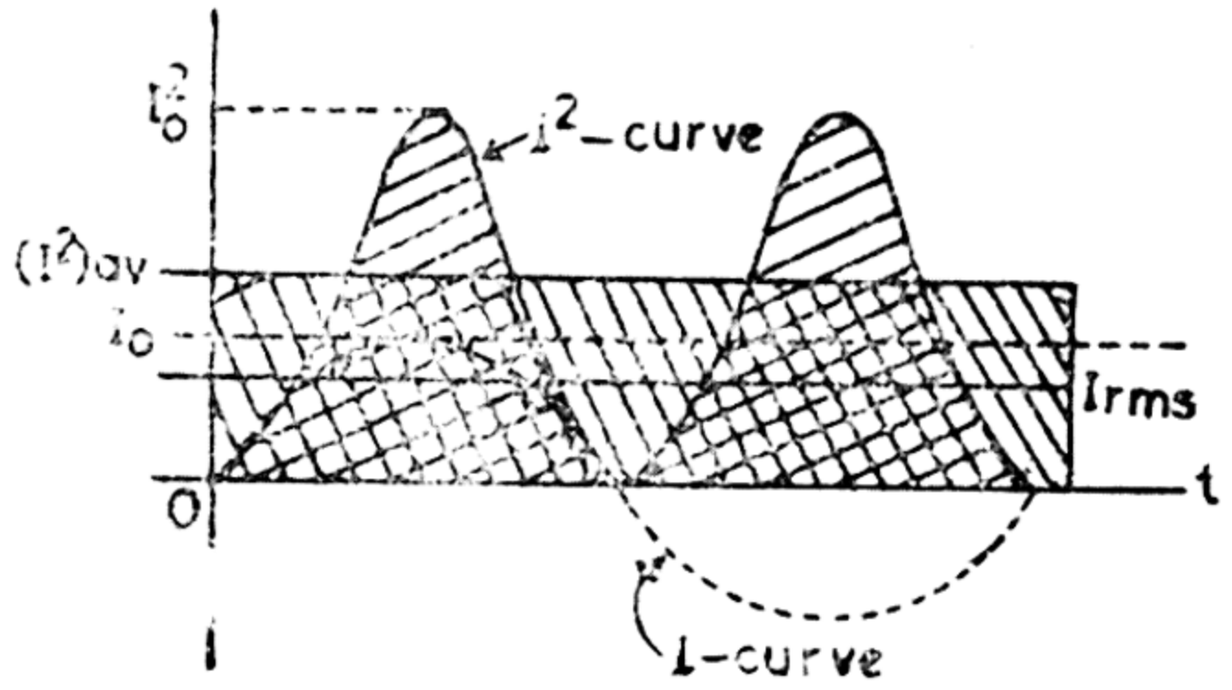


Fig. 12.1.

12.2. ACTUAL ELEMENTS

Resistors, capacitors and inductors are named as circuit elements. Practically these elements are not ideal or pure. A resistor has capacitance between its various parts, which can be represented as a small capacitance between its ends and a small series inductance, which is extremely small when a non-inductive winding is used. An inductance coil is similar but has a large quality ratio. A capacitor has in general a small leakage conductance, representable by a large resistance in parallel, and dielectric losses, the effect of which can be represented by a small series resistance as these losses are associated with the same current that charges the condenser.

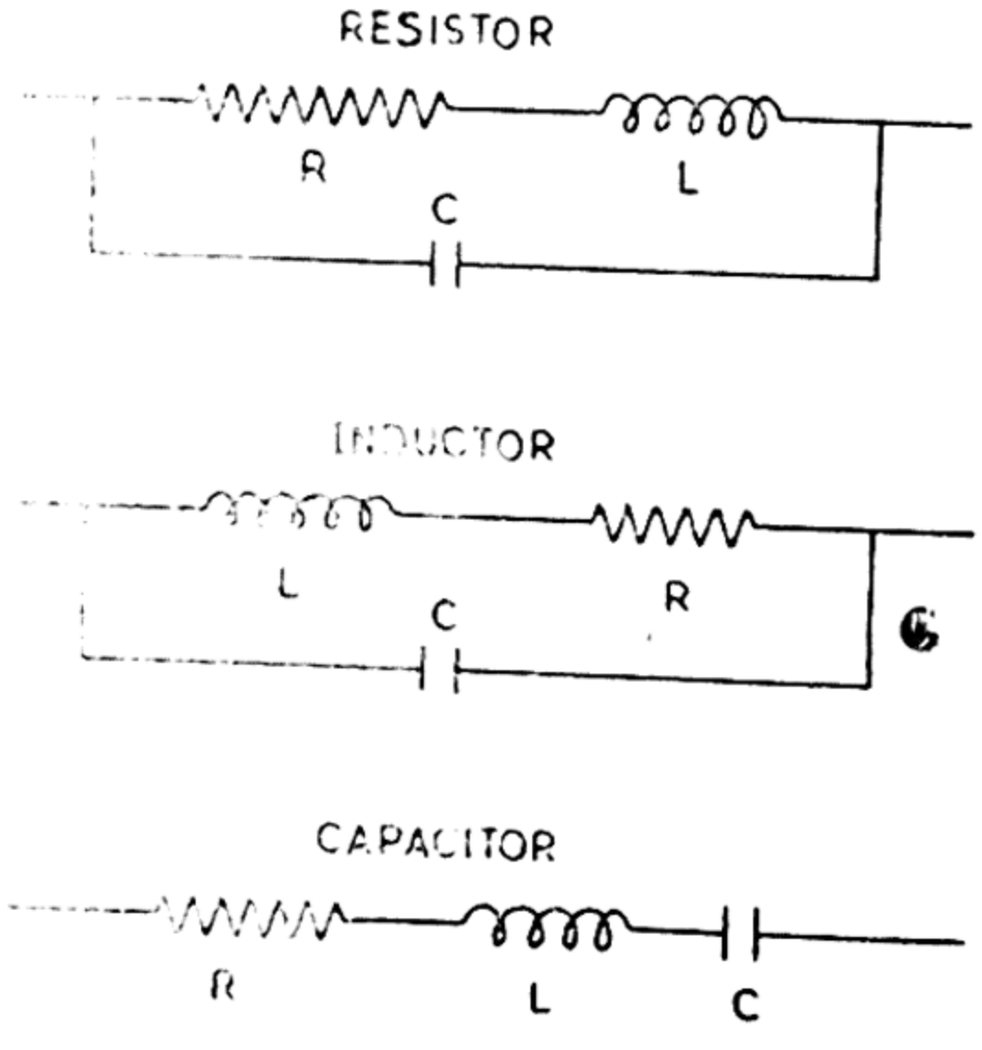


Fig. 12.2.

Load is a device or element which will draw current from a voltage source. The connection of a load reduces the voltage. A *short circuit* between two points is a direct connection. To short out a component a thick wire is connected in parallel with it. Load is infinite for open circuit.

12.3. CURRENT AND POTENTIAL RELATIONS

Before proceeding to the circuit analysis let us know the relations between current and potential when the sinusoidal voltage is applied to the three circuit elements, resistance, inductance and capacitance, separately.

(*a*) **Pure Resistance R.** Suppose that an alternating potential difference $v = V_o \sin \omega t$ is applied to a non-inductive resistance R. Then the instantaneous value of the current i passing through this resistance is given by the relation.

$$v = V_o \sin \omega t = iR,$$

or

$$i = (V_o/R) \sin \omega t = I_o \sin \omega t. \quad \ldots(9)$$

The quantity V_o/R, the maximum current or the current amplitude I_o, is same as obtained with direct current. Current and voltage both vary with sin ωt, hence are in *same phase.*

(*b*) **Pure Inductance.** Suppose that an alternating potential difference $v = V_o \sin \omega t$ is applied across a pure inductor having self inductance L. If i is the instantaneous current at any instant t, then

$$v = V_o \sin \omega t = L \frac{di}{dt} \text{ or } di = \frac{V_o}{L} \sin \omega t \, dt.$$

Its integration gives

$$i = -(V_o/\omega L) \cos \omega t = I_o \sin (\omega t - \pi/2). \quad \ldots(10)$$

The quantity $V_o/\omega L$ is equal to the maximum current I_o. This relation shows that the effective *ac* resistance, *i.e.*, the *inductive reactance* of inductor, $X_L = \omega L$ and the maximum current $I_o = V_o/X_L$. The unit of inductive reactance is also ohm. The reactance is greater for the greater inductance and for the higher frequency or the angular velocity ω. It is also clear from equation (10) that the *current in the inductor lags voltage by 90° or the voltage across the inductor leads the current passing through it by 90°.*

(*c*) **Pure Capacitance** C. If a capacitor of capacitance C is connected across the alternating source, the instantaneous charge on the capacitor

$$q = Cv = CV_o \sin \omega t,$$

and the instantaneous current i passing through it is given by

$$i = \frac{dq}{dt} = CV_o \omega \cos \omega t = \frac{V_o}{1/\omega C} \sin (\omega t + \pi/2)$$

$$= I_o \sin (\omega t + \pi/2), \quad \ldots(11)$$

This relation shows that the quantity $1/\omega C$ is the effective *ac* resistance or the *capacitive reactance* of the capacitor and is represented as Xc. It has unit as ohm. It varies inversely as the capacitance C and as the angular velocity ω or the frequency of the *ac*

source *f*. It is also clear that the *current leads the voltage by 90° or the potential drop across the capacitor lags the current passing through*

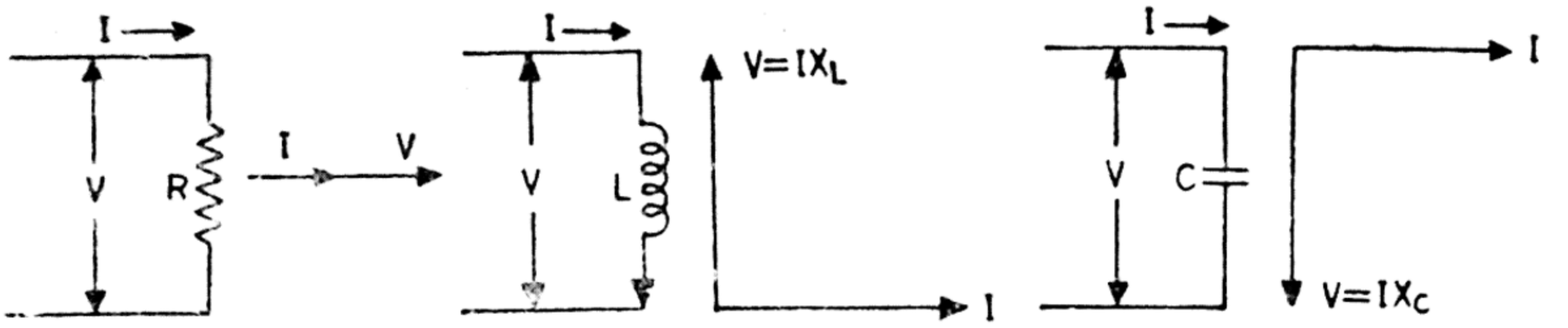


Fig. 12.3.

it by 90°. The relation between *i* and *v* through *R*, *L* or *C* is vectorially represented in Fig. 12.3.

12.4. PHASOR ALGEBRA

We know that the complex quantities, normally employed in *ac* circuit analysis, can be added and subtracted like coplaner vectors. Such coplaner vectors which represent sinusoidally time varying quantities are now known as *phasors*.

If *x*-axis and *y*-axis projections of a phasor *A* are known as x_A and y_A respectively, then its magnitude *A* is given as

$$A=\sqrt{(x_A^2+y_A^2)}. \qquad ...(12)$$

The angle between the direction of phasor **A** and the direction of the positive *x*-axis is

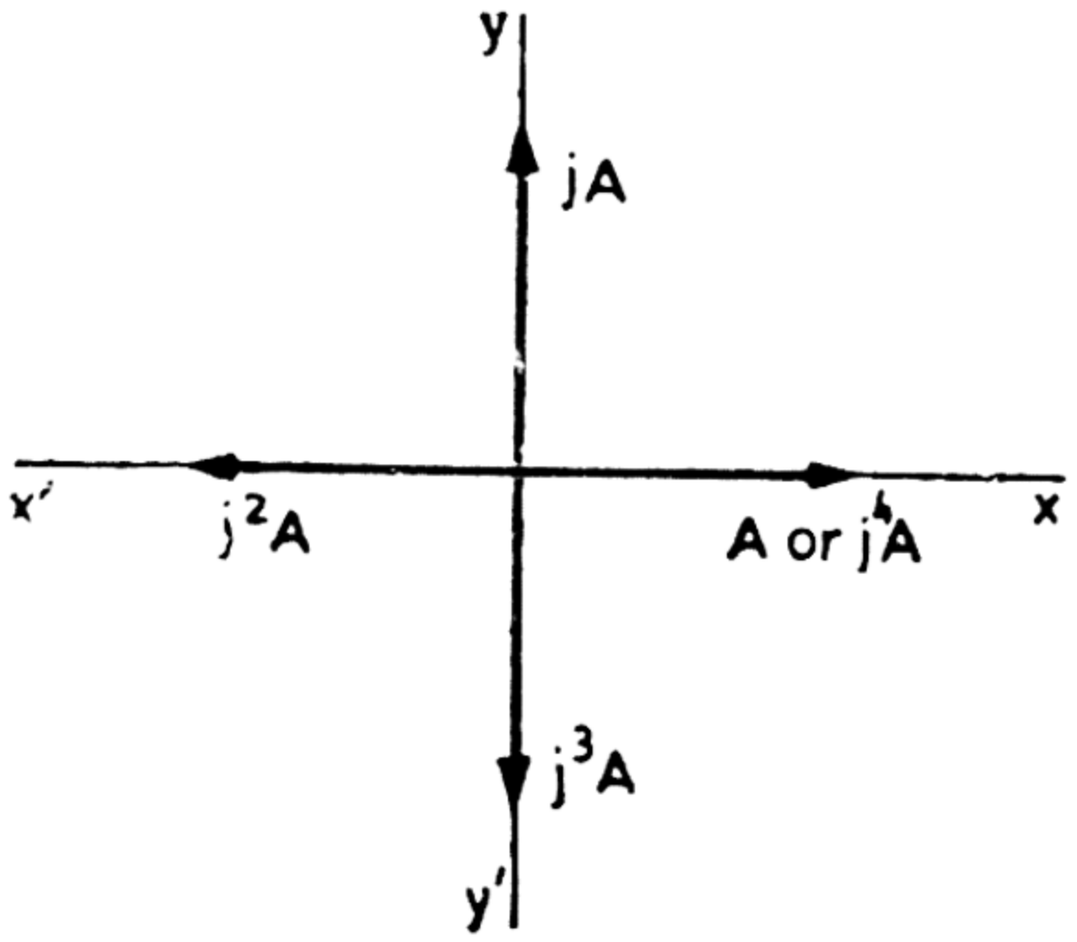


Fig. 12.4

$$\theta=\tan^{-1}(y_A/x_A). \qquad ...(13)$$

To distinguish between the *x*-axis projection and *y*-axis projection a convenient operator is used which will, when applied to a phasor, rotate it 90° counterclock wise without changing its magnitude. Let *j* be the such operator. When a given phasor **A**, the direction of which is along the *x*-axis, is multiplied by the operator *j*, a new phasor *j* **A** is obtained which will be 90° counter clock-wise from **A** *i.e.*, along *y*-axis. If the operator *j* is applied now to the phasor *j***A**, a new phasor $j^2\mathbf{A}$ is obtained which will be along —*x*-axis and having same magnitude as of **A**. Thus $j^2\mathbf{A}=-\mathbf{A}$.

$$\therefore \qquad j^2=-1 \text{ or } j=\sqrt{(-1)}. \qquad ...(14)$$

Thus in **cartesian form**, a phasor **A** can be written as

$$\mathbf{A}=a+jb,$$

where *a* is the *x*-component and *b* the *y*-component of phasor **A**.

In **polar form** the phasor can be written as

$$\mathbf{A}=A\cos\theta+jA\sin\theta=A(\cos\theta+j\sin\theta). \qquad ...(15)$$

Here $A\cos\theta$ is the x-component of the phasor and $A\sin\theta$ the y-component. If the angle is measured in the clockwise direction from the x-axis, the phasor **A** can be written as

$$\mathbf{A}=A(\cos\theta-j\sin\theta). \qquad ...(16)$$

Thus we see that $\cos\theta+j\sin\theta$ is an operator which when applied with a phasor A, along the x-direction, rotates it through an angle θ from its initial position. Similarly the operator $\cos\theta-j\sin\theta$ rotates the original phasor through an angle $-\theta$. It can also be proved that this operator rotates the phasor irrespective of the initial position of the phasor.

In **exponential form** the phasor **A** can thus be written as

$$\mathbf{A}=Ae^{\pm j\theta}=A(\cos\theta\pm j\sin\theta)=A\ \underline{/\pm\theta}. \qquad ...(17)$$

Therefore $e^{\pm j\theta}=\cos\theta\pm j\sin\theta=\underline{/\pm\theta}.$

Addition of phasors. The sum of two phasors is a phasor which is defined in magnitude and phase position by the diagonal of parallelogram formed by two phasors.

$$\mathbf{A}+\mathbf{B}=\mathbf{C}=\mathbf{B}+\mathbf{A}. \qquad ...(18)$$

Subtraction of phasors. In phasor algebra, phasor to be subtracted is rotated through 180° and the addition is done by completing the parallelogram.

Multiplication of phasors. The product of two phasors is a third phasor having magnitude equal to the product of the magnitudes and the phase position (phasor angle) with respect to the reference axis equal to the algebraic sum of the individual phasor angles of these phasors with respect to the same reference axis.

$$\mathbf{A}=Ae^{j\theta_A}=A\ \underline{/\theta_A} \quad \text{and} \quad \mathbf{B}=Be^{j\theta_B}=B\ \underline{/\theta_B}$$

$$\therefore \quad \mathbf{AB}=ABe^{j(\theta_A+\theta_B)}=AB\ \underline{/\theta_A+\theta_B}. \qquad ...(19)$$

Division of Phasors. When one phasor A is divided by another phasor **B**, it results in a phasor having magnitude A/B and the phase position with respect to the reference axis is the algebraic difference between the individual phase angles of these phasors with respect to the same reference axis.

$$\text{Thus} \quad \mathbf{A}\div\mathbf{B}=\mathbf{C}=\frac{Ae^{j\theta_A}}{Be^{j\theta_B}}=\frac{A}{B}\,e^{j(\theta_A-\theta_B)}$$

$$=\frac{A}{B}\ \underline{/\theta_A-\theta_B} \qquad ...(20)$$

Raising a Power. A phasor may be raised to a given power n, an given integer, by multiplying the phasor by itself n times.

$$(\mathbf{A})^n = A^n \angle n\theta_{\mathbf{A}}. \qquad \ldots(21)$$

If we want to write potential difference across the inductance or the capacitance in the form of phasor, we can write it as under :

(*i*) Since the voltage across the inductor leads the current passing through it by 90°, hence the inductive reactance ωL can be written as $j\omega L$.

(*ii*) Since the voltage across the capacitor lags the current passing through it by 90°, hence the capacitive reactance $1/\omega C$ can be written as

$$-j/\omega C = 1/j\omega C.$$

12.5 KIRCHHOFF'S LAWS

For the analysis of complicated a.c. circuits, Kirchhoff's laws are used. These laws are applied as in d.c. circuits (discussed in chapter 6), but are defined in complex terms as :

(1) As there is no accumulation of electric charge in a system of conductors, the sum of complex currents entering any junction equals the sum of the complex currents leaving that junction.

(2) The sum of the complex emf's in any mesh equals the sum of the complex potential drops across all elements of that mesh. Thus we can add series impedances as

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \ldots \mathbf{Z}_n. \qquad \ldots(22)$$

The corresponding formula for a number of impedances in parallel is found by writing the total current as the sum of the currents through the impedance elements. Thus we can write the net impedance as given by

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots \frac{1}{\mathbf{Z}_n}. \qquad \ldots(23)$$

The reciprocal of the impedance is called the **admittance Y**, thus for the combination of parallel impedances we can write

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \ldots\ldots \mathbf{Y}_n. \qquad \ldots(24)$$

In general **Y** is a complex quantity. Thus

$$\mathbf{Y} = G + jS = \frac{1}{\mathbf{Z}} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2}.$$

$$\therefore \qquad G = R/(R^2 + X^2) \text{ and } S = -X/(R^2 + X^2).$$

The G is known as *conductance* and S the *susceptance*.

12.6. IMPERFECT INDUCTANCE (L & R IN SERIES)

If an alternating potential $v=V_0 \sin \omega t$ is applied to the circuit containing inductance and resistance in series (Fig. 12.5). This problem can be considered in three ways.

(*a*) **To write down emf equation for the circuit and obtain a solution.** If i is the instantaneous current in the circuit containing resistance R and inductance L, then at any instant t,

$$v-L\, di/dt=iR$$

or $$L\, di/dt+iR=v=V_0 \sin \omega t \quad \ldots(25)$$

There are two parts of the solution of this equation and the complete solution is the sum of these two : (*a*) the solution of the equation $L\, di/dt+iR=0$ and (*b*) any solution, which satisfies the equation (25) and contains two integration constants.

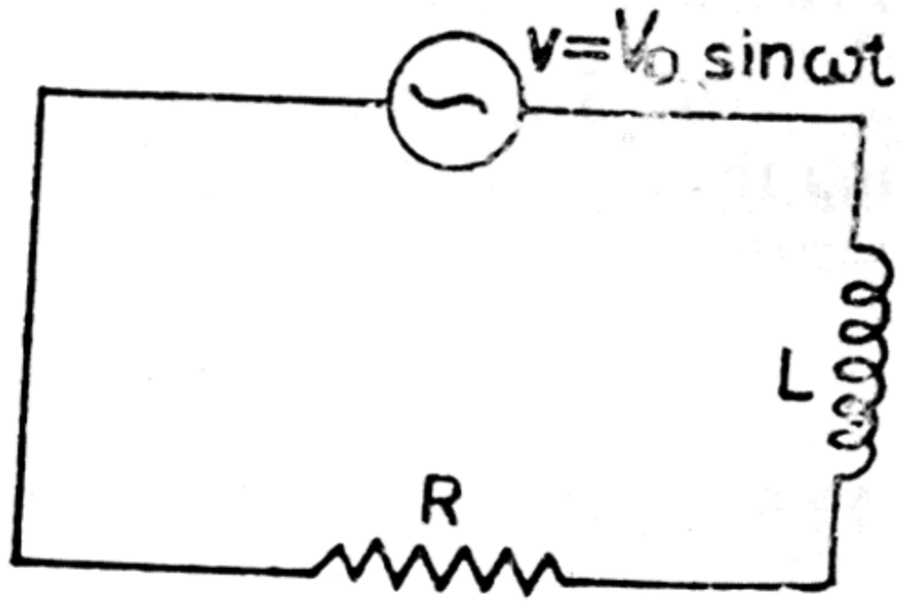


Fig. 12.5.

As discussed in chapter 9, the solution of the first part is given by the relation $i=I_0 e^{-Rt/L}$. It becomes zero after a considerable time. For the latter, we may assume that the periodicity of the current will be same as that of the applied emf, but can forecast nothing about its amplitude and phase. Let the solution be

$$i=A \sin (\omega t+\theta), \quad \ldots(26)$$

where A and θ are constants.

Substituting for i and di/dt in Eq. (25) from Eq. (26), we get

$$L\omega A \cos (\omega t+\theta)+RA \sin (\omega t+\theta)=V_0 \sin \omega t=V_0 \sin (\omega t+\theta)-\theta]$$

$$=V_0 [\sin (\omega t+\theta) \cos \theta-\cos (\omega t+\theta) \sin \theta].$$

This is true for all values of t. Hence equating coefficients of $\sin (\omega t+\theta)$ and $\cos (\omega t+\theta)$ separately, we get

$$L\omega A=-V_0 \sin \theta \text{ and } RA=V_o \cos \theta$$

$$\therefore \quad \tan \theta=-L\omega/R \text{ and } A=V_o/\sqrt{(R^2+\omega^2L^2)}.$$

Hence equation for the current at time t is given as

$$i=\frac{V_0}{\sqrt{(R^2+\omega^2L^2)}} \sin (\omega t+\theta). \quad \ldots(27)$$

The angle θ is obviously negative, so that the best way of writing the expression is

$$i=\frac{V_0}{\sqrt{R^2+\omega^2L^2)}} \sin (\omega t-\phi) \text{ where } \tan \phi=\omega L/R. \quad \ldots(28)$$

The amplitude of the current is $I_0 = V_0/\sqrt{(R^2+\omega^2L^2)}$. The quantity $\sqrt{(R^2+\omega^2L^2)}$ plays the same roll in *ac* circuits as the ohmic resistance does in *dc* circuits. It is termed as Z and known as the *impedance* of the circuit. For very small frequencies, this reduces to R, the ordinary resistance and for very large values it becomes ωL and is termed as the *reactance* of the circuit. Relation (28) indicates that the current is not in phase but lags behind the applied voltage by an angle ϕ, or a fraction $\phi/2\pi$ of a period.

(*b*) **To draw a vector diagram and solve by simple geometrical or analytical method.** We know that the voltage across the resistance is in phase with the current through it, the voltage across the inductance leads the current in phase by 90°. The current must obviously be in the same phase in the inductance as in the resistance in series circuit. We thus may represent these voltages by vectors Fig. (12.6). If the potential difference across the resistance is represented by the vector **OA**. This vector will be in the same direction as a vector representing the current in the circuit. The vector **AB** representing the magnitude and direction of the *pd* across the inductance is drawn 90° to the current vector **OA**. The positive angle is best drawn in an anti-clockwise direction.

The resultant vector **OB** will represent the combined *pd* across the inductance and resistance in series. This will be ZI, where Z is the impedance (or the total *ac* resistance) of the circuit.

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB}.$$

$$OB = \sqrt{(OA^2+AB^2)}$$

or $$Z = \sqrt{(R^2+\omega^2L^2)}$$

and $\tan\phi = \omega L/R$,

where ϕ is the phase angle of lead of the applied emf over the current. These relations are same as obtained earlier Eq. (28).

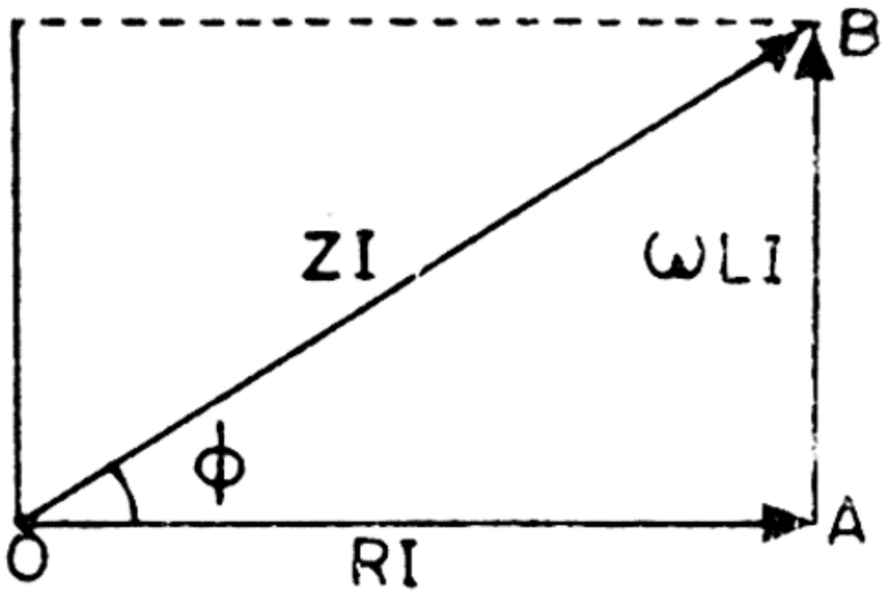


Fig. 12.6.

(*c*) **By use of the operator j.** The use of the operator **j** is of very great importance in *ac* calculations. In cartesian coordinates a unit vector is taken along +ve x-axis. As the multiplication by the imaginary quantity **j** denotes the rotation of the positive unit vector counter clockwise through +90°. Thus **j** is along y-axis.

Let us write the applied peak emf across the resistance and inductance in series as equal to the sum of the peak *pds* across the resistance and inductance separately. Thus

$$\mathbf{V} = R\mathbf{I} + j\omega L\mathbf{I},$$

$$\therefore \qquad \mathbf{I} = \frac{\mathbf{V}}{R+j\omega L} = \frac{\mathbf{V}}{\mathbf{Z}}, \qquad \ldots(29)$$

where $\mathbf{Z}=R+j\omega L$ is known as the complex impedance. The real part is often called the resistive *or inphase component* and the imaginary part as the reactive or *quadrature component.* The circuit impedance is given by the modulus of the complex impedance operator and the phase angle by its argument.

$$\mathbf{Z}=R+j\omega L=Ze^{j\phi}=Z\ (\cos\phi+j\sin\phi).$$

$$\therefore \qquad Z=\sqrt{(R^2+\omega^2L^2)} \text{ and } \tan\phi=\omega L/R.$$

Thus the current

$$\mathbf{I}=\frac{\mathbf{V}}{\mathbf{Z}}=\frac{V_0e^{j\omega t}}{Ze^{j\phi}}=\frac{V_0}{Z}\ e^{j(\omega t-\phi)}. \qquad \ldots(30)$$

This also represents Eq. (28).

12.7. IMPERFECT CONDENSER (*i.e.*, R & C IN SERIES)

An imperfect condenser may be regarded as a capacitance C in series with a resistance R. If an instantaneous emf is applied to this circuit consisting of R and C in series then at any instant t, when charge on the condenser C is q, we have

$$iR+q/C=v=V_0\sin\omega t.$$

As $i=dq/dt$, therefore we get

$$R\frac{dq}{dt}+\frac{q}{C}=V_0\sin\omega t. \qquad \ldots(31)$$

This equation is identical in form with that given by Eq. (25). The solution will be sum of two parts, one known as complimentary function which is the solution of Eq. $Rdq/dt+q/C=0$ and other known as particular integral. The solution of the first part is $q=q_0e^{-t/RC}$, which decreases with time and becomes negligible after some time. As done in earlier case, let us assume the solution of second part as

$$q=A\sin(\omega t+\theta). \qquad \ldots(32)$$

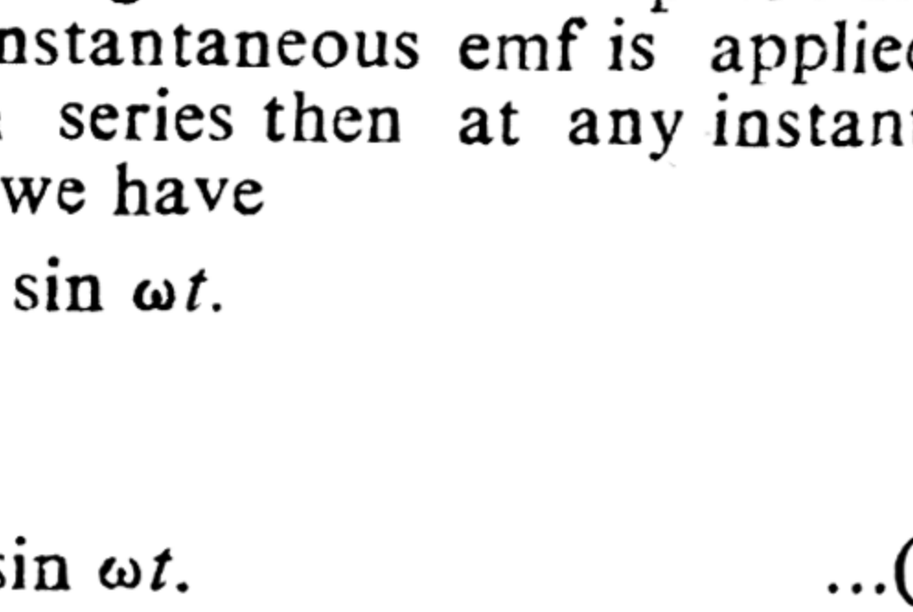


Fig. 12.7.

Substituting for q and dq/dt in Eq. (31), we have

$$RA\omega\cos(\omega t+\theta)+(A/C)\sin(\omega t+\theta)=V_0\sin[(\omega t+\theta)-\theta]$$

Equating coefficients of $\cos(\omega t+\theta)$ and $\sin(\omega t+\theta)$, we have

$$RA\omega=-V_0\sin\theta \text{ and } A/C=V_0\cos\theta.$$

From these relations, we have

$$A=\frac{V_0}{\sqrt{(\omega^2R^2+1/C^2)}}=\frac{V_0C}{\sqrt{(1+\omega^2R^2C^2)}}$$

and $\tan\theta = -\omega RC.$

$$\therefore \quad q = \frac{V_0 C}{\sqrt{(1+\omega^2 R^2 C^2)}} \sin(\omega t - \theta), \qquad \ldots (33)$$

where $\tan\theta = R\omega C,$

or $$i = \frac{dq}{dt} = \frac{V_0 C\omega}{\sqrt{(1+\omega^2 R^2 C^2)}} \cos(\omega t - \theta)$$

$$= \frac{V_0}{\sqrt{(R^2 + 1/C^2\omega^2)}} \sin(\omega t + \phi) \qquad \ldots(34)$$

where $\phi = \pi/2 - \theta.$

$$\therefore \quad \tan\phi = \tan(\pi/2-\theta) = \cot\theta = 1/\omega RC. \qquad \ldots(35)$$

The current is thus in advance of the applied emf by a phase angle of ϕ or $\phi/2\pi$ of a period. The amplitude of the current $I_0 = V_0/\sqrt{(R^2 + 1/C^2\omega^2)}$ and the impedance is thus given as

$$Z = \sqrt{(R^2 + 1/C^2\omega^2)}. \qquad \ldots(36)$$

Since i increases as ω increases, hence at high frequencies the circuit transmits considerable current. The impedance is infinite for zero frequency, *i.e.*, for *dc* source.

It is very easy to solve the problem by using **vector treatment.** We know that the voltage across the resistance is in phase with the current through it and the voltage across the capacitance lags the current in phase by 90°. The current is obviously in the same phase in the resistance and capacitance. If the potential difference across the resistance R is represented by a vector **OA** and that across the capacitance by **AB**, then the resultant potential difference

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB}$$

$$Z\mathbf{I} = R\mathbf{I} + (1/\omega C)\,\mathbf{I}.$$

$$OB = Z = \sqrt{(R^2 + 1/\omega^2 C^2)}$$

and $\tan\phi = -1/\omega CR \qquad \ldots(37)$

Here −ve sign indicates that the voltage lags the current by this angle ϕ.

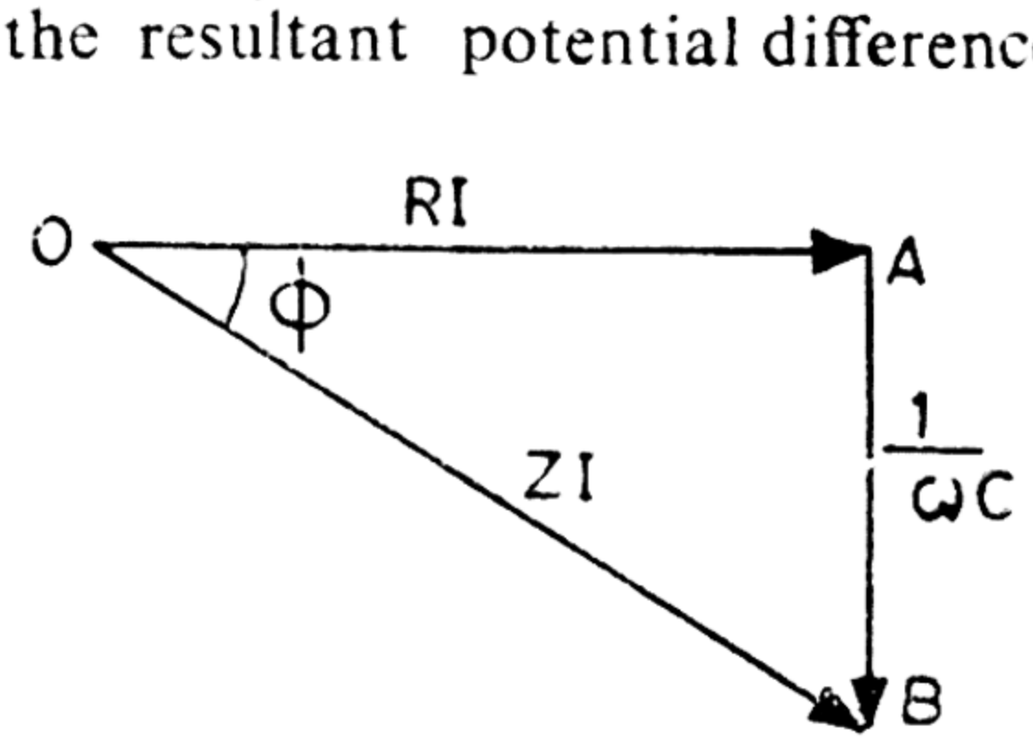


Fig. 12.8.

Problem can also be solved by **using operator** j. The applied *pd* is across the resistance and capacitance in series. Therefore

$$\mathbf{V} = R\mathbf{I} - (j/\omega C)\,\mathbf{I}$$

or $$\mathbf{I} = \frac{\mathbf{V}}{R - j/\omega C} = \frac{\mathbf{V}}{\mathbf{Z}}.$$

Hence the impedance operator
$\mathbf{Z} = Ze^{j\theta} = Z(\cos\theta + j\sin\theta) = R - j/\omega C.$ Therefore

$$\therefore \quad Z = \sqrt{(R^2 + 1/\omega^2 C^2)} \text{ and } \theta = \tan^{-1}(-1/R\omega C).$$

Thus current $\mathbf{I}=\frac{\mathbf{V}}{Z}=\frac{V_0 e^{j\omega t}}{Ze^{j\theta}}=\frac{V_0}{Z}e^{j(\omega t-\theta)}=\frac{V_0}{Z}e^{j(\omega t+\phi)}$,

where $-\theta=\phi=\tan^{-1}(1/R\omega C)$.

Thus we see that the current leads the voltage or the voltage lags the current by the angle ϕ, given by the relation $\tan\phi=1/R\omega C$.

12.8. CIRCUIT CONTAINING INDUCTANCE AND CAPACITANCE

Let a condenser of capacitance C, and an inductance L of negligible resistance are connected in series. When the emf $V_0 \sin \omega t$ is applied, we have at any time t,

$$L\frac{di}{dt}+\frac{q}{C}=V_0 \sin \omega t.$$

Since current is defined as given by $i=dq/dt$, hence this equation becomes

$$\therefore \quad L\frac{d^2q}{dt^2}+\frac{q}{C}=V_0 \sin \omega t$$

or $$\frac{d^2q}{dt^2}+\frac{q}{LC}=\frac{V_0}{L}\sin \omega t. \qquad ...(38)$$

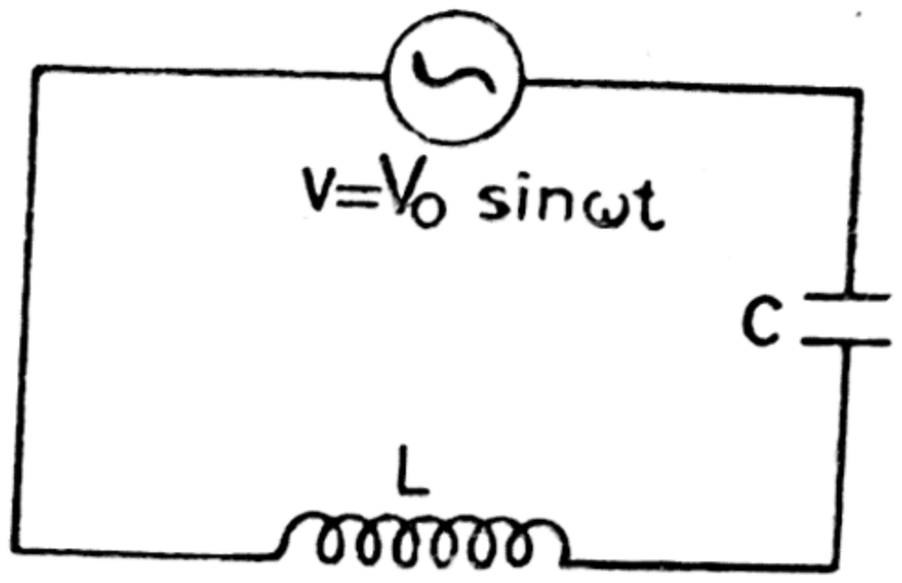


Fig. 12.9.

This is a linear differential equation of second order, the solution of such type of equation consists of two parts The first part, known as complementary function, the solution of Eq. $d^2q/dt^2+q/LC=0$, which comes out to be $q=q_0 \sin [t/\sqrt{(LC+\pi/2}]$. Thus the motion is simple harmonic and the frequency $f=1/2\pi\sqrt{(LC)}$, known as the *natural frequency* of the LC circuit. The second part, known as particular integral, can the found by trial method, assuming the frequency as that of applied emf. For the second part, let us assume that the solution of Eq. (38) is

$$q=A\ sin\ (\omega t+\theta). \qquad ...(39)$$

Substituting values of q and its derivatives in Eq. (38), we get

$$-\omega^2 A\ sin\ (\omega t+\theta)+\frac{A}{LC}\ sin\ (\omega t+\theta)=\frac{V_0}{L}\ sin\ \omega t$$

$$=\frac{V_0}{L}\ sin\ [(\omega t+\theta)-\theta]$$

$$=\frac{V_0}{L}\left[\ sin\ (\omega t+\theta)\cos\ \theta-cos\ (\omega t+\theta)\ \sin\ \theta\ \right]$$

Equating coefficients of sin $(\omega t+\theta)$ and cos $(\omega t+\theta)$ separately, we get

$$-\omega^2 A + A/LC = (V_0/L)\cos\theta \text{ and } 0 = -(V_0/L)\sin\theta.$$

Thus $\theta = 0$ and $A = \dfrac{V_0/L}{(1/LC - \omega^2)}$

$$\therefore \quad q = \frac{V_0/L}{1/LC - \omega^2}\sin\omega t. \qquad \ldots(40)$$

or $$i = \frac{dq}{dt} = \frac{V_0\omega/L}{1/LC - \omega^2}\cos\omega t = \frac{V_0}{(1/C\omega - L\omega)}\sin(\omega t + \pi/2) \ldots(41)$$

Thus the current is exactly $\pi/2$ out of phase with the applied emf. The amplitude of current becomes infinite when $\omega = 1/\sqrt{(LC)}$, *i.e.*, when the frequency of the applied emf is exactly equal to the natural frequency of the circuit. This condition is called *resonance*.

The same result may be obtained by **vector method**. We know that the current through the inductance or the capacitance is same ; the voltage across the inductance leads the current by 90° and the inductive reactance is ωL, when the frequency of the applied emf is $\omega/2\pi$; the voltage across the capacitance lags behind the current by 90° and the capacitative reactance is $1/\omega C$. Thus we can represent potential differences across the inductance and capacitance by vectors, as given in Fig. 12.10. From this figure it is clear that the resultant potential difference will be $|\mathbf{V}_L - \mathbf{V}_C|$. It will lead if $V_L > V_C$ and lag if $V_C > V_L$ the current by an angle $\pi/2$. In other words we can say that the combination is equivalent to a single element, either an inductor, or a capacitor, depending on whether the quantity $(\omega L - 1/\omega C)$ is +ve or −ve.

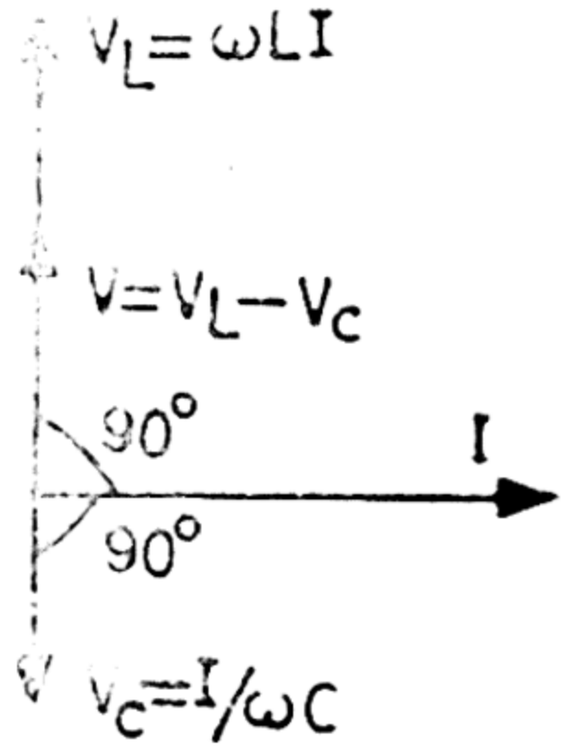


Fig. 12.10.

Perfect Choke. When an alternating emf $v = V_0 \sin\omega t$ is applied across inductance and capacitance in parallel. If I be the current flowing at any time t at a junction, it will divide in two parts one being along the inductor and other along the capacitor. Using phasor algebra, we get

total current $\mathbf{I} = \mathbf{I}_L + \mathbf{I}_C$.

$$\therefore \quad \frac{\mathbf{V}}{\mathbf{Z}} = \frac{\mathbf{V}}{j\omega L} + \frac{\mathbf{V}}{1/j\omega C}$$

or $$\frac{\mathbf{1}}{\mathbf{Z}} = \frac{1}{j\omega L} + j\omega C.$$

Total current **I** will be zero for all values of t, when

$$1/j\omega L + j\omega C = 0$$

or $$\omega = 1/\sqrt{(LC)}. \qquad \ldots(42)$$

Thus the circuit acts as a perfect choke for oscillations of its own natural period, *i.e.*, the oscillations of frequency ω are *prevented* from

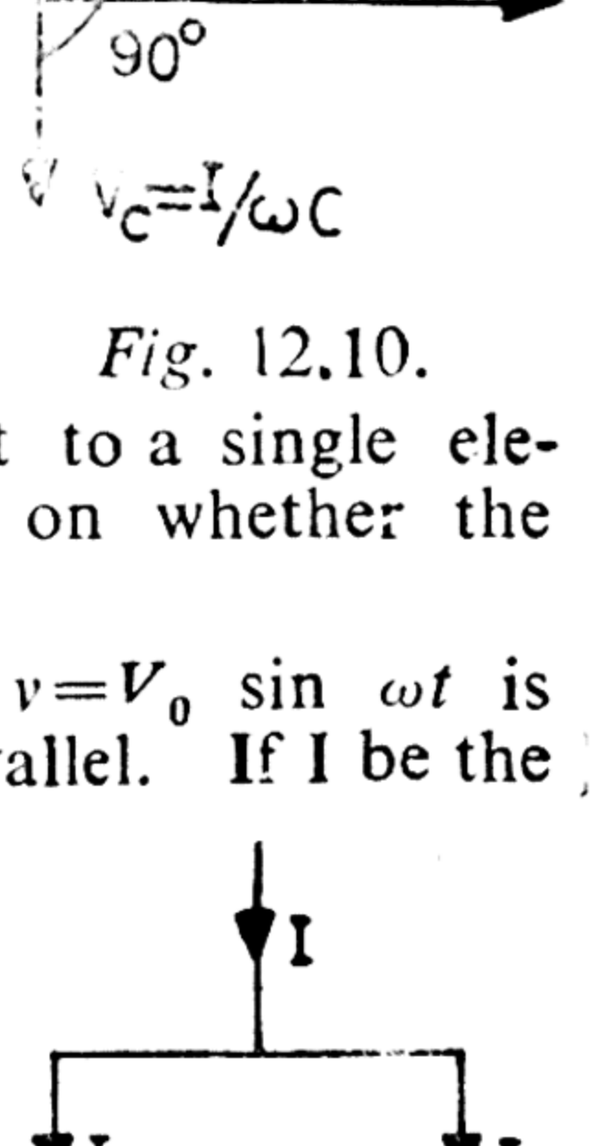


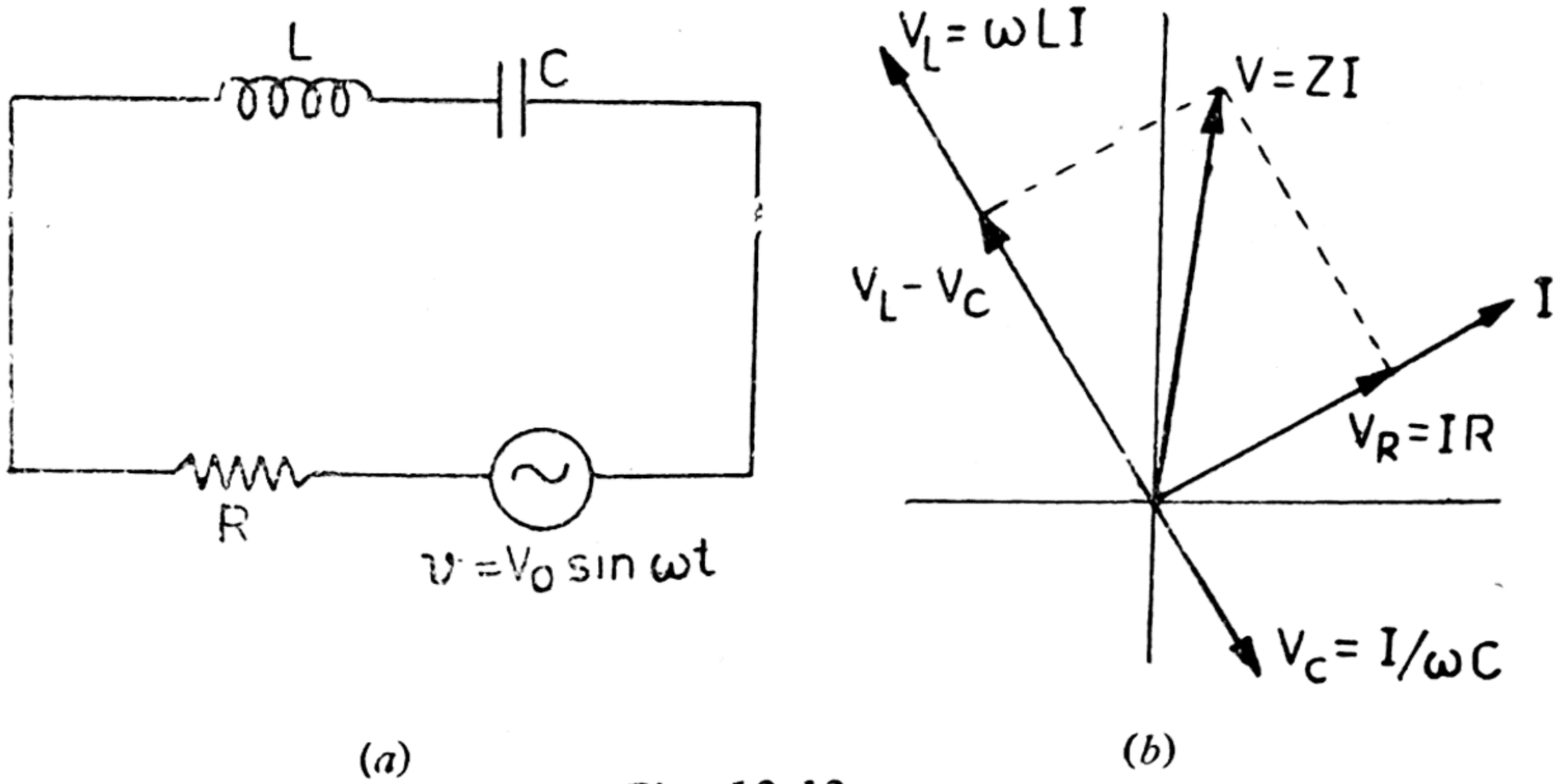
Fig. 12.11.

passing through the circuit consisting inductance and capacitance in parallel, when this frequency equals the natural frequency of this circuit.

Hence a *filter circuit can be designed to transmit or suppress any required frequency by connecting L and C, of required values, in series for the first case and in parallel for the other.*

12.9. CIRCUIT CONTAINING INDUCTANCE, CAPACITANCE AND RESISTANCE

(*a*) **Series Circuit (Accepter Circuit).** Let an alternating voltage $V_0 \sin \omega t$ be applied to an inductance, a resistance and a capacitance, in series. If i is the instantaneous current flowing at any instant,



(*a*) (*b*)

Fig. 12.12.

the emf sets up in the inductance L is $-L(di/dt)$, and the voltage across the capacitor is q/C, where q is the charge on the condenser at that instant. Thus we have

$$V_0 \sin \omega t - L(di/dt) - q/C = Ri,$$

or

$$L(di/dt) + Ri + q/C = V_0 \sin \omega t.$$

Since $i = dq/dt$, hence this equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t. \qquad \ldots(43)$$

This is a differential equation whose solution consists of two parts. The first of these, known as the complementary function, is found by solving the equation obtained by setting the right hand side term equal to zero. If q_0 is the initial charge, this solution is obtained as in article 9.24 and is given by

$$q = \tfrac{1}{2}q_0\, e^{(-R/2L)t}\Big[(1 + R/\sqrt{R^2 - 4L/C})\, e^{\sqrt{(R^2/4L^2 - 1/LC)}\,t} + (1 - R/\sqrt{(R^2 - 4L/C)})\, e^{-(R^2/4L^2 - 1/LC)t}\Big] \qquad \ldots(44)$$

Three conditions thus arise :

(*i*) If $R^2/4L^2 > 1/LC$, the square roots are real and the discharge is *non oscillatory* or *heavily damped.*

(*ii*) If $R^2/4L^2 = 1/LC$, bracket terms become 1 and the discharge is *critically damped*, given by relation $q = q_0 e^{-(R/2L)t}$.

(*iii*) If $R^2/4L^2 < 1/LC$, the square roots are imaginary and the *discharge is oscillating* due to the terms within the brackets and damped, as the amplitude is $q_0 e^{-(R/2L)t}$.

Thus we see that the first solution becomes negligible within a few seconds. As the alternating emf is applied continuously, we may assume that the period of alternation of the current or the discharge will be that of the applied emf. As we can not say about the amplitude and phase. Let us thus assume the second part of the solution, known as particular integral, as

$$q = A \sin(\omega t + \theta). \qquad \ldots(45)$$

Substituting values of q and its derivatives in Eq. (43), we get

$$-L\omega^2 A\sin(\omega t+\theta) + R\omega A \cos(\omega t+\theta) + (A/C)\sin(\omega t+\theta) = V_0 \sin \omega t$$
$$= V_0[\sin(\omega t+\theta)\cos\theta - \cos(\omega t+\theta)\sin\theta]$$

Equating coefficients of $\sin(\omega t+\theta)$ and $\cos(\omega t+\theta)$, we get

$$-L\omega^2 A + A/C = V_0 \cos\theta \text{ and } R\omega A = -V_0 \sin\theta.$$

Solving these equations, we get

$$A = \frac{V_0}{\sqrt{[R^2\omega^2 + (1/C - L\omega^2)^2]}} \text{ and } \tan\theta = -\frac{R\omega}{1/C - L\omega^2} = \frac{R}{L\omega - 1/C\omega}.$$

$$\therefore \quad q = \frac{V_0}{\sqrt{[R^2\omega^2 + (1/C - L\omega^2)^2]}} \sin(\omega t+\theta). \qquad \ldots(46)$$

or
$$i = \frac{dq}{dt} = \frac{V_0}{\sqrt{[R^2 + (1/C\omega - L\omega)^2]}} \cos(\omega t+\theta).$$

$$= \frac{V_0}{\sqrt{[R^2 + (1/C\omega - L\omega)^2]}} \sin(\omega t+\theta'),$$

$$= \frac{V_0}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}} \sin(\omega t - \phi),$$

where $\theta' = \tan^{-1}\frac{1/\omega C - \omega L}{R}$ and $\phi = \tan^{-1}\frac{\omega L - 1/\omega C}{R}$. $\qquad \ldots(47)$

Eq. (47) can also be obtained by **using vector** or by *constructing the vector rotor diagram* for the LCR series circuit. If the frequency is not too high, the instantaneous current is the same in all parts of the circuit. Even in the capacitor we see that the growth of positive

charge on one plate requires an equal growth of opposite charge on the other plate, which is equivalent to equal currents on both sides. Thus a single rotor **I**, of length proportional to the current amplitude $\mathbf{I}_0$ is sufficient to represent the current in each circuit elements. Since the sum of sinusoidal voltages of the same frequency is always another sinusoidal voltage, hence we can say that the resultant voltage across all three elements is just the vector sum of the individual generating voltages and is equal to the applied voltage.

We know that the potential difference across the resistor R is in phase with the current in the resistor and that its maximum value $V_R = IR$. Thus V_R is along $\mathbf{I}_R$ or **I**. The voltage across the inductor leads the current through it by 90°. The voltage amplitude is $V_L = I_L X_L = I\omega L$. The voltage across the capacitor lags the current through it by 90°. The voltage amplitude is $V_C = I_C X_C = I/\omega C$. These voltage vectors are represented by the Fig. 12.12 (*b*).

Since $\mathbf{V}_L$ and $\mathbf{V}_C$ always lie in the same straight line but in opposite directions, hence we should first subtract the smaller one (say $\mathbf{V}_C$) from the greater one (say $\mathbf{V}_L$) giving the vector $\mathbf{V}_L - \mathbf{V}_C$, which is at right angles to the vector $\mathbf{V}_R$. Thus the sum of vectors **V** can be found by using vector diagram, as

$$\mathbf{V} = \mathbf{V}_R + (\mathbf{V}_L + \mathbf{V}_C).$$

Its magnitude $V = \sqrt{[V_R^2 + (V_L - V_C)^2]} = I\sqrt{[R^2 + (X_L - X_C)^2]}$

$$= I\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}.$$

or
$$I_0 = \frac{V_0}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}}. \qquad \ldots(48)$$

The current in the circuit lags the voltage applied by ϕ, the tangent of which is given as

$$\tan\phi = \frac{I(X_L - X_C)}{IR} = \frac{\omega L - 1/\omega C}{R}. \qquad \ldots(49)$$

Hence we can write the instantaneous value of current i as

$$i = \frac{V_0}{\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}} \sin(\omega t - \phi).$$

This equation is similar to Eq. (47).

Using the j operator method, the differential equation of emf's in the circuit can be rewritten as

$$\mathbf{V} = R\mathbf{I} + j\omega L\mathbf{I} - (j/\omega C)\mathbf{I}, \qquad \ldots(50)$$

where **I** is the instantaneous value of current when an instantaneous voltage **V** is applied. As we know that operator j precedes the *pd* $\omega L\mathbf{I}$ across the inductance because this *pd* leads the current by 90°, whereas j precedes the *pd* $\mathbf{I}/\omega C$ across the capacitance because this *pd* lags on the current by 90°.

$$\mathbf{V} = [R + j(\omega L - 1/\omega C)]\mathbf{I}.$$

Impedance $\mathbf{Z}=\mathbf{V}/\mathbf{I}=R-j(\omega L-1/\omega C)$. ...(51)

Let us represent this complex quantity $\mathbf{Z}$ as

$$\mathbf{Z}=Ze^{j\phi}=Z(\cos\phi+j\sin\phi). \quad ...(52)$$

From equations (51) and (52), we get

$$Z=\sqrt{\left[R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right]} \text{ and } \tan\phi=\frac{\omega L-1/\omega C}{R} \quad ...(53)$$

Three cases will thus arise :

(*i*) If $\omega L>1/\omega C$, tan ϕ becomes positive and the applied voltage thus leads the current by the phase angle ϕ.

(*ii*) If $\omega L<1/\omega C$, tan ϕ becomes negative and the applied voltage thus lags on the current by the phase angle ϕ.

(*iii*) If $\omega L=1/\omega C$, tan ϕ becomes zero and the applied voltage becomes in the same phase as that of the current. The circuit is thus purely resistive in its effect. Such a condition is known as *Resonance* and frequency, known as *resonance frequency*, is given by

$$\omega L=1/\omega C \text{ or } \omega=1/\sqrt{(LC)}$$

$$\therefore \quad f=\omega/2\pi=1/2\pi\sqrt{(LC)}. \quad ..(54)$$

The current in such a case is maximum and is given by $I=V/R$, because the alternating pds across the inductance and the capacitor are then equal and 180° out of phase and therefore cancel out.

Voltage Magnification in Series Resonance Circuit. At resonance the potential differences across the inductor and the capacitor are equal and 180° out of phase and therefore cancel out hence the applied emf is merely to overcome the resistance opposition only. If an inductance or capacitance is connected of very large impedance is connected the potential across it increases to a very high value. The ratio is known as voltage magnification and is given as

$$\frac{pd\ across\ the\ inductance\ (or\ capacitance)}{Applied\ emf}=\frac{\omega LI_{rms}}{RI_{rms}}=\frac{\omega L}{R} \quad ...(55)$$

which is greater than unity. Thus for high frequencies $\omega L/R$ can be designed to magnify the voltage to any desired value.

Mechanical Analogy. Mechanical analogies can be constructed for any of the circuits described so far. For this purpose, let us consider a mass M, suspended at the free end of spiral spring. Let a force F be applied, producing an acceleration a in the direction of original extension. If v is the velocity at an instant, when displacement is x, then the equation of restoring force is

$$F-\mu x-kv=Ma,$$

or

$$M\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\mu x=F. \quad ...(56)$$

Comparing this with the Eq. (43), we get

(*i*) *the inductance L in the electrical problem corresponds to mass M in the mechanical one.*

(*ii*) *the resistance R in the electrical problem corresponds to k, the restoring force (or the viscous damping force for unit velocity,*

(*iii*) *the reciprocal of capacitance (1/C) in the electrical problem corresponds to μ, the restoring force per unit displacement.*

(*iv*) *the mechanical counterpart of the charge itself is displacement and that of applied emf is mechanical force.*

In both cases the nature of the solution depends on the resistance term. If k or $R=0$, the motion is simple harmonic and undamped, while increasing k or R produces first damped harmonic motion, then critical damped and finally heavy damped for progressively larger values of this term.

Response Curves of Series Circuit

The impedance of an *LCR* series circuit depends on the frequency. The dependence is shown in Fig. 12.13, where the frequency is taken in logarithmic form because of its wide range.

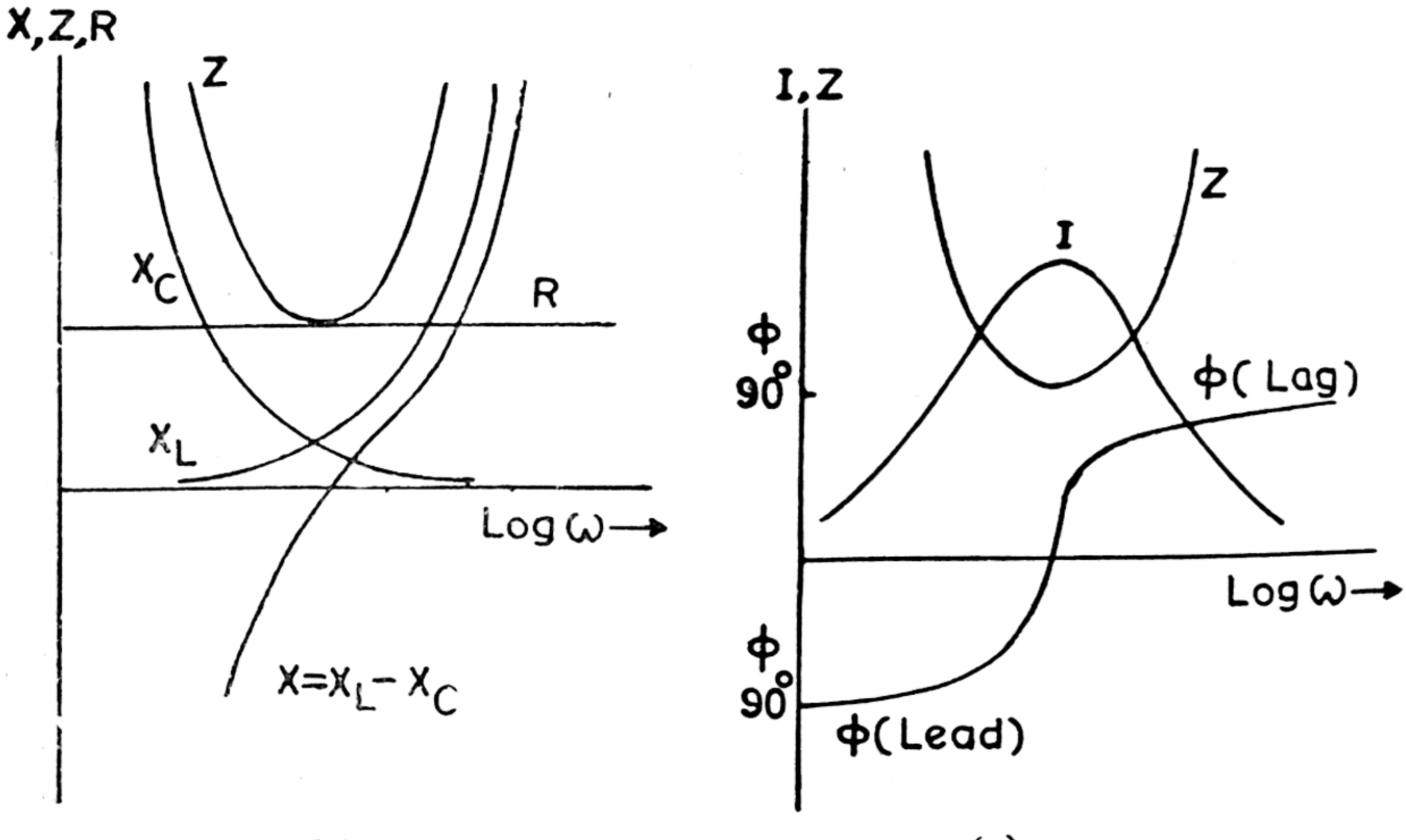


Fig. 12.13.

Fig. 12.13 (*a*) shows that there is one particular frequency at which X_L and X_C are numerically equal. At this frequency $X_L - X_C$ is zero and the impedance Z is equal to R, and hence minimum. Fig. 12.13 *b*) shows that the current is maximum at this frequency and is doubled when the resistance is reduced to its half value. Hence the sharpness of the curve increases with the decrement of R. The

sharpness of such a curve is measured by its half-width, which is defined as the *frequency difference between two frequencies, each of which corresponds to a current amplitude of one half of its maximum.* The sharpness of resonance is large if the half width is smaller, hence it is numerically defined as the *reciprocal of the half width.*

The phase difference ϕ is also shown in the Fig. 12.13 (*b*). It is clear from the curve that at low frequencies, which capacitive reactance X_C predominates, the current leads the voltage; at high frequencies, where inductive reactance X_L predominates, the current legs the voltage. The current and voltage are in phase, at the frequency where $X_B = X_C$.

Acceptor Circuit. If the frequency of the *ac* supply is varied, or if the *ac* supply contains number of frequency components, (*i.e., radio or television signal*), then a series *LCR* circuit across the supply will give a maximum response, *i.e.*, pass a maximum current and having a maximum potential difference across its inductance for that component of frequency for which $f = 1/2\pi\sqrt{(LC)}$. This is the procedure by which a radio or television receiving set may be turned to receive the signal from a desired station sending signals at a particular frequency. The circuit is thus known as an *acceptor circuit,* especially in radio communications.

B. PARALLEL CIRCUIT (REJECTOR CIRCUIT)

Let us consider an alternating source connected across an inductance L in parallel with a capacitor C. If resistance in series with the inductance is R and with the capacitor is zero (as the power factors of good quality capacitors are very small). Let the instantaneous value of emf applied be V and the corresponding current be I, the current through the inductance be I_L and through the capacitance I_C. These latter currents (I_L and I_C) will be almost in antiphase if R is very small. The total current

$$\mathbf{I} = \mathbf{I}_L + \mathbf{I}_C$$

or
$$\frac{\mathbf{V}}{Z} = \frac{\mathbf{V}}{R + j\omega L} + \frac{\mathbf{V}}{1/j\omega C}$$

$$\therefore \quad \frac{\mathbf{1}}{Z} = \frac{1}{R + j\omega L} + j\omega C \qquad \ldots(57)$$

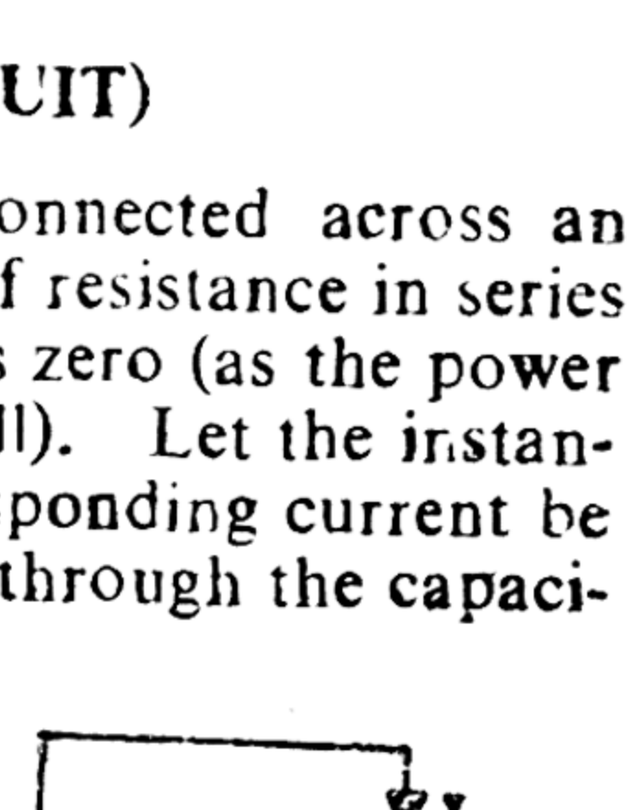


Fig. 12.14

or
$$\mathbf{Y} = \frac{1}{Z} = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$= \frac{R + j(\omega C R^2 + \omega^3 L^2 C - \omega L)}{R^2 + \omega^2 L^2}.$$

$\therefore$ The magnitude of the admittance

$$Y = \frac{1}{Z} = \frac{\sqrt{[R^2 + (\omega C R^2 + \omega^3 L\ C - \omega L)^2]}}{R^2 + \omega^2 L^2}. \qquad \ldots(58)$$

The admittance will be minimum, when

$$\omega CR^2+\omega^3L^2C-\omega L=0,$$

or

$$\omega=\omega_0=\sqrt{(1/LC-R^2/L^2)}. \quad ...(59)$$

It gives the condition of resonance and the corresponding frequency

$$f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)} \quad ...(60)$$

is known as *resonance frequency*.

For the values of L, C, R and ω satisfying Eq. (60), the reactive component of Y vanishes or Y is real. As such a minimum admittance, *i e*, maximum impedance, the circuit current is minimum. Thus the parallel circuit does not allow this frequency from the source to pass in the circuit. Due to this reason the circuit with such a frequency is known as *rejector circuit*.

The reciprocal of the admittance at resonance is called the *parallel resistor* or the *dynamic resistance*. At the time of resonance the current and voltage are in phase. The dynamical resistance is thus reciprocal of the real part of the admittance.

$$Dynamic\ resonance=\frac{R^2+\omega^2L^2}{R}$$

$$=\frac{R^2+L^2(1/LC-R^2/L^2)}{R}=\frac{L}{CR}.$$

At resonance, the peak current from the supply, known as

$$make\ up\ current=V_0/(L/CR)=CRV_0/L.$$

The *forced oscillatory current* is the current passing through capacitor and inductor circuit or simply through capacitor. The peak current through capacitor is

$$I_0=V_0/(1/\omega C)=\omega CV_0. \quad ...(61)$$

This current will lead the applied voltage by $\pi/2$.

At resonance

$$\frac{Amplitude\ of\ oscillatory\ current}{Amplitude\ of\ make\ up\ current\ from\ supply}=\frac{V_0\omega C}{V_0CR/L}$$

$$=\frac{\omega L}{R}. \quad ...(62)$$

This ratio is known as Q– factor, which is the measure of the current magnification. Thus the *rejector circuit at resonance exhibits current magnification of $\omega L/R$, similar to the voltage magnification of the same ratio exhibited by the series acceptor circuit at resonance.*

12.10. POWER IN AN AC CIRCUIT

The rate of doing work or the activity of a current, in the case of a steady current, is given by VI. In an alternating current both

V and I vary harmonically and differ in phase. If $v=V_0 \sin \omega t$ is the applied emf across the circuit at any instant t and the corresponding current is given by $i=I_0 \sin (\omega t-\phi)$, where ϕ is the phase difference between the emf and the current. If emf and current are in volts and amperes respectively, then the instantaneous power in watts is given by

$$vi=V_0 I_0 \sin \omega t \sin (\omega t-\phi)$$
$$=V_0 I_0 [\sin^2 \omega t \cos \phi-\tfrac{1}{2} \sin 2\omega t \sin \phi].$$

The mean rate of doing work or the average power

$$P=\frac{1}{T}\int_0^T vidt=\frac{1}{2} V_0 I_0 \cos \phi=\frac{V_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \cos \phi$$
$$=Virtual\ volts \times Virtual\ amperes \times cos\ \phi, \qquad ...(63)$$

where the term $\cos \phi$ is known as *power factor*. It is said to be leading if current leads voltage, lagging if current lags voltage. Thus a power factor of 0.8 lagging means that current lags the voltage by $\cos^{-1} 0.8=36.8°$.

If the rms voltage V_{rms} and the rms current I_{rms}, also known as virtual values, are measured separately in an *ac* circuit, then the product $V_{rms} \times I_{rms}$ does not give the true power dissipated in the circuit as does the product VI for a *dc* circuit and thus gives *apparent power*. The *true power* or the *actual power* dissipitated is given by

$$True\ power = Apparent\ power \times Power\ factor. \qquad ...(64)$$

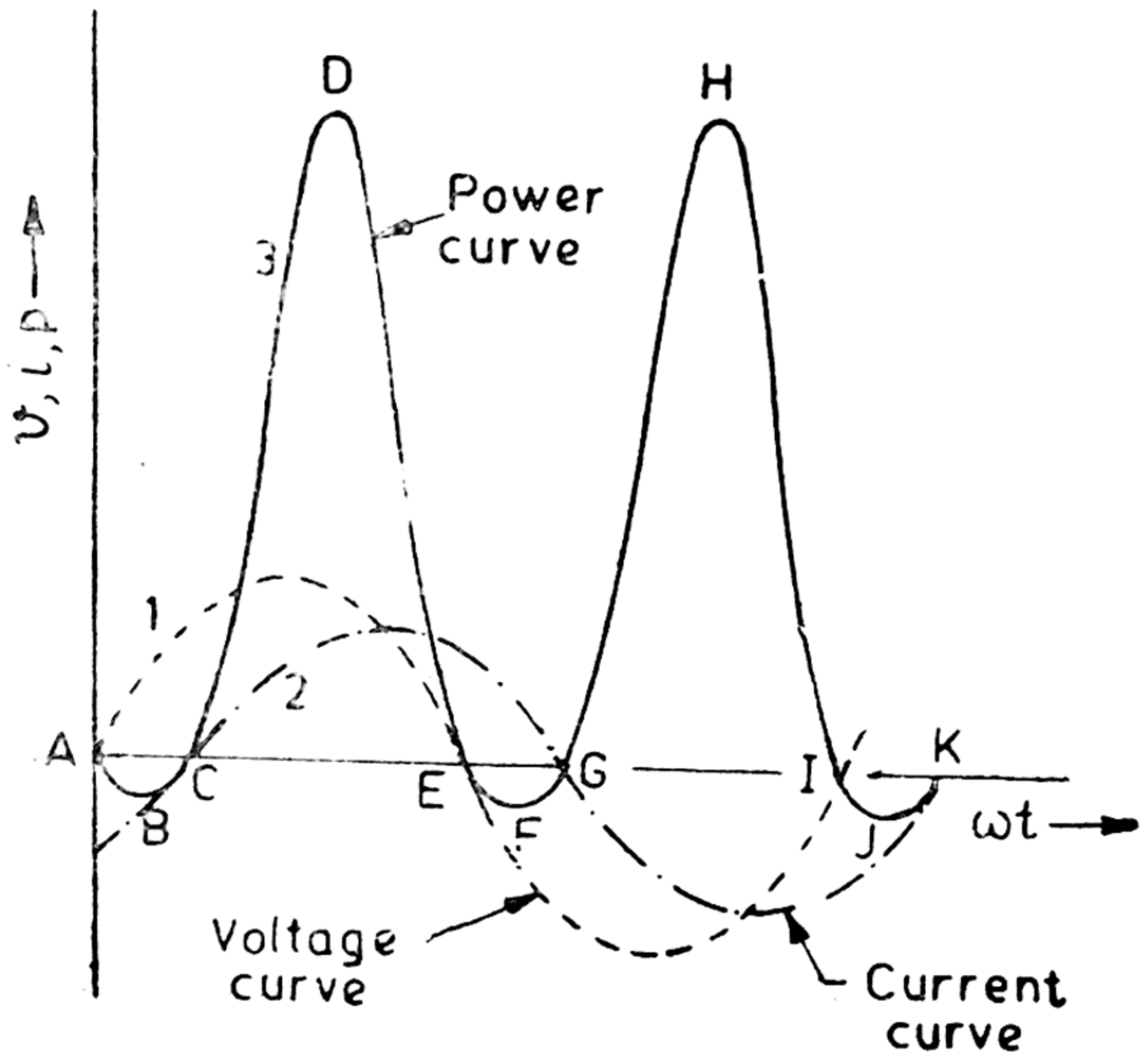


Fig. 12.15.

If the curves $v=V_0 \sin \omega t$ and $i=I_0 \sin (\omega t-\phi)$ are drawn on the same graph, the power curve can be obtained by plotting ordinates corresponding to the product vi at all times during the cycle. For the parts of the cycle CE and GI, the voltage and current are in the same direction and the *source of supply is doing work on the circuit.* For the parts AC, and EG of the cycle, the voltage and current are in opposite directions, the work is negative, *i.e.*, the *circuit is doing work on the source supply.* The difference between the positive and negative areas during the time T gives the work done in the cycle, which is equal to $V_{rms}\, I_{rms} \cos \phi$.

For $\phi=0^c$, the current and voltage are in phase and the negative portions of the graph of power (vi) against time vanish. The power is thus maximum $(V_{rms} \,.\, I_{rms})$. If $\phi=90°$, the positive areas of the power curve will equal to the negative areas. The work done is zero. The current is then stated as *wattless.* Such a case will arise when the circuit is purely inductive or capacitative. The case is similar to that of a frictionless pendulum, where the total work done by gravity upon the pendulum in a cycle is zero.

12.11. CHOKING COIL

Let us consider a coil of large inductance L and of low resistance R. The power foctor for such a coil is given by

$$\cos \phi=\frac{R}{\sqrt{(R^2+L^2\omega^2)}}=\frac{R}{L\omega} \qquad (\text{As } R \ll L\omega) \qquad ...(65)$$

As $R \ll L\omega$, $\cos \phi$ is very small and thus ϕ is nearly equal to $\pi/2$. Thus the power absorbed by the coil $V_{rms}, I_{rms} \cos \phi$ is very small. On account of its larger impedance $[\sqrt{(R^2+\omega^2L^2)}]$, the current passing through the coil is very small. Such a coil is usually known as *choking coil and is used largely in alternating current circuits for the purpose of adjusting current to any required value without waste of energy.* The only waste of energy is due to the hysteresis loss in the iron core, which is much less than the waste of energy in the resistance that can also reduce the current if placed instead of the choking coil.

Choking coils are very much used in electronic circuits, mercury lamps and sodium vapour lamps.

Quality Factor Q. The reciprocal of the power factor $\cos \phi$ is known as the quality factor and most often referred as 'Q'. It is also called as quality ratio, as it is defined as

$$Q=2\pi \frac{\textit{Maximum energy stored}}{\textit{Energy dissipated per cycle}}$$

$$= \omega \frac{\textit{Maximum energy stored}}{\textit{Average power dissipated}}. \qquad ...(66)$$

Thus, for an inductor with impedance L and series resistance R, Q value is given as

$$Q_L=\omega\ \frac{LI_{rms}^2}{RI_{rms}^2}=\frac{\omega L}{R}. \qquad ...(67)$$

Similarly, the quality factor of a capacitor can be shown to be

$$Q_C=1/\omega CR. \qquad ...(68)$$

Values of Q_L and Q_C are same as that of the values of cos ϕ of good quality of inductor and capacitor respectively. In practice no capacitor or a inductor is pure, hence the power factor is different from zero.

12.12. SHARPNESS OF RESONANCE IN SERIES CIRCUIT

To find the sharpness two values of ω for which R/Z is $1/\sqrt{2}$ times its maximum value and their difference is determined. The value of Z will be $\sqrt{2}R$ when $X^2=R^2$, or $X=\pm R$. Hence we can write two values of ω as

$$X=\omega_1 L-1/\omega_1 C=R \text{ and } X=\omega_2 L-1/\omega_2 C=-R.$$

$$\therefore \qquad (\omega_1+\omega_2)L-\frac{\omega_1+\omega_2}{\omega_1\omega_2 C}=0 \quad \text{or} \quad \omega_1\omega_2=\frac{1}{LC}=\omega_0^2,$$

and $$(\omega_1-\omega_2)L+\frac{\omega_1-\omega_2}{\omega_1\omega_2 C}=2R \text{ or } \omega_1-\omega_2=\frac{R}{L}.$$

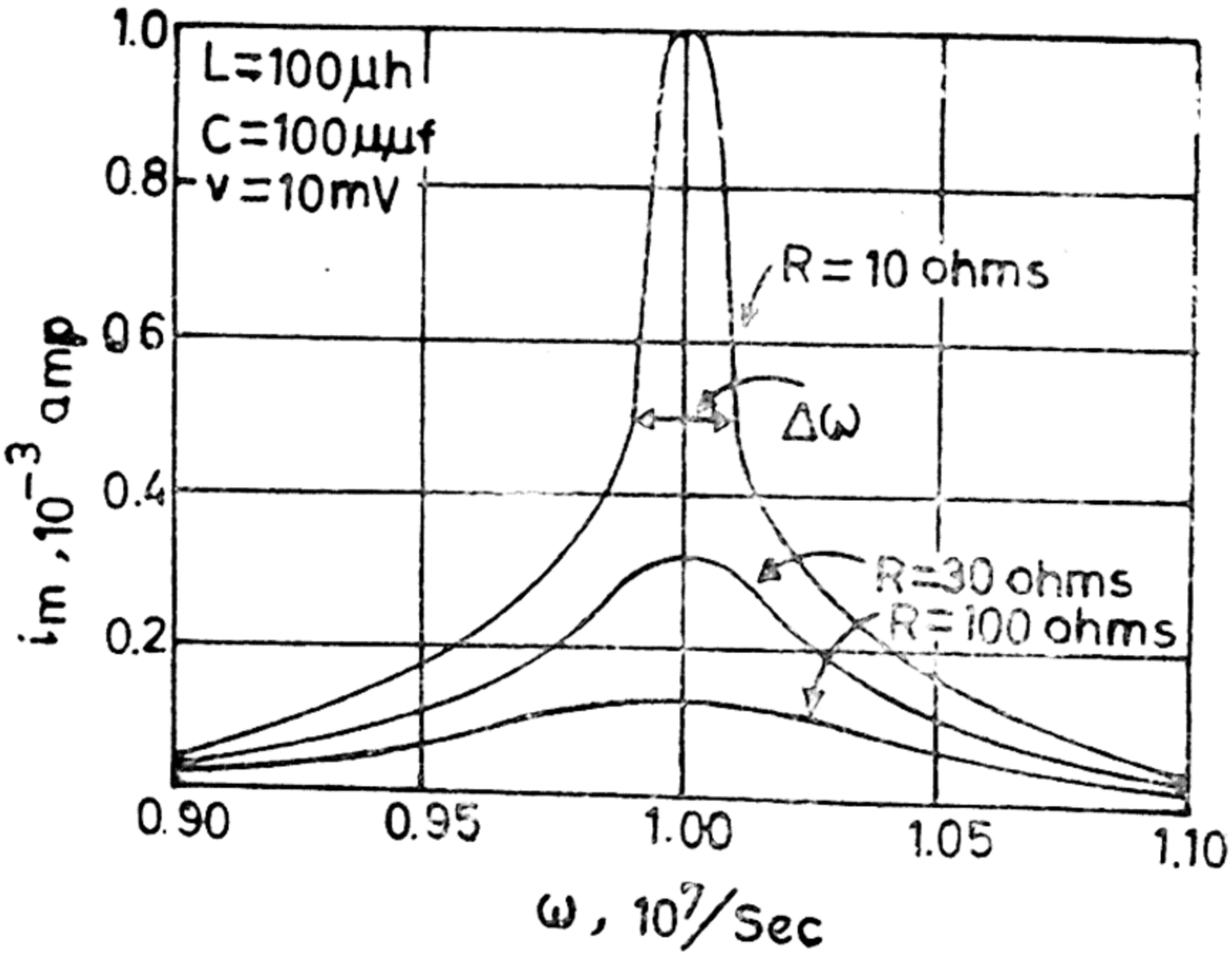


Fig. 12.16.

Hence we can write $$\frac{\omega_1-\omega_2}{\omega_2}=\frac{R}{\omega_0 L}=\frac{1}{Q_0}. \qquad ...(69)$$

Thus for the demand of sharp resonance curves (*sharp tuning*) a large Q_0 is desirable.

The difference between two half power frequencies (f_1-f_2) is known as the *band width* of the resonant curve. Thus

$$B.W=f_1-f_2=f_0/Q_0=R/2\pi L. \qquad ...(70)$$

It might be thought that the Q value of a coil could be made very large at high frequencies, but it is not the case. The time constant L/R can be made as large as 1 second by the use of magnetic core materials. But effective permeability decreases as frequency increases. At high frequency the skin effect increases R and the effects of distributed capacitance between the windings make the coil act like a smaller effective inductance.

12.13. COUPLED CIRCUIT

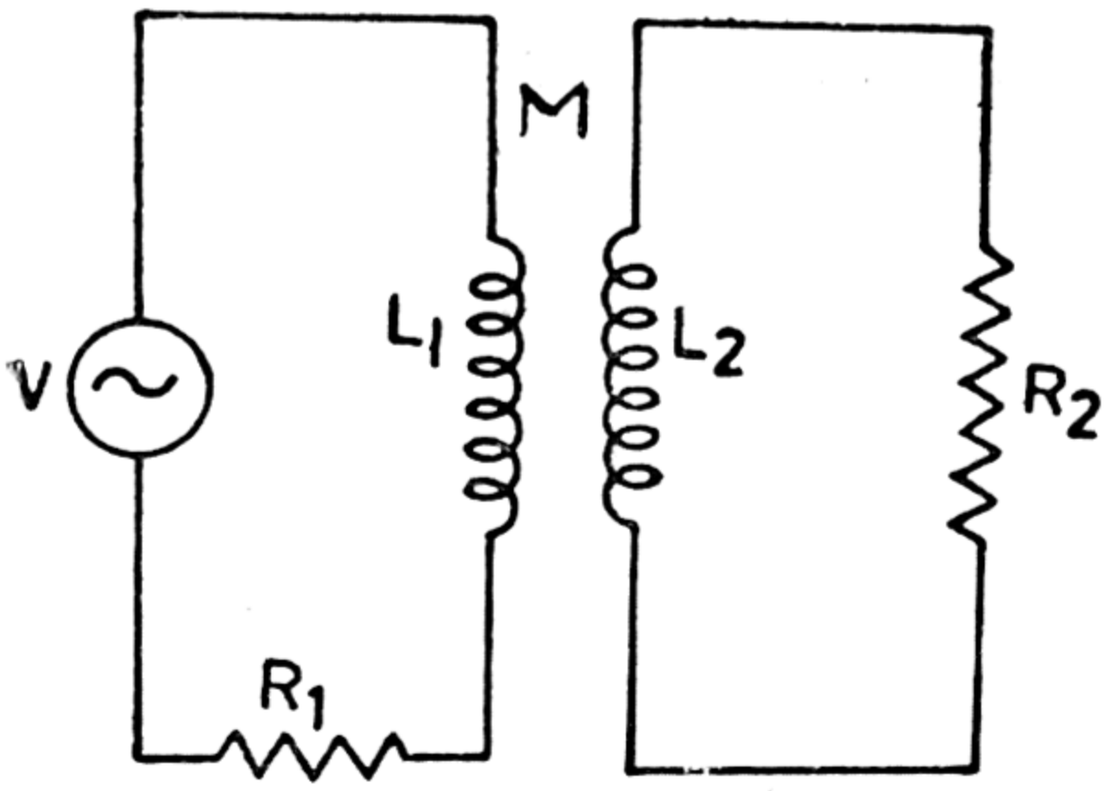


Fig. 12.17.

Let us consider an alternating emf, represented by $v=V_0e^{j\omega t}$, applied to the primary of mutual inductance M, the primary of which has a self inductance L_1 and a resistance R_1 and secondary is a closed circuit of inductance L_2 and a resistance R_2. If $\mathbf{I}_1$ and $\mathbf{I}_2$ be the currents in primary and secondary respectively, then

$$\mathbf{V}=(R_1+j\omega L_1)\mathbf{I}_1+j\omega M\mathbf{I}_2$$
$$=\mathbf{Z}_1\mathbf{I}_1+j\omega M\mathbf{I}_2 \quad \ldots(71)$$

and $$0=(R_2+j\omega L_2)\mathbf{I}_2+j\omega M\mathbf{I}_1=\mathbf{Z}_2\mathbf{I}_2+j\omega M\mathbf{I}_1. \quad \ldots(72)$$

On solving these equations, we get

$$\mathbf{I}_1=\frac{\mathbf{V}}{\mathbf{Z}_1+\omega^3M^2/\mathbf{Z}_2}=\frac{\mathbf{V}}{\mathbf{Z}_1'}, \quad \ldots(73)$$

and $$\mathbf{I}_2=-\frac{j\omega M\mathbf{V}}{\mathbf{Z}_1\mathbf{Z}_2+\omega^2M^2}=\frac{\mathbf{V}}{\mathbf{Z}_2'}, \quad \ldots(74)$$

where $\mathbf{Z}_1'$ and $\mathbf{Z}_2'$ are two effective impedance operators for the primary and secondary circuits.

$$\mathbf{Z}_1'=R_1+j\omega L_1+\frac{\omega^2M^2}{R_2+j\omega L_2}\cdot\frac{R_2-j\omega L_2}{R_2-j\omega L_2}$$
$$=R_1+\frac{\omega^2M^2}{R_2^2+\omega^2L_2^2}R_2+j\omega\left[L_1-\frac{\omega^2M^2}{R_2^2+\omega^2L_2^2}L_2\right]$$

or $$\mathbf{Z}_1'=R_1'+j\omega L_1'$$

where R_1' is the effective primary resistance which is greater than R_1 by $\omega^2M^2R_2/(R_2^2+\omega^2L_2^2)$ due to the so-called *reflected resistance* from the secondary and L_1' is the effective primary inductance which is less than L_1 by $\omega^2M^2L_2/(R_2^2+\omega^2L_2^2)$ due to the *reflected inductance* from the secondary. The reflected resistance is large and reaches a maximum value for the value of $R_2=\omega L_2$. It is also clear that the effective self inductance of primary can never becomes negative as even for $R_2\to0$, $L_1'\to(L_1-M^2/L_2)$ or a positive quantity as $M^2\ll L_1L_2$ always. Thus the primary current

$$\mathbf{I}_1=\mathbf{V}/(R_1'+j\omega L_1')$$

or
$$I_1=\frac{V_0e^{j\omega t}}{(R_1'^2+\omega^2L_1'^2)^{1/2}\,e^{j\theta}}=\frac{V_0e^{j(\omega t-\theta)}}{(R_1'^2+\omega^2L_1'^2)^{1/2}}$$

$$=\frac{V_0}{Z_1'}\;e^{i(\omega t-\theta)},$$

where $\tan\theta=\omega L_1'/R_1'$.

Thus we see that $\mathbf{I}_1$ always legs behind **V**. The phase lag of current on voltage for the primary alone is $\tan^{-1}\omega L_1/R_1$. On coupling with the circuit, L_1 and R_1 are replaced by L_1' and R_1' respectively. As $R_1'>R_1$ and $L_1'<L_1$, the phase lag reduces when such coupling takes place. When $R_2\to0$, this lag decreases and the coupling becomes more tight. The cosine of the phase angle becomes larger as phase angle becomes smaller and thus the *power factor and so the power taken from an ac source increases on coupling the secondary to the primary.*

The current in the secondary

$$\mathbf{I}_2=\frac{-j\omega M}{Z_2}\quad\mathbf{I}_1=\frac{-j\omega M}{R_2+j\omega L_2}\quad\mathbf{I}_1=\frac{\omega Me^{-j\pi/2}}{\sqrt{(R_2^2+\omega^2L_2^2)}\;e^{j\varphi}}\mathbf{I}_1$$

$$=\frac{\omega M}{\sqrt{(R_2^2+\omega^2L_2^2)}}\;e^{-j(\pi/2+\phi)}\,\mathbf{I}_1, \qquad \ldots(77)$$

where $\tan\phi=\omega L_2/R_2$.

Thus we see that the secondary current lags the primary current by a phase angle of $(\pi/2+\phi)$.

12.14. TRANSFORMERS

The transformer consists essentially of a magnetic field circuit linking with two distinct windings. When one winding is connected to an a.c. supply an alternating flux will be set up in the core and this flux linking with the another will induce an alternating emf in it. The former winding is known as *primary* and another as *secondary*.

Transformers are used for converting an *ac* of high voltage and low current into a current of lower voltage and high current and *vice versa* with small energy loss without using moving parts. The former is known as *step-down* and the latter *step-up* transformer.

Types of Transformers. There are three main types of transformers depending on the structure. (1) *Core type* with single magnetic circuit, (2) *Shell type* with double magnetic circuit and (3) *Berry type* with distributed magnetic circuit.

The *cores* are built up of sheet iron or alloyed steel. The small transformer cores have rectangular cross sections, but the large sizes have circular cross sections. The *windings* are either of the *cylindrical* or *sandwich* type. The former type is mainly used on core transformers and the latter type on the shell type.

There are three types of transformers used mainly, depending on the frequency of the source in the circuit.

(*a*) **Mains frequency transformers.** These employ an iron core, laminated to avoid eddy current loss and occur in a very wide range of sizes from small laboratory components to the big transformers used at power stations. The frequency of *ac* mains is 50 c/s in Britain and India and is 60 c/s in U.S.A.

(*b*) **Audio-frequency transformers.** These are used chiefly in radio communication and electronic circuits and contain a laminated iron core to increase coupling efficiency. They operate over the frequency range from 50 to 20,000 c/s.

(*c*) **Radio-frequency transformers.** These can not have iron cores because of large eddy current losses. Transformers without core, known as aircored also suffer from the disadvantages that the coupling coefficient and efficiency are low. More efficient device employs a *dust core*, consisting of a fine powder of iron mixed with an insulating binder and compressed into a core of the desired shape. These transformers are used at frequencies from 20 kc/s to 30 Mc/s.

Transformer losses. In practice the power output in the secondary is always less than the power input in the primary because of the losses due to imperfection of design. These losses are :

(*i*) *Copper losses.* The losses are due to the heating effects of inevitable resistances of the primary and secondary windings.

(*ii*) *Core losses.* The hysteresis and eddy current heating losses in the core constitutes core losses. The former are minimized by the use of iron having a narrow hysteresis loop or small expenditure of energy and the latter losses are minimized by using the core of laminated sheets of soft iron which are mutually insulated.

(*iii*) *Magnetic leakage.* These losses are due to the leakage of magnetic flux between primary and secondary. In other words all the lines of magnetic flux produced by the primary current do not cut the secondary winding and the flux produced by the secondary current do not cut the primary winding. The leakage is reduced to a minimum if the two coils are interwound. For the good insulation between primary and secondary, winding should be done side by side.

(*iv*) *Self capacity of winding.* The capacitances exist (*i*) between each winding of a transformer, due to the inter turns and inter layers and (*ii*) between every pair of winding. These capacitances have a large effect on the audiofrequency transformers and have practically no effect at low frequencies (power frequencies).

Efficiency. On account of the losses, the power taken from the secondary of a transformer cannot exceed that taken by the primary from the source. The efficiency of a given transformer is thus defined as

$$\eta = \frac{\text{Power output from the secondary}}{\text{Power delivered by the source}} \times 100\%$$

$$= \frac{\text{Power output}}{\text{Power input} + \text{Power lost}} \times 100\% \qquad \ldots(78)$$

The ideal transformer, with no losses, has efficiency one.

From a constructional point of view, a *transformer in its simplest form is a mutual inductance.* Using the results of the article 12.13. Let us consider following two cases :

(*a*) **Open circuit secondary.** In an open circuit secondary, there is no secondary current, *i.e.*, $I_2 = 0$, thus equation (71) gives

$$\because \qquad \mathbf{V} = \mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1$$

or
$$I_1 = V_1/\sqrt{(R_1^2 + \omega^2 L_1^2)}. \qquad \ldots(79)$$

The open circuit secondary emf is therefore given by

$$V_2 = \omega M I_1 = \omega M V_1/\sqrt{R_1^2 + \omega^2 L_1^2}). \qquad \ldots(80)$$

Since $R_1^2 \ll \omega^2 L_1^2$ and coupling coefficient $k = M/\sqrt{(L_1 L_2)}$, hence we have

$$\frac{V_2}{V_1} = \frac{\omega M}{\sqrt{(R_1^2 + \omega^2 L_1^2)}} = \frac{M}{L_1} = k\sqrt{\left(\frac{L_2}{L_1}\right)}.$$

If N_1 and N_2 be the numbers of primary and secondary turns respectively, then $L_1 \propto N_1^2$ and $L_2 \propto N_2^2$ and hence

$$V_2/V_1 = k\sqrt{(L_2/L_1)} = kN_2/N_1. \qquad \ldots(81)$$

For a perfect coupling, $k = 1$. It means that the primary and secondary coils both enclose the whole of the magnetic flux produced by the primary. Iron cored transformer is used in practice to produce so high value of k, with nearry 1 and with large values of L_1 and L_2. Really tne value of k reaches to 0.99 only. Thus for *a perfect coupling and an open circuited secondary the transformation ratio* $V_2/V_1 = N_2/N_1$, *the ratio of the number of turns in the secondary coil to that in the primary.* If the resistance R_1 of the primary coil is small in comparison to the reactance ωL_1 of its inductance, the current is *wattless and the power loss is zero. Thus a transformer with open circuited secondary and a low resistance primary dissipates negligible energy.*

(*b*) **Closed circuited secondary.** When the secondary of the transformer is connected to a load resistance R_2, the problem becomes that of coupled circuit (article 12.13), and the effective values of the resistance and inductance of the primary circuit are given by

$$R_1' = R_1 + \frac{\omega^2 M^2 R_2}{R_2^2 + \omega^2 L_2^2} \quad \text{and} \quad L_1' = L_1 - \frac{\omega^2 M^2 L_2}{R_2^2 + \omega^2 L_2^2}.$$

If the resistance R_2 is small compared with ωL_2, then for the iron cored transformer for which the coupling coefficient $k=1$ and $M=\sqrt{(L_1L_2)}$, we get

$$R_1'=R_1+\frac{L_1L_2}{L_2^2} \qquad R_2=R_1+\frac{L_1}{L_2}R_2$$

and
$$L_1'=L_1-\frac{L_1L_2}{L_2}=0. \qquad \ldots(82)$$

Thus we see that with a load of low resistance across a secondary of transformer, the effective impedance of primary is purely resistive in nature.

Now from Eqs. (73) and (74), the ratio of the secondary to the primary current is given as

$$\frac{\mathbf{I}_2}{\mathbf{I}_1}=\frac{j\omega M}{R_2+j\omega L_2}.$$

For a transformer with a perfect coupling, $M=\sqrt{L_1L_2}$ and the resistances of the two coils of the transformer are nearly zero. In addition the reactances of the primary and secondary coils are so large in comparison with the generator and load impedances. Therefore

$$\frac{\mathbf{I}_2}{\mathbf{I}_1}=\frac{j\omega M}{j\omega L_2}=\frac{M}{L_2}=\sqrt{\frac{L_1}{L_2}}=\frac{N_1}{N_2}. \qquad \ldots(83)$$

Further the ratio of voltage drops across the primary and secondary coils are

$$\frac{\mathbf{V}_2}{\mathbf{V}_1}=\frac{R_2\mathbf{I}_2}{\mathbf{V}-\mathbf{I}_1R_1}$$

$$=\frac{j\omega MR_2\mathbf{V}}{[(R_1+j\omega L_1)(R_2+j\omega L_2)+\omega^2M^2][\mathbf{V}-\mathbf{I}_1R_1]}.$$

Since $R_1<j\omega L_1$, hence

$$\frac{\mathbf{V}_2}{\mathbf{V}_1}=\frac{j\omega MR_2}{j\omega L_1R_2-\omega^2L_1L_2+\omega^2M^2}$$

$$=\frac{j\omega MR_2}{j\omega L_1R_2}=\frac{M}{L_1}$$

$$=\sqrt{(L_2/L_1)}=N_2/N_1. \qquad \ldots(84)$$

Thus we see that for the transformation ratio (V_2/V_1) is equal to the ratio of the turns in the secondary to that in the primary. For $N_2>N_1$ we get $V_2>V_1$ and $I_1>I_2$. Thus, the step-up transformer increases the voltage and decreases the current whereas a step-down transformer reduceds the voltage and increases the current.

The equations (83) and (84) give

$$I_1V_1 = I_2V_2$$

i.e., *Input power = Output power.*

Thus an ideal transformer introduces no loss and has 100% efficiency. In practice only an efficiency of about 95% can be achieved in *power transformers.*

In the preceeding article, the problem was simplified by making the L's the total inductances, *i.e.*, including both leakage flux and mutual flux. Let us now discuss the problem in which the various fluxes are regarded as separate. The various components of the primary applied voltage are as follows :

(*i*) The voltage $j\omega L_1\mathbf{I}_1$ to overcome the self-induced emf due to the true leakage reactance L_1.

(*ii*) The voltage $j\omega M\mathbf{I}_2$ to overcome the induced emf due to the changes in current in the secondary.

(*iii*) The voltage $j\omega(M/\rho)\mathbf{I}_1$ to overcome the induced emf due to the changes in current in the primary, where ρ is the turns ratio N_2/N_1.

Hence the emf in the primary is given as

$$\mathbf{V}_1 = R_1\mathbf{I}_1 + j\omega L_1\mathbf{I}_1 + j\omega(M/\rho)\mathbf{I}_1 + j\omega M\mathbf{I}_2. \qquad \ldots(85)$$

The emf induced in the secondary $-j\omega M\mathbf{I}_1$ is used to overcome (*i*) the self inductance of the secondary $j\omega\rho M\mathbf{I}_2$ and (*ii*) the total impedance drop of both secondary and load. Therefore, we have

$$\mathbf{V}_2 = -j\omega M\mathbf{I}_1$$

$$= R_2\mathbf{I}_2 + j\omega L_2\mathbf{I}_2 + j\omega\rho M\mathbf{I}_2.$$

$$\therefore \quad \mathbf{I}_2 = -j\omega M\mathbf{I}_1/(R_2 + j\omega L_2 + j\omega\rho M)$$

or

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = -\left(\rho + \frac{R_2 + j\omega L_2}{j\omega M}\right).$$

Substituting the value of $\mathbf{I}_2$ in equation (85), we have

$$\mathbf{V}_1 = \left[R_1 + j\omega L_1 + \frac{j\omega M}{\rho} + \frac{\omega^2M^2}{R_2 + j\omega L_2 + j\omega\rho M}\right]\mathbf{I}_1$$

$$= \left[R_1 + j\omega\left(L_1 + \frac{M}{\rho}\right) + \frac{\omega^2M^2[R_2 - j\omega(L_2 + \rho M)]}{R_2^2 + \omega^2(L_2 + \rho M)^2}\right]\mathbf{I}_1$$

$$= \mathbf{I}_1\left\{R_1 + \frac{\omega^2M^2R_2}{R_2^2 + \omega^2(L_2 + \rho M)^2} + j\omega\left[L_1 + \frac{M}{\rho} - \frac{\omega^2M^2(L_2 + \rho M)}{R_2^2 + \omega^2(L_2 + \rho M)^2}\right]\right\}$$

$$= \mathbf{I}_1[R' + j\omega L'] \qquad \ldots(86)$$

This shows that R' and L' are the equivalent resistance of the whole circuit and the equivalent inductance of the whole circuit respectiveiy.

The ideal transformer, having no losses and in which all of the flux is confined to the iron core is shown schematically in Fig. 12.18. A primary winding of N_1 turns and a secondary winding of N_2 turns both encircle the core in the same sense. As the secondary is open, there is no secondary current. In the vector diagram, vector $\mathbf{V}_1$ represents the primary voltage As the resistance in primary is assumed to be zero, the primary current known as the magnetizing

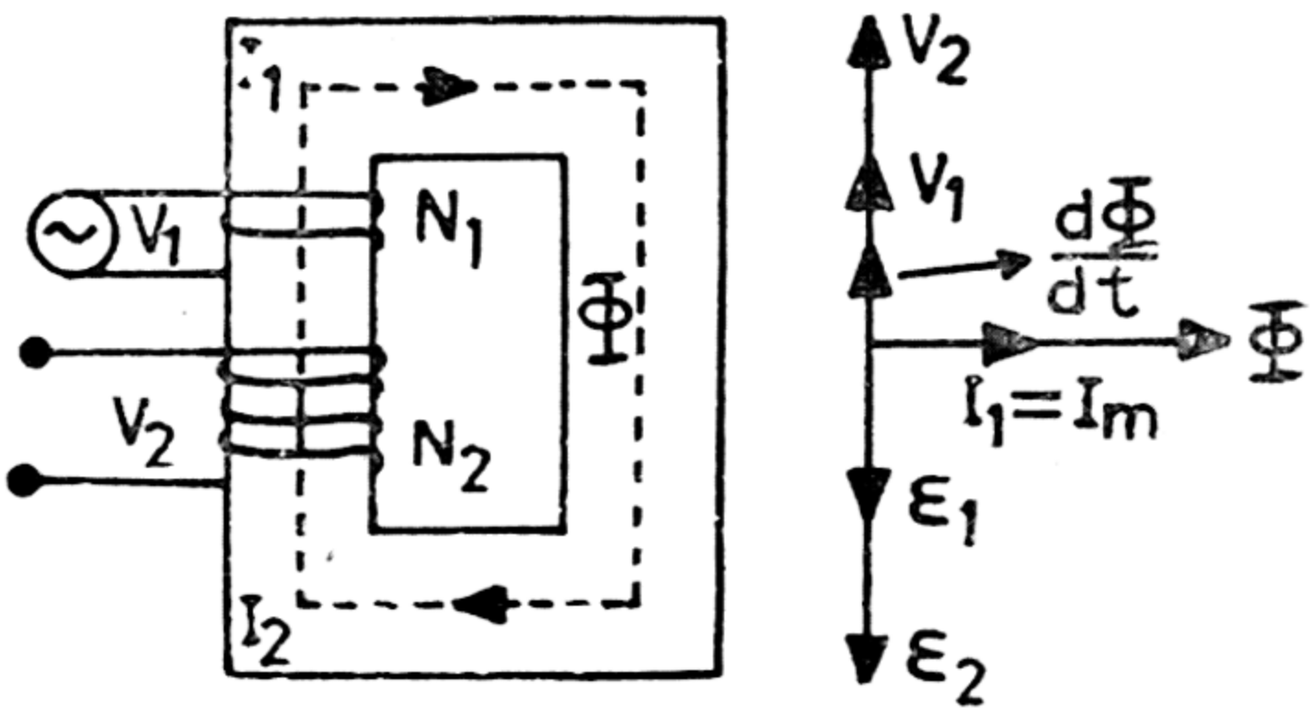


Fig. 12.18.

current I_m lags the primary voltage by 90° having magnitude V_1/X_1, where X_1 is the reactance of the primary. This current sets up a flux of a amplitude Φ in the core, in phase with the current. The rate of change of flux $d\Phi/dt$ will lead Φ by 90°. By Faraday's law, we know that the induced emf $\mathcal{E} \propto -d\Phi/dt$. This —ve sign indicates that this emf is out of phase with $d\Phi/dt$. In the diagram, $\mathcal{E}_1$ and $\mathcal{E}_2$ represent the induced voltages at the primary and secondary terminals respectively.

If a resistance R is connected across the secondary. In this case as the primary voltage V_1 is fixed, the rate of change of flux $d\Phi/dt$ must be the same, as shown in Fig. 12.19. Hence the core flux Φ and the secondary voltage V_2 must also be the same. The core flux is now set up both by the primary current I_1 and by the secondary current I_2. The secondary current I_2 is in phase with V_2 and of magnitude V_2/R. The flux Φ_2, set up by the secondary current is opposite in phase to this current. The flux Φ_1 due to the primary current is the vector sum of Φ and Φ_2. The primary current I_1 is in phase with Φ_1.

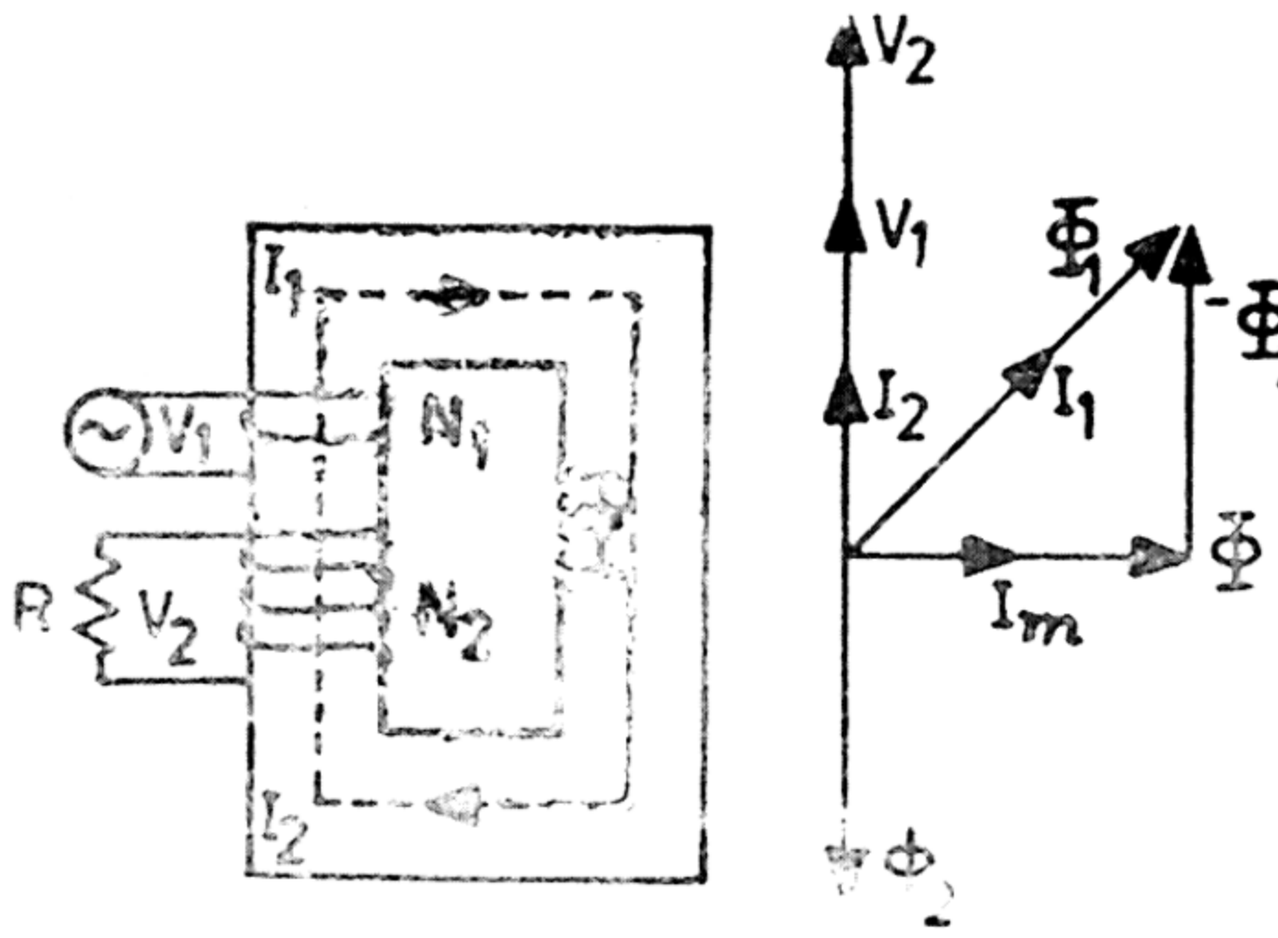


Fig. 12.19.

Uses of Transformers

(1) The *step-up and step-down transformers* are used in a.c. electrical power distribution for the domestic and industrial purposes.

(2) The *audio-frequency transformers* are used in radio-receivers, radio-telephony, radio-telegraphy and in televisions.

(3) The *radio frequency transformers* are used in radio-communications at frequencies of the order of mega-cycles.

(4) The *impedance transformers* are used for impedance matching, which is necessary to get maximum power from a source, of two independent circuits.

(5) The *current transformers* are used for measuring large alternating currents. The current to be measured is passed through the primary and the secondary being connected to the ammeter.

(6) The *pressure transformers* are used for measuring very high pressures. The pressures to be measured is stepped down by means of such a transformer, the low tension secondary of it is connected to the voltmeter.

(7) For measuring power on a high pressure system, both current and pressure transformers are used.

(8) The *welding transformers* are used for welding as well as in induction furnace The ends of one or two turns of thick copper wire, forming the secondary of the transformer ; are connected to a pair of rods to be welded. When the current is on, the tips of the rods get red hot and started welding.

(9) *Long distance power transmission* From the point of view of both efficiency and economy, the power must be transmitted at high voltage [suppose a given power say, 44,000 watts) is transmitted from a generating station to a long distant city, it can

be transmitted at a voltage 220 volts and a current of 200 amp or at a voltage of 22000 volts and a current of 2 amp. In the former case the large amount of energy (i^2Rt) will be lost as heat during transmission, a voltage drop (iR) along the line will be greater, and a very thick line wire will be required to carry the high current]. Hence the step-up transformer should be used at the generating station before transmission and the step-down transformer should be used at receiving station before its supply to the city.

12.15. ROTATING MAGNETIC FIELD AND INDUCTOR MOTOR

A rotating magnetic field is one in which the flux rotates round a fixed axis. Its magnitude remains constant while the direction changes at a constant rate. It can be produced by (*i*) merely rotating a magnet or a current carrying coil or (*ii*) a system of stationary coils supplied with single, two, three or polyphase currents.

Let us consider two coils C_1 and C_2 carrying alternating current and placed at right angles to each other. If the currents in these coils are respectively I_1 and I_2 which are equal in frequency ω, but differing in phase by θ, then we have

$$I_1 = I_{01} \sin (\omega t + \theta)$$

and

$$I_2 = I_{02} \sin \omega t. \qquad \ldots(87)$$

The magnetic fields due to these current carrying coils will be in phase with the currents and may be represented as

$$B_1 = B_{01} \sin (\omega t + \theta)$$

and

$$B_2 = B_{02} \sin \omega t. \qquad \ldots(88)$$

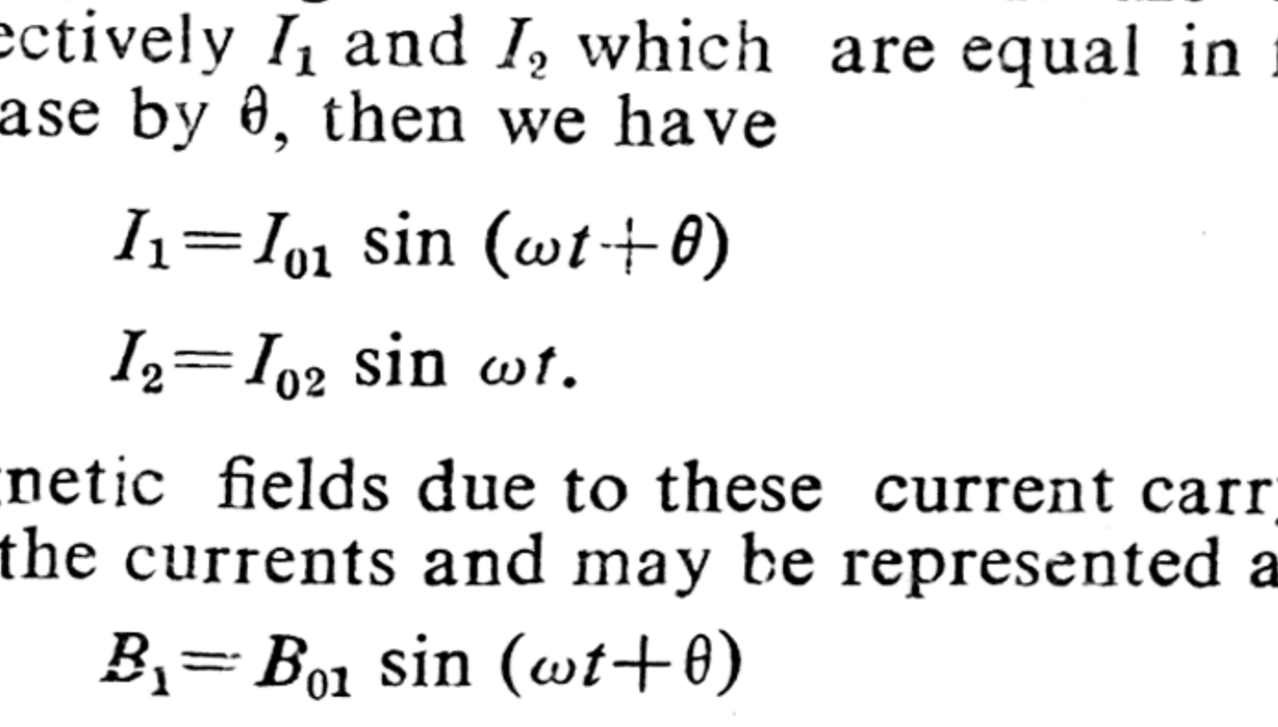


Fig. 12.20.

Since the coils are at right angles to each other, the resultant magnetic field is given by

$$B = \sqrt{B_1^2 + B_2^2}. \qquad \ldots(89)$$

It makes an angle ψ with B_1, such that $\tan \psi = B_2/B_1$.

The resultant field vary periodically as :

At $t=0$ $B_2=0$, $B=B_1=B_{01}\sin\theta$

$t=-\theta/\omega$, $B_1=0$, $B=B_2=B_{02}\sin(-\theta)$
$=-B_{02}\sin\theta$

$t=\pi/\omega$ $B_2=0$, $B=B_1=B_{01}\sin\theta$

$t=(\pi-\theta)/\omega$, $B_1=0$, $B=B_2=B_{02}\sin(\pi-\theta)$
$=-B_{02}\sin\theta$.

Thus the *resultant magnetic field B rotates (in an anticlockwise direction) and at the same time varies periodically in magnitude.*

The most important instant is when the magnetic fields have equal amplitudes and are in quadrature, *i.e.*, $B_{01}=B_{02}=B_0$ (say) and $\theta=\pi/2$. Under such conditions

$$B=B_0\sqrt{\sin^2(\omega t+\pi/2)+\sin^2\omega t}=B_0$$

and $$\tan\psi=\sin\omega t/\sin(\omega t+\pi/2)=\tan\omega t,$$

or $$\psi=\omega t. \qquad \ldots(90)$$

Thus the resultant magnetic field B is constant in magnitude and rotates with constant angular velocity ω, in anti-clockwise direction if the fields B_1 and B_2 are of same amplitude and frequency and B_1 is $\pi/2$ ahead of B_2 in phase.

If, on the other hand B_1 lags behind B_2 by $\pi/2$ in phase, *i.e.*, $\theta=-\pi/2$, we have

$$B_1=B_0\sin(\omega t-\pi/2)=-B_0\cos\omega t$$

and $$B_2=B_0\sin\omega t.$$

$\therefore$ $$B=\sqrt{B_1^2+B_2^2}=B_0$$

and $$\tan\psi=-\frac{\sin\omega t}{\cos\omega t}=-\tan\omega t$$

or $$\psi=-\omega t$$

and the rotation is thus clock-wise.

Couple acting on a coil placed in a rotating magnetic field. It has been shown above that a rotating magnetic field may be produced in a region surrounding a fixed point O by passing currents, equal in amplitude and frequency, but differing in phase by $\pi/2$, in two coils placed at right angles to each other. If a small coil of N turns and effective area A is placed in the rotating magnetic field such that the centre of the coil coincides the point O, the induced current will be produced in coil. In accordance with Lenz's law the magnetic effects of such induced currents and the original field will interact in such a manner as to reduce the relative motion between the coil and the field. Thus a couple acts on a coil, tending it to rotate in the direction of the resultant field.

Let us calculate the couple experienced by the coil when placed in a rotating magnetic field. The total magnetic flux Φ linked with the coil at any instant t is given by

$$\Phi = NAB \sin \omega t,$$

where ωt is the angle between the direction of magnetic field B and the plane of the coil.

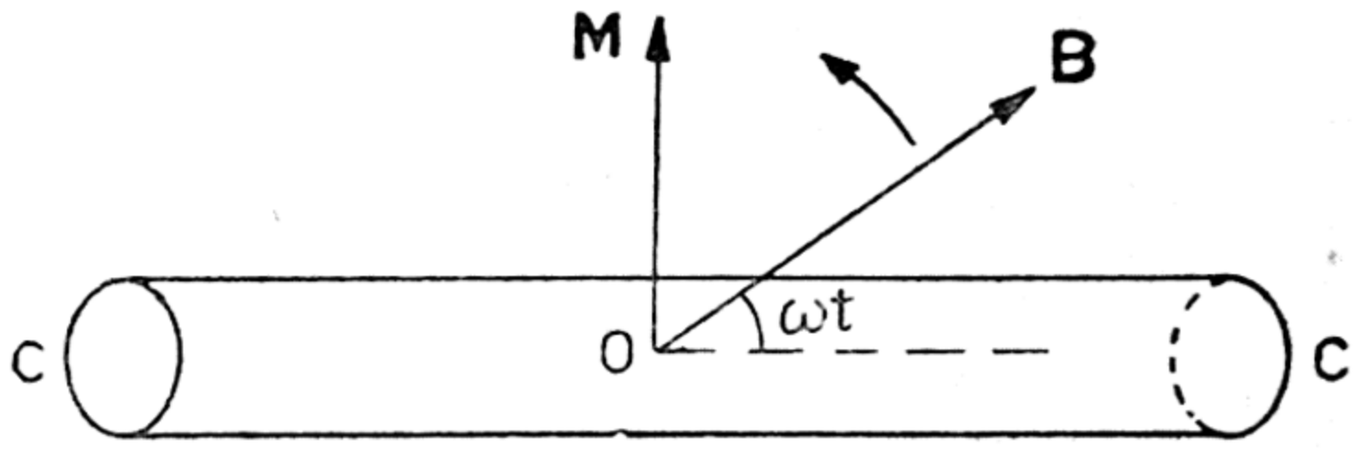


Fig. 12.21.

Due to the change of flux, the induced emf in the coil will be

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\ (NAB \sin \omega t)$$

$$= NAB\ \omega \cos (\omega t - \pi).$$

If L is the self inductance and R the resistance of the coil, then the induced current in the coil will be

$$i = \frac{NAB\omega}{\sqrt{R^2 + \omega^2 L^2}} \cos (\omega t - \pi - \alpha), \qquad \ldots (91)$$

where $\tan \alpha = \omega L/R$.

This current gives rise to a magnetic field and the coil will behave as a magnetic shell of magnetic moment NAi in the direction perpendicular to its plane. The couple acting on the coil will turn it into the direction of **B**. The moment of the couple is given by

$$\tau = \mathrm{M} \times \mathbf{B}$$

or
$$\tau = MB \sin (\omega t - \pi/2) = -BNAi \cos \omega t$$

$$= -\frac{N^2A^2B^2\omega}{\sqrt{R^2 + \omega^2 L^2}} \cos (\omega t - \pi - \alpha) \cos \omega t$$

$$= \frac{N^2A^2B^2\omega}{\sqrt{R^2 + \omega^2 L^2}} [\cos^2 \omega t \cos \alpha + \sin \omega t \cos \omega t \sin \alpha].$$

Since the average value of $\cos^2 \omega t$ over one cycle is $\frac{1}{2}$ and that of $\sin \omega t \cos \omega t$ is zero. Therefore, the mean couple acting on the coil is

$$\tau_{av} = \frac{1}{2}\ \frac{N^2A^2B^2\omega}{\sqrt{R^2 + \omega^2 L^2}} \cos \alpha = \frac{1}{2}\ \frac{N^2A^2B^2\omega R}{(R^2 + \omega^2 L^2)}. \qquad \ldots (92)$$

The direction of the mean couple will be the direction of rotation of the field.

The above relation shows that the mean couple depends on the value of the relative angular velocity of the field with respect to the coil. It is zero if $\omega = 0$ or ∞. If a coil is free to rotate, its angular velocity in the direction of the field rotation will increase till the rate of doing work against the friction of all kinds is equal to the rate of work done by the rotating field. When the coil is released its angular velocity increases at first. This will decrease the relative angular velocity ω of the field with respect to the coil. Thus the couple experienced by the coil will increase, which will further speed up the coil and decrease ω. However, the angular velocity of the coil will never be equal to that of the field, as in that case $\omega = 0$ and $\tau = 0$. The average couple is maximum for same value of ω between zero and infinity, can be obtained by differentiating τ_{av} w.r.t. ω and putting it as zero, *i.e.*,

$$\frac{d\tau_{av}}{d\omega} = \frac{N^2A^2B^2R}{2} \frac{d}{d\omega}\left(\frac{\omega}{R^2+\omega^2L^2}\right)=0$$

or
$$\frac{N^2A^2B^2R}{2} \cdot \frac{R^2+\omega^2L^2-\omega\,(2\omega L^2)}{(R^2+\omega^2L^2)^2}=0$$

or $L^2\omega^2 = R^2$ or $\omega = R/L.$ ...(93)

As the double differentiation of τ_{av} w.r.t. ω is negative for $\omega = R/L$, hence τ_{av} is maximum for $\omega = R/L$.

Again $\tan \alpha = \omega L/R = 1$ or $\alpha = 45°$. Thus the value of α for maximum couple is $45°$ and the value of maximum couple is

$$\tau_{av,\,max} = \frac{N^2A^2B^2\,(R/L)\,R}{2\,(R^2+R^2)} = \frac{N^2A^2B^2}{4L}. \quad ...(94)$$

Induction Motor. The motion of a conductor in a rotating magnetic field has been used in the construction of electromotors besides many other partial applications. In an induction motors two pairs of electromagnets AB and CD carry alternating current, differing in phase ; this results in a rotating magnetic field. One of the coils AB is called the main magnetising coil and the other CD is called the shunt coil, the two being at right angles to each other. As the coils are supplied ac by the same source, a phase difference can be introduced by putting an extra inductance L into circuit CD or a capacity C into the circuit AB or both. If the electromagnets are similar, then the magnetic fields will have same amplitudes and the resultant magnetic field will be a rotating field.

A mass of metal or a system of closed coils, known as rotor, is mounted upon an axle in the rotating field. It will experience a deriving couple as discussed above. There are two types of rotor winding, the *squirrel cage* and the *phase wound*. The former consists of a number of copper bars threaded through slots in a

laminated rotor core, with their ends connected to strong copper end rings, and is used only for comparatively small motors. The

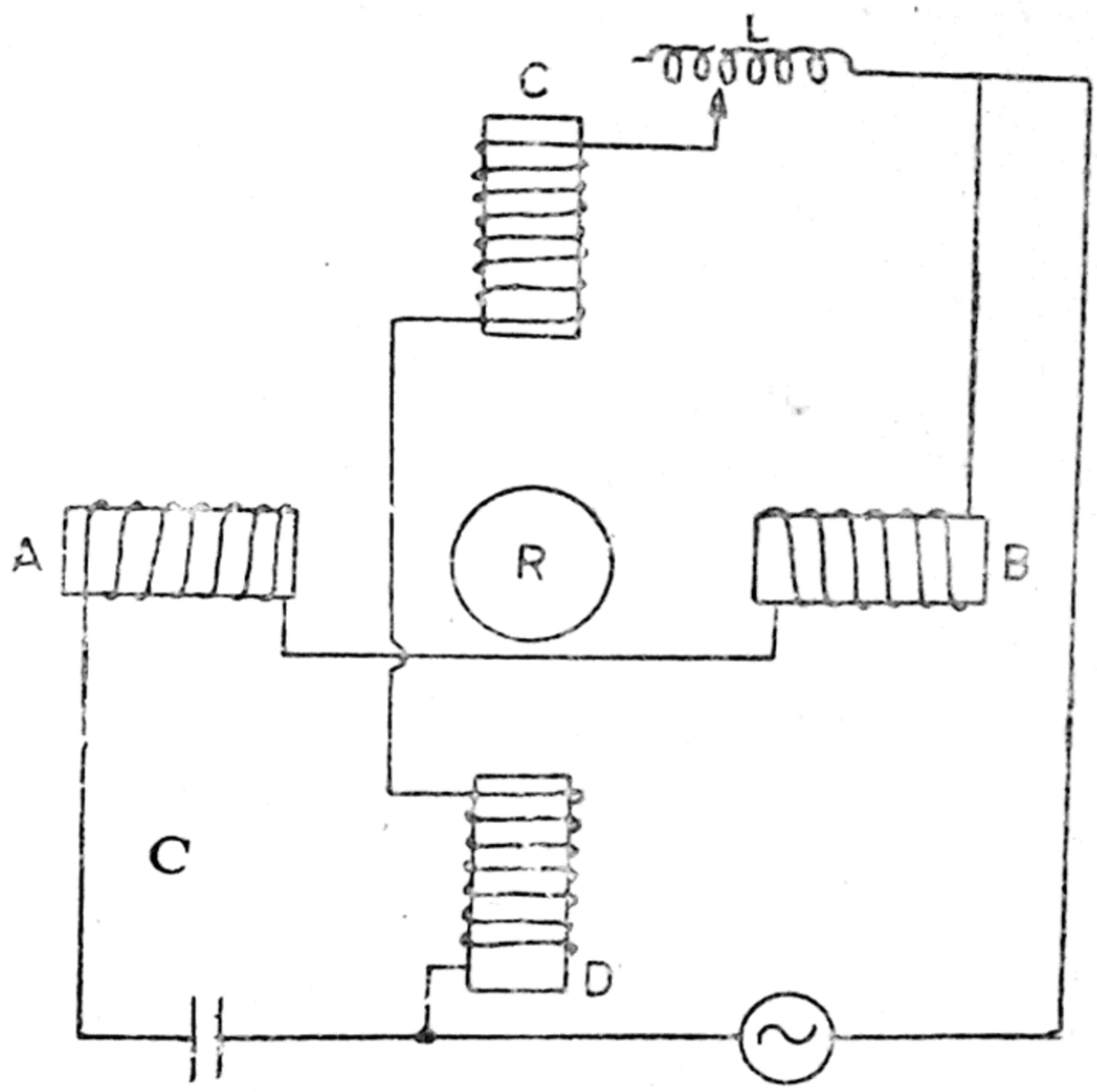


Fig. 12.22.

latter is provided with a distributed winding. The separate phases are generally connected in star and the three free ends connected each to one of three insulated slip rings mounted on the motor shaft.

In polyphase induction motor, the current in its rotor is not drawn from the supply, but is induced by relative motion of the rotor and the magnetic field produced by the stator currents. The stator is built exactly like an alternating stator, carries a polyphase winding similar to an alternating winding. An ac induction motor in which its speed is strictly proportional to the frequency of the current is called a *synchronous motor*. In such a motor, the coils of the rotor are not short circuited but connected through slip rings to a dc source and the stator is supplied an ac. The rotor is thus driven at exactly the same speed as the rotating field.

12.16. SKIN EFFECT

The current density **j** remains constant over any given section when the steady current passes through the uniform wire. But when the alternating voltage is applied between the ends of the wire, the distribution of current is no longer uniform. This uneven distribution of current is due to the emf induced in the wire itself owing to changing flux. Since the magnetic flux is always present inside and outside the current carrying conductor. A change in current produces a greater change in flux across the inner core than across the outer layer (as $B \propto 1/r^2$). Hence the induced emf due to this change of flux and therefore the effective resistance is greater at the centre than in the outer layers and the current is concentrated in the outer layers. When the frequency is very large,

the current is almost entirely confined to the surface layer. Such a phenomenon is known as *skin effect*. As the central parts of thick wires would not carry any appreciable part of the current and would therefore be useless, hence the conductors required to carry high frequency ac are made up of a number of stands of fine wire, insulated from each other, in order to have a large surface for any given area of cross section.

ALTERNATING CURRENT BRIDGES

12.17. IMPEDANCE BRIDGE

The theory of the Wheatstone bridge for the comparison of resistances, using dc source, has already been discussed in chapter 6. The theory of ac bridges is very important as the capacitance and inductance measurements are now made by the aid of these bridges. A simple bridge consisting of four impedances, arranged as in the Wheatstone bridge is known as *impedance bridge*. The cell and galvanometer of the familiar dc bridge are replaced by an ac generator and a suitable detector D respectively. The detector is generally a pair of headphones, giving a minimum sound at balance with an ac source of about 1 kc/sec. A vibration galvanometer is also used as a detector using the mains with a step-down transformer as the source of supply.

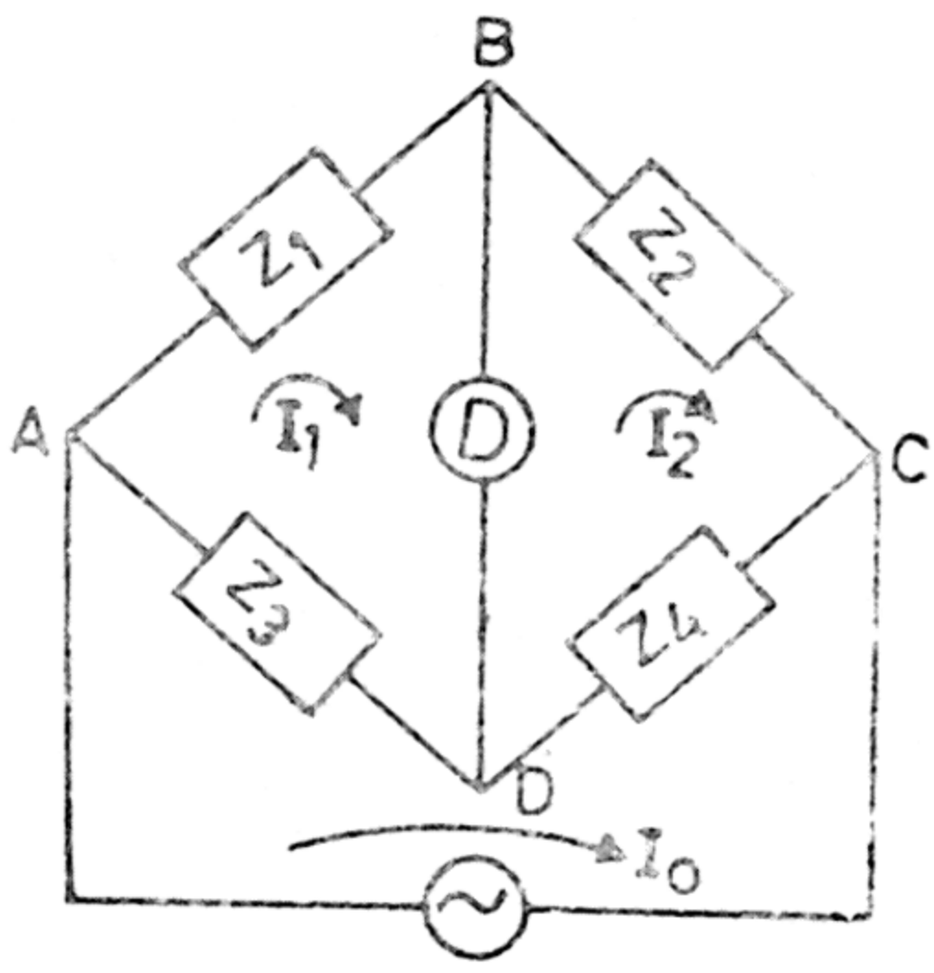


Fig. 12.23.

To deduce the condition for balance, let us assume that $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_0$ be the cyclic currents in the loops indicated in Fig. 12.23.

Applying Kirchhoff's laws for the mesh $ABDA$

$$\mathbf{I}_1(Z_1+Z_D+Z_3)-\mathbf{I}_2Z_D-\mathbf{I}_0Z_3=0,$$

for the mesh $BCDB$

$$\mathbf{I}_2(Z_2+Z_4+Z_D)-\mathbf{I}_1Z_D-\mathbf{I}_0Z_4=0.$$

Solving these equations the current through the detector is given by

$$\mathbf{I}=\mathbf{I}_1-\mathbf{I}_2=\frac{\mathbf{I}_0(Z_1Z_4-Z_2Z_3)}{Z_D(Z_1+Z_2+Z_3+Z_4)+(Z_1+Z_3)(Z_2+Z_4)}.$$

If the detector and generator are interchanged, the current through the detector

$$\mathbf{I}=\frac{\mathbf{I}_0(Z_2Z_3-Z_1Z_4)}{Z_D(Z_1+Z_2+Z_3+Z_4)+(Z_1+Z_2)(Z_3+Z_4)}.$$

The bridge is balanced when the current I is zero, *i.e.*,

$$Z_1Z_4=Z_2Z_3 \quad \text{or} \quad Z_1/Z_2=Z_3/Z_4. \qquad \ldots(95)$$

The result remains identical when detector and generator are interchanged as the denominator has no effect at the time of balance.

All these Z's are complex impedances and are comprise of, in general, resistances, inductances and capacitances. Writing each Z in the from $R+jX$, we have

$$(R_1+jX_1)(R_4+jX_4)=(R_2+jX_2)(R_3+jX_3)$$

$$R_1R_4-X_1X_4+j(X_1R_4+R_1X_4)=R_2R_3-X_2Y_3+j(X_2R_3+X_3R_2).$$

Equating reals and unreals on each side, we get

$$R_1R_4-X_1X_4=R_2R_3-X_2X_3 \text{ or } R_1R_4=R_2R_3 \text{ and } X_1X_4=X_2X_3, \qquad \ldots(96)$$

$$\text{and} \quad R_1X_4+X_1R_4=R_3X_3+X_2R_3 \quad \text{or} \quad \frac{R_1}{X_1}+\frac{R_4}{X_4}=\frac{R_2}{X_2}+\frac{R_3}{X_3}.$$

$$\text{or} \qquad \phi_1+\phi_4=\phi_2+\phi_3. \qquad \ldots(97)$$

These are two conditions to be satisfied in order to obtain a continuous balance. Except in special circumstances, all ac bridges involve a double balance. The former is usually a balance of the resistive components of the bridge and could be made with a dc source and moving coil galvanometer if the bridge arms contained no capacitors. It the balance conditions obtained are independent of the frequency of the supply, then the use of a supply with non-sinusoidal waveform does not introduce serious errors as the harmonic components will be balanced out as well as the fundamental. However a single frequency is desirable because the value of the electrical components involved may be frequency conscious.

12.18. MEASUREMENT OF INDUCTANCE

(*a*) Maxwell's Impedance Bridge. It is a simple type of ac bridge for the comparison of an inductance with a variable standard inductance. The ratio arms R_3 and R_4 are non-inductive resistances. The variable standard self inductance of constant resistance R_2 is L_2. L_1 is the unknown inductance with resistance R_1 and R is an adjustable noninductive resistance which may be placed in series with this inductance, if required, by changing the contact lead from A to a. Between terminals AC and BD are the source (ac or dc) and the detector (head-phone or galvanometer) respectively.

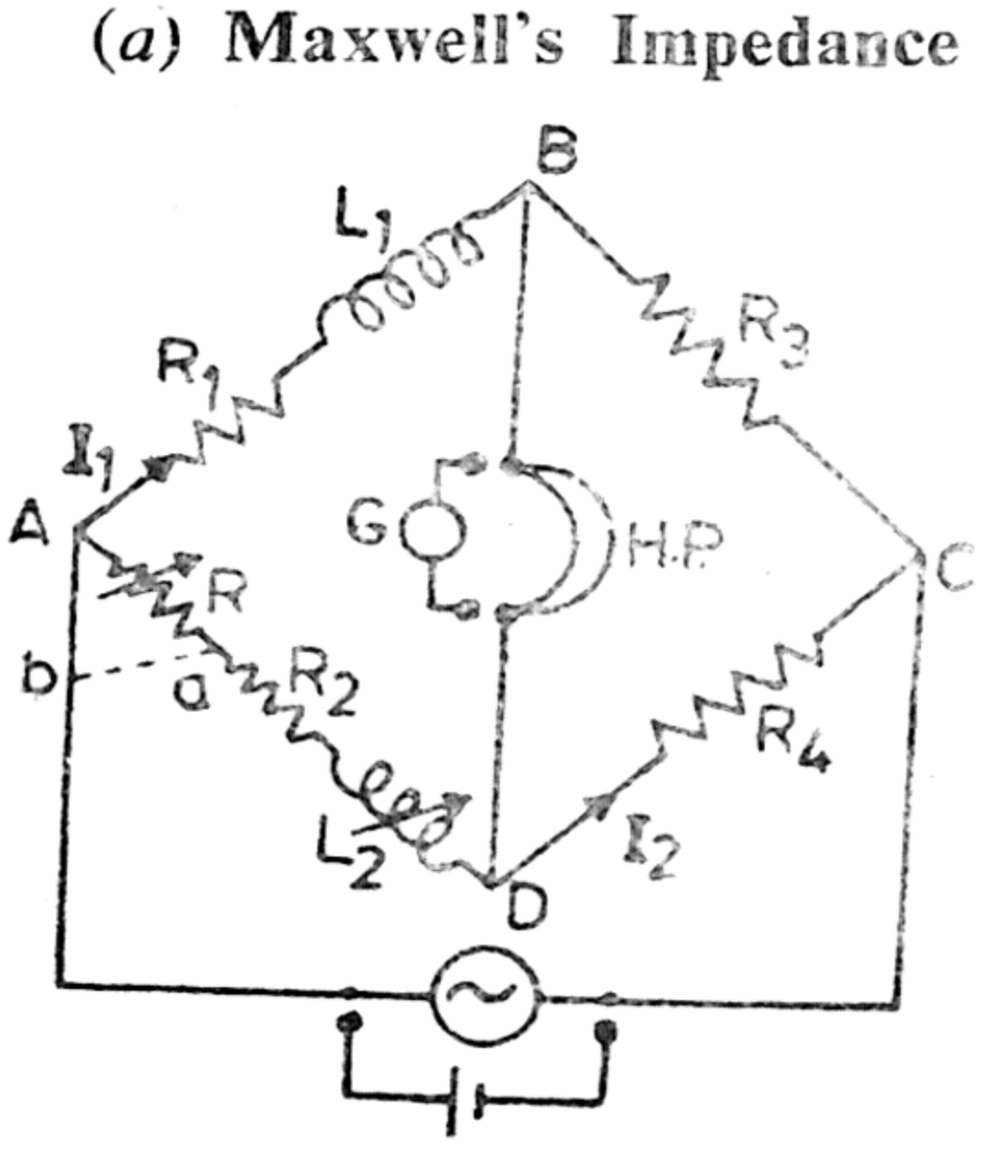


Fig 12.24

At the time of balance, using Eq. (95), we have

$$\frac{R_1+j\omega L_1}{R_2+j\omega L_2}=\frac{R_3}{R_4}$$

or $$R_1R_4+j\omega L_1R_4=R_2R_3+j\omega L_2R_3. \quad ...(98)$$

This equation is complex and is really equivalent to two equations. Thus the bridge may be balanced when the sum of all the horizontal components, *i.e.*, the real terms is zero and the sum of all the vertical components, *i.e.*, the imaginary terms is zero. Hence

$$R_1R_4-R_2R_3=0 \quad \text{or} \quad R_1/R_2=R_3/R_4, \quad ...(99)$$

and $$j\omega(L_1R_4-L_2R_3)=0 \quad \text{or} \quad L_1=L_2R_3/R_4. \quad (100)$$

Therefore a perfect balance implies two things—*that the ohmic resistances are balanced and that the inductances are in the ratio of the resistances in corresponding bridge arms.* The first balance is obtained as in the ordinary Wheatstone bridge by connecting a small variable resistance R in series with the inductance of larger value.

This bridge suffers from the *disadvantage* that the balance conditions cannot be satisfied independently of one another. If L_2 is constant, the two conditions of balance are semi-independent as a change in R_3 affects a change in R. If L_2 is variable and $R_3 \approx R_4$, the conditions are again semi-independent as a change in L_2 affects a change in R. Conditions can be made independent of each other only if R is kept constant and L_2 and R_3 are varying.

L_1 and L_2 are kept away to avoid mutual inductance M. Let us examine the effect of M between coils L_1 and L_2. If $\mathbf{I}_1$ and $\mathbf{I}_2$ are the current vectors for the arms ABC and ADC of the balanced bridge, then we have

$$(R_1+j\omega L_1+j\omega M)\,\mathbf{I}_1=(R_2+j\omega L_2+j\omega M)\,\mathbf{I}_2.$$

and $$R_3\mathbf{I}_1=R_4\mathbf{I}_2.$$

Eliminating $\mathbf{I}_1$ and $\mathbf{I}_2$, we have

$$R_1R_4+j\omega R_4(L_1+M)=R_2R_3+j\omega R_3(L_2+M).$$

Thus the required conditions are

$$R_1R_4=R_2R_3 \quad \text{or} \quad R_1/R_2=R_3/R_4. \quad ...(101)$$

and $$R_1(L_1+M)=R_3(L_2+M) \quad \text{or} \quad (L_1+M)/(L_2+M)=R_3/R_4. \quad ...(102)$$

Combining these equations, we get

$$\frac{L_1+M}{L_2+M}=\frac{R_3}{R_4}=\frac{R_1}{R_2}. \quad ...(103)$$

If $R_3=R_4$ and L_2 is varied to L_1, then the balance point is unaffected by mutual inductance M.

In a properly constructed bridge, the resistance coils should be free from inductance and capacitance. In practice in all *ac* bridges residual reactances in R_3 or R_4 are always present. To avoid this effect, we adjust $L_1/L_2=R_1/R_2=R_3/R_4=1$.

If the order of magnitude of the inductance is entirely unknown, one may begin with $R_3=R_4$, and obtain balance by varying R. The resistance R may be transferred to the other side of the bridge, if required. Now *dc* source is replaced by *ac* source and moving coil galvanometer is replaced by the headphone or vibration galvanometer. The balance position is obtained by adjusting variable inductance L_2. On account of the limited range of the inductance L_2, the chances are of not obtaining of exact balance. The bridge is then rebalanced for direct and then for alternating currents. A good balance may be obtained after two or three trails. If for $R_3=R_4$, the sound is apparently the same, a large change in the ratio is required, say for example $R_3/R_4=10$. If it does not give results. $R_3/R_4=1/10$ may be tried. A final attempt to obtain a perfect balance may be made by altering R and L_2 slightly.

Eqs. (99) and (100) show that the conditions are free from ω. However the balance conditions are affected by variation of ω due to *eddy current and skin effects. Self capacitances of* R_1 *and* R_2 *are effective specially at higher* ω. *The earth capacitance, effective at high frequencies, is avoided by shielding all the branches from one another.*

(*b*) **Maxwell's L/C bridge.** It is one of the earliest forms of bridge for the measurement of self inductance. With its aid a self inductance is measured in terms of a capacitance and two resistances. The arm DC contains the coil under test in series with a variable non-inductive resistance box R_4. The arms AB, BC and AD are non-inductive resistances, or more exactly resistances of very low residual inductances. A standard variable condenser C is placed in parallel with the resistance R_1, opposite to the arm containing L.

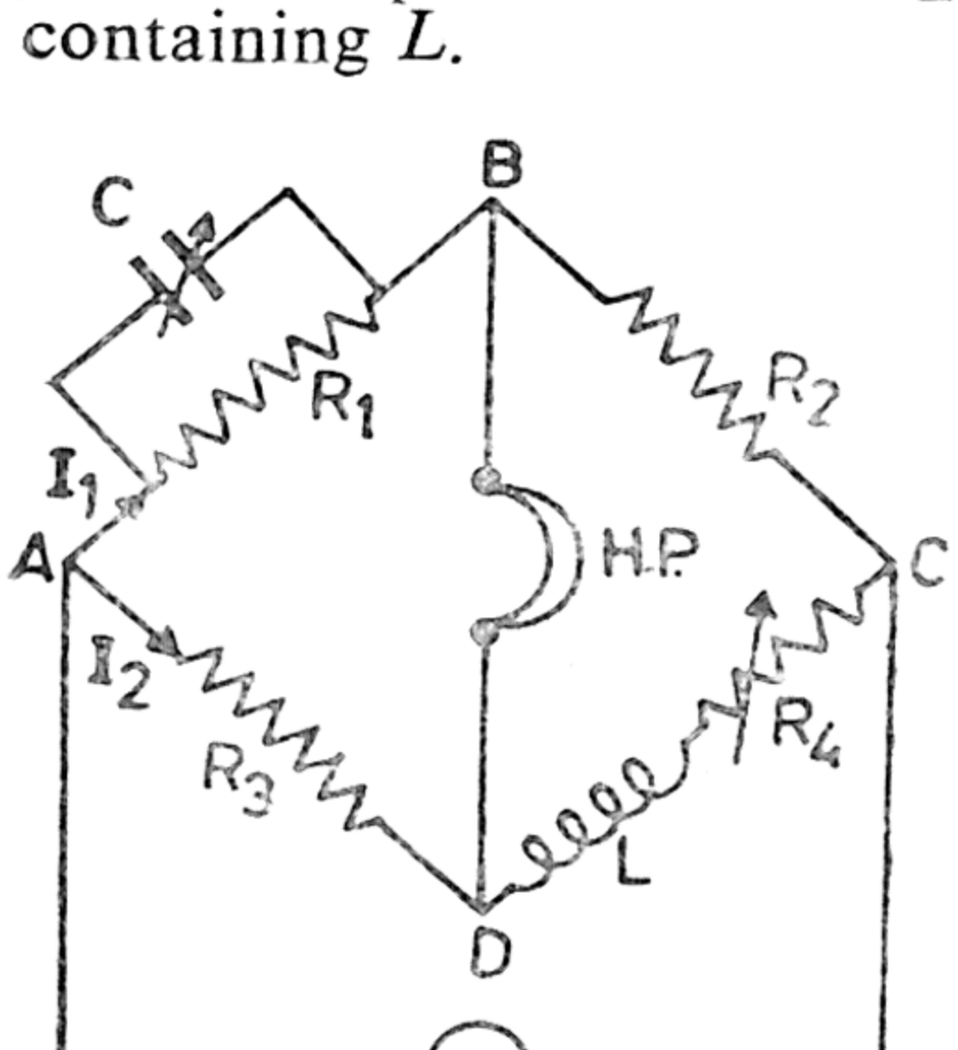


Fig. 12.25.

The bridge is balanced, when

$$\frac{Z_1}{Z_2}=\frac{Z_3}{Z_4} \quad \text{or} \quad \frac{Z_1}{Z_2}=\frac{R_3}{R_4+j\omega L},$$

Here R_4 includes the resistance of the inductance L.

$$\frac{1}{Z_1}=\frac{1}{R_1}+j\omega C \quad \text{or} \quad Z_1=\frac{R_1}{1+j\omega CR_1}.$$

$$\therefore \quad \frac{R_1}{R_2(1+j\omega CR_1)}=\frac{R_3}{R_4+j\omega L}$$

$$\text{or} \quad R_1R_4+j\omega LR_1=R_2R_3+j\omega CR_1R_2R_3.$$

Equating real and imaginary parts, we have

$$R_1R_4=R_2R_3$$

$$\text{or} \quad R_1/R_2=R_3/R_4, \qquad \ldots(104)$$

and $$j\omega LR_1=j\omega CR_1R_2R_3 \quad \text{or} \quad L=CR_2R_3. \qquad \ldots(105)$$

Thus we see that the balance in this bridge is independent of the frequency. This bridge does not permit the two conditions of balance to be attained independently if the two variable quantities are resistances. These two conditions of balance are completely independent if a variable condenser is used. Good mica condensers arranged in decade dials over the range $0-1.11\ \mu f$, to which a variable air condenser of capacitance $0.001\ \mu f$ may be added in parallel, are of practical importance for the continuous variation of capacitance.

In practice R_1/R_2 is best kept as a simple ratio, whilst R_4 is varied first to obtain in phase minimum, then C is varied to obtain a quadrature minimum. Two or three alternate variations of R_4 and C should give a good balance. The variable resistance in series with L should include decade subdivided ohm for the fine adjustment, preferably a small carbon plate rheostat may be used as the value of R_4 is not required for the calculation of L. *The coil L will always have some self capacity so it will not be possible to obtain a perfect balance.*

In practice C is always imperfect, it is accompanied by either a small resistance r in series or a high resistance R in parallel. Assuming r in series, the balance conditions are given by

$$Z_1/R_2=R_3/(R_4+j\omega L).$$

Here Z_1 is given by

$$\frac{1}{Z_1}=\frac{1}{R_1}+\frac{j\omega C}{1+j\omega Cr}=\frac{1+j\omega Cr+j\omega CR_1}{R_1(1+j\omega Cr)}.$$

$$\therefore\quad \frac{R_1(1+j\omega Cr)/[1+j\omega C(R_1+r)]}{R_2}=\frac{R_3}{R_4+j\omega L}$$

or $$R_1R_4+j\omega LR_1+j\omega CrR_1R_4-\omega^2CrR_1L=R_2R_3+j\omega CR_2R_3(R_1+r).$$

Equating real and imaginary parts, we get

$$R_1R_4-\omega^2CrR_1L=R_2R_2$$

and $$LR_1+CrR_1R_4=CR_2R_3(R_1+r)$$

$$\therefore\quad \frac{R_4-\omega^2rLC}{R_3}=\frac{R_2}{R_1}\text{ and }\frac{L}{R_3}+\frac{rCR_4}{R_3}=CR_2+\frac{rCR_2}{R_1}.$$

Eliminating R_2/R_1, we get

$$L=CR_2R_3/(1+\omega^2C^2r^2). \qquad \ldots(106)$$

This expression shows that the correction term involves the power factor of the capacitor (ωCr). For a high class mica condenser the power factor is less than 10^{-4}, thus the correction is less (1 in 10^8) and may be taken to be zero.

A perfect balance is not possible due to the presence of harmonics in the source, since even the formulae at balance are independent

of frequency. Where the highest accuracy is required, the effects of stray capacitance must be eliminated by the use of the Wagner earth, to be discussed latter.

(*c*) **Owen's Bridge.** This bridge is used to measure self inductance in terms of a standard capacitance C_2 and two resistances R_2 and R_3. The bridge is arranged as in Fig. 12.26. Assuming that the condensers are without losses and that all the resistances are perfectly non-reactive, the balance conditions are given by

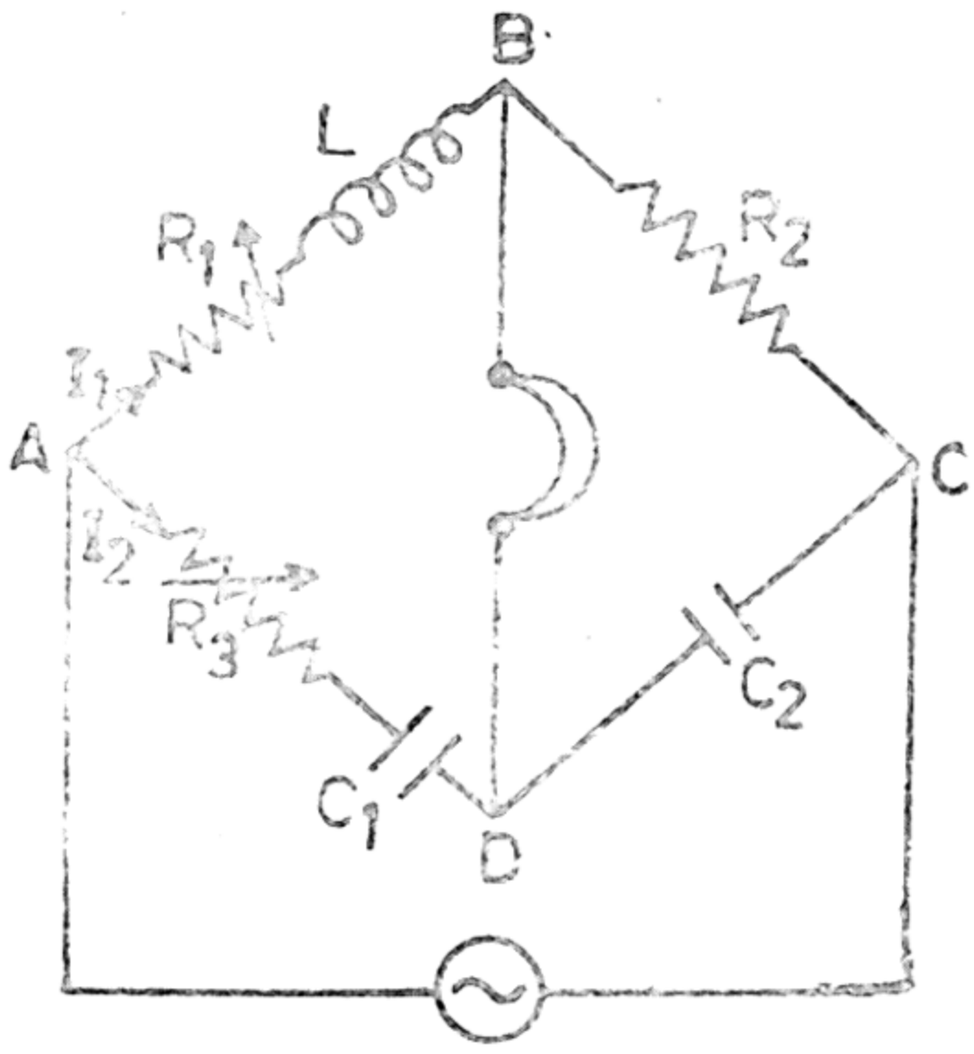


Fig. 12.26.

$$\frac{R_1+j\omega L}{R_2}=\frac{R_3+1/j\omega C_1}{1/j\omega C_2},$$

or $$R_2R_3+R_2/j\omega C_1=L/C_2+R_1/j\omega C_2,$$

where R_1 includes coil resistance R_2.

Equating real and imaginary parts, we get

$$L=C_2R_2R_3, \qquad ...(107)$$

and

$$C_1/C_2=R_2/R_1. \qquad ...(108)$$

Both the conditions of balance are independent of each other and of the frequency. It can be shown that before the actual balance, one component of the current through the detector depends upon ω (varies inversely as ω) and the other does not. It is true also for all bridges in which final balance is independent of frequency.

Vector diagram of the bridge. The voltage V across the bridge is represented by EF (Fig. 12.27). Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be the current vectors for the arms ABC and ADC respectively. When the bridge is balanced $\mathbf{I}_1$ will lag behind V through ϕ_1 (say) and $\mathbf{I}_2$ will lead through ϕ_2 (say). These ϕ_1 and ϕ_2 are given by the relations.

$$\tan\phi_1=\frac{\omega L}{R_1+R_2} \text{ and } \tan\phi_2=\frac{1}{\omega CR_3},$$

where $$1/C=1/C_1+1/C_2.$$

$$\therefore \quad \tan\phi_1 \tan\phi_2=\frac{L}{(R_1+R_2)R_3C}=\frac{L}{R_3(R_1+R_2)}\left[\frac{1}{C_1}+\frac{1}{C_2}\right]$$

$$=\frac{L}{R_3(R_1+R_2)}\left[\frac{R_1}{R_2C_2}+\frac{1}{C_2}\right]=\frac{L}{R_2R_3C_2}$$

$$=1.$$

or $$\phi_1+\phi_2=\pi/2. \qquad ...(109)$$

If $\mathbf{EG}=R_3\mathbf{I}_2$ and **GH** is normal to **EG** and of length $\mathbf{I}_2/\omega C_1$, then **EH** gives the potential drop across arm AD of the bridge. This must be equal to the potential drop across the arm AB of the bridge, *i.e.*, $\mathbf{EI}+\mathbf{IH}$, where $\mathbf{EI}=R_1\mathbf{I}_1$ and $\mathbf{IH}=j\omega L\mathbf{I}_1$. **IH** is normal to **EI**. We have also proved that $\phi_1+\phi_2=\pi/2$, *i.e.*, **EI** is normal to **EG** or the currents $\mathbf{I}_1$ and $\mathbf{I}_2$, are in quadrature. Now **HF** is $R_2\mathbf{I}_1$ and since it is also $\mathbf{I}_2/j\omega C_2$, it must be normal to $\mathbf{I}_2$. Thus GHF is a straight line.

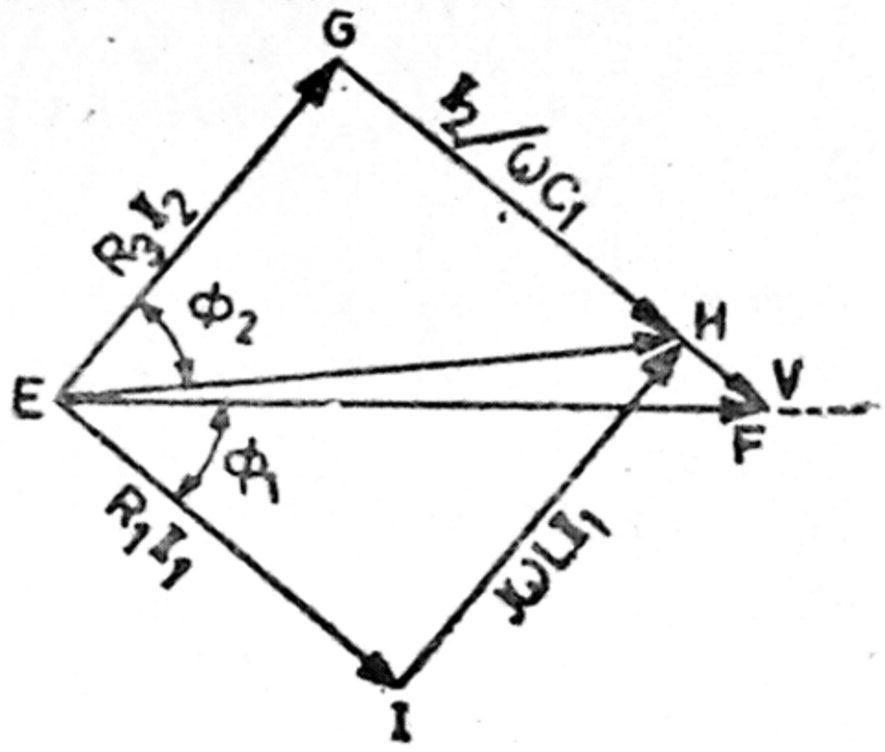


Fig. 12.27.

Condition (107) may be secured by varying R_3 and condition (108) either by varying R_1, or by varying C_1, the former is preferable. This bridge is extremelyconvenient and possesses the great practical advantage as the balance is obtained by adjusting resistances only. As it consists only one inductance, under test, all mutual reactance between inductances are avoided.

Owen has shown that even with limited apparatus the range of this bridge is very wide. Thus with $C_1=C_2=0.3\ \mu f$ and values of R_2 from 1 to 200Ω, self inductances from $2\mu H$ to $0.5\ H$ can be measured accurately. The residuals due to the inductance of leads and terminals may be easily eliminated by obtaining a balance first with the inductance L and then with a short circuit across L. If R_3 and R_3' are the respective values of the R_3 for these two balances, then we have to a close approximation.

$$L=C_2R_2(R_3-R_3'). \qquad ...(110)$$

This second balance is necessary for large values of L and can be avoided for low values of L.

(*d*) **Anderson's Bridge.** This is the modified form of Maxwell's bridge. The double balance is obtained by adjusting resistances only, the standard condenser being of a fixed value. The modification consists in placing a non-inductive resistance r in series with the condenser, the combination being in parallel with R_1, the other components and the detector are placed as shown in Fig. 12.28.

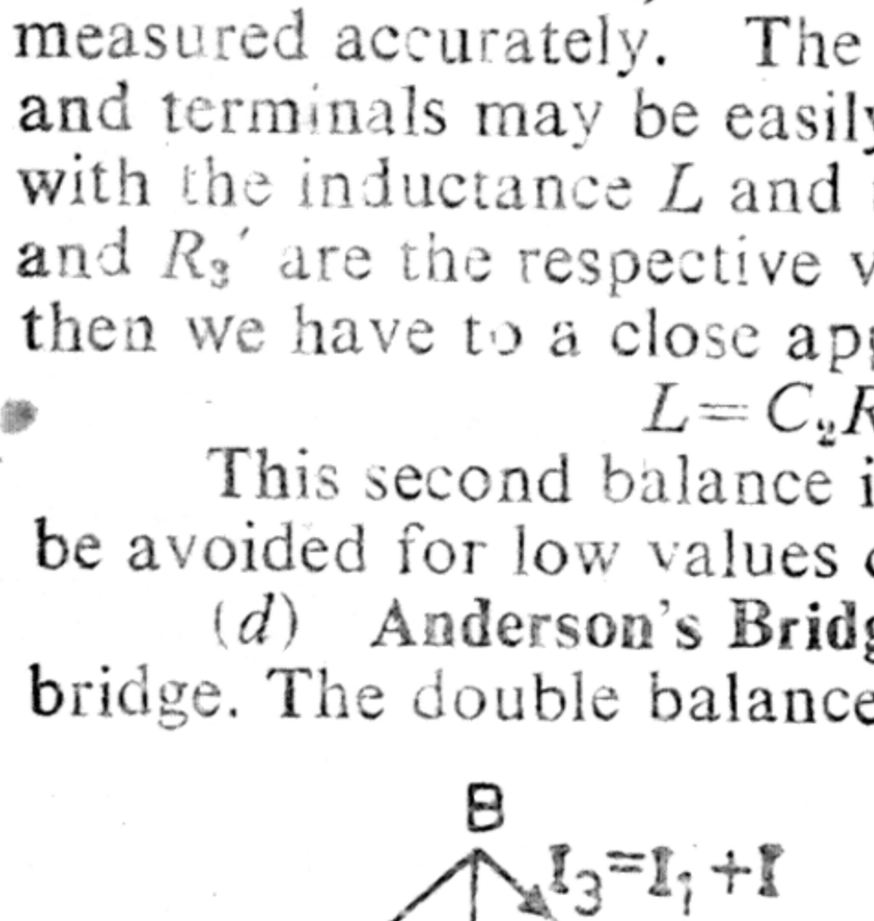


Fig. 12.28.

At the time of balance the potential at E is equal to the potential at D. Applying Kirchhoff's laws we have

for mesh $ABEA$

$$R_1\mathbf{I}_1-(r+1/j\omega C)\,\mathbf{I}=0, \qquad ...(111)$$

for mesh $AEDA$

$$(1/j\omega C)\mathbf{I}-R_3\mathbf{I}_2=0. \qquad ...(112)$$

and for mesh $BCDB$

$$R_2(\mathbf{I}_1+\mathbf{I})-(j\omega L+R_4)\,\mathbf{I}_2+r\mathbf{I}=0. \quad \ldots(113)$$

Substituting values of $\mathbf{I}_1$ and $\mathbf{I}_2$ from Eqs. (111) and (112) in Eq. (113), we have

$$\frac{R_2}{R_1}\left(r+\frac{1}{j\omega C}\right)\mathbf{I}+R_2\mathbf{I}-\frac{(j\omega L+R_4)}{j\omega CR_3}\,\mathbf{I}+r\mathbf{I}=0. \quad \ldots(114)$$

Equating real and imaginary parts, we get

$$\frac{R_2}{R_1}r+R_2-\frac{L}{CR_3}+r=0$$

or
$$L=CR_3\left[r\left(1+\frac{R_2}{R_1}\right)+R_2\right] \quad \ldots(115)$$

and
$$\frac{R_2}{R_1\omega C}-\frac{R_4}{R_3\omega C}=0, \text{ or } \frac{R_1}{R_2}=\frac{R_3}{R_4}. \quad \ldots(116)$$

These conditions are independent of the frequency of the source and can be satisfied independently of each other by adjusting R_4 (the variable resistance in series with the inductance coil L) in the latter case and r in the former case. Eq. (115) shows that the ac balance is possible only when $L>CR_2R_3$, otherwise r will be negative. Since there is no other inductance, all mutual reaction between neighbouring inductances is avoided.

It is best to use $R_1 : R_2=1 : 1$. The formulae then reduce to

$$R_3=R_4 \text{ and } L=CR_3\,[2r+R_2]. \quad \ldots(117)$$

Vector Diagram. The vector diagram of Anderson's bridge is shown in Fig. 12.29. In it **EF** is **V**, the potential vector for the applied source and is also equal to the potential difference between terminals A and C. If $\mathbf{I}_2$ represents the current in ADC, then $\mathbf{EG}=R_3\mathbf{I}_2$. $\mathbf{GH}=j\omega L\,\mathbf{I}_2$ and $\mathbf{HF}=R_4\mathbf{I}_2$, so that **HF** is $\parallel$ to **EG** and **GH** is $\perp$ to **EG** or **HF**. Since the potential of point E is the same as that of D. hence $\mathbf{V}_{AE}=\mathbf{V}_{AD}$ or $(1/j\omega C)\mathbf{I}=R_3\mathbf{I}_2$, This relation shows that **I** is in quadrature with $\mathbf{I}_2$. If we draw $\mathbf{GI}=r\mathbf{I}$, then the vector **EI** is $R_1\,\mathbf{I}_1$. The vector **IF** is $R_2\mathbf{I}_3=R_2(\mathbf{I}_1+\mathbf{I})$. The directions of $\mathbf{I}_1$ and $\mathbf{I}_3$ are thus determined from this diagram.

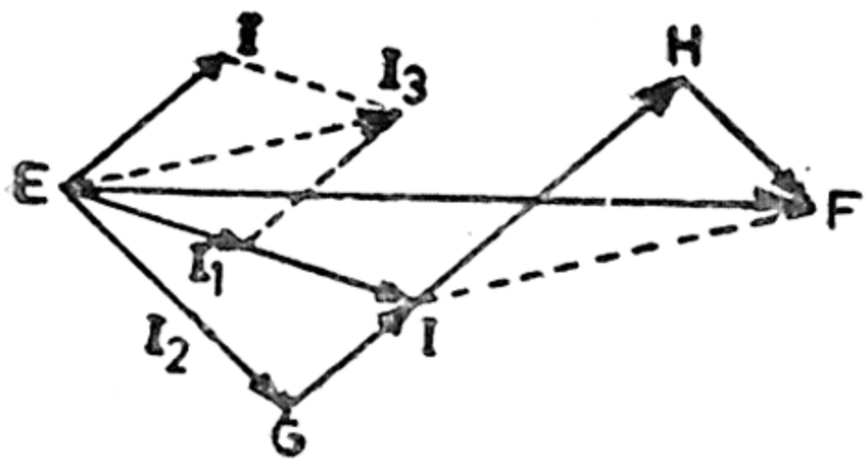


Fig. 12.29.

First of all the circuit is arranged as a simple Wheatstone's bridge connecting the inductance in the unknown arm CD. The resistance R_1 and R_2 are fixed to 10 ohm each and R_3 is varied until the galvanometer deflection changes direction. The expected value is calculated with the formula (116).

The experiment is repeated with $R_1/R_2=10$ and 100 and the actual resistance in the inductance arm R_L is calculated. Now R_1 and R_2, are re-adjusted to the ratio 10 : 10 and a fractional resistance R' is placed in the series with the inductance. Now R_3 and R' are varied for no deflection in galvanometer. In this case $R_L+R'=R_4=R_3$. To increase the sensitivity resistances are so arranged that $R_1=R_2=\frac{1}{2}R_3$.

Now galvanometer and dc source are replaced by headphones and ac source (oscillator) respectively and capacitance C and resistor r (0 to 10 $k\Omega$) are inserted as shown in Fig. 12.28. R_1, R_2, R_3 and R_4 are kept as such and the bridge is balanced for minimum sound with the help of resistance r. The experiment may be repeated by taking different capacitances. Substituting these values in Eq. (117), self inductance of a given coil can be calculated.

Relative Merits of Anderson's and Owen's bridges when condensers are slightly imperfect. To examine the effect of an imperfect capacitor let us assume that such a defect may be represented by a high resistance R_0 in parallel with capacitor C. When the bridge is balanced let the current vectors be as indicated in Fig. 12.30. Applying Kirchhoff's laws, for the meshes *ABEA*, *AEDA* and *BCDB*, we have

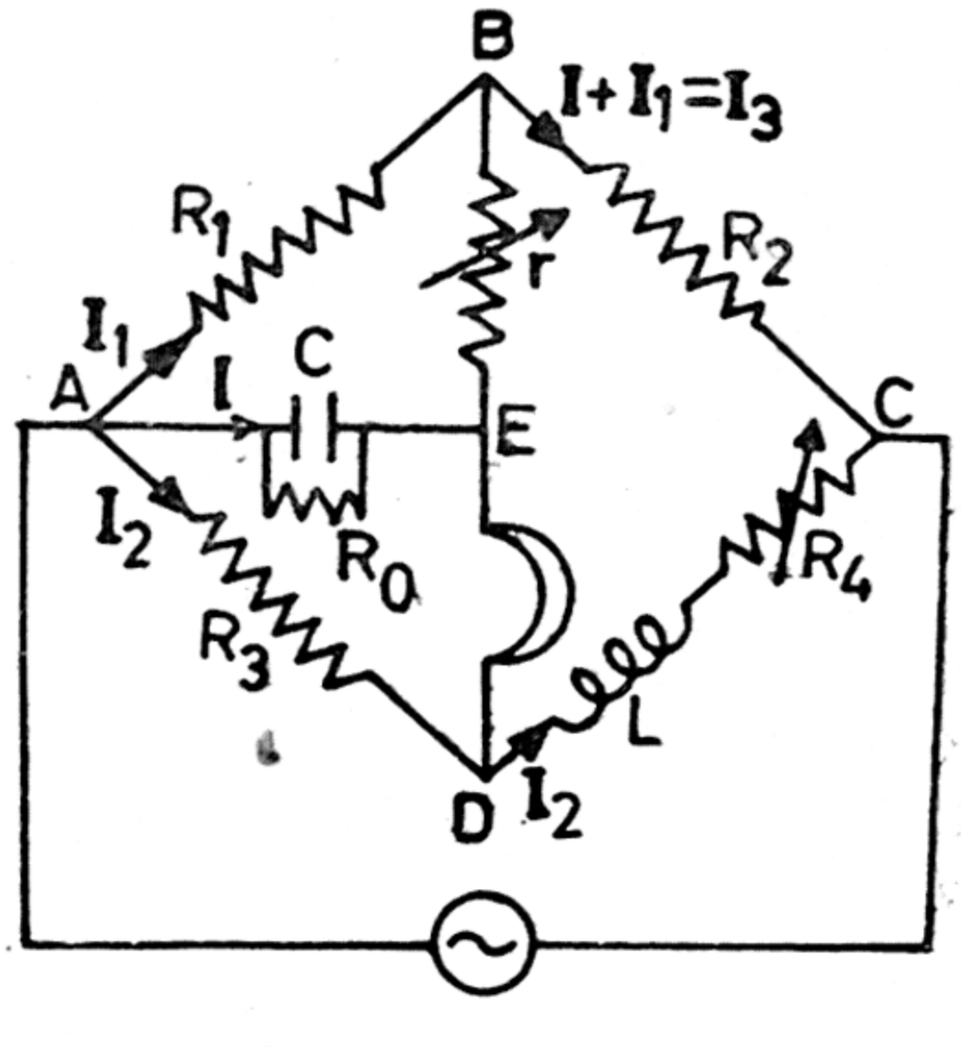


Fig. 12.30.

$$R_1\mathbf{I}_1-\left[\, r+\frac{R_0}{1+j\omega CR_0}\,\right]\mathbf{I}=0, \quad \ldots(118)$$

$$\frac{R_0}{1+j\omega CR_0}\,\mathbf{I}-R_3\mathbf{I}_2=0 \qquad \ldots(119)$$

and $$R_2\,(\mathbf{I}+\mathbf{I}_1)-j\omega L\mathbf{I}_2-R_4\mathbf{I}_2+r\mathbf{I}=0. \qquad \ldots(120)$$

Substituting values of I_1 and I_2 from Eq. (118) and (119) respectively in Eq. (120) and equating the real and imaginary parts, we get

$$R_1R_3+rR_3+\frac{R_3R_1}{R_2}\,r+R_0R_3=\frac{R_0R_1R_4}{R_2} \qquad \ldots(121)$$

and $$L=CR_3\,[r\,(1+R_2/R_1)+R_2]. \qquad \ldots(122)$$

Thus we see that the imperfection of capacitor has not effect on the measured value of the self inductance L and has a small effect on the measured value of R_4.

To examine the effect of an imperfect capacitor in Owen's bridge, let us assume that the effect may by represented by a small series resistance R_4. The conditions for balance are given by

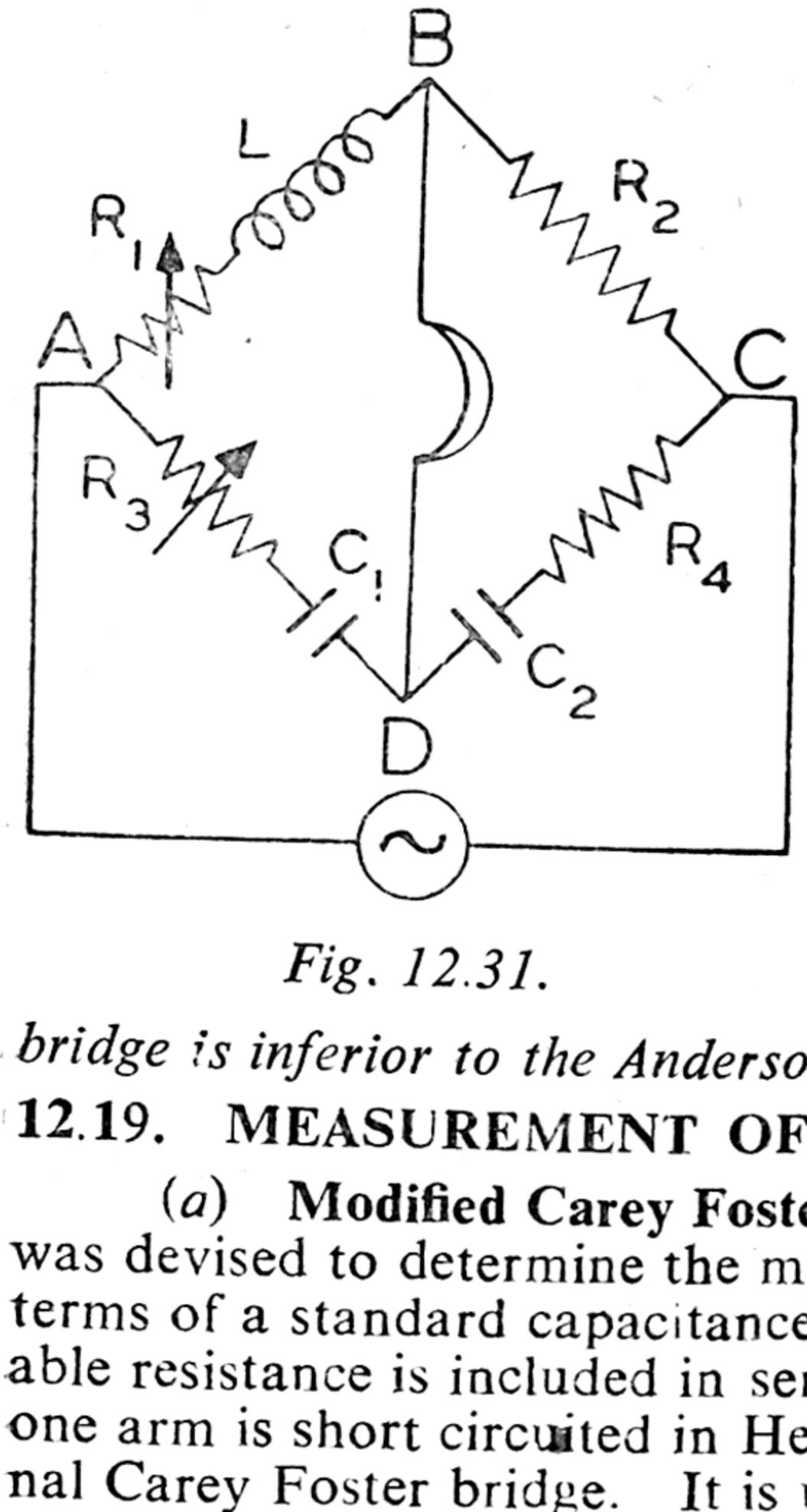


Fig. 12.31.

$$\frac{R_1+j\omega L}{R_2}=\frac{R_3+1/j\omega C_1}{R_4+1/j\omega C_2}$$

Solving this relation and equating real and imaginary parts, we have

$$R_1R_4+L/C_2=R_2R_3 \qquad \ldots(123)$$

and $$\omega LR_4-R_1/\omega C_2+R_2/\omega C_1=0 \qquad \ldots(124)$$

Thus both conditions for balance are effected by the imperfectness. The second condition involves ω, showing that the source must be free from harmonics. In this respect, *Owen's bridge is inferior to the Anderson's bridge.*

12.19. MEASUREMENT OF MUTUAL INDUCTANCE

(*a*) **Modified Carey Foster Method.** The Carey Foster bridge was devised to determine the mutual inductance between two coils in terms of a standard capacitance and two known resistances. A variable resistance is included in series with the standard capacitor and one arm is short circuited in Heydweiller's modification of the original Carey Foster bridge. It is undesirable to classify this circuit as a bridge, but it is the usual practice.

When R_2 is zero the resistance R_1 is obviously connected directly across the supply (neglecting the primary of M). The primary of the mutual inductance is connected so that the voltage induced by it in arm CD neutralizes the drop due to the current $\mathbf{I}_2$ in this branch. When $R_2=0$, the drop across the arm CD must be zero for the balance position. The balance is obtained by varying R_3 and R_4.

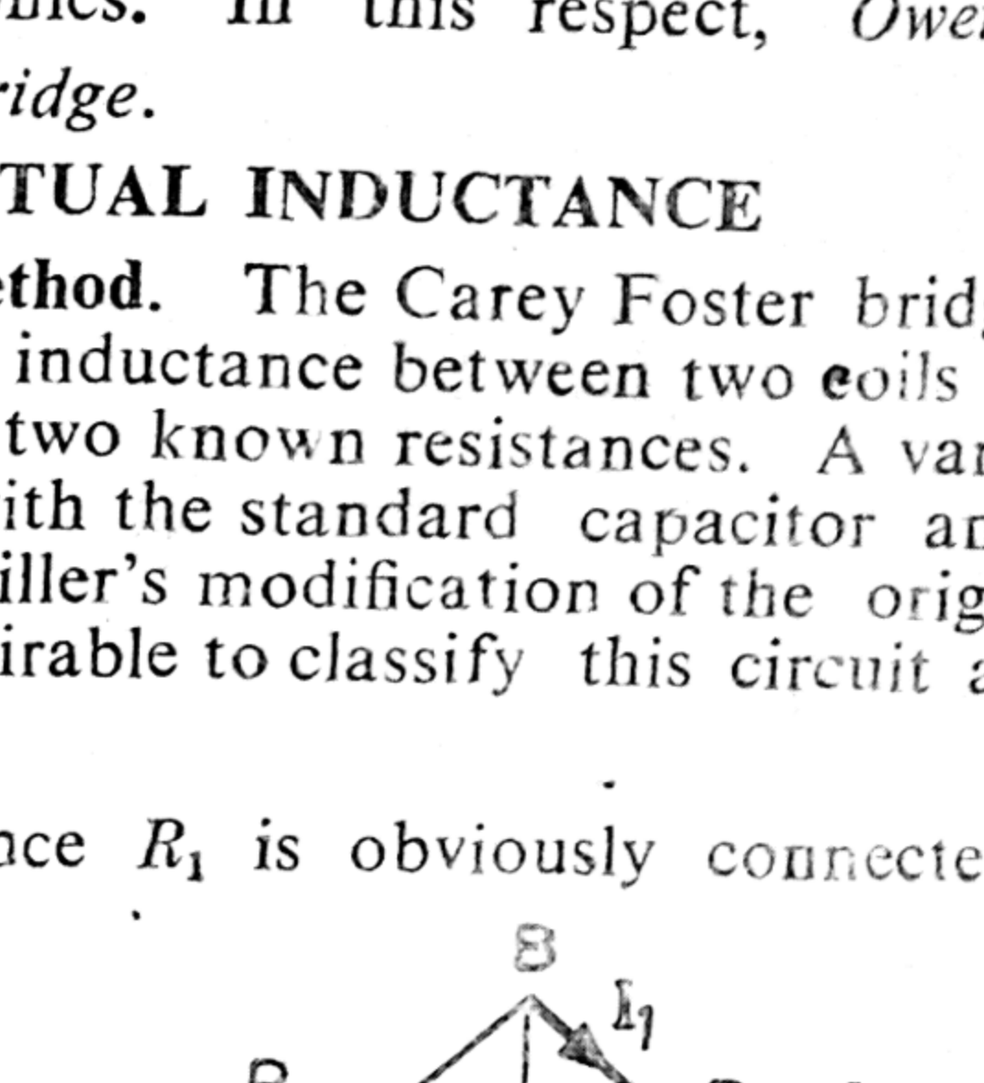


Fig. 12.32.

As it is not of the Wheatstone bridge type, hence the general relation $Z_1/Z_2=Z_3/Z_4$, can not be used. Using Kirchhoff's laws we thus have

$$\mathbf{I}_1=\mathbf{I}+\mathbf{I}_2, \qquad \text{(first law)} \qquad \ldots(125)$$

$$R_1\mathbf{I}_1-(R_3+1/j\omega C)\,\mathbf{I}_2=0 \quad \text{(for mesh } ABDA\text{)} \qquad \ldots(126)$$

and $$(R_4+j\omega L)\mathbf{I}_2+j\omega M(\mathbf{I}_1+\mathbf{I}_2)=0. \quad \text{(for mesh } BCDB\text{)} \qquad \ldots(127)$$

Substituting value of $\mathbf{I}_1$ from Eq. (126) in Eq. (127), we get

$$R_4\mathbf{I}_2+j\omega \mathrm{L}\ \mathbf{I}_2+j\omega\,\frac{M}{R_1}\left(\ R_3+\frac{1}{j\omega C}\ \right)\mathbf{I}_2+j\omega M\mathbf{I}_2=0.$$

Equating real and imaginary parts we have

$$R_4+M/CR_1=0 \text{ or } M=-CR_1R_4, \qquad \ldots(128)$$

and $$L+MR_3/R_1+M=0 \text{ or } L=-M(R_3/R_1+1). \qquad \ldots(129)$$

These equations represent the conditions of balance. These conditions require that M shall be negative. Thus we have

$$M=CR_1R_4 \text{ and } L=M(R_3/R_1+1). \qquad \ldots(130)$$

These balance conditions are independent of one another and frequency of the source, can be obtained by varying R_4 and R_3 and keeping R_1 constant. Second condition shows that $L>M$. The balance cannot be obtained if the leads of the secondary or the primary are wraped in the wrong way.

If the condenser is imperfect, $R_3=R+r$, where R is the known resistance and r is the contribution of imperfectness, then we have

$$L=M[1+(R+r)/R_1] \text{ or } R=R_1[L/M-1]-r. \qquad \ldots(131)$$

The power factor for the condenser ωCr can be calculated by plotting R against R_1. Since R_4 includes the resistance of the coil (secondary of the mutual inductance), we can write $R_4=R'+r'$, where R' is the known resistance. Then

$$M=CR_1(R'+r') \text{ or } R'=M/CR_1-r'. \qquad \ldots(132)$$

A graph between R' and $1/R_1$ gives us the value of M, if C is known. The value of r' can be calculated by the intersection.

In the derivation of balance conditions we have assumed $R_2=0$. If the arm BC has a small but finite resistance ΔR, then the balance conditions, obtained by using Kirchhoff's laws, are

$$R_3\Delta R+M/C=R_1\ R_4 \qquad \ldots(133)$$

and $$L=M[1+R_3/R_1]-\Delta R/\omega^2CR_1. \qquad \ldots(134)$$

These equations show that the balance is independent of frequency only if $\Delta R=0$.

(*b*) **Mexwell's Method.** This method is used for the comparision of two unequal mutual inductances. One M_2 is a standard variable and other is measured in terms of it. L_1 and L_2 are the self inductances of their secondary windings. R_1 and R_2 are the total resistances of two balances. To obtain exact balance a variable self inductance L is inserted in either branch so that it is in series with L_1 or L_2.

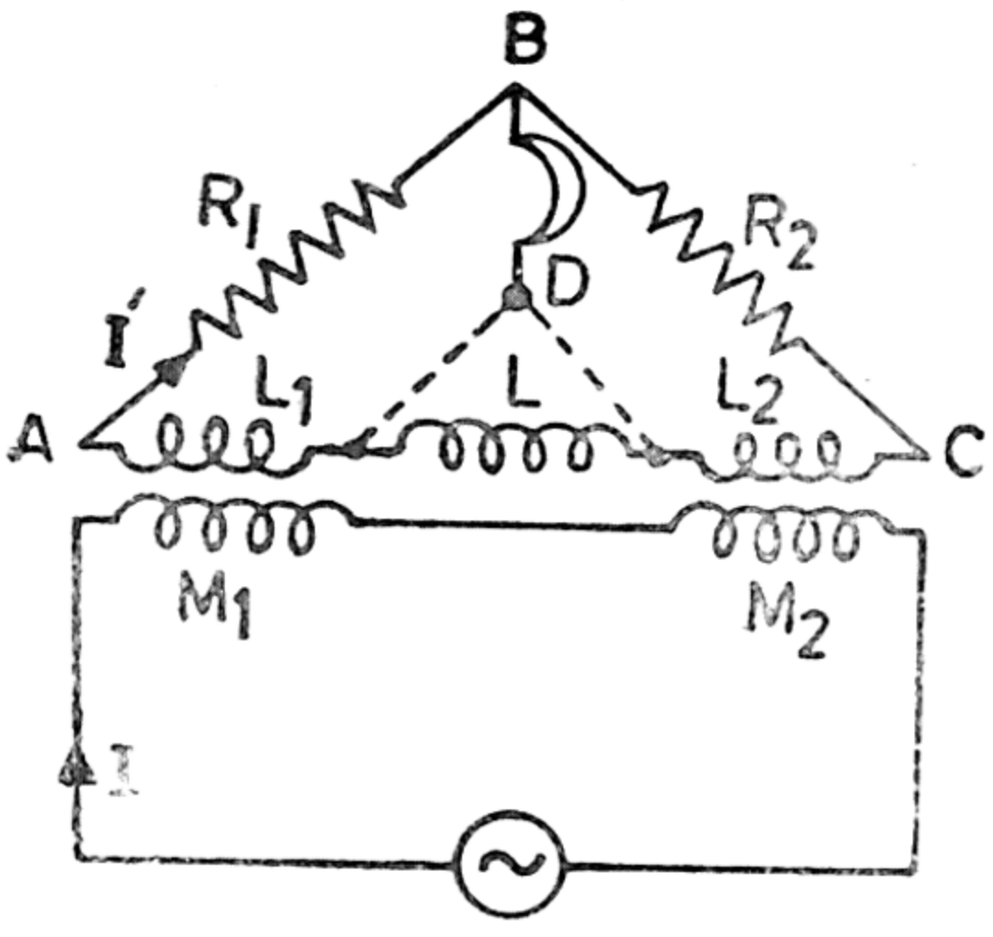


Fig. 12.33.

At the time of a balance, if **I** is the current in the source circuit and **I**′ in the loop $ABCDA$, and inductance L is in series with L_1, then

$$R_1\mathbf{I}'+j\omega(L_1+L)\mathbf{I}'+j\omega M_1\,\mathbf{I}=0. \text{ (for mesh } ABDA) \quad \ldots(135)$$

and $$R_2\mathbf{I}'+j\omega L_2\mathbf{I}'+j\omega M_2\mathbf{I}=0. \text{ (for mesh } BCDB) \quad \ldots(136)$$

Substituting value of **I** from Eq. (135) to Eq. (136), we get

$$(R_2+j\omega L_2)\mathbf{I}'-\frac{M_2}{M_1}[R_1+j\omega(L_1+L)\mathbf{I}'=0.$$

Equating real and imaginary parts we have

$$R_2-M_2R_1/M_1=0 \text{ or } M_1/M_2=R_1/R_2$$

and $$L_1+L-M_1L_2/M_2=0 \text{ or } M_1/M_2=(L_1+L)/L_2.$$

$$\therefore \quad M_1/M_2=R_1/R_2=(L_1+L)/L_2. \quad \ldots(137)$$

12.20. MEASUREMENT OF CAPACITANCE

The methods of determination of self and mutual inductance in terms of standard capacitance and resistances may equally well be applied conversely to the measurement of capacitance if the standards of inductance are available. Other few methods are :

(*a*) **De Sauty's Bridge**—This method is the simplest way of comparing two capacitances. In this bridge two capacitors whose capacitances are to be compared or one unknown and other of known capacitance are connected as shown in Fig. 12.34. Audio frequency oscillator and headphone are connected in the gaps AC and BD respectively.

If the power factors of the capacitors are neglected, then at balance we have

$$\frac{Z_1}{Z_2}=\frac{Z_3}{Z_4}, \quad \therefore \quad \frac{R_1}{R_2}=\frac{1/j\omega C_1}{1/j\omega C_2}.$$

Thus $$R_1/R_2=C_2/C_1 \text{ or } C_1=C_2R_2/R_1. \quad \ldots(138)$$

A comparison of capacitances is therefore made in terms of

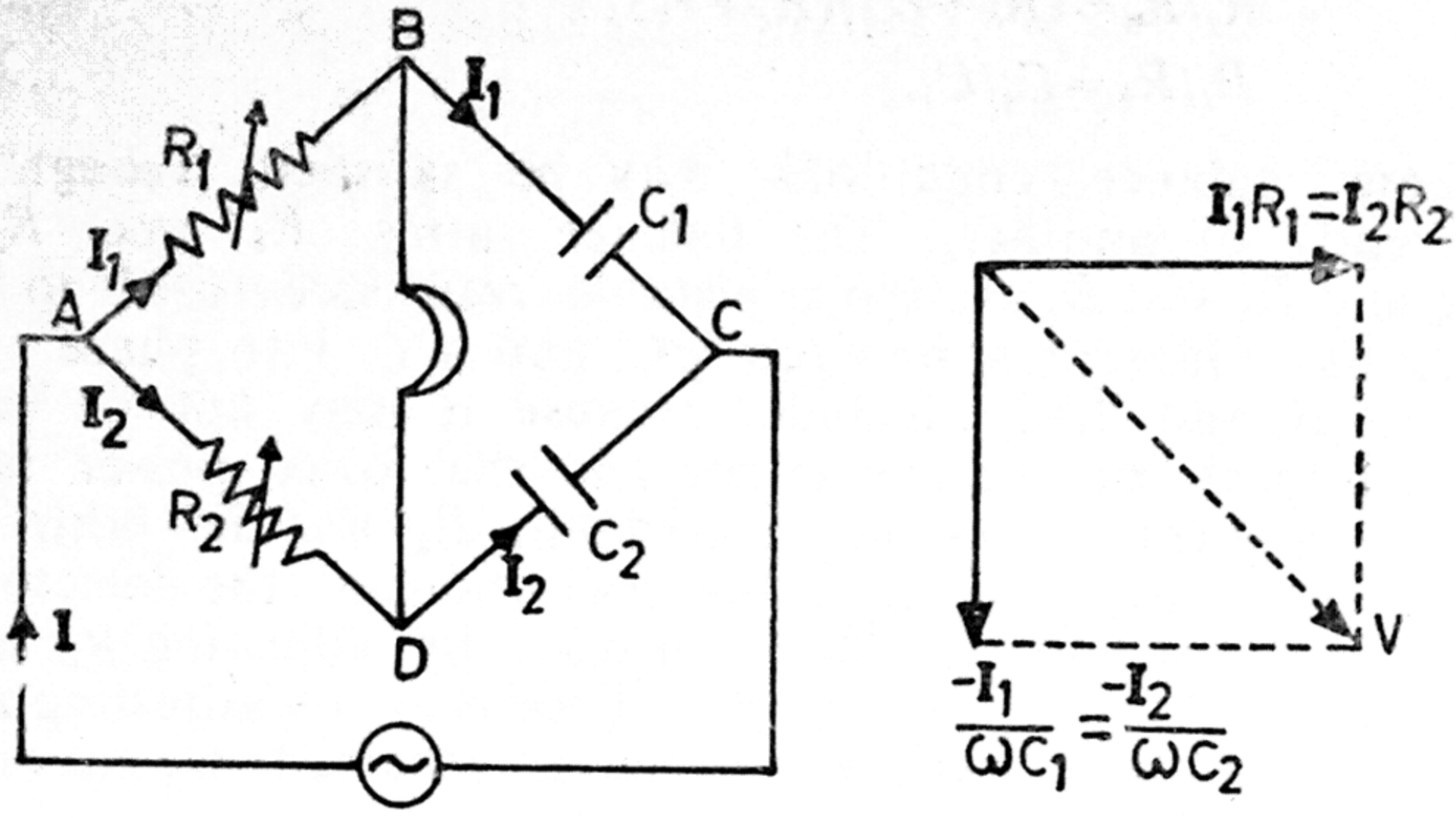


Fig. 12.34.

variable, *non inductive resistances* R_1 and R_2. The balance is also independent of frequency. For maximum sensitivity C_2 should be equal to C_1. The perfect balance can be obtained only in the case of air capacitor.

In practice the comparison of ordinary condenser is not so simple, as the capacitors are not both free from dielectric lossess. As the behaviour of a condenser with an imperfect dielectric depends on the frequency, it is important that the correct periodicity be employed. Imperfect capacitor may be regarded as made up of a capacitance associated with a resistance, either a small resistace in a series or a large resistance in parallel. To obtain an accurate balance, it is necessary to balance out the associated resistance.

(*b*) **Wien's Bridge.** It is a direct modification of De Sautys' bridge. In it C_1 and C_2, the capacitors to be compared, are assumed as perfect capacitors in series with small hypothetical resistances r_1 and r_2 respectively, where r_1 and r_2 are the loss components of the capacitors. R_3 and R_4 are the resistances required to obtain exact balance. All the resistances in the bridge are supposed to be non-inductive. At the time of balance

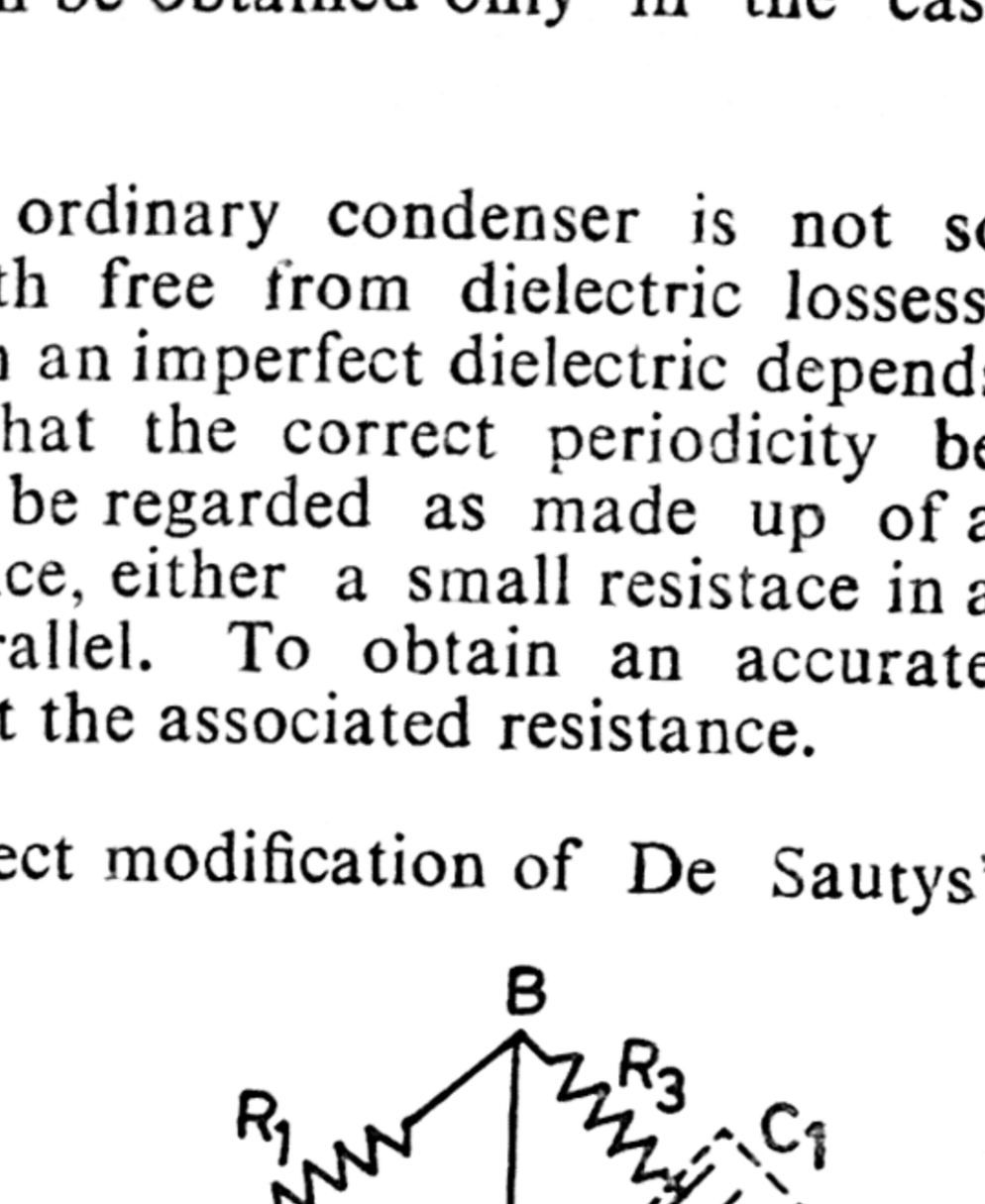


Fig. 12.35.

$$Z_1/Z_2=Z_3/Z_4,$$

or
$$\frac{R_1}{R_2}=\frac{R_3+r_1+1/j\omega C_1}{R_4+r_2+1/j\omega C_2},$$

or $$R_1\,(R_4+r_2+1/j\omega C_2)=R_2\,(R_3+r_1+1/j\omega C_1)$$

Equating real and imaginary parts, we have

$$R_1/R_2=(R_3+r_1)/(R_4+r_2) \qquad ...(139)$$

and $$R_1/R_2=C_2/C_1. \qquad ...(140)$$

These balance conditions may be satisfied, though not independently of another. The balance arms R_1 and R_2 are variable and R_3 and R_4 are the adjustable resistances used to bring the potential differences between BC and CD into phase. Two resistances (R_3 and R_4) are included because it may not be known at the start which of the capacitors has the lower power factor. Only one of the resistances can be used with R_3 and R_4 both zero, the ratio R_1/R_2 is adjusted until the indication of the detector is a minimum. The balance is then improved by adjusting R_3 or R_4 as the case may be and still further improved by adjusting R_1/R_2 and so on. Thus a perfect balance is obtained by successive adjustments.

(*c*) **Schering Bridge**. Schering bridge is the most accurate bridge method for the determination of (small) capacitances of standard capacitors. It is also used for the measurement of permittivities, *i.e.*, dielectric constants. *It is now most widely used for measuring dielectric loss and power factor and in the testing of cables and insulators at high voltages.*

In its simplest form the bridge is shown in Fig. 12.36. The condenser under test C_1 is placed in the arm AB of the bridge. R is an imaginary resistance representing its dielectric loss component .A standard capacitor C_2 is placed in the arm AD. The remaining arms contain non-inductive resistances R_2 and R_3. In parallel with R_3 a continuously variable calibrated air capacitor C_3 is connected. Such an air capacitor will have negligible power factor. The impedances of arms AB and AD are usually very high and that of arms BC and CD are usually small compared with those of the former. Due to this reason the detector (vibration galvanometer) and the resistances are atpotential only a few volts even when a high voltage supply ($\sim 10^5$ volts) is used.

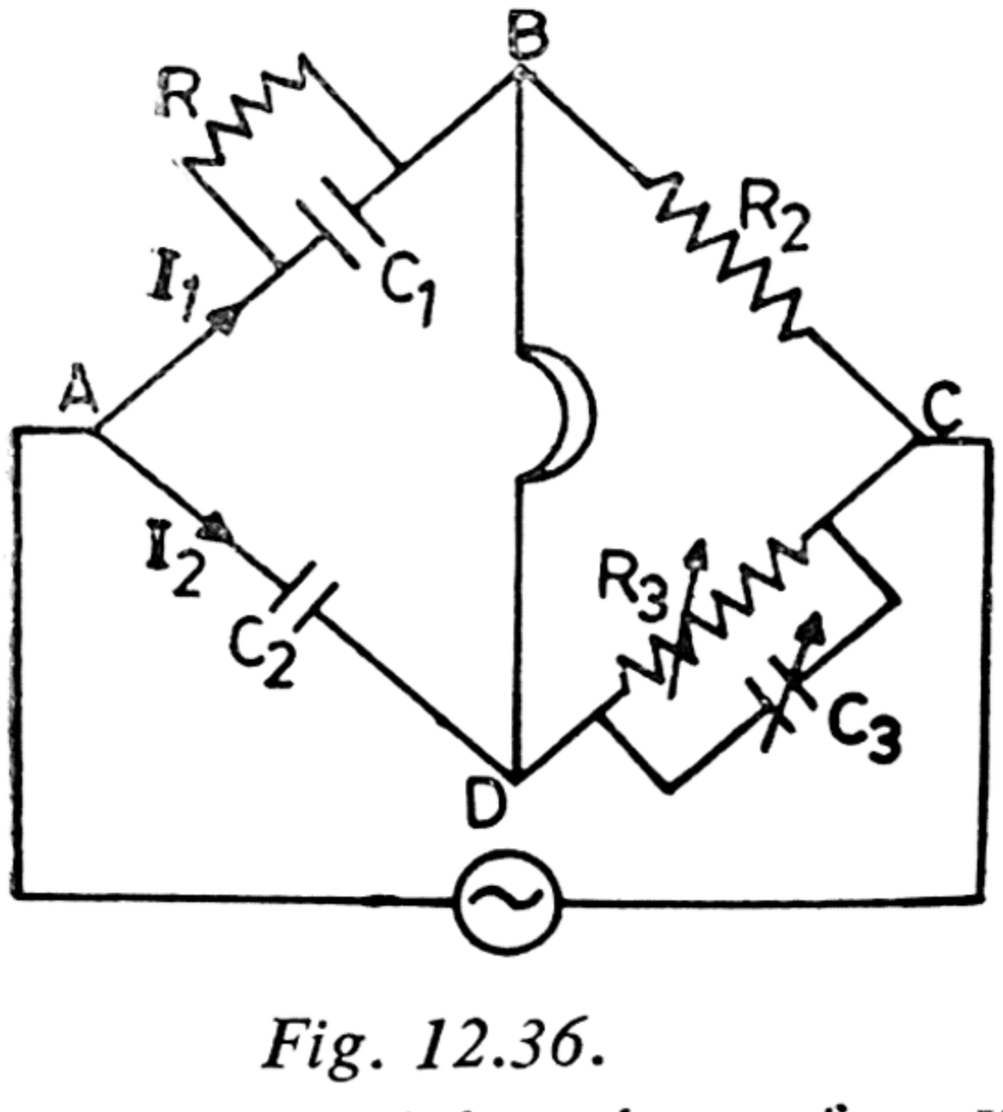


Fig. 12.36.

At the time of balance, we have

$$Z_1/Z_2=Z_3/Z_4.$$

Here $$\frac{1}{Z_1}=\frac{1}{R}+\frac{1}{1/j\omega C_1} \quad \text{or} \quad Z_1=\frac{R}{1+j\omega C_1 R},$$

$$Z_2=R_2,\ Z_3=1/j\omega C_3 \text{ and } Z_4=R_3/(1+j\omega C_3R_3).$$

$$\therefore \quad \frac{R}{R_2(1+j\omega C_1R)}=\frac{1/j\omega C_3}{R_3/(1+j\omega C_3R_3)}$$

or
$$\frac{R(1-j\omega C_1R)}{R_2(1+\omega^2C_1^2R^2)}=-j\frac{(1+j\omega C_3R_3)}{\omega C_2R_3}=\frac{\omega C_3R_3-j}{\omega C_2R_3}.$$

Equating real terms, we get

$$R/(1+\omega^2C_1^2R^2)=C_3R_2/C_2. \qquad \ldots(141)$$

Fig. 12.37 represents vector diagram. **OC** is the potential applied to the bridge from the supply. $\mathbf{OA}=\mathbf{V_1}$, the pd across *AB*

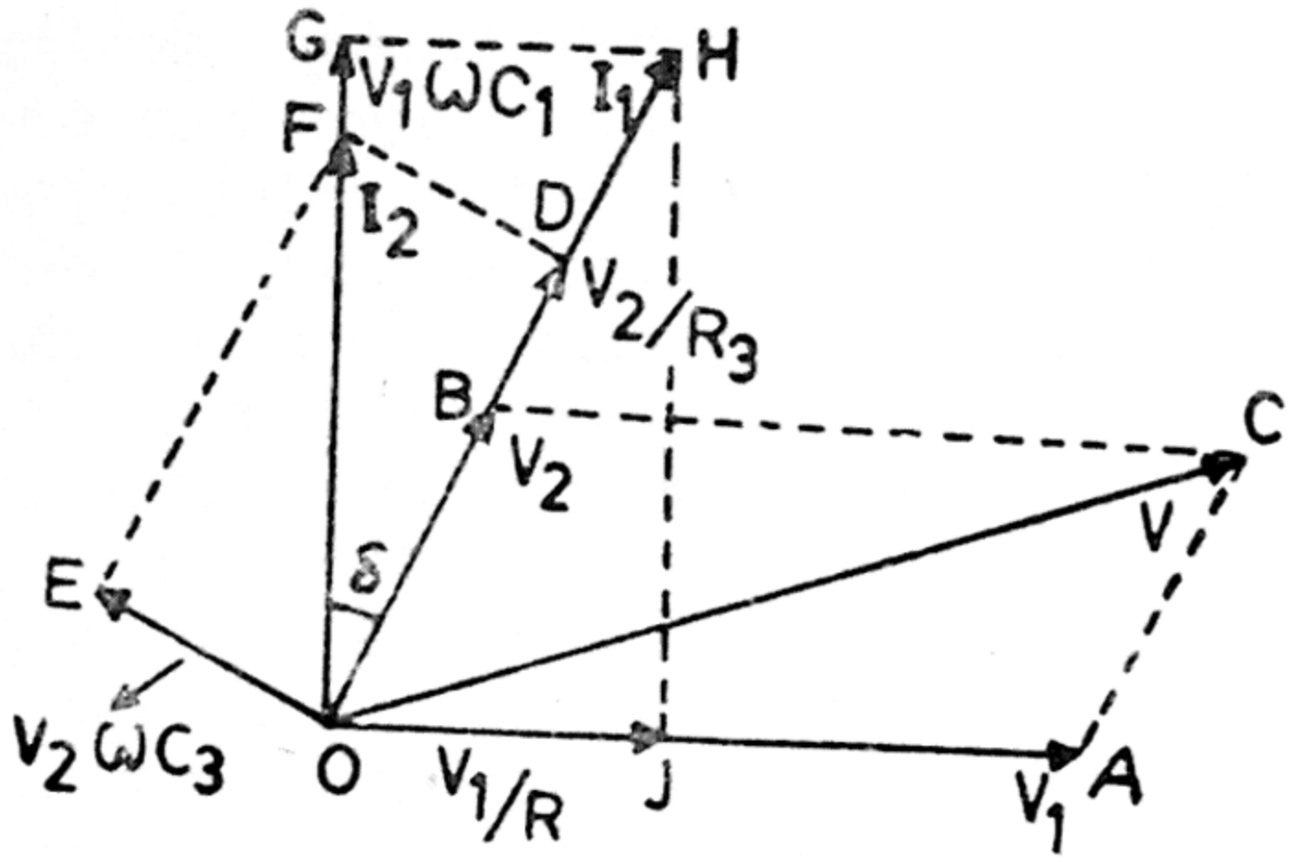


Fig. 12.37.

(or *AD*) and vector $\mathbf{OB}=\mathbf{V_2}$, the pd across *BC* (or *DC*). The current $\mathbf{I_1}$ flowing in arms *AB* and *BC* is represented by vector **OH**, while **OF** represents the current $\mathbf{I_2}$ flowing in *AD* and *DC*. **OG** and **OJ** represent the component parts of current $\mathbf{I_1}$ through C_1 and R respectively. Similarly **OD** and **OE** represent the components of current I_2 through R_3 and C_3 respectively. It is clear from figure that δ, the angle between current and potential vectors through capacitor C_1 is given by

$$\cos\delta=\frac{V_1\omega C_1}{V_1\sqrt{(1/R^2+\omega^2C_1^2)}}=\frac{\omega C_1R}{\sqrt{(1+\omega^2C_1^2R^2)}} \qquad \ldots(142)$$

Thus Eq. (141) reduces to

$$\cos^2\delta/\omega^2C_1^2R=C_3R_2/C_2 \text{ or } C_1=C_2\cos^2\delta/\omega^2C_1C_3RR_2$$

From triangles *GOH* and *EFO*, we have

$$\tan\delta=\frac{V_1/R}{V_1\omega C_1}=\frac{V_2\omega C_3}{V_2/R_3},$$

hence
$$1/\omega C_1R=\omega R_3C_3 \text{ or } R_3=1/\omega^2C_1C_3R.$$

Thus the expression for C_1 reduces to

$$C_1=C_2R_3\cos^2\delta/R_2$$
$$=C_2R_3/R_2, \text{ since } \delta \text{ is small.} \qquad \ldots(143)$$

To obtain balance, R_2, C_2 and C_3 are kept fixed and R_3 is adjusted for minimum sound. If the sound is not minimized C_3 is also adjusted first. By doing alternate adjustments of C_3 and R_3 a perfect balance is obtained.

The power factor of C_1 is cos δ, we may write it as

$$\cos \delta = \omega C_1 R. \qquad \ldots(144)$$

12.21. MEASUREMENT OF FREQUENCY (LOW)

In *ac* bridge used for the measurement of inductance or capacitance the balance conditions are independent of frequency that is why the freedom of the source from harmonics is not an essential requirement. The situation is entirely different in the measurement of the source frequency as the balance is to hold only at one frequency. If harmonics are there the balance point is blurred when a telephone is used as a detector, a vibration galvanometer is also impracticable as this would have to be turned afresh to every new frequency.

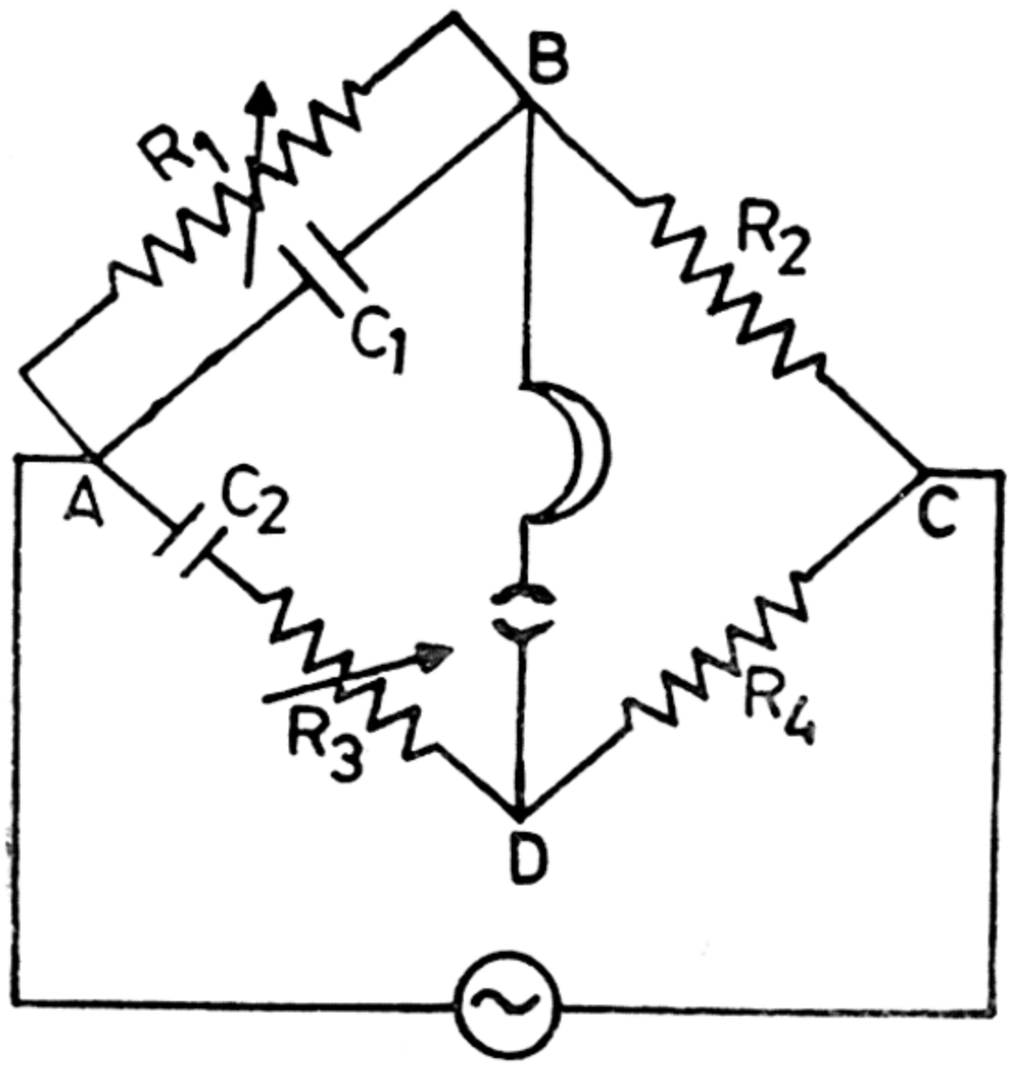


Fig. 12.38.

Robinson's Frequency Bridge. It is the modified form of the Wein's capacitance bridge. It consists two high class standard capacitors C_1 and C_2, preferably of equal value, and four non-inductive resistors R_1, R_2, R_3 and R_4. R_1 is in parallel with C_1, R_3 in series with C_2 and R_2 and R_4 constitute the third and fourth arms of the bridge. For balance we require

$$\left(\frac{1}{R_1}+j\omega C_1\right)\left(R_3+\frac{1}{j\omega C_2}\right)=\frac{R_4}{R_2}.$$

Equating in turn the real and imaginary parts, we get

$$R_3/R_1+C_1/C_2=R_4/R_2, \qquad \ldots(145)$$

and $$\omega C_1 R_3+(-1)/\omega C_2 R_1=0 \text{ or } \omega^2=1/C_1C_2R_1R_3. \qquad \ldots(146)$$

Robinson simplified these conditions for balance by making $C_1=C_2=C(say)$ and $R_1=R_3=R$ (*say*). Now above conditions become

$$R_4=2R_2, \qquad \ldots(147)$$

and $$\omega=1/CR \text{ or frequency } f=1/2\pi CR. \qquad \ldots(148)$$

Thus the bridge can be used for the measurement of frequency by adjusting $R_1=R_3$, $R_4=2R_2$ and $C_1=C_2$.

For C constant, $f \propto 1/R$ (or $\propto$ conductance), thus the scale of the instrument becomes linear when resistance boxes are replaced by conductance boxes. This bridge has the great merit that there is no inductive coupling in any part of the bridge.

Any harmonic present in the input voltage will appear in the detector arm at balance as the conditions for balance consist ω. A sensitive ear can find out balance position with the fundamental even in the presence of harmonics.

12.22. WAGNER EARTHING DEVICE

At high frequencies stray capacitances arise error in the results obtained with the bridges previously described. Let us suppose that

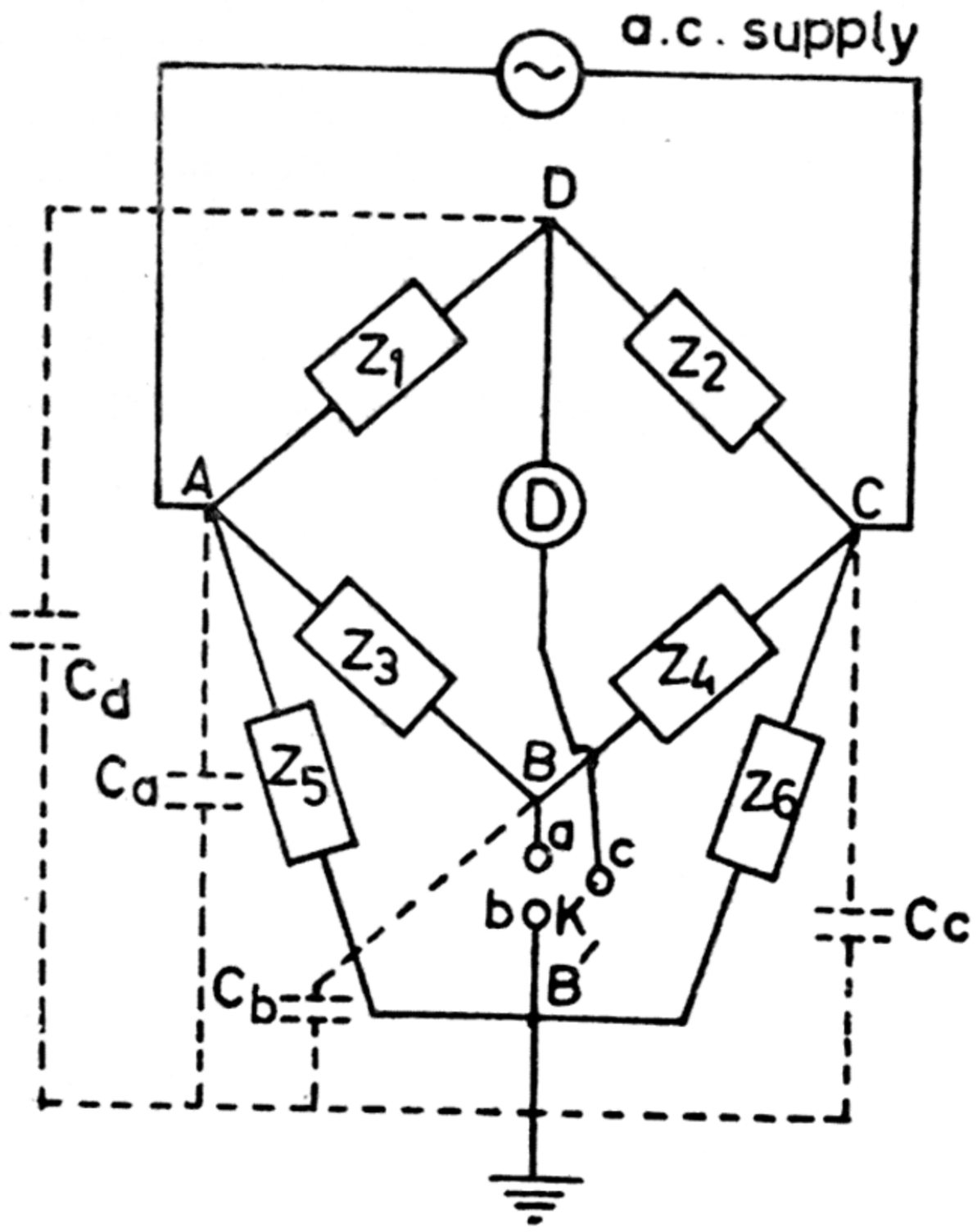


Fig. 12.39.

Z_1, Z_2, Z_3 and Z_4 comprise the four impedances forming an *ac* bridge and one of them is on test. Z_5 and Z_6 are the two variable impedances of the Wagner earth branch connected across the *ac* supply, the central point of which is earthed. The detector is connected by a two-way key either to B or to B', the common terminals of impedances Z_3 and Z_4, and Z_5 and Z_6, respectively. The stray capacitances are represented by the broken lines in the figure. They will connect points A, B, C and D to earth and current will flow from the bridge to earth. The flow of current from the points A and C to earth can not affect the balance of the bridge but the currents through Z_1 and Z_2 (or Z_3 and Z_4) cannot be equal if currents flow from the points B and D to earth. If the point B is earthed, the currents Z_3 and Z_4 are unequal even when the detector current is zero. It means that we do not have natural balance. Wagner over-

come this difficulty by bringing points B and D at earth potential without connecting actually to earth. It was done by using subsidiary bridge $AB'CD$.

In Wagner device, the main bridge $ABCD$ is first balanced by adjusting Z_3 or Z_4 or both. As we obtain best obtainable balance, the detector is connected to earthed point B' and the balance is obtained by adjusting Z_5 and Z_6. As B' is at earth potential, at balance, point D is at earth potential but because of the current to earth at B' it cannot be said that $Z_1/Z_2 = Z_5/Z_6$. The main bridge is balanced again by adjusting slightly Z_3 and Z_4. These processes are repeated till perfect balance is obtained. At this time detector current is zero either we connect the detector to B or to B'. Thus points D, B and B' are at the same potential. In this way points B and D are brought at earth potential without connecting to earth actually. The stray capacitances C_b and C_d have no effect on the balance conditions and the relation $Z_1/Z_2 = Z_3/Z_4$ is satisfied.

12.23. MEASUREMENT OF DIELECTRIC CONSTANT

Wagner earth device is needed when the impedance to be found is high (*e.g.*, small capacitance). The dielectric constant can be measured by measuring capacitance of a condenser, first with air as dielectric medium and the second with the substance under investigation as dielectric medium. The latter is naturally greater than the previous one. The ratio of these capacitances gives the dielectric constant, Since such capacitors have a high reactance, hence Wagner earth device should be used in that bridge, used for the capacitance measurement. A simple bridge used for this purpose is a de-Sauty bridge with a Wagner earth (Fig. 12.40). C is a variable wireless receiving capacitor of maximum capacitance 500 pf. It is carried below a polystyrene plate supported on the upper edge of a cylindrical glass jar of ample size. On the top of the plate there are two terminals connected to the fixed and moving vanes respectively. This plate is placed in the arm AD of the bridge. C_1 and C_2 are two variable air condensers of the maximum value $0.0005\mu f$ (500 pf). R_1, R_2 and R are non inductive resistance boxes. The source is a valve oscillator supplying about 15 volts at a frequency of 1200 c/s. The detector is a pair of earphones.

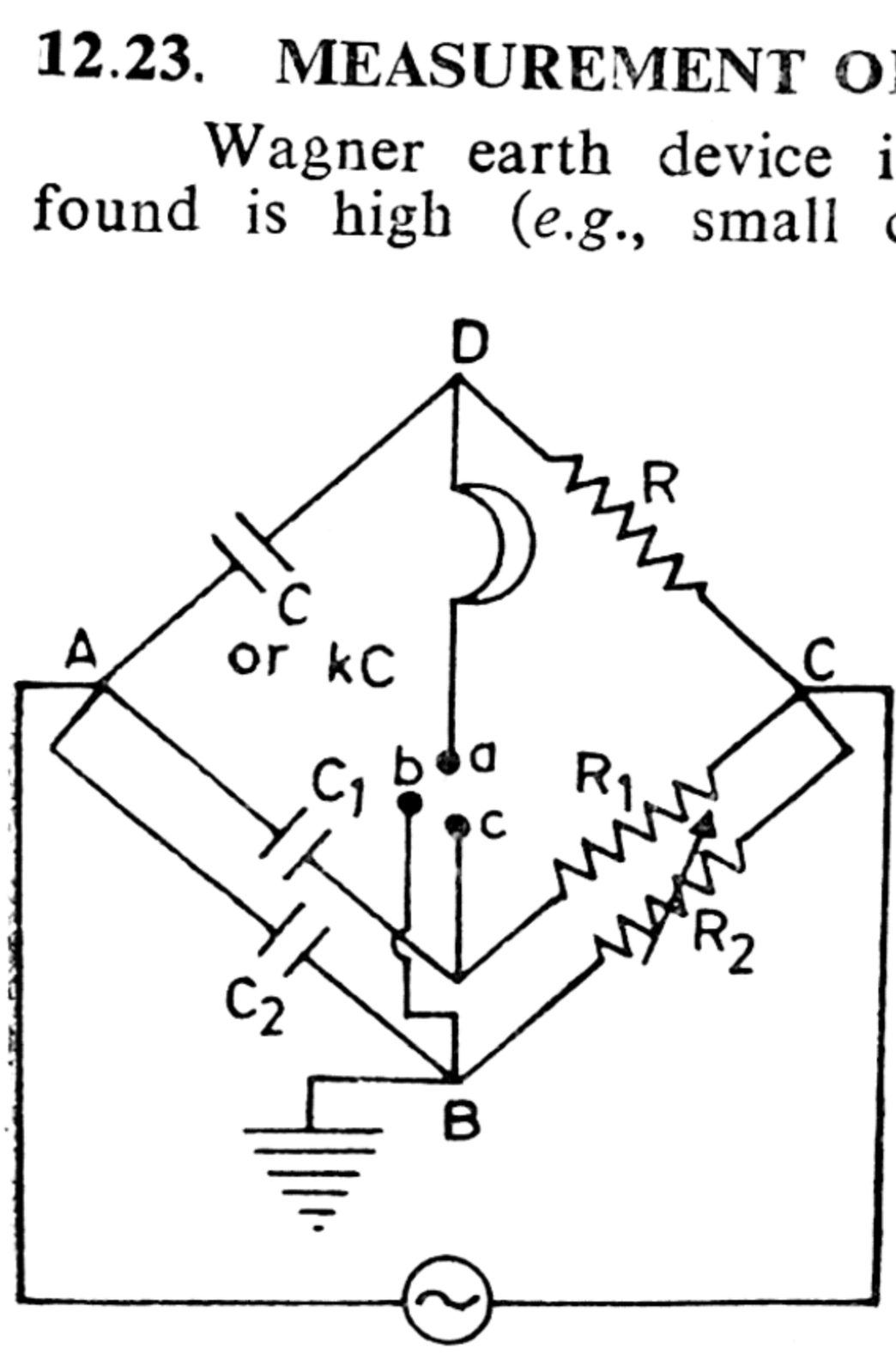


Fig. 12.40.

Starting with air as medium, alternate balances for the main bridge and then for the Wagner earth circuit are found by varying

R_1 and R_2. The vessel (jar) is filled with the liquid under test. The balances are again found alternately by adjusting R_1 and R_2 as done before. Let R' be the new value of R, then

$$CR=kCR' \quad \text{or} \quad k=R/R'. \qquad \ldots(149)$$

The error is 5%, if the Wagner earth in not used.

12.24. MEASUREMENT OF SPECIFIC CONDUCTIVITY

Kohlraush's bridge. The specific conductivity of the electrolyte is determined by using Kohlraush's bridge, the modified form of the Wheatstone's bridge in which the effect of polarisation is reduced by using an *ac* source. The back emf is reduced by using electrodes of large area.

The non-inductive resistances R_1 and R_2 are connected in the ratio arms. R_3 and a capacitor C are connected in parallel and thus form the resistance arm. The voltameter is connected in the unknown arm. A supply in the range from 10 to 20 V at frequencies upto 1 kc/sec and high impedance telephones (used as a detector) are connected in the source arm AC and detector arm BD respectively (Fig. 12.41)

At the balance condition

$$\frac{Z_1}{Z_2}=\frac{Z_3}{Z_4}.$$

Here $Z_1=R_1$, $Z_2=R_2$

$$Z_3=\frac{R_3}{1+j\omega CR_3}, Z_4=\frac{R_x}{1+j\omega C_xR_x}$$

$$\therefore \quad \frac{R_1}{R_2}=\frac{R_3(1+j\omega C_xR_x)}{R_x(1+j\omega CR_3)}$$

or $$R_1R_x+j\omega CR_1R_3R_x = R_2R_3+j\omega C_xR_xR_3R_2.$$

Equating real and imaginary parts, we get

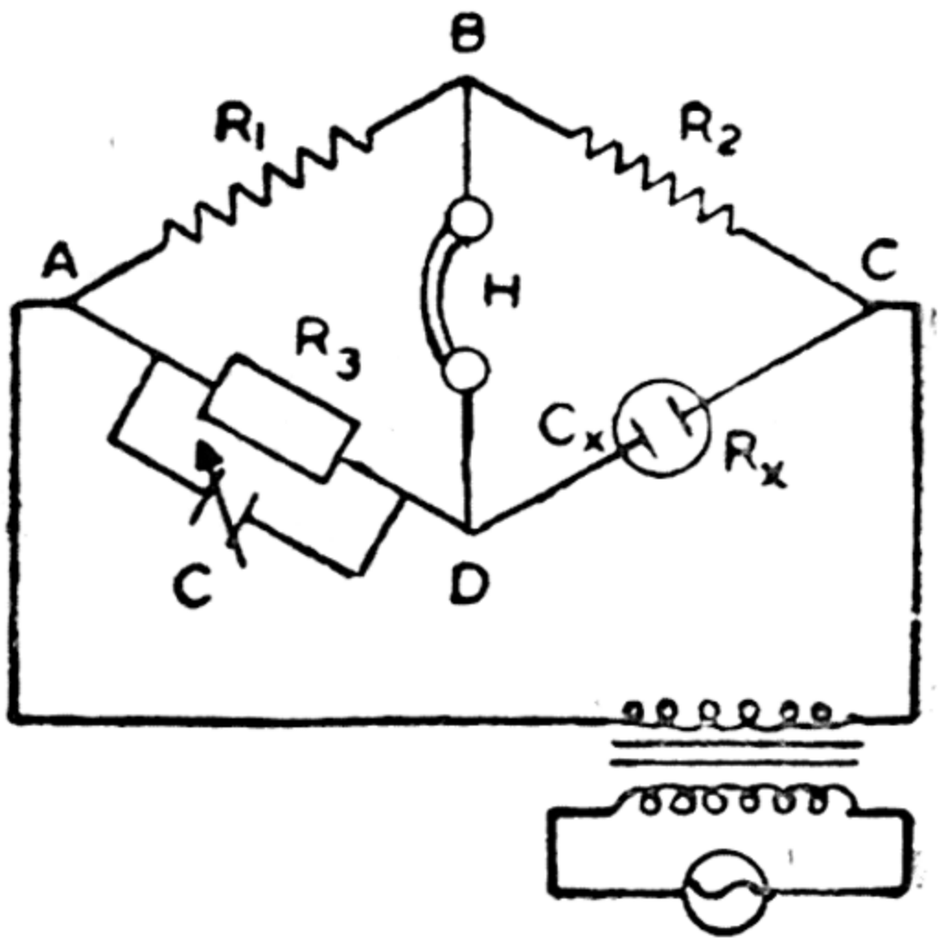


Fig. 12.41.

$$R_1R_x=R_2R_3 \quad \text{or} \quad R_x=R_2R_3/R_1. \qquad \ldots(150)$$

and $\omega CR_1R_3R_x=\omega C_xR_xR_2R_3$ or $$C_x=CR_1/R_2. \qquad \ldots(151)$$

The R_3 and C are adjusted alternatively for minimum sound in the detector. With the help of above equations, the inherent capacitance C_x is measured.

Kohlrausch from his experimental results suggested that the dissociation theory is only tenable for the *weak electrolytes*, *e.g.*, ammonia and acetic acid and most of the organic acids and salts. For *strong electrolytes*, *e.g.*, inorganic salts and acids, α is practically unity for all concentrations. The variation of Λ with concentration

is mainly due to variation of the ionic mobilities with the concentration. Experimental results show that the equivalent conductivity decreases at high concentrations and becomes practically constant at low concentrations. Debye and Huckel gave a theory applicable to strong electrolytes. The students are advised to read it from **Electricity and Magnetism**, vol I, by *J.H. Fewkes and John Yarwood.*

Exercises

Example 1. *A circular coil, having 100 turns and of radius 25 cm rotates uniformly 10 times per sec about a vertical diameter in a place where the angle of dip is 60° and the resultant field strength is 30 amp/meter. Assuming self inductance of the coil as negligible, find (a) the maximum emf in volts, (b) the maximum current in amperes and (c) the rate of production of heat per second in the coil of resistance 10 Ω.*

The vertical and horizontal components of magnetic field are 30 sin 60° and 30 cos 60° respectively. The vertical component is always ‖ to the vertical coil, hence no emf is induced in the coil as the resultant flux through the coil is zero. The horizontal component gives the induced emf only. The flux associated with the coil at an angle θ is

$$\therefore \qquad \Phi = BAn \cos\theta = \mu_0 nAH \cos\theta.$$

$$\therefore \text{ Induced emf } \mathcal{E} = -d\Phi/dt = \mu_0 nAH \sin\theta \, (d\theta/dt).$$

Hence $\mathcal{E}$ will have maximum value V_0, when $\sin\theta = 1$.

$$\therefore \quad V_0 = \mu_0 nAH \, (d\theta/dt) = \mu_0 nA \, (30 \cos 60^\circ) \, (d\theta/dt)$$

$$= 4\pi \times 10^{-7} \times 100 \times \pi \times (0.25)^2 \times 30 \times (1/2) \times 20\pi$$

$$= 2.32 \times 10^{-2} \text{ volt.}$$

$$\text{The maximum current } I_0 = \frac{V_0}{R} = \frac{2.32 \times 10^{-2}}{10}$$

$$= 2.32 \times 10^{-3} \text{ amp.}$$

$$\therefore \quad \text{Rate of production of heat} = I_{rms}^2 \, R = \tfrac{1}{2} \, I_0^2 R$$

$$= 2.65 \times 10^{-5} \text{ joule/sec.}$$

Example 2. *A coil of self inductance 0.7 henry is joined in series with a non-inductive resistance of 50 Ω. Calculate the wattless and power components of current when connected to a supply of 200 volts at a frequency of 50 cycles/sec.*

$$\text{Impedance } Z = \sqrt{(R^2 + \omega^2 L^2)} = \sqrt{[50^2 + (2\pi \times 50 \times 0.7)^2]}$$

$$= 225.4 \text{ ohms.}$$

$$\therefore \text{ Current } I = \frac{V}{Z} = \frac{200}{225.4}$$

$$= 0.887 \text{ amp.}$$

and $$\tan \phi = \frac{\omega L}{R} = \frac{2\pi \times 50 \times 0.7}{50}$$

$$= 4.4.$$

or $$\phi = 77°12'.$$

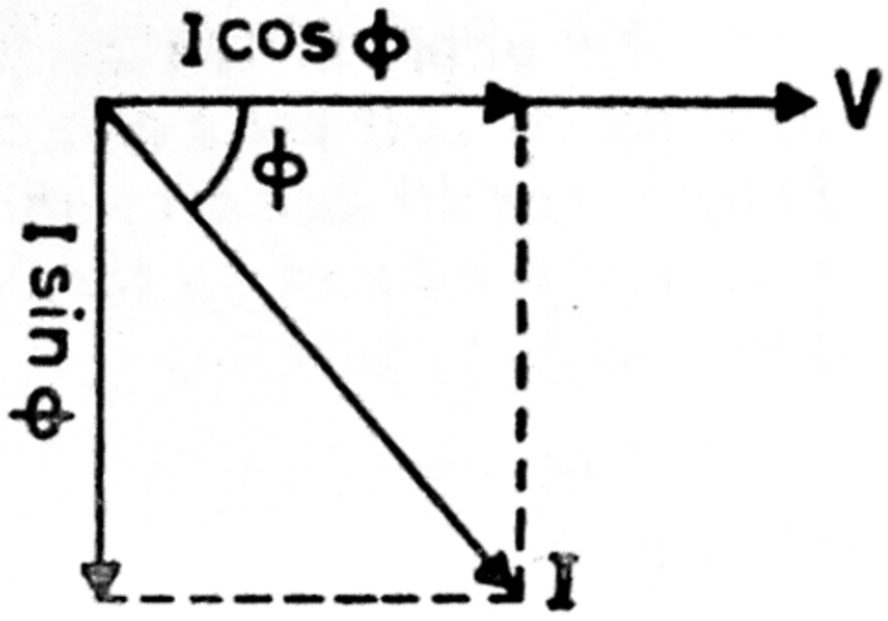


Fig. 12.42.

The current lags behind the *ac* emf by an angle ϕ. Thus the current vector **I** will make angle ϕ in clockwise direction with the emf vector **V**, The component $I \cos \phi$ in phase with **V** is responsible for the power dissipated and is known as the power component of **I.** The component of $I \sin \phi$, 90° out of phase with **V**, does not contribute to the power and is known as wattless current.

$\therefore$ Power component of current $= I \cos \phi = 0.887 \times \cos 77° 12'$

$$= 0.197 \text{ amp.}$$

and wattless component of the current

$$= I \sin \phi = 0.887 \times \sin 77° 12'$$

$$= 0.865 \text{ amp.}$$

Epample 3. *Show that when a coil o inductance L and resistance R is attached to two terminals at which an emf* $v = V_0 \sin \omega t$ *is maintained, the average rate of consumption of energy is*

$$\tfrac{1}{2} V_0^2 R/(R^2 + L^2\omega^2).$$

This problem is same as given in article 12.6. The current flowing through the circuit is given by

$$i = \frac{V_0}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \phi), \text{ where } \phi = \tan^{-1} \frac{L\omega}{R}.$$

As the energy will only be dissipated in the circuit resistance, hence the average rate of consumption of energy

$$= \frac{\int_0^{T=2\pi/\omega} i^2 R\, dt}{\int_0^{T=2\pi/\omega} dt} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \frac{V_0^2 R}{R^2 + \omega^2 L^2} \sin^2(\omega t - \phi)\, dt$$

$$= \frac{V_0^2 R\omega}{2\pi(R^2 + \omega^2 L^2)} \frac{1}{2} \int_0^{2\pi/\omega} [1 - \cos 2(\omega t - \phi)]\, dt$$

$$= \frac{V_0^2 R\omega}{4\pi(R^2 + \omega^2 L^2)} \left[t - \frac{\sin 2(\omega t - \phi)}{2\omega} \right]_0^{2\pi/\omega} = \frac{V_0^2 R}{2(R^2 + \omega^2 L^2)}.$$

Example 4. *An ac arc lamp needing 150 volts is put across main at a pd. of 200 volts and a chocking coil in series with the lamp. The lamp takes 10 amperes and the frequency is 50 per second. Find (i) the pressure produced by the chocking coil reaction and (ii) the self inductance of the chocking.*

Total emf applied $\mathbf{V}$=pd $\mathbf{V_1}$ across the lamp+pd $\mathbf{V_2}$ across choke coil.

$$\therefore \quad V=\sqrt{(V_1^2+V_2^2)}$$

or $$V_2=\sqrt{(V^2-V_1^2)}=\sqrt{(200^2-150^2)}=132.3 \text{ volts.}$$

Current will be same in the circuit, hence the pd across the choke of self inductance L is

$$V_2=\omega LI=2\pi fIL$$

or $$L=\frac{V_2}{2\pi fI}=\frac{132.3}{2\times 3.14\times 50\times 10}=0.042 \text{ henry.}$$

Example 5. *In an oscillating LC circuit, calculate the charge in terms of maximum charge on the capacitor when the whole energy is shared between electric field and magnetic field. Assuming $L=10$ mH and $C=1.0\mu f$, find the time required for this condition.*

Let q_m be the maximum charge on the capacitor.

$\therefore$ Maximum energy associated with the capacitor $=q_m^2/2C$. If q be the charge on the capacitor, when half of the energy is converted into electric energy linked with the inductor, then

$$q^2/2C=\tfrac{1}{2}\, q_m^2/2C \text{ or } q=q_m/\sqrt{2}.$$

To find time, assuming $\phi=0$,

$$q=q_m \cos \omega t=q_m/\sqrt{2}.$$

$\therefore$ $\cos \omega t=1/\sqrt{2}$, $\omega t=45^\circ=\pi/4$ or $t=\pi/4\omega$.

As $$\omega=\frac{1}{\sqrt{(LC)}}=\frac{1}{\sqrt{(10\times 10^{-3}\times 10^{-6})}}=10^4 \text{ radians/sec.}$$

$$\therefore \quad t=\frac{\pi}{4\times 10^4}=7.85\times 10^{-5} \text{ sec.}$$

Example 6. *A 20 volt 5 watt lamp is to be used on a.c. mains of 200 volts 50 cycles/sec. Calculate the (i) capacitor, (ii) inductor to be put in series to run the lamp. How much pure resistance should be included in place of the above devises so that the lamp can run on its rated voltage. Which of the above arrangements will be more economical and why ?*

The current required by the lamp

$$=\frac{\text{Power}}{\text{Voltage}}=\frac{5}{20}=0.25 \text{ amp.}$$

$\therefore$ The resistance of the lamp = Voltage/current = 20/0.25

= 80 ohms.

If the lamp is to run at 200 volts, 50 cycles/sec. a.c. mains, a capacitor (or choke) must be placed in series with the lamp in order to increase its effective resistance so that the current through it is 0.25 amp only.

(*i*) Let C be the capacitor of the required capacitor, then the impedance of the circuit becomes

$$Z=\sqrt{(R^2+X_C^2)}=\sqrt{(R^2+1/\omega^2C^2)}=\sqrt{[80^2+1/(2\pi\times 50)^2C^2]}.$$

The current in the circuit must be 0.25 amp, hence

$$I=V/Z=200/\sqrt{(80^2+1/100^2\pi^2C^2)}=0.25$$

or $$80^2+1/(100\pi)^2C^2=800^2$$

i.e., $$C=4.0\ \mu f.$$

(*ii*) Let L be the inductance of the choke placed in series with the lamp to set the current at 0.25 amp, then the impedance of the circuit becomes

$$Z=\sqrt{(R^2+X_L^2)}=\sqrt{(R^2+\omega^2L^2)}=\sqrt{[80^2+(2\pi\times 50)^2L^2]}$$

The current in the circuit is

$$I=V/Z=200/\sqrt{(80^2+100^2\pi^2L^2)}=0.25$$

or $$80^2+100^2\pi^2L^2=800^2$$

i.e., $$L=2.54 \text{ henry.}$$

(*iii*) Let r be the resistance placed in series with the lamp in place of a choke, then the current in the circuit

$$I=V/(R+r)=200/(80+r)=0.25$$

or $$r=720 \text{ ohms.}$$

(*iv*) In the last case, *i.e.*, the insertion of a resistance of 720 ohm to run the lamp, the dissipation of energy is maximum, while in the case of a choke in series the power consumption is practically zero ($R \ll X_L$ or X_C). Hence it will be more economical to use inductance or capacitance in series with the lamp to run it.,

Example 7. *A 200-ohm resistor and a 1μf capacitor are connected in series across a 120 volts (r.m.s.), 60 cycles/sec. line. (a) What is the impedance ? (b) What is the r.m.s. value of the current. (c) What is the power dissipated in the circuit ? (d) What will be the reading of an a.c. voltmeter-connected across the resistor and across the capacitor ?*

(*a*) The capacitive reactance $X_C = 1/\omega C = 1/2\pi fC$

$= 1/2 \times 3.14 \times 60 \times 1 \times 10^{-6}$

$= 2653.9$ ohm.

When a capacitor C is connected in series with a resistance R, the impedance of the circuit becomes

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{2000^2 + 2653.9^2} = 3323.2 \text{ ohms.}$$

(*b*) The r.m.s. value of the circuit is given by

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{120}{3323.2} = 0.036 \text{ amp.}$$

(*c*) In the circuit containing resistance and capacitor, the phase lag is given by ϕ and the power factor

$$\cos \phi = R/Z = 2000/3323.2 = 0.602.$$

Therefore the power dissipated in the circuit

$$P = V_{rms} I_{rms} \cos \phi$$

$$= 120 \times 0.036 \times 0.602 = 2.6 \text{ watts}$$

(*d*) The potential differences across the resistor and the capacitor are given by

$$V_R = I_{rms} \times R = 0.036 \times 2000 = 72 \text{ volts}$$

$$V_C = I_{rms} \times X_C = 0.036 \times 2653.9 = 95.5 \text{ volts.}$$

Thus the a.c. voltmeter will read 72 and 95.5 volts, when connected across the resistor and capacitor respectively.

Example 8. *A series resonant circuit has $L = 1.01$ H, $C = 20\mu f$ and $R = 70$ ohms to which an emf of 100 V, (r.m.s.) is applied. At which frequency of the applied emf will it resonate with maximum response.*

In a LCR series resonance circuit, the impedance of the circuit is minimum and is equal to the resistance of the circuit, *i.e.*

$$Z = R = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

i.e., $\omega L = 1/\omega C$ or $\omega^2 = 1/LC$.

Thus the frequency of the applied emf for the maximum response

$$f = \omega/2\pi = 1/2\pi\sqrt{LC.}$$

$$= 1/2 \times 3.14 \, (1.01 \times 20 \times 10^{-6})^{1/2} = 35.43 \text{ } c/\text{sec.}$$

Example 9. *If in the LCR-series circuit a.c. voltmeter reads 30 volts across the resistance, 80 volts across the inductance and 40 volts across the condenser, what is the applied emf?*

The voltmeter reads the r.m.s. value of the voltage. The total applied emf may be obtained by a vector method. The voltage across the inductance V_L leads by 90° on V_R and across the capacitor V_C lags behind by 90°, therefore the resultant voltage is the vector sum of V_R and $(V_L \sim V_C)$ which are mutually perpendicular to each other.

$$\therefore \quad V_{rms}^2 = V_R^2 + (V_L - V_C)^2$$

$$= 30^2 + (80-40)^2 = 900 + 1600$$

or $$V_{rms} = 50 \text{ volts.}$$

Another method is to use the relations

$$V_R = RI_{rms},\ V_L = \omega L I_{rms},\ V_C = I_{rms}/\omega C.$$

As $$V_{rms} = I_{rms}\, Z = I_{rms}\sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

$$= \sqrt{(I_{rms}\, R)^2 + (I_{rms}\, \omega L - I_{rms}/\omega C)^2}$$

$$= \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{30^2 + (80-40)^2}$$

$$= 50 \text{ volts.}$$

Example 10. *A coil of resistance 60 ohm and inductance 3 henry is connected in series with a capacitor of 4 μf and an a.c. supply of 200 volt and 50 c/s. Calculate (i) the impedance in the circuit, (ii) the phase difference between current and voltage and (iii) the potential difference across the inductor and capacitor.*

(*i*) The impedance of the *LCR*-series-circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2},$$

Given $R = 60$ ohms, $X_L = \omega L = 2\pi f L = 2 \times 3.14 \times 50 \times 3 = 942$,

$$X_C = 1/\omega C = 1/2\pi f C = 1/2 \times 3.14 \times 50 \times 4 \times 10^{-6}$$

$$= 796.2 \text{ ohms}$$

$$\therefore \quad Z = \sqrt{60^2 + (942 - 796.2)^2} = 157.7 \text{ o ,m.}$$

(*ii*) The phase difference between current and voltage is given by

$$\phi = \tan^{-1} \frac{X_L - X_C}{R} = \tan^{-1} \frac{942 - 796.2}{60} = \tan^{-1} 2.43.$$

(*iii*) The r.m.s. value of current passing through the circuit

$$I_{rms} = V_{rms}/Z = 200/157.7 = 1.27 \text{ amp.}$$

$\therefore$ The potential difference across the inductor

$$V_L = I_{rms}\, X_L = 1.27 \times 942 = 1194.7 \text{ volts}$$

and across the capacitor

$$V_C = I_{rms}\, X_C = 1.27 \times 796.2 = 1009.8 \text{ volts.}$$

Example 11. *In an a.c. circuit an inductance of 1 H, a capacitor of 0.2 μf and a resistance of 3000 ohms are connected in series to a 10 volt a.c. source of $1000/2\pi$ cycles/sec. Find the peak current in the circuit and power consumed.*

In a series LCR circuit, the peak current is given by

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}.$$

In this problem, $R = 3000$, $\omega = 2\pi f = 1000$, $L = 1$, $C = 0.2 \times 10^{-6}$. and $V_{rms} = 10$ volts. Therefore $V_0 = \sqrt{2}\ V_{rms} = 10\sqrt{2}$. Substituting these values in the above relation, we get

$$I_0 = 10\sqrt{2}\sqrt{3000^2 + (1000 \times 1 - 1/1000 \times 0.2 \times 10^{-6})^2}$$

$$= 2.83 \text{ milliampere.}$$

The power factor is given by

$$\cos\phi = R/\sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

$$= 3000/\sqrt{3000^2 + (1000 - 1/1000 \times 0.2 \times 10^{-6})^2}$$

$$= 0.6$$

$$I_{rms} = I_0/\sqrt{2} = 2.83 \times 10^{-3}/\sqrt{2} = 2 \times 10^{-3} \text{ amp.}$$

$\therefore$ Power consumed in the circuit

$$P = I_{rms}\, V_{rms} \cos\phi$$

$$= 2 \times 10^{-3} \times 10 \times 0.6 = 1.2 \times 10^{-2} \text{ watt.}$$

Example 12. *A 60 cycles AC-circuit has a resistance of 2 ohms and inductance of 10 millihenries. What is the power factor? What capacitance placed in the circuit will make the power factor unity?*

Power factor of a LR-circuit is given by

$$\cos\theta = R/\sqrt{R^2 + (\omega L)^2}$$

In this problem, given that

$$R = 2 \text{ ohms}, \quad L = 10 \times 10^{-3} \text{ henry}$$

and $$\omega = 2\pi f = 2 \times 3.14 \times 60.$$

$\therefore$ $$\cos\theta = 2/\sqrt{2^2 + (2 \times 3.14 \times 60 \times 10^{-2})^2}$$

$$= 0.469.$$

When the capacitance is placed in the circuit in series with the resistance and inductance, the power factor

$$\cos\theta = R/\sqrt{R^2+(\omega L - 1/\omega C)^2}$$

It will be unity, if

$$R = Z = \sqrt{R^2+(\omega L - 1/\omega C)^2}$$

$$\therefore \quad \omega L - 1/\omega C = 0 \quad \text{or} \quad C = 1/\omega^2 L = 1/4\pi^2 f^2 L.$$

Substitution of numerical values gives

$$C = 1/4\times(3.14)^2\times 60^2\times 10^{-2} = 704\ \mu f.$$

Example 13. *The alternating emf of rms value 100 volts and 50 cycles is applied to a circuit containing the inductance of 5 henries and the capacitor of 2 microfarads and resistance of 1000 ohms in series. What rms current flows ? What would be the effect of (a) doubling and (b) halving the frequency ?*

The maximum value of emf $= V_0 = V_{rms}\sqrt{2} = 100\sqrt{2}$.

The current in LCR circuit is given as

$$i = \frac{V_0}{\sqrt{[R^2+(\omega L - 1/\omega C)^2]}}\sin(\omega t - \phi),$$

where $$\phi = \tan^{-1}\frac{\omega L - 1/\omega C}{R}$$

$\therefore$ Maximum value of current $I_0 = V_0/\sqrt{[R^2+(\omega L - 1/\omega C)^2]}$

$$= \frac{100\sqrt{2}}{\sqrt{[1000^2+(2\pi\times 50\times 5 - 1/2\pi\times 50\times 2\times 10^{-6})^2]}}$$

$$= 0.09998\sqrt{2} = 0.1414 \text{ amp.}$$

or rms value of current $= I_{rms} = I_0/\sqrt{2} = 0.09998$ amp.

This current is nearly same as due to the resistance only as the bracket term is very small. It will lag behind the applied emf by a phase

$$\phi = \tan^{-1}\frac{\omega L - 1/\omega C}{R}$$

$$= \tan^{-1}\frac{2\pi\times 50\times 5 - 1/2\pi\times 50\times 2\times 10^{-6}}{1000}$$

$$= \tan^{-1}(-0.02) = -0.02 \text{ radian (−ve for lag)}$$

(Because the angle is very small hence $\tan\phi = \phi$ in radian)

$\therefore$ Time advance $= \frac{0.02}{2\pi}\frac{1}{50} = 6.4\times 10^{-5}$ sec.

Rest part of the example is left as an exercise.

Example 14. *Find the voltage across the various elements, i.e., resistance, capacitance and inductance, which are in series and having values 1000Ω, 1 μf and 2.0 henry respectively.*

Given emf as $v=(100\sqrt{2})$ *sin 1000 t volts.*

The rms value of voltage across the source $=100\sqrt{2}/\sqrt{2}=100$ volts and $\omega=2\pi f=1000$ sec^{-1}.

$$\therefore \quad I_{rms}=V_{rms}/Z=V_{rms}/\sqrt{[R^2+(\omega L-1/\omega C)^2]}$$

$$=\frac{100}{\sqrt{[1000^2+(1000\times 2-1/1000\times 1\times 10^{-6})^2]}}$$

$=0.0707$ amp.

The current will be same everywhere in the circuit, therefore

PD across resistor $=V_R=0.0707\times 1000=70.7$ volts

PD across inductor $=V_L=0.0707\times 1000\times 2=141.4$ volts

PD across capacitor $=V_C=0.0707\times(1/1000\times 10^{-6})=70.7$ volts.

The rms voltages do not add directly, as

$$V_R+V_L+V_C=282.8 \text{ volts},$$

which is not the source voltage 100 volts. The reason is that these voltages are not in phase and can be added by vector or by phasor algebra.

Example 15. *A circuit consists of a non inductive resistance of 50Ω, a coil of a inductance 0.3H and resistance 2Ω, and a capacitor of 40 μf in series and is supplied with 200 volts rms at 50 cycles/sec. Find the current lag or lead and the power in the circuit.*

Total resistance in the circuit $=50+2=52$ ohms.

Inductive reactance $X_L=\omega L=2\times 3.14\times 50\times 0.3=94.2$ ohms.

Capacitive reactance $X_C=1/\omega C=1/2\times 3.14\times 50\times 40\times 10^{-6}$

$=79.6$ ohms.

$$\therefore \quad I_{rms}=\frac{V_{rms}}{\sqrt{[R^2+(X_L-X_C)^2]}}=\frac{200}{\sqrt{[52^2+(94.2-79.6)^2]}}$$

$=3.7$ amp.

As $X_L>X_C$, hence the current lags behind the applied emf by an angle ϕ, given by

$$\tan\phi=\frac{X_L-X_C}{R}=\frac{94.2-79.6}{52}=0.28.$$

$$\phi=15°5'.$$

Power factor $\cos\phi=\dfrac{R}{\sqrt{[R^2+(X_L-X_C)^2]}}=\dfrac{52}{\sqrt{[52^2+(14.6)^2]}}$

or $\cos\phi=0.96.$

$\therefore$ Apparent power $= I_{rms}\, V_{rms} = 3.7 \times 200 = 740$ watts,

and true power $= I_{rms}\, V_{rms} \cos\phi = 740 \times 0.96 = 710.4$ watts.

Same value is also obtained by the relation

true power $= I_{rms}^2\, R = (3.7)^2 \times 52 = 711.9$ watts.

Example 16. *A coil of inductance 2mH and resistance 15Ω is connected in parallel with a capacitor of magnitude 0.001 μf. Find (a) the frequency at which the current from an a.c. supply is minimum, (b) the peak value of this make up current if $V_0 = 2$ volts and (c) the peak oscillatory current under these conditions.*

We know that in this type of circuit the current from the supply is minimum at the resonant frequency of the circuit. The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} = \frac{1}{2\times 3.14}\sqrt{\left(\frac{1}{2\times 10^{-3}\times 10^{-9}} - \frac{15^2}{(2\times 10^{-3})^2}\right)}$$

$= 112.5$ kilo cycles/sec.

Make up current peak value at resonance

$$= \frac{\text{Peak value of emf}}{\text{Dynamic resistance}} = \frac{V_0}{L/CR} = \frac{V_0 CR}{L} = \frac{2\times 10^{-9}\times 15}{2\times 10^{-3}}$$

$= 15 \times 10^{-6}$ amp. $= 15\mu$A.

The peak oscillatory current at resonance

$= V_0/(1/\omega C) = V_0 \omega C = 2 \times 2\pi \times 112.5 \times 10^3 \times 10^{-9}$

$= 1.41$ mA.

Example 17. *In an a.c. circuit given that $X_C = 200\ \Omega$, $R = 100\ \Omega$ and the applied voltage $V_{rms} = 100$ volts. If the inductive reactance X_L is varied over a wide range by varying L, find the condition for resonance and the condition for maximum choke drop.*

We know that the current in a *LCR*-series circuit is given by

$$I_{rms} = V_{rms}/\sqrt{[R^2 + (X_L - X_C)^2]}$$

Thus the potential difference across the choke is given by

$$V_L = X_L\, I_{rms} = X_L\, V_{rms}/\sqrt{[R^2 + (X_L - X_C)^2]}.$$

The condition for resonance is

$$X_L = X_C = 200\ \Omega \quad \text{and } Z = R = 100\ \Omega$$

$\therefore$ Current $I_{rms} = V_{rms}/R = 100/100 = 1$ amp,

and the potential difference across the choke

$$V_L = X_L\, I_{rms} = 200 \times 1 = 200 \text{ volts.}$$

The potential drop across the choke will be maximum, when

$$dV_L/dX_L=0$$

$$\text{i.e., } \frac{[R^2+(X_L-X_C)^2]^{1/2}V_{rms}-X_L V_{rms}\frac{1}{2}[R^2+(X_L-X_C)^2]^{-1/2}\times 2(X_L-X_C)}{R^2+(X_L-X_C)^2}$$

$$=0$$

$$\text{or} \qquad [R^2+(X_L-X_C)^2]^{1/2}=X_L(X_L-X_C)[R^2+(X_L-X_C)^2]^{-1/2}$$

$$\text{or} \qquad R^2+(X_L-X_C)^2=X_L(X_L-X_C)$$

$$\therefore \qquad X_L=(R^2+X_C^2)/X_C.$$

In the present problem, $X_C=200\ \Omega$ and $R=100\ \Omega$

$$\therefore \qquad X_L=(100^2+200^2)/200=250\ \Omega.$$

Hence the current in the circuit

$$I_{rms}=\frac{V_{rms}}{\sqrt{[R^2+(X_L-X_C)^2}}=\frac{100}{[100^2+(250-200)^2]^{1/2}}$$

$$=0.89 \text{ amp.}$$

and $$V_L=X_L\, I_{rms}=250\times 0.89$$

$$=223.3 \text{ volts.}$$

Example 18. *A circuit containing resistance R_1, inductance L_1 and capacitance C_1, connected in series gives resonance at the same frequency as the second similar combination R_2, L_2 and C_2. If the two circuits are connected in series, show that the whole circuit will resonate with the same frequency f.*

In the series-resonant circuit, the frequency is given by the relation $f=1/2\pi\sqrt{LC}$. In this problem these two sets of LCR series circuit have the same frequency, hence

$$f=\frac{1}{2\pi\sqrt{L_1C_1}}=\frac{1}{2\pi\sqrt{L_2C_2}}$$

$$\therefore \qquad L_1C_1=L_2C_2. \quad \text{or} \quad L_1=L_2C_2/C_1.$$

When these two circuits are connected in series, the total inductance $L=L_1+L_2$ and the total capacitance C is given by

$$\frac{1}{C}=\frac{1}{C_1}+\frac{1}{C_2} \quad \text{or} \quad C=\frac{C_1C_2}{C_1+C_2}.$$

The resonant frequency of the whole (or combined) circuit is given by

$$f'=\frac{1}{2\pi\sqrt{LC}}=\frac{1}{2\pi\sqrt{(L_1+L_2)C_1C_2/(C_1+C_2)}}$$

$$f'=\frac{1}{2\pi\sqrt{\left(\frac{L_2C_2}{C_1}+L_2\right)\frac{C_1C_2}{C_1+C_2}}}=\frac{1}{2\pi\sqrt{L_2\left(\frac{C_1+C_2}{C_1}\right)\frac{C_1C_2}{C_1+C_2}}}$$

$$=\frac{1}{2\pi\sqrt{L_2C_2}}=f\text{ (frequency of each circuit).}$$

Example 19. *A capacitor of 250 pf is connected in parallel with a coil having inductance of 16 mH and effective resistance 20 Ω. Calculate (i) the resonance frequency and (ii) circuit impedance at resonance.*

The resonant frequency of the *LCR*-parallel circuit is given by

$$f=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}\ .$$

In this problem, given $L=0.16\times10^{-3}$ henry, $C=250\times10^{-12}$ farad, and $R=20$ ohms. Therefore

$$f=\frac{1}{2\times3.14}\sqrt{\frac{1}{0.16\times10^{-3}\times250\times10^{-12}}-\frac{20^2}{(0.16\times10^{-3})^2}}$$

$=7.96\times10^5$ hertz.

(*ii*) The circuit impedance at resonance is given by

$$Z=\frac{L}{CR}=\frac{0.16\times10^{-3}}{250\times10^{-12}\times20}$$

$=3.2\times10^4$ ohms.

Example 20. *An A.C. circuit consists of a 220 ohms resistance and a 0.7 henry choke. Find the power absorbed from 220 volt 50 cycle mains connected to this circuit if the resistance and choke are joined (i) in series and (ii) in parallel.*

(*i*) When the resistance R and choke of inductance L are connected in series, then the impedance Z of the a c. circuit is given as

$$Z=\sqrt{R^2+\omega^2L^2}=\sqrt{R^2+(2\pi fL)^2}$$

Substituting numerical values, we get

$$Z=\sqrt{(220)^2+(2\times3.14\times50\times0.7)^2}=310.99\text{ ohm}$$

$$\therefore\quad I_{rms}=V_{rms}/Z=220/310.99=0.707\text{ amp.}$$

Therefore the power absorbed in the circuit

$$P=I_{rms}\,V_{rms}\cos\phi=I_{rms}\,V_{rms}\,(R/Z)$$

$$=0.707\times220\times(220/310.99)=110.1\text{ watts.}$$

(*ii*) When the resistance and choke are parallel, the entire power will be absorbed in the resistance, as the choke does not absorb any power.

$\therefore$ Power absorbed $P = I_{rms}^2 R = V_{rms}^2/R$

$= 220^2/220 = 220$ watts.

Example 21. *An emf $V_0 \sin \omega t$ is applied to a circuit which consists of a self inductance L of negligible resistance in series with a variable capacitor C. The capacitor is shunted by a variable resistance R. Find the value of C for which the amplitude of the current is independent of R.*

To make the problem easy, let us make use of phasor algebra. The complex impedance of the circuit (Fig. 12.43).

$$\mathbf{Z} = j\omega L + \mathbf{Z}',$$

where Z′ is due to C and R in parallel and is given by

$$\frac{1}{\mathbf{Z}'} = \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R}$$

or $$\mathbf{Z}' = \frac{R}{1 + j\omega CR} = \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

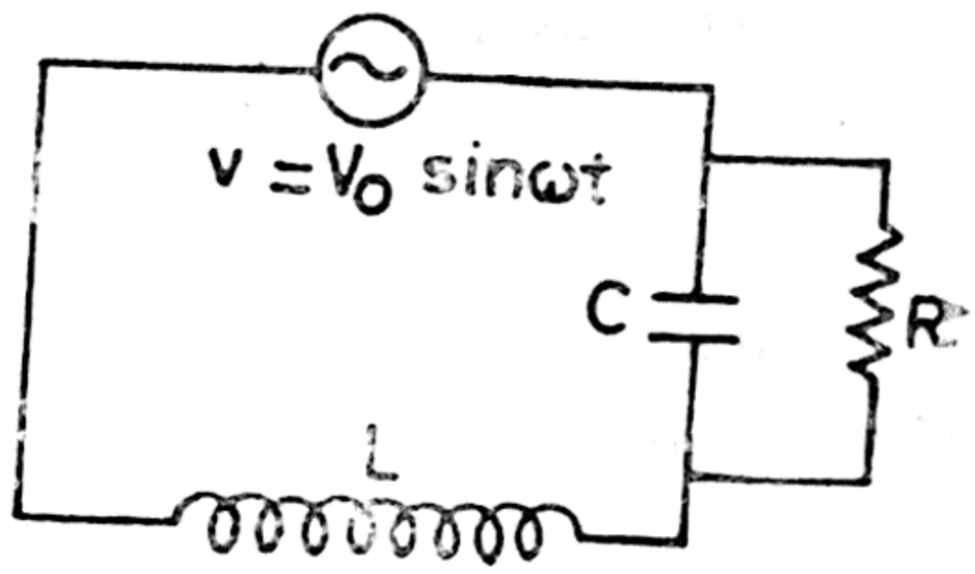


Fig. 12.43.

$$\therefore \quad \mathbf{Z} = j\omega L + \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$= \frac{R}{1 + \omega^2 C^2 R^2} + j\left(\omega L - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2} \right)$$

The magnitude of $\mathbf{Z}$ is thus given by

$$Z = \sqrt{\left[\frac{R^2}{(1 + \omega^2 C^2 R^2)^2} + \left(\omega L - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2} \right)^2 \right]}$$

or $$Z^2 = \frac{R^2}{(1 + \omega^2 C^2 R^2)^2} + \omega^2 L^2 + \frac{\omega^2 C^2 R^4}{(1 + \omega^2 C^2 R^2)^2} - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2}$$

$$= \frac{R^2 - 2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2} + \omega^2 L^2.$$

The peak value of current will be independent of R, if Z or Z^2 is also independent of R. It is possible when

$$R^2 - 2\omega^2 LCR^2 = 0, \quad \text{or} \quad C = 1/2\omega^2 L.$$

Example 22. *A series LCR circuit has inductor of 1 mH, a capacitor of 0.1 μf and a resistor of resistance 10 Ω. Calculate the resonant frequency of the circuit. What is the separation between the two half power points, the power factor for the circuit and the frequency corresponding to a half power point ?*

The resonant frequency of the LCR circuit

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{(LC)}} = \frac{1}{2 \times 3.14\ (1 \times 10^{-3} \times 0.1 \times 10^{-6})^{1/2}}$$

$f = 1.592 \times 10^4$ cycles/sec.

The angular frequency separation for the two half power points
$$2\Delta\omega = R/L = 10/10^{-3} = 1\times10^4 \text{ radians/sec,}$$

which is quite large, showing that the resonance curve is flat.

If R decreases, $2\Delta\omega$ decreases showing the sharpness of the resonance. The resonance is very sharp for $R=0.1$ ohm, as for this half power point separation is only 10^2 radians/sec.

The separation in frequency
$$2\Delta f = 2\Delta\omega/2\pi = 1.592\times10^3 \text{ cycles/sec.}$$
$\therefore$ Frequencies corresponding to half power points are
$$f+\Delta f = 1.672\times10^4 \text{ cycles/sec}$$
and $f-\Delta f = 1.512\times10^4$ cycles/sec.

The power factor at resonance $= R/Z_{\text{Resonance}}$
$$= R/R = 1$$
and at half power points $= R/Z_{\text{hpp}}$
$$= R/R\sqrt{2} = 1/\sqrt{2}.$$

Example 23. *A set-up transformer works on 220 volts and gives 2 ampere to an external resistor. The turn ratio between the primary and secondary coils is 2 : 25. Assuming 100% efficiency, find the secondary voltage, primary current and power delivered.*

If I_p and I_s be the currents in the primary and secondary respectively and V_p and V_s the respective voltages across them. Therefore for 100% efficiency, we have power loss zero and
$$I_pV_p = I_sV_s$$
or
$$I_p/I_s = V_s/V_p = N_s/N_p.$$

In this problem, given the turn ratio $N_s/N_p = 25/2$, $V_p = 220$ volts and $I_s = 2$ amp.

$\therefore$ $V_s/220 = 25/2$ or $V_s = 2750$ volts

and $I_p/2 = 25/2$ or $I_p = 25$ amp.

$\therefore$ Power delivered $= I_sV_s = 2\times2750 = 5500$ watts.

Example 24. *Calculate the currents in the primary and secondary circuits with no-load and on load conditions, in an iron cored transformer having*

$R_1 = 1\Omega$, $L_1 = 0.006$ H, $R_2 = 50\Omega$, $L_2 = 0.080$, $M = 0.5$ H, *turns ratio* $N_s/N_p = \rho = 2$, $V_1 = 100$ V *at* $f = 50$ c/s.

The iron cored transformer may be considered as a mutually—inductive circuits. Each winding has its own separate resistance R_1 and R_2 and total inductance L_1 and L_2. M is the mutual inductance of the circuits.

In the first case, with no load, the secondary resistance R_2 should be considered as infinity and the primary current $I_{1,0}$ may be given by

$$\mathbf{V}_1=\mathbf{I}_{1,0}\,(R_1+jL_1\omega)+jM\omega\mathbf{I}_2=\mathbf{I}_{1,0}\,[R_1+j\omega(L_1+M/\rho]$$

or $$100=\mathbf{I}_{1,0}\,[1+j\times 2\times 3.14\times 50\,(0.006+0.5/2)]$$

$$=\mathbf{I}_{1,0}\,[1+j\,80.384]$$

$$\therefore\quad \mathbf{I}_{1,0}=\frac{100}{(1+j\,80.384)}=\frac{100(1-j\,80.384)}{1^2+80.384^2}$$

$$=0.0155-j\,1.2438$$

$I_{1,0}=1.244$ amp. and $\phi_0=\tan^{-1}1.2438/0.0155=270°43'$.

In the second case, with a load $R_2=50\Omega$, we have relation for V_1 and I_1 with on load condition, as given by equation (86). Substituting the numerical values, we get

$$100=\mathbf{I}_1\,[11.4887+j\,9.2452].$$

or $$\mathbf{I}_1=100(11.4887-j\,9.2452)/217.464$$

$$\mathbf{I}_1=5.283-j\,4.251$$

$$\therefore\quad \mathbf{I}_1=6.781 \text{ amp. and } \phi_1=\tan^{-1}\frac{4.251}{5.283}=321°11'$$

The current in the secondary with on-load condition is given by

$$\mathbf{I}_2=j\omega M\mathbf{I}_1/(R_2+j\omega L_2+j\omega\rho M)$$

Substituting the numerical values, we get

$$\mathbf{I}_2=\left(\frac{j\,157}{50+j\,25.12+j\,314}\right)\left(5.283-j\,4.251\right)$$

$$=\frac{j\,157(5.283-j\,4.251)(50-j\,339.12)}{50^2+339.12^2}$$

$$=2.6776-j\,1.5732$$

$$\therefore\quad I_2=3.105 \text{ amp.} \quad \text{and} \quad \phi_2=329°34'.$$

Oral Questions

1. Give the physical reasons behind the facts that impedance of an inductor increases with frequency while the impedance of a capacitor decreases with frequency.

2. How can one separate sinusoidal voltages of frequencies 10^5 and 10 Hz by inserting either a $2\mu f$ capacitor or a 20 mH inductor in series with the power source ?

3. Which of the two combinations of inductor and a capacitor is required to (*a*) transmit a required frequency, (*b*) to suppress it ?

4. Point out as many similarities as you can between a LC-resonant circuit and the vibration of a point mass at the end of a spring.

5. How can one measure the leakage resistance of the direction in a condenser ?

6. Under what conditions can current be supplied to an *ac* circuit without delivering power to it ?

7. Which will be an ideal chocking coil ?

8. What are the conditions for resonance in LCR circuit ?

9. Why the LCR-series circuit is called an accepter circuit ?

10. What are the possible ways of reducing current in an a.c. circuit ? Explain which one is better ?

11. What happened to the voltage and current if the turns in the secondary of the transformer is doubled ?

12. How emf is developed in the secondary of the transformer ? How the primary is affected by increase of load in the secondary ?

13. A transformer has n_1 primary turns and n_2 secondary turns, 60% of the magnetic flux due to current in the primary goes out as leakage. Find the ratio of primary to secondary voltage with no load on the secondary.

14. How can you explain two balance conditions in *ac* bridges ?

15. Why the Owen's bridge is inferior to Anderson's bridge ?

16. What do you expect if no *ac* balance is obtained in Anderson's bridge ?

17. What are the relative merits of the *ac* bridges used for capacitance measurement ?

18. Whether a 6 volt step down transformer may be used to perform bridge experiments with headphone as a detector ? Give reasons.

Problems :

1. A circular coil of negligible self inductance, having 100 turns and of radius 25 cm, rotates uniformly five times per second about a vertical diameter in a plane where the magnetic dip is 60° and the resultant magnetic field strength is 38 amp/meter. Calculate (*a*) V_0, (*b*) I_0 and (*c*) the rate of production of heat in the coil, if its resistance is 10 ohms.

[1.172×10^{-2} *V*, 1.172×10^{-3} amp., 6.88×10^{-6} joule/sec.]

2. Find the rms value of current in the circuit, consisting of an inductance of 100 mH and a resistance of 2000 ohms in series along with a *ac* source of emf 50 *V* at a frequency of 1000 cycles/sec. Also calculate pd across the inductance and across the capacitor. (*24 mA*, *15 V*, *47.6 V*)

3. A 500-Ω light bulb is being operated at 40 watts by an alternating sine voltage. Find both the rms and peak values for this voltage.
(*141 V*, *200 V*)

4. An alternating emf 200 virtual volts at 50 cycles is connected to a circuit of resistance 1 ohm and inductance 0.01 henry. What is the current and lag in phase between the emf and the current ? (*60.3 amp*, *72°*)

5. An *ac* voltage of 220 rms volts at a frequency of 40 cycles per sec is applied to a circuit containing a pure inductance of 0.01 henry and a pure resistance of 6 ohms in series. Find (*a*) the current, (*b*) the pd across the circuit elements, (*c*) the lag in time between the voltage and the current.
(*38.83 amp.*, *202.98 V*, *96.83 V 0.00158 sec.*)

6. A 100 volts, 50 watt light bulb is to be run by 200 volts 50 cycle supply. Calculate the capacity of the capacitor to be used in series with the bulb.
(9.18 μf)

7. A resistance and inductance are connected in series across a voltage $v=283 \sin 314t$. The current is found to be $4 \sin (314t-\pi/4)$. Find the values of the inductance and resistance.
(0.16 henry, 50 ohm)

8. A generator having emf $282 \sin (100\pi t)$, and internal resistance $2k\Omega$ is connected in series with an inductance $(40/\pi H)$, capacitance $(10/\pi \mu f)$ and resistance $(2k\Omega)$. Calculate (*i*) the rms value of current flowing in the circuit, (*ii*) the rms value of the voltage across the inductor and (*iii*) the frequency at which the impedance of the circuit will be purely resistive.
(0.35 amp, 141 volts, 25 hertz)

9. An arc lamp consumes 10 amp. at 40 volts. Calculate the power factor when it is connected with a suitable value of choke coil required to run the arc lamp on ac mains of 200 volts (rms) and 50 cycles per sec. Calculate also the energy wasted per second if it was to run with a resistance on 200 volts d.c.
(0.2, 1600 watts)

10. An electric lamp which runs at 100 volts dc and consumes 10 amp. current is connected to ac mains at 150 volts 50 cycles with a choke coil in series. Calculate the inductance and drop of voltage across the choke. Neglect the resistance of choke.
(0.036 H. 111.8 volts)

11. A circuit operating at $360/2\pi$ Hz contains a $1\ \mu f$ capacitor and a $20\ \Omega$ resistor How large an inductor must be added in series to make the phase angle for the circuit zero ? Calculate the current in the circuit if the applied voltage is 120 volts.
(7.7 henry, 6 A)

12. If a 0.03 henry inductor, a $10\ \Omega$ resistor and a $2\mu f$ capacitor are connected in series. At what frequency will they resonate ? What will be the phase angle at resonance ?
(650 Hz, 0)

13. A series circuit consists of a resistance of 10Ω, an inductance 5 mH and a capacitor whose capacitance is 800 pf. An emf of amplitude 5 volts and of variable frequency is connected across the above series combination. Find values for the frequencies at which the amplitude of the current will be (*a*) a maximum, (*b*) one half its maximum value.
[(*a*) 7.96×10^4 *cycles/sec.* ; (*b*) 7.99×10^4 *and* 7.3×10^4 *cycle/sec.*]

14. A series LCR circuit has $L=1$mH, $C=0.1\mu f$ and $R=10\ \Omega$. Calculate resonance frequency of the circuit.
(1.59×10^4 c/s.)

15. A series circuit contaning $R=300$ ohm, $C=2\mu f$, and $L=0.02$ H is connected to a 9 volt a.c. supply. Find (*a*) the resonant frequency, (*b*) the current in the circuit at resonance and (*c*) the power absorbed at resonance. If the applied frequency is twice that at resonance, then what will be the absorbed power ?

16. A resistance of 10 ohm is jointed in series to an inductance of 0.5 henry. What capacitance should be put in series with the combination to obtain maximum current ? What will be the potential difference across the resistance, inductance and capacitance ? The current is being supplied by a 200 volt and 50 cycle a.c. mains.
(2μf, 200 volts, 3140 volts, 3140 volts)

17. Two coils, one of resistance $2\ \Omega$ and self inductance 0.015 H, the other having a resistance of $1\ \Omega$ and a self inductance of 0.08 H are arranged in parallel on a 200 volt 50 cycles/sec frequency circuit. Calculate the currents flowing in each circuit and through the whole system.

18. When a potential difference of 200 volts and 50 hertzs is applied to an inductive circuit, the current in it is 10 amp and the power consumed is 1000 watts. Find the power factor, resistance and inductance of the circuit. *(0.5, 10Ω, 3.04 mH)*

19. A resistance of 40 Ω is joined in parallel with a reactance of 30 Ω, across an *ac* supply. Obtain values for the resistance and reactance which, when connected in series with the same supply, would take the same current and have the same power factor. *(14.4 Ω, 19.2 Ω)*

20. A average power to the plant is 250 kW and the power factor is 0.707 lagging. The generators with a capacitor in parallel is connected across the plant. If the generator is sinusoidal with ω=377 radians/sec. and the effective value is 2300 volts, determine the value of the capacitance such that the power factor is 1. *(125 μf)*

21. The internal resistance of an *ac* source is 10 *k*Ω. (*a*) What should be the turns ratio of a transformer to match the source to a load of resistance 10 ohms? (*b*) If the voltage amplitude of the source is 100V, what is the voltage amplitude of the secondary on open circuit?
(0.0316, 3.16 V)

22. A current of 6 amperes at 220 volts flow in the primary of a transformer. In the secondary of the same transformer, a current of 1 ampere at 100 volts is available. Find the power loss in the transformer.
(1000 watts)

13

Network Analysis and Network Theorems

13.1. NETWORK

A *network* is a circuit containting *circuit elements* or *branches*. The *circuit elements* are of two types : (*a*) *passive* and (*b*) *active*. The inductor, resistor and capacitor with two terminals are regarded as *passive elements* as they are not fundamental sources of power. The quantitative measure of these elements are inductance, resistance and capacitance respectively. The *active elements* are those which feed the power into the given circuits. These are of two types : (*a*) *voltage source* and (*b*) *current source*. An *ideal voltage source* is an active device which can maintain a constant voltage across its terminals irrespective of the current supplied by it. It is also called *constant voltage source*. It is always equal to the *open-circuit voltage*. In an actual voltage source, the voltage across its terminals decreases as the load resistance is decreased and the more current is supplied by the source. This voltage is highest when the circuit is an open circuit, *i.e.*, when the source does not supply any current. If the load resistance reduces to zero, *i.e.*, if the output terminals of the voltage source are *short-circuited*, the current will be infinite. A *constant current source* is an active device which can supply the constant current to any load resistance that is connected across its terminals. In such a case the terminal voltage is directly proportional to the load resistance.

A group of elements in series is called *branch*. The junction of two or more branches is known as a *node* or *vertex*. A set of branches forming a closed path in a network is called a *mesh*. The network is said to be a linear network if it consists of linear impedances. A linear impedance is any impedance that obey's Ohm's law. The resistors, inductors and capacitors fall into this category.

13.2. NETWORK ANALYSIS

Besides the various circuit theorems, to be discussed in the next article, that aid the analysis of circuits, the following several methods are used for solving network problems.

1. Series Parallel Connections. When the circuit consists of series and parallel branches, the first step is to simplify each series and parallel branch separately. For this purpose the assistance of Kirchhoff's laws is needed.

2. T-and π-Networks (Star and Delta networks). The above simple principle alone is not sufficient to solve all problems. Any complicated electrical network can be represented, at a single frequency, by a T-or π-network.

Convenient impedance values which may be chosen are with open circuit and short circuit. The impedance measured across the 1, 1 terminals with 2, 2 terminals open circuited are designated as Z_{1oc} and the measurement with 2, 2 terminals short circuited as Z_{1sc}. Similar measurements are taken at the 2, 2 terminals and are designated as Z_{2oc} and Z_{2sc}.

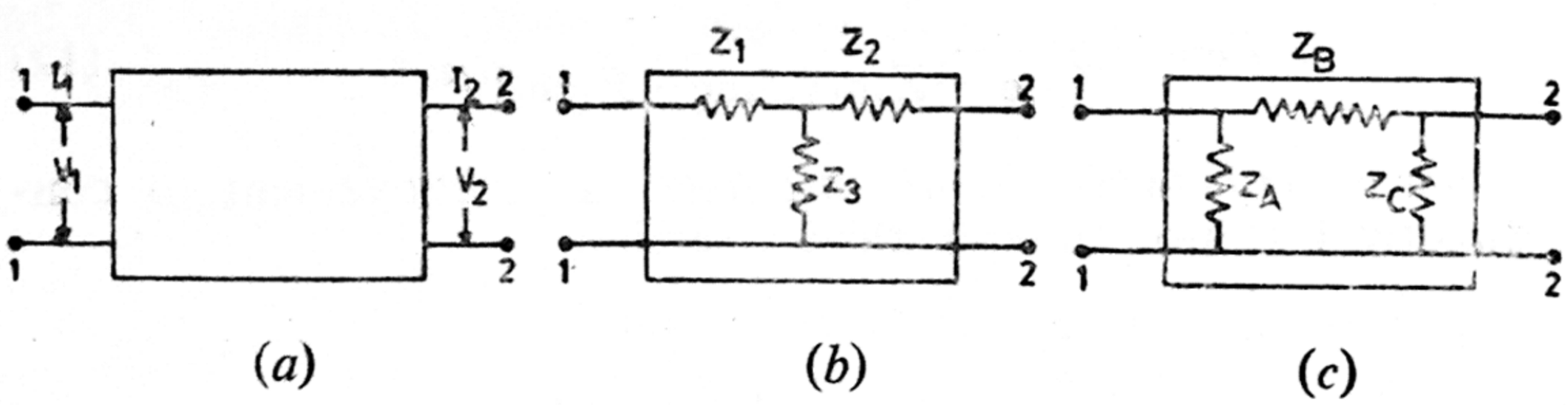


Fig. 13.1.

There are eight quantities, the magnitudes and phase angles of V_1, V_2, I_1 and I_2, which must be established by the equivalent network. If a load Z_R is connected between terminals 2, 2 then with the relation $V_2=I_2Z_R$, one magnitude and one phase angle will be determined and we left with other three magnitudes and three phase angles to be determined. Thus we require at least three adjustable impedances to set up an equivalent network. These three impedances can be arranged in two different ways as T and π networks.

T-network would be said to be equivalent to the network at Fig. 13.1 (*a*), if Z_1, Z_2 and Z_3 are selected to have the same external currents and voltages as the original network (*a*), and hence the open and short-circuit measurements must be identical. Therefore for T-network

$$Z_{1oc}=Z_1+Z_3. \qquad \ldots(1)$$

$$Z_{1sc}=Z_1+Z_2Z_3/(Z_2+Z_3). \qquad \ldots(2)$$

$$Z_{2oc}=Z_2+Z_3. \qquad \ldots(3)$$

$$Z_{2sc}=Z_2+Z_1Z_3/(Z_1+Z_3). \qquad \ldots(4)$$

From Eqs. (1) and (2), we get

$$Z_{1oc}-Z_{1sc}=Z_3-Z_2Z_3/(Z_2+Z_3)=Z_3^2/(Z_2+Z_3)$$

or

$$Z_3=\pm[Z_{2oc}(Z_{1oc}-Z_{1sc})]^{1/2}. \qquad \ldots(5)$$

From Eqs. (1) and (3), we get

$$Z_1 = Z_{1OC} - Z_3 \text{ and } Z_2 = Z_{2OC} - Z_3. \qquad \ldots(6)$$

Similarly for an equivalent π-section [Fig. 13.1 (*c*)], we have

$$Z_{1OC} = Z_A(Z_B + Z_C)/(Z_A + Z_B + Z_C). \qquad \ldots(7)$$

$$Z_{1SC} = Z_A\ Z_B/(Z_A + Z_B). \qquad \ldots(8)$$

$$Z_{2OC} = Z_C(Z_A + Z_B)/(Z_A + Z_B + Z_C). \qquad \ldots(9)$$

Solving these relations, we have

$$Z_A = \frac{Z_{2OC}\ Z_{1SC}}{Z_{2OC} - [Z_{2OC}(Z_{1OC} - Z_{1SC})]^{1/2}} \qquad \ldots(10)$$

$$Z_B = \frac{Z_{2OC}\ Z_{1SC}}{[Z_{2OC}(Z_{1OC} - Z_{1SC})]^{1/2}} \qquad \ldots(11)$$

$$Z_C = \frac{Z_{2OC}\ Z_{1SC}}{Z_{1OC} - [Z_{2OC}(Z_{1OC} - Z_{1SC})]^{1/2}}. \qquad \ldots(12)$$

Conversion between T-and π-sections. It is convenient to convert directly T-section to equivalent π-section.

From relations (1-3) and (7-9), we get

$$Z_1 + Z_3 = Z_A(Z_B + Z_C)/(Z_A + Z_B + Z_C),$$
$$Z_1 + Z_2Z_3/(Z_2 + Z_3) = Z_AZ_B/(Z_A + Z_B),$$
$$Z_2 + Z_3 = Z_C(Z_A + Z_B)/(Z_A + Z_B + Z_C).$$

All branches of the T-network is thus given as

$$Z_1 = \frac{Z_AZ_B}{Z_A + Z_B + Z_C},\ Z_2 = \frac{Z_BZ_C}{Z_A + Z_B + Z_C},\ Z_3 = \frac{Z_CZ_A}{Z_A + Z_B + Z_C} \qquad \ldots(13)$$

If the impedances of the π-circuit are known, the above equations may be used to determine the equivalent T-network.

The inverse transformation can be obtained, if we have values of Z_A, Z_B and Z_C in terms of Z_1, Z_2 and Z_3 as

$$Z_A = \square/Z_2,\ Z_B = \square/Z_3,\ Z_C = \square/Z_1, \qquad \ldots(14)$$

where $\square = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.

3. **Nodal Equations.** The alternate method of solving complicated network problems is the *nodal method.* In this method admittances (Y) are used instead of impedances and the voltage of the node is represented by E_a, E_b, etc, From any reference point *a* currents are going to its neighbouring points, because of its own potential. Similarly owing to potentials of the neighbouring points currents are coming to this point. As there is no accumulation of charge at this point, hence

$$I_1 - I_2 - I_4 = 0. \qquad \ldots(15)$$

For the solution of the problem some convenient junction between elements is chosen as a reference node (point O in Fig. 13.2)

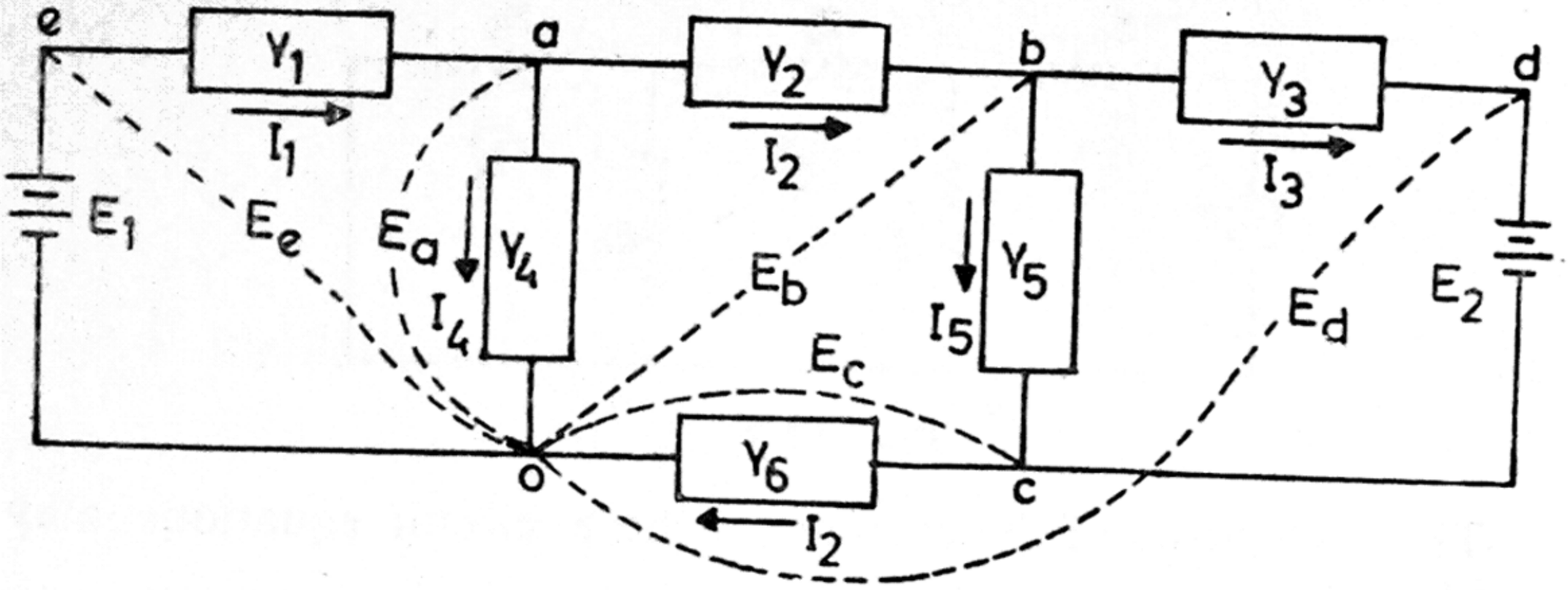


Fig. 13.2.

and the voltages at different nodes are assumed with respect to this reference point. The figure shows that

$$E_e = E_1 \quad \text{and} \quad E_d = E_c + E_2. \qquad \ldots(16)$$

Current summation *at junction a* can be written as

$$(E_1 - E_a)Y_1 - (E_a - E_b)Y_2 - E_a Y_4 = 0. \qquad \ldots(17)$$

At junction b $\qquad I_2 - I_3 - I_5 = 0, \qquad \ldots(18)$

or $\qquad (E_a - E_b)Y_2 - (E_b - E_2 - E_c)Y_3 - (E_b - E_c)Y_5 = 0 \qquad \ldots(19)$

At junction c $\qquad I_3 + I_5 - I_6 = 0, \qquad \ldots(20)$

or $\qquad (E_b - E_2 - E_c)Y_3 + (E_b - E_c)\ Y_5 - E_c Y_6 = 0. \qquad \ldots(21)$

The equations may be written as

$$(-Y_1 + Y_2 + Y_4)E_a + Y_2 E_b = -Y_1 E_1 \qquad \ldots(22)$$

$$Y_2 E_a - (Y_2 + Y_3 + Y_5)E_b + (Y_3 + Y_5)E_c = -Y_3 E_2 \qquad \ldots(23)$$

$$(Y_3 + Y_5)E_b - (Y_3 + Y_5 + Y_6)E_c = Y_3 E_2. \qquad \ldots(24)$$

From these relations we can get potential information at any junction. As at high frequencies voltmeters are more commonly used and are more accurate than ammeters, therefore nodal method is very important.

4. Maxwell's Method. In this method matrices and determinants are used. Maxwell suggested cycle currents to flow through different loops and the two following terms : (1) *Self Resistance*—Sum of resistances through which any cycle current is passing. (2) *Mutual Resistance* —The sum of resistances through which two current loops are passing. The latter is negative if the loop currents are passing in opposite directions and is zero when there is no common path.

Let us consider a general network such as Fig. 13.3. The three equations may be written as

$$E = (Z_1 + Z_3)I_1 - Z_3 I_2. \qquad \ldots(25)$$

$$0 = -Z_3 I_1 + (Z_2 + Z_3 + Z_4) I_2 - Z_4 I_3 \qquad \ldots(26)$$

$$0 = -Z_4 I_2 + (Z_4 + Z_5) I_3. \qquad \ldots(27)$$

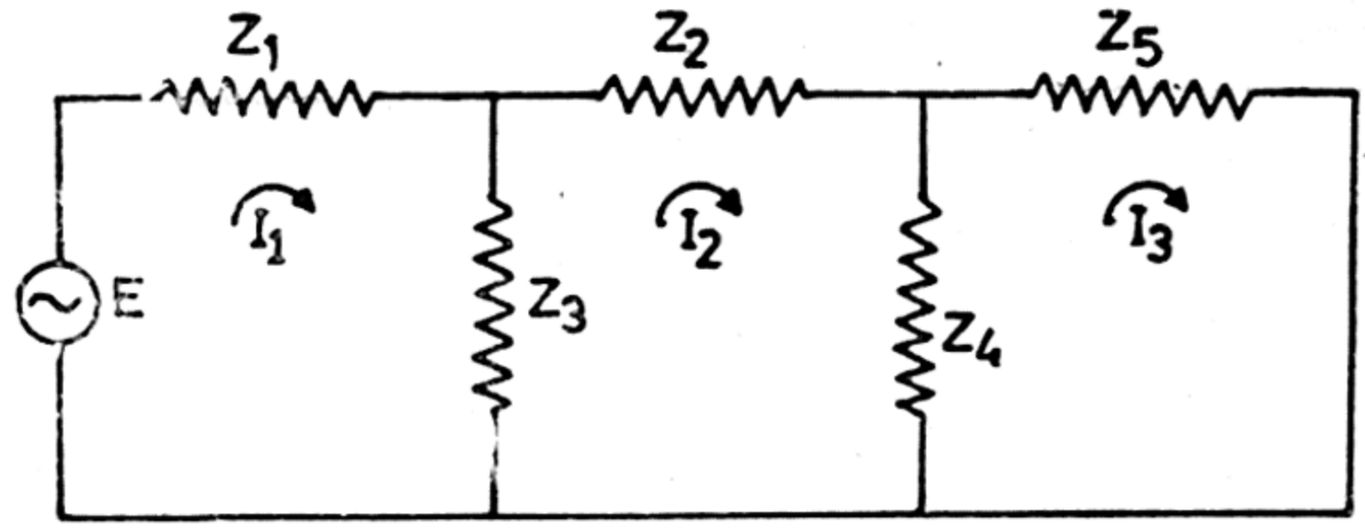


Fig. 13.3.

The coefficients of the currents in these circuit equations may be written as

$$[Z] = \begin{bmatrix} Z_1 + Z_3 & -Z_3 & 0 \\ -Z_3 & Z_2 + Z_3 + Z_4 & -Z_4 \\ 0 & -Z_4 & Z_4 + Z_5 \end{bmatrix} \qquad \ldots(28)$$

This *impedance matrix* may be written as

$$[Z] = \begin{bmatrix} Z_{11} & Z_{12} & 0 \\ Z_{12} & Z_{22} & Z_{23} \\ 0 & Z_{23} & Z_{33} \end{bmatrix}, \qquad \ldots(29)$$

where $Z_{11} = Z_1 + Z_3$ the impedance of first mesh, $Z_{22} = Z_2 + Z_3 + Z_4$ the impedance of second mesh, $Z_{33} = Z_4 + Z_5$ the impedence of third mesh. Also $-Z_3 = Z_{12}$ the mutual impedance between meshes 1 and 2, $-Z_4 = Z_{23}$ the mutual impedance between meshes 2 and 3. The original mesh equations (25-27) may thus be written as

$$\begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & 0 \\ Z_{12} & Z_{22} & Z_{23} \\ 0 & Z_{23} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}. \qquad \ldots(30)$$

The currents can be calculated by simple algebra. We thus get

$$I_1 = \frac{[(Z_2 + Z_3)(Z_4 + Z_5) + Z_4 Z_5] E}{(Z_4 + Z_5)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) + (Z_1 + Z_3) Z_4 Z_5}, \qquad \ldots(31)$$

$$I_2 = \frac{Z_3 (Z_4 + Z_5) E}{(Z_4 + Z_5)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) + (Z_1 + Z_3) Z_4 Z_5}, \qquad \ldots(32)$$

$$I_3 = \frac{Z_3 Z_4 E}{(Z_4 + Z_5)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) + (Z_1 + Z_3) Z_4 Z_5}. \qquad \ldots(33)$$

The determinant of Z may be written as

$$\Delta_z = (Z_1 + Z_3) \begin{vmatrix} Z_2 + Z_3 + Z_4 & -Z_4 \\ -Z_4 & Z_4 + Z_5 \end{vmatrix} + Z_3 \begin{vmatrix} -Z_3 & -Z_4 \\ 0 & Z_4 + Z_5 \end{vmatrix}$$

$$= (Z_1 + Z_3)[(Z_2 + Z_3 + Z_4)(Z_4 + Z_5) - Z_4^2] - Z_3^2 (Z_4 + Z_5)$$

$$= (Z_4 + Z_5)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3) + (Z_1 + Z_3) Z_4 Z_5. \qquad \ldots(34)$$

If the first column of the matrix be replaced by the left hand sides E column matrix, then the determinant

$$\Delta_1 = \begin{vmatrix} E & -Z_3 & 0 \\ 0 & Z_2+Z_3+Z_4 & -Z_4 \\ 0 & -Z_4 & Z_4+Z_5 \end{vmatrix}$$

$$= E[(Z_2+Z_3)(Z_4+Z_5)+Z_4Z_5] \quad \ldots(35)$$

Similarly we have

$$\Delta_2 = \begin{vmatrix} Z_1+Z_3 & E & 0 \\ -Z_3 & 0 & -Z_4 \\ 0 & 0 & Z_4+Z_5 \end{vmatrix}$$

$$= EZ_3(Z_4+Z_5) \quad \ldots(36)$$

and

$$\Delta_3 = \begin{vmatrix} Z_1+Z_3 & -Z_3 & E \\ -Z_3 & Z_2+Z_3+Z_4 & 0 \\ 0 & -Z_4 & 0 \end{vmatrix}$$

$$= Z_3Z_4\,E. \quad \ldots(37)$$

Thus we see that the currents can be written as

$$I_1 = \frac{\Delta_1}{\Delta_z}, \; I_2 = \frac{\Delta_2}{\Delta_z} \text{ and } I_3 = \frac{\Delta_3}{\Delta_z}. \quad \ldots(38)$$

Thus we see that to find expression for I_r, we have to write determinant Δr formed by replacing the r_{th} column with the mesh voltage column and then divide it by the matrix determinant Δ_z.

13.3. NETWORK THEOREMS

Network theorems are very powerful tools for solving advanced and complicated network problems. Some of the important theorems are :

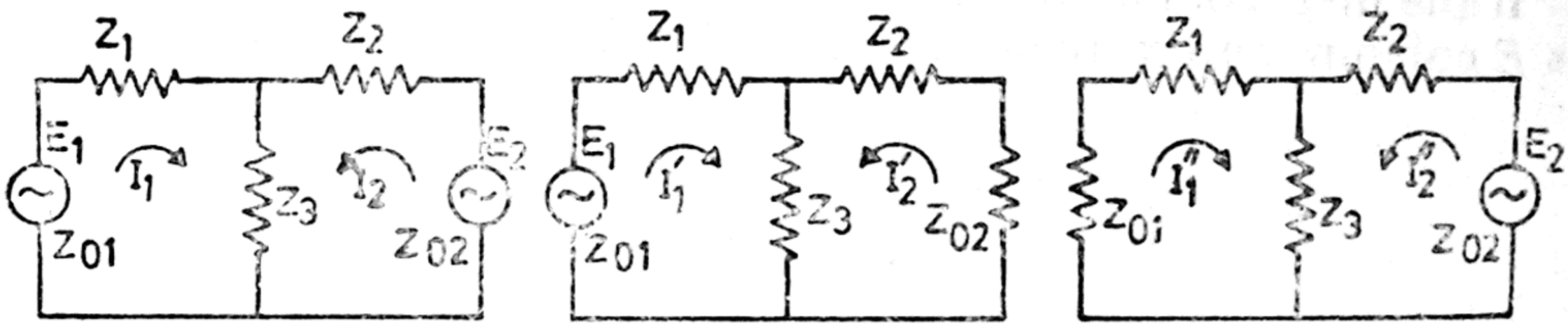
1. **Superposition Theorems.** *In any linear network containing linear impedances and potential sources, the current flowing in any element is the vector sum of the currents that would flow in that element by each potential source, all other sources being replaced at that time by their internal impedances.*

Consider a simple case of network consisting two meshes with two potential sources E_1 and E_2. For this network, the mesh currents are as shown in Fig. 13.4. The mesh equations may thus by written as

$$E_1 = (Z_1+Z_3+Z_{01})\,I_1 + Z_3I_2 \quad \ldots(39)$$

and

$$E_2 = Z_3I_1 + (Z_2+Z_3+Z_{02})\,I_2, \quad \ldots(40)$$



(a) (b) (c)

Fig. 13.4.

The currents may be obtained by using Maxwell's method

$$I_1=\frac{\begin{vmatrix} E_1 & Z_3 \\ E_2 & Z_2+Z_3+Z_{02} \end{vmatrix}}{\begin{vmatrix} Z_1+Z_3+Z_{01} & Z_3 \\ Z_3 & Z_2+Z_3+Z_{02} \end{vmatrix}}$$

or $$I_1=[(Z_2+Z_3+Z_{02})\ E_1-Z_3\ E_2]/\Delta z \qquad ...(41)$$

and $$I_2=[-Z_3E_1+(Z_1+Z_3+Z_{01})\ E_2]/\Delta z, \qquad ...(42)$$

where Δz is the impedance matrix.

To prove superposition theorem, let us first find I_1' and I_2' by short circuiting emf E_2 (but not its impedance) and then I_1'' and I_2'' by short circuiting emf E_1 (but not its impedance). Using Fig. 13.4 (*b*) we get

$$I_1'=\frac{(Z_2+Z_3+Z_{02})\ E_1}{\Delta z} \text{ and } I_2'=\frac{-Z_3E_1}{\Delta z} \qquad ...(43)$$

Using Fig. 13.4 (*c*), we get

$$I_1''=\frac{-Z_3\ E_2}{\Delta z} \text{ and } I_2''=\frac{(Z_1+Z_3+Z_{01})\ E_2}{\Delta z}. \qquad ...(44)$$

Equations (41) to (44) show that

$$I_1=I_1'+I_1'' \quad \text{and} \quad i_2=I_2'+I_2''. \qquad ...(45)$$

Hence we have proved superposition theorem. This theorem can also be used if voltages of different frequencies are introduced.

2. **Reciprocity Theorem.** *In any linear network of conductors the current in one mesh due to an emf in another mesh is equal to the current in the latter when the same emf is placed in the former.* In other words the interchange of an ideal current source and an ideal voltage source in any network will not change the voltmeter and ammeter readings.

This theorem can be proved very easily with the help of Fig. 13.4. Eqs. (43) show the currents in two meshes when the voltage source E_1 is in the first mesh. If the source in Fig. 13.4 (*c*) is replaced by the source in Fig. 13.4 (*b*), *i.e.* $E_2=E_1$ and $Z_{02}=Z_{01}$, then the current in the first mesh in the modified Fig. 13.4 (*c*).

$$I_1''=-Z_3\ E_1/\Delta z.$$

It is same as I_2', the current in second mesh when the same voltage source was in the first mesh. It is same as required by the theorem.

Eqs. (43) and (44) show that

$$I_2'/E_1 = I_1''/E_2, \qquad \ldots(46)$$

i.e., the ratio of response transform to excitation transform is invariant to an interchange of the position in the network of the excitation and the response. It is the another way to state the theorem.

3. **Thevenin's Theorem.** *Any two terminal linear network containing generators (voltage sources) and impedances may be replaced with an equivalent circuit consisting of voltage source E_0 in series with an impedance Z_0. The value of E_0 is the open circuit (when there is no load) voltage between the terminals of the network and Z_0 is the impedance appearing across the terminals when all generators in the network have been replaced by their internal impedances.*

Consider two terminal network containing two meshes, one active and other passive. If load impedance Z_L is placed between the terminals, then the mesh equations are written as

$$E = (Z_1 + Z_3 + Z_i) I_S - Z_3 I_L \qquad \ldots(47)$$

$$0 = -Z_3 I_S + (Z_2 + Z_3 + Z_L) I_L. \qquad \ldots(48)$$

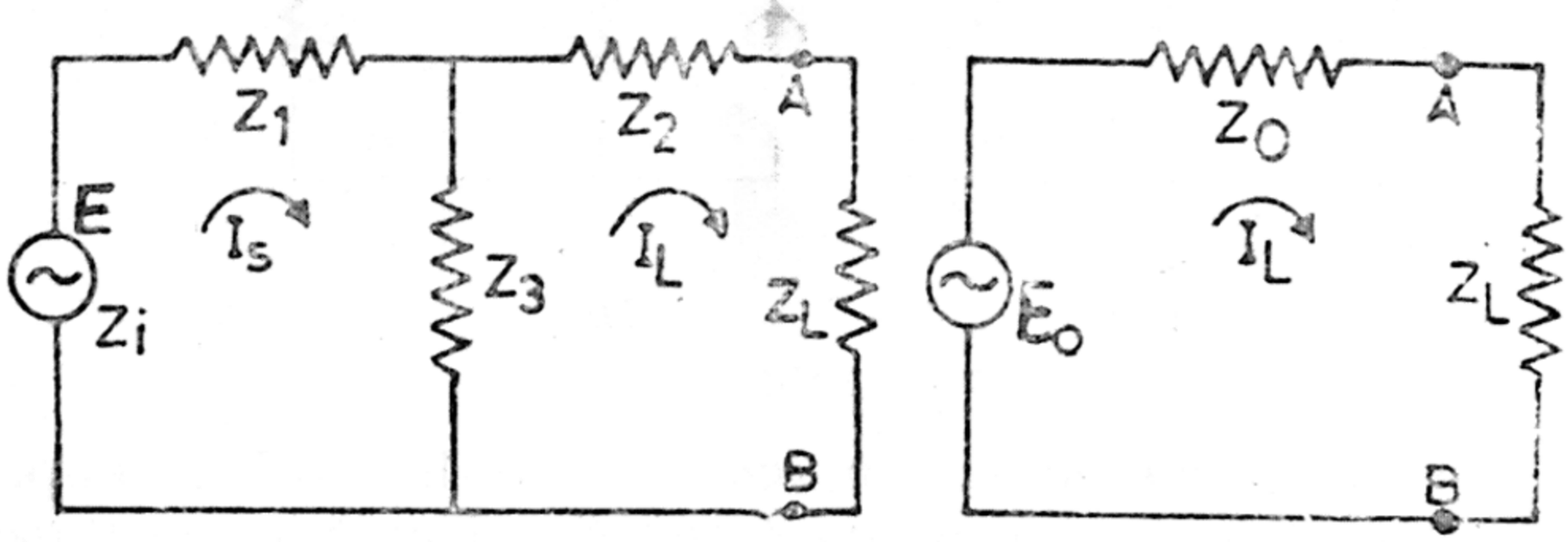


Fig. 13.5.

Solving these Eqs. for the currents, we get

$$I_L = \frac{\begin{vmatrix} Z_1+Z_3+Z_i & E \\ -Z_3 & 0 \end{vmatrix}}{\begin{vmatrix} Z_1+Z_3+Z_i & -Z_3 \\ -Z_3 & Z_2+Z_3+Z_L \end{vmatrix}}$$

$$= \frac{EZ_3}{(Z_2+Z_L)(Z_1+Z_3+Z_i)+Z_3(Z_1+Z_i)}$$

$$= \frac{EZ_3/(Z_1+Z_3+Z_i)}{Z_2+Z_L+Z_3(Z_1+Z_i)/(Z_1+Z_3+Z_i)}. \qquad \ldots(49)$$

As the open circuit voltage across AB terminals is

$$E_0 = EZ_3/(Z_1+Z_3+Z_i), \qquad \ldots(50)$$

and the total impedance between terminals when the generator is replaced by its internal impedance

$$Z_0=Z_2+\frac{1}{1/Z_3+1/(Z_1+Z_i)}=Z_2+\frac{Z_3(Z_1+Z_i)}{Z_1+Z_i+Z_3}. \quad ...(51)$$

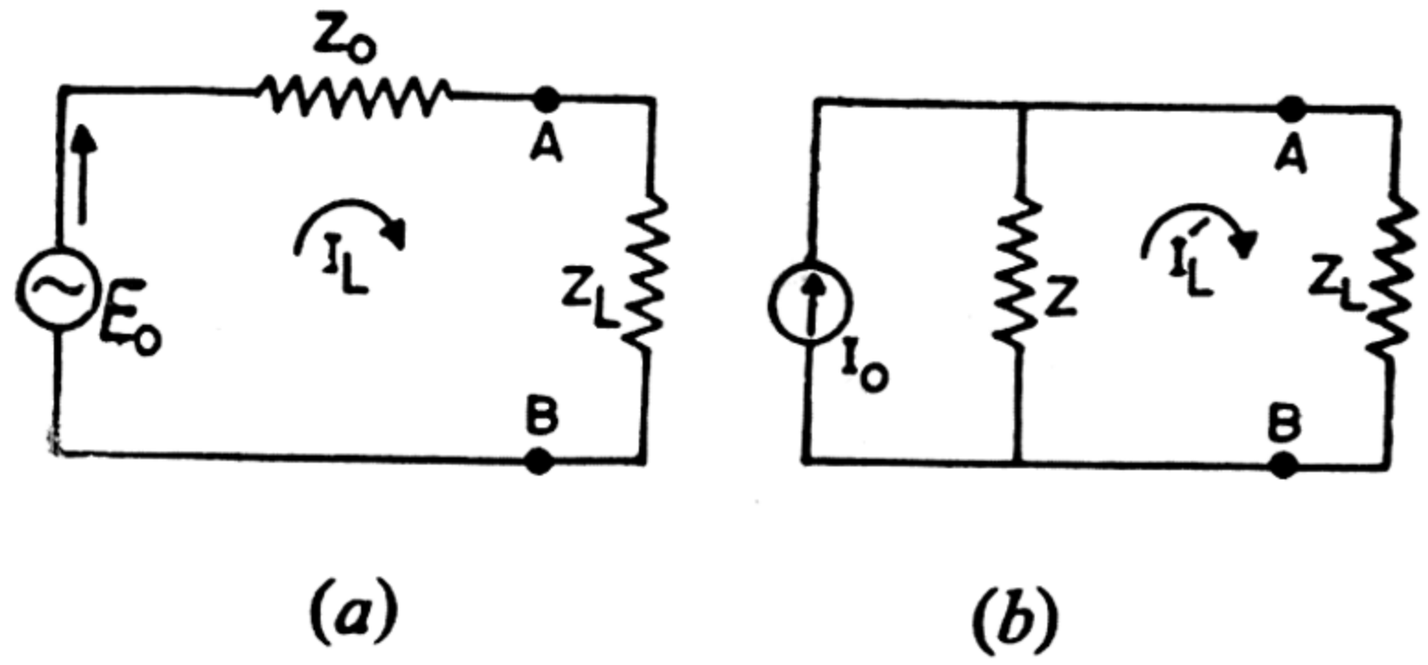
These quantities E_0 and Z_0 are called *Thevenin's components.* Thus the Eq. (49) reduces in terms of these components, as

$$I_L=E_0/(Z_0+Z_L). \quad ...(52)$$

This current is same as obtained from an equivalent circuit, known as *Thevenin's equivalent network*, consisting a voltage source E_0 in series with an impedance Z_0, hence is the theorem. It may be generalized to any number of generators by using superposition theorem.

(4) **Norton's Theorem.** *This theorem states that two terminal active network, consisting of linear impedances and generators may be replaced by a current generator in parallel with an impedance. The current is equal to that obtained through a short circuit across the two terminals and the impedance is that viewed across the two terminals when all the generators are replaced by their internal impedances.*

This theorem may be easily proved by considering a Thevenin's equivalent network shown in Fig. 13.6 (*a*). Now this may be replaced



(*a*) (*b*)

Fig. 13.6.

by a circuit consisting current source I_0 in parallel with an impedance appearing between two terminals *A* and *B*. The current flowing in the network (*a*) is given by

$$I_L=E_0/(Z_0+Z_L). \quad ...(53)$$

Applying the current division law in network Fig. 13.6 (*b*), we have

$$I_L'=ZI_0/(Z+Z_L). \quad ...(54)$$

If $Z=Z_0$. The load current I_L' can be made equal to I_L. Thus from above relations we have

$$I_0=E_0/Z_0, \quad ...(55)$$

which is the short circuit current of the Thevenin's equivalent network. Thus we see that interchange of voltage and current sources with the help of Thevenin's and Norton's theorems gives a method of network analysis. The main difference between these is *that in the former case a voltage source is removed from a circuit by short circuiting its emf while in the latter case current source is removed by opening circuit.*

(5) Maximum Power-Transfer Theorem. *This theorem states that the power received by one network from another connected network is maximum when the impedance of the receiving network is conjugate to the impedance of the connected network looking at its two terminals. The process of bringing about this condition is called impedance matching.*

Let us consider a two terminal active linear network connected to a two terminal passive linear network. The former can be replaced by Thevenin's equivalent active network consisting of open circuit potential and the total impedance when all the generators are replaced by their internal impedances and the latter network is replaced by equivalent impedance Z_R. Let the impedances Z_0 and Z_R are of complex nature, *i.e.*

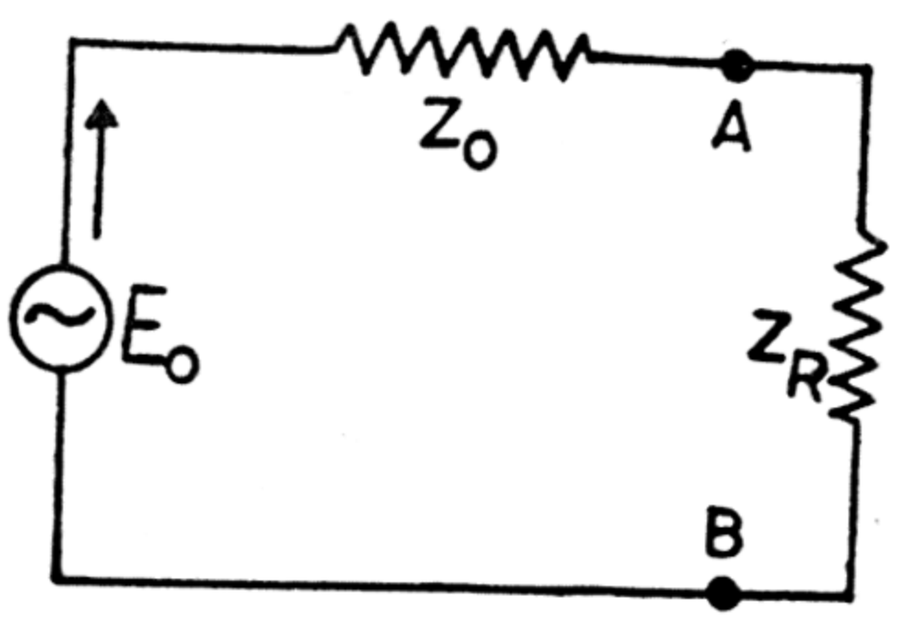


Fig. 13.7.

$$Z_0=R_0+jX_0 \quad \text{and} \quad Z_R=R_R+jX_R. \qquad \ldots(56)$$

Thus the current flowing in the network

$$I=\frac{E_0}{Z_0+Z_R}=\frac{E_0}{R_0+R_R+j\,(X_0+X_R)}. \qquad \ldots(57)$$

The power delivered to the passive network or to the impedance Z_R

$$P=\frac{E_0^2 R_R}{(R_0+R_R)^2+(X_0+X_R)^2}. \qquad \ldots(58)$$

If the impedance of receiving network, *i.e.*, $Z_R(=R_R+jX_R)$ is varied then the maximum power transfer may be calculated first by considering X_R variable and then R_R only.

(*i*) If X_R is varying, then power will be maximum when

$$\frac{\partial P}{\partial X_R}=\frac{-E_0^2 R_R\,(2)\,(X_0+X_R)}{[(R_0+R_R)^2+(X_0+X_R)^2]^2}=0.$$

or $\quad X_R+X_0=0 \quad$ or $\quad X_R=-X_0.$

$$\therefore \quad P_{max}=E_0^2\,R_R/(R_R+R_0)^2. \qquad \ldots(59)$$

(*ii*) If R_R is varying, then power transfer will be maximum, when

$$\frac{\partial P}{\partial R_R}=\frac{E_0^2\,[(R_R+R_0)^2-2R_R\,(R_R+R_0)]}{(R_R+R_0)^4}=0.$$

or $\quad R_R=R_0.$

$$\therefore \quad P_{max}=E_0^2/4R_0. \qquad \ldots(60)$$

Hence the total impedance of the passive network

$$Z_R=R_R+jX_R=R_0-jX_0, \qquad \ldots(61)$$

which is conjugate to the impedance of the active network looking at two terminals, *i.e.*, to the equivalent impedance $Z_0 = R_0 + jX_0$.

$$\text{Power generated from the generator} = \frac{E_0^2}{\text{Total resistance}}$$

$$= \frac{E_0^2}{R_0 + jX_0 + R_0 - jX_0} = \frac{E_0^2}{2R_0}. \qquad \ldots(62)$$

Thus we see that the maximum power is transferred to the passive network when the impedance of this network is conjugate to the impedance of the active network. The amount of this power is half to the total power generated. In other words we can say that the power lost in the internal generator is equal to the power delivered to the receiving network.

6. Compensation Theorem. *Any impedance having an emf across its terminals may be replaced by a generator of zero internal impedance and of emf equal to the emf across the impedance at every instant.*

This theorem applies to the effect produced on the network currents by a change $\triangle Z$ in the impedance of a network branch. This effect may be found by replacing $\triangle Z$ with a generator of emf $-I\triangle Z$, where I is the original current which flowed in the branch.

7. Millman Theorem. It is just an extension of Norton's theorem and is very useful in simplifying electronic circuits. The each voltage source can be converted into a current source parallel to an admittance. Hence the combination of voltage sources E_1, E_2, E_3 with admittances Y_1, Y_2, Y_3 respectively is equivalent to a circuit consisting of a source of emf

$$E = \frac{E_1Y_1 + E_2Y_2 + E_3Y_3}{Y_1 + Y_2 + Y_3}. \qquad \ldots(63)$$

and admittance $Y = Y_1 + Y_2 + Y_3$ in series across the terminals.

This is known as *Millman's Theorem.*

13.4. FOUR-TERMINAL NETWORKS

Most filter sections take the form of a four terminal network, as they possess two pairs of terminals one pair is called input and the other output. Assuming the circuit elements linear. If Z_{11} and Z_{22} are the self impedances of the meshes and Z_{12} and Z_{21} the mutual impedances, then

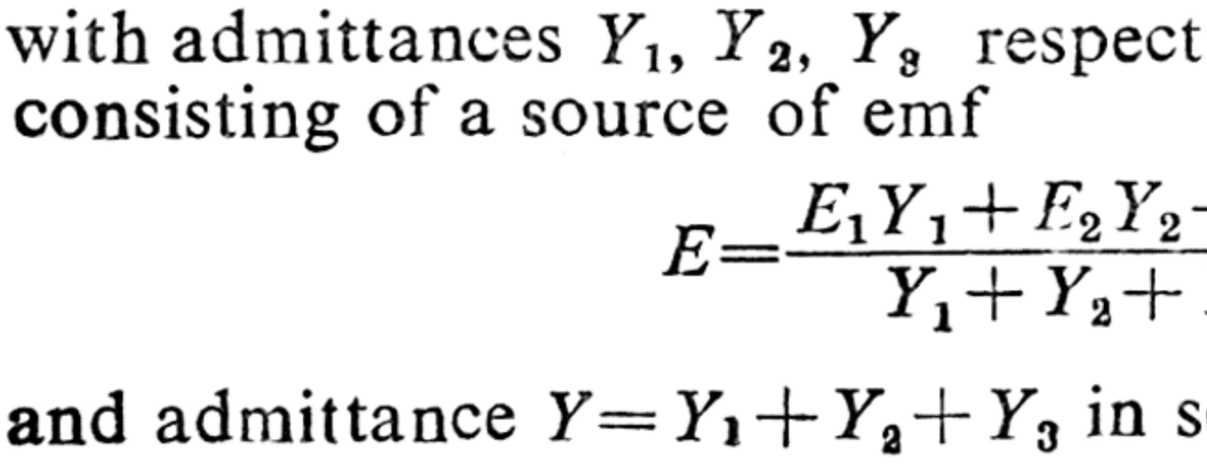


Fig. 13.8.

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \qquad \ldots(64)$$

where $Z_{11}=E_1/I_1$, when $I_2=0$, the impedance between the terminals 1-1 with the terminals 2-2 open circuited,

$Z_{22}=E_2/I_2$, when $I_1=0$, the impedance between the terminals 2-2 with the terminals 1-1 open circuited,

$Z_{12}=E_1/I_2$, when $I_1=0$, the transfer impedance with 1-1 open circuited,

and $Z_{21}=E_2/I_1$, when $I_2=0$, the transfer impedance with 2-2 open circuited.

The above relation can be written as

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \ldots(65)$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ I_1 \end{bmatrix}. \qquad \ldots(66)$$

These different forms are called Z, Y and A matrix forms. These parameters are called *linear parameters*. These parameters are related to each other as,

$$A_{11}=Z_{11}/Z_{21},\ A_{12}=Z_{12}-Z_{22}\times Z_{11}/Z_{21},\ A_{21}=1/Z_{21},$$

$$A_{22}=-Z_{22}/Z_{21}.$$

In the case of cascaded networks, we have

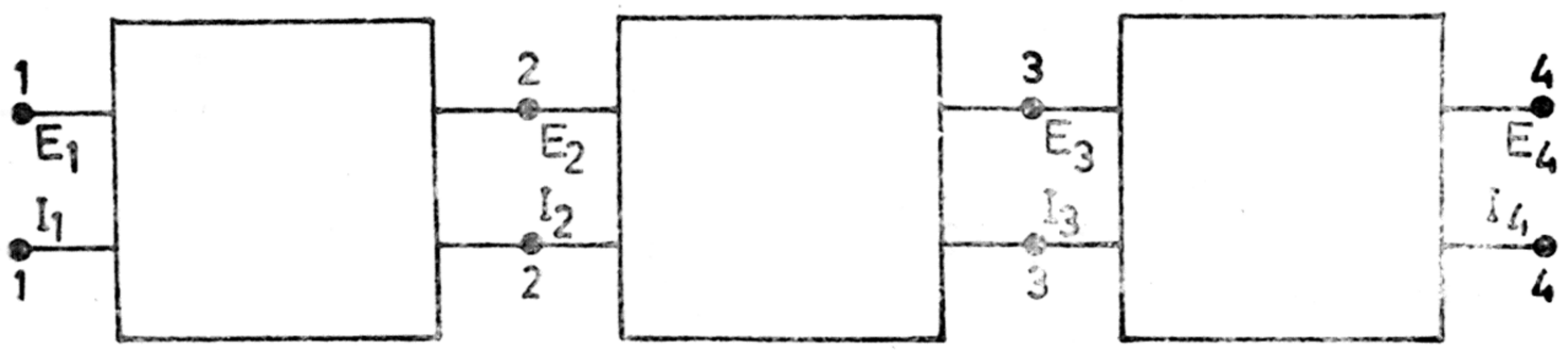


Fig. 13.9.

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix}, \quad \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} E_3 \\ I_3 \end{bmatrix},$$

$$\begin{bmatrix} E_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} E_4 \\ I_4 \end{bmatrix}, \quad \begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} E_4 \\ I_4 \end{bmatrix}$$

or

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}. \qquad \ldots(67)$$

For a network shown in Fig. 13.10 (*a*), we have

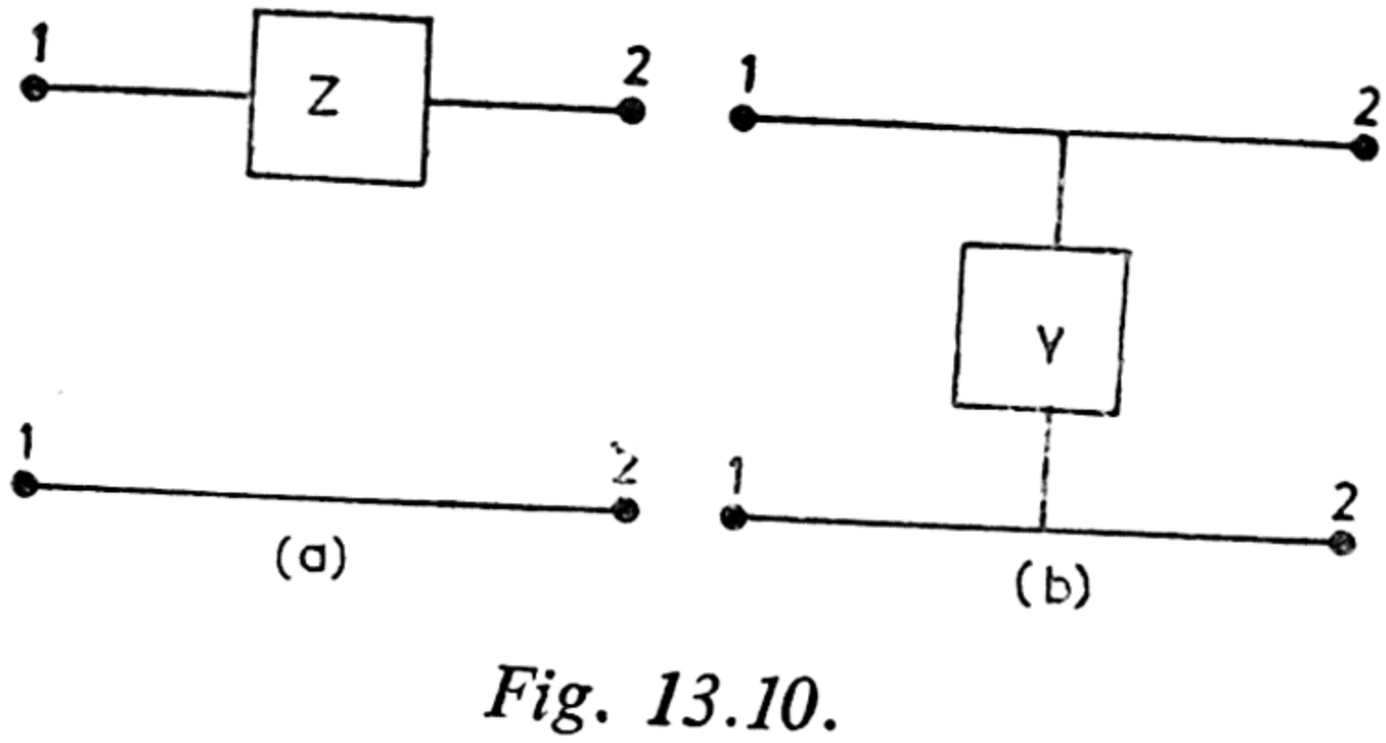


Fig. 13.10.

$$E_1=E_2+ZI_2,\ I_1=I,$$

and $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}=\begin{vmatrix} 1 & Z \\ 0 & 1 \end{vmatrix}$.

Similarly for the network Fig. 13.10 (*b*)

$$E_1=E_2,\ I_1=YE_2+I_2$$

and $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}=\begin{vmatrix} 1 & 0 \\ Y & 1 \end{vmatrix}$

Exercises

Example 1. *A resistance of 10 Ω is in series with an inductive reactance of 6.28 Ω and the combination is in parallel with a circuit consisting of a resistance of 20Ω in series with a capacitance of 100 μf. The whole is in series with a resistance of 15 Ω and an inductive reactance 12.56 Ω. Calculate the equivalent resistance, reactance and impedance of the whole circuit.*

Since the frequency of the supply is not given, let us assume it as 50 c/s. The reactance of the capacitor

$$X_2=1/j\omega C=-j/314\times10^{-4}=-j\ 31.85.$$

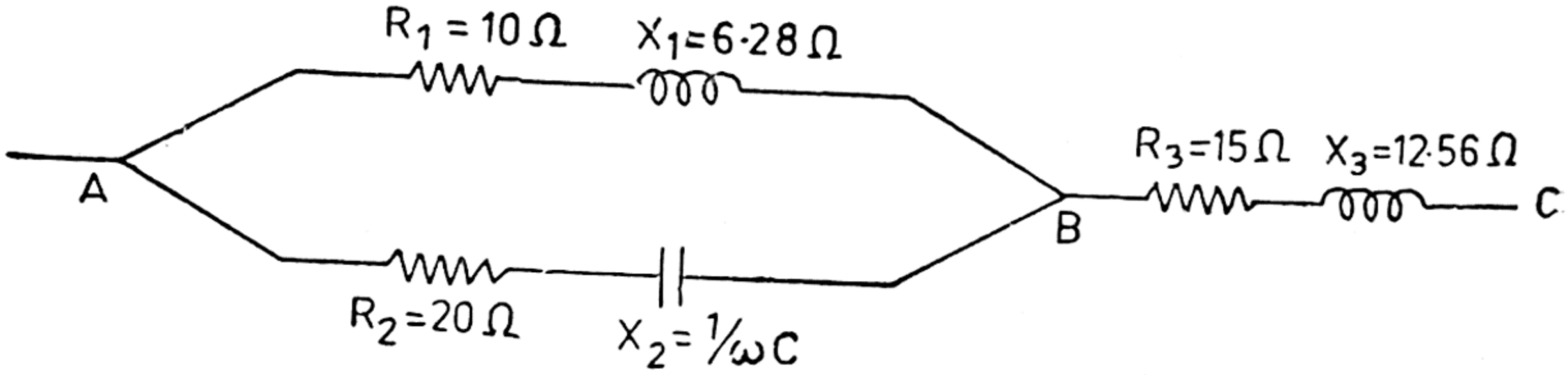


Fig. 13.11.

For branch I

$$Z_1=10+j\ 6.28$$

and for branch II

$$Z_2=20-j\ 31.85.$$

Since these branches are in parallel, the total impedance of the combination in parallel is given as

$$\frac{1}{Z_c}=\frac{1}{Z_1}+\frac{1}{Z_2} \quad \text{or} \quad \frac{1}{Z_c}=\frac{1}{10+j\ 6.28}+\frac{1}{20-j\ 31.85}$$

or $$Z_c=\frac{(10+j\ 6.28)(20-j\ 31.85)}{30-j\ 25.57}=\frac{(400-j\ 192.9)(30+j\ 25.57)}{30^2+25.57^2}$$

$$=10.90+j\ 2.86.$$

It is in series with an impedance

$$Z_3=15+j\ 12.56.$$

$$\therefore\ Z_{\text{total}}=Z_c+Z_3=25.90+j\ 15.42$$

$$=R_{\text{total}}+j\,X_{\text{total}}.$$

$\therefore$ Total resistance$=25.90\ \Omega$, total reactance$=15.42\Omega$ and total impedance$=(25.90^2+15.42^2)^{1/2}=30.14\ \Omega$.

Example 2. *Design a* 'Π' *network, which is equivalent to the undermentioned 'T' network.*

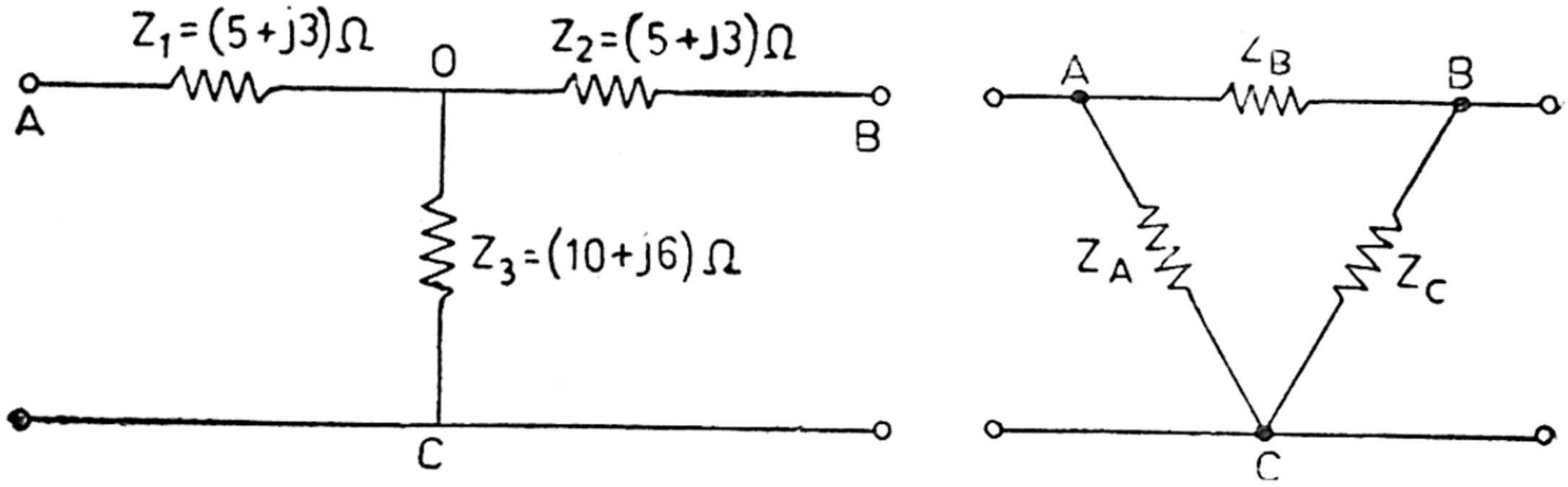


Fig. 13.12.

We know that the impedances of the Π-circuit may be determined if we have the impedances of the T-network with the following relations

$$Z_A=\square/Z_2,\ Z_B=\square/Z_3 \text{ and } Z_C=\square/Z_1,$$

where $\square=Z_1Z_2+Z_2Z_3+Z_3Z_1$.

In this problem $Z_1=5+j\,3$, $Z_2=5+j\,3$ and $Z_3=10+j\,6$.

$$\therefore \quad \square=(5+j\,3)(5+j\,3)+(5+j\,3)(10+j\,6)+(10+j\,6)(5+j\,3)$$
$$=16+30\,j+32+j\,60+32+j\,60=80+j\,150.$$

Therefore $Z_A=\dfrac{80+j\,150}{5+j\,3}=\dfrac{(80+j\,150)(5-j\,3)}{(5+j\,3)(5-j\,3)}=25+j\,15$

$$Z_B=\frac{80+j\,150}{10+j\,6}=\frac{(80+j\,150))(10-j\,6)}{(10+j\,6)(10-j\,6)}$$
$$=12.5+j\,7.5$$

and $$Z_C=\frac{80+j\,150}{5+j\,3}=\frac{(80+j\,150)(5-j\,3)}{(5+j\,3)(5-j\,3)}=25+j\,15.$$

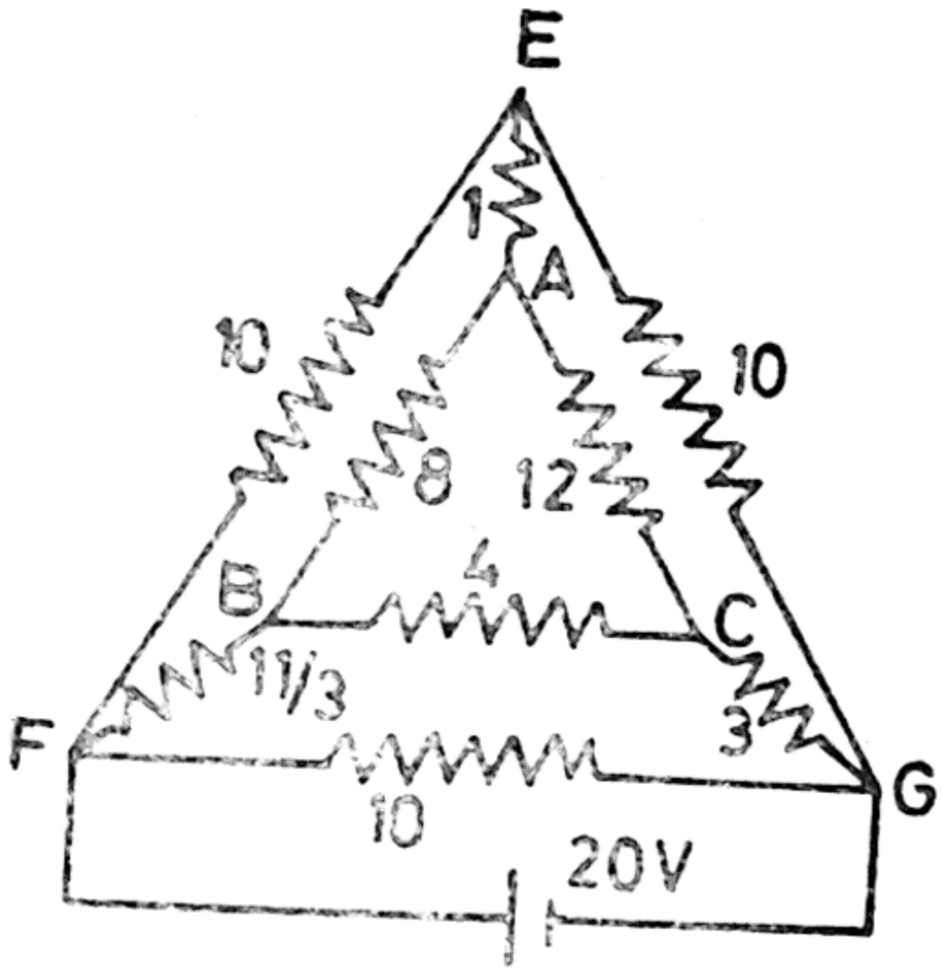
Example 3. *Find the current drawn from the battery of 20 volts connected to a circuit as shown in Fig. 13.13 (a).*

Let us first reduce the delta connection ABC to a star connection as shown in Fig. 13.13 (b). If O be the central point of the star (or T)-circuit, then the resistances OA, OB and OC may be obtained by using equations (13), as given below. Since the circuit consists of pure resistances, hence the impedances Z's are to replaced by R's.

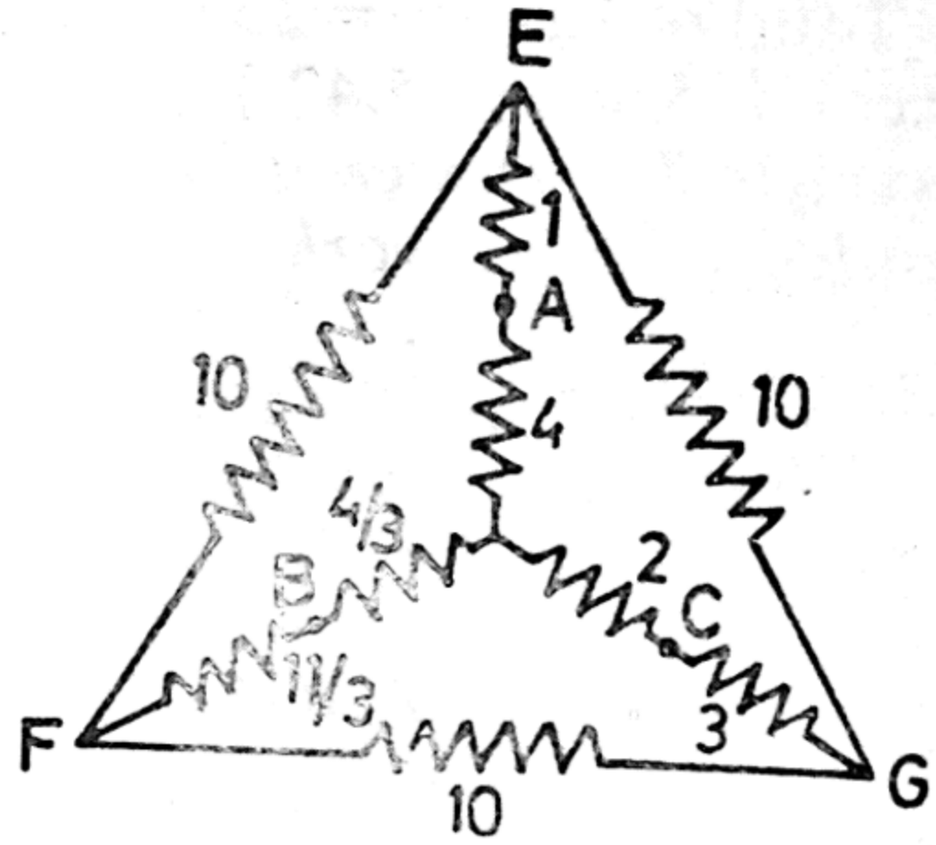
$$\therefore \quad R_{OA}=\frac{R_{AC}\times R_{AB}}{R_{AC}+R_{AB}+R_{BC}}=\frac{12\times 8}{12+8+4}=4\ \Omega$$

$$R_{OB}=\frac{R_{AB}\times R_{BC}}{R_{AC}+R_{AB}+R_{BC}}=\frac{8\times 4}{12+8+4}=\frac{4}{3}\ \Omega$$

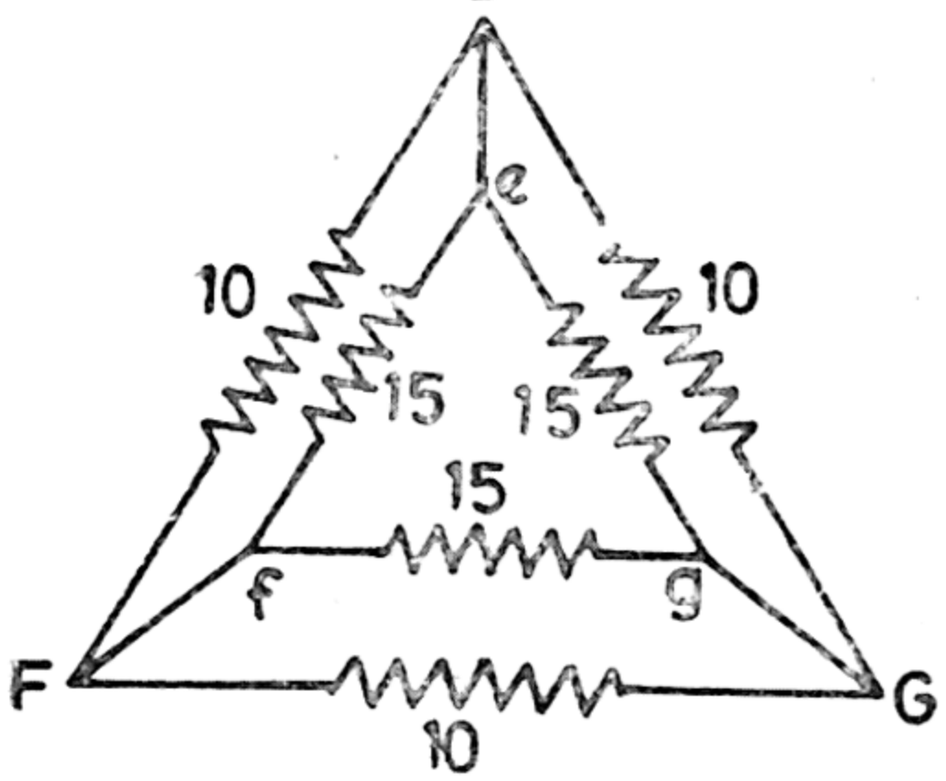
and $$R_{OC}=\frac{R_{BC}\times R_{AC}}{R_{AC}+R_{AB}+R_{BC}}=\frac{4\times 12}{12+8+4}=2\ \Omega.$$



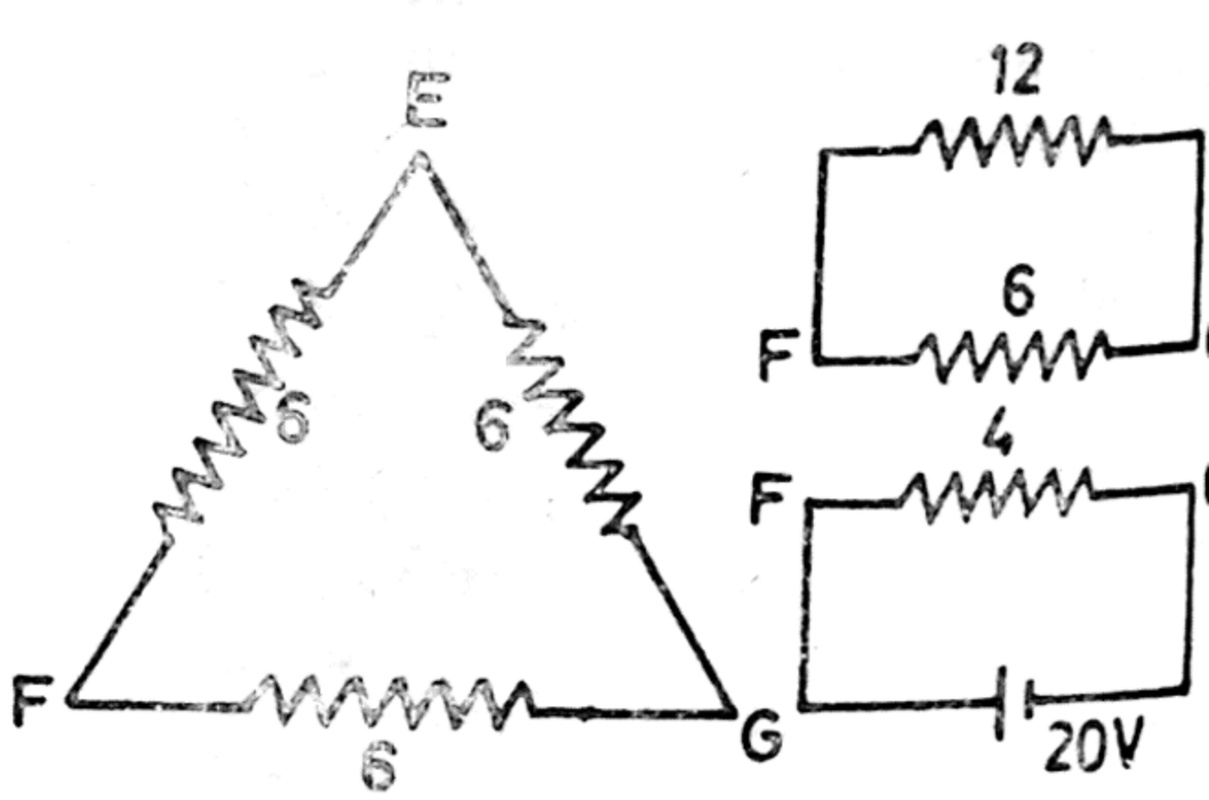
(a)



(b)



(c)



(d)

Fig. 13 13.

These resistances are added with the series resistances originally present in the circuit. Now these resistances 5, 5, 5 make star network. This star network is now converted into a delta as shown in Fig. (*c*).

$$R_{ef}=\frac{5\times5+5\times5+5\times5}{5}=15,$$

$R_{eg}=15$ and $R_{fg}=15$.

Since these are in parallel with originally present side of resistance of 10Ω each, the effective value of the sides of *EFG* are 6Ω each.

$$\left[\frac{1}{R}=\frac{1}{10}+\frac{1}{15}=\frac{1}{6} \text{ or } R=6 \text{ ohm}\right]$$

This glves two parallel resistances *FG* and (*FE*+*EG*), *i.e.*, 6 and (6+6). The combination $12\times6/(12+6)=4\,\Omega$ acts across the battery of voltage 20 volts. Hence the current drawn

$I=V/R=20/4=5$ amp.

Example 4. *In a network given below, calculate the mesh currents I_1, I_2 and I_3 by the method of determinants. Hence obtain the voltages V_1 and V_2 and also the voltage across the 6Ω resistor.*

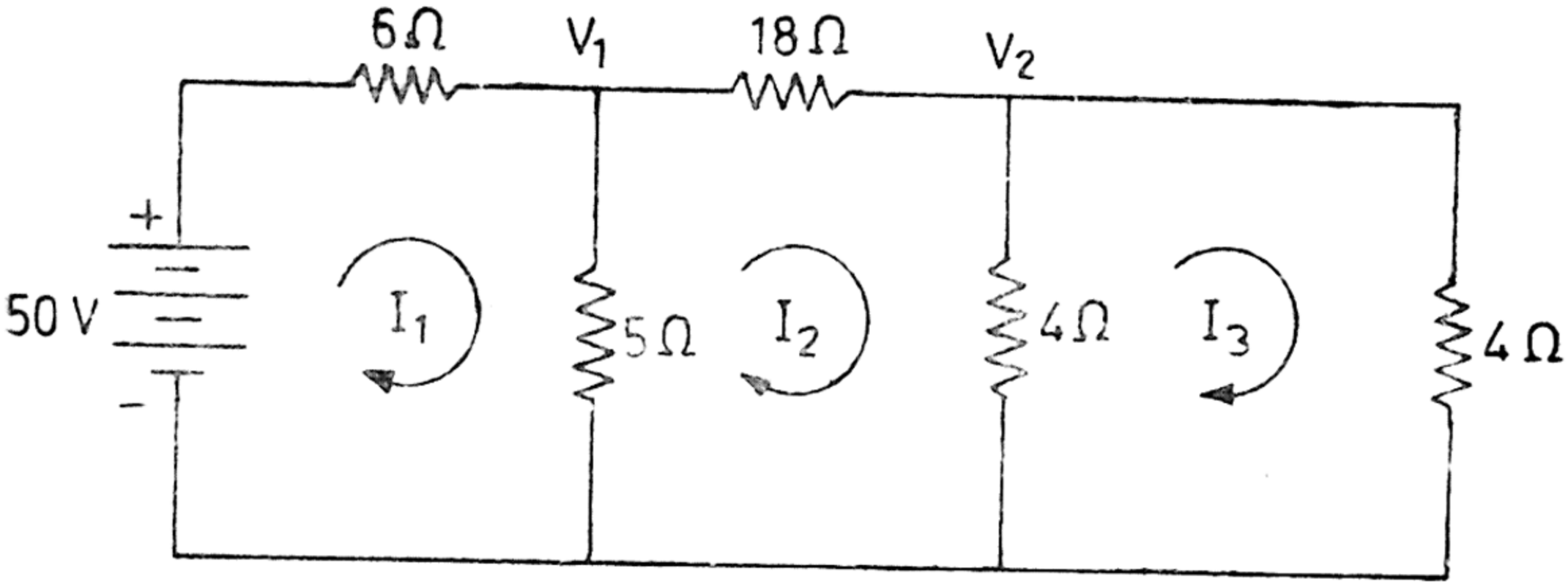


Fig. 13.14.

The three equations for the three meshes may be written as

$$50 = (6+5)\ I_1 - 5\ I_2 \qquad \ldots(i)$$

$$0 = -5\ I_1 + (5+18+4)\ I_2 - 4I_3 \qquad \ldots(ii)$$

$$0 = -4\ I_2 + (4+4)\ I_3. \qquad \ldots(iii)$$

These equations may be written in the form of determinants as

$$\begin{vmatrix} 50 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 11 & -5 & 0 \\ -5 & 27 & -4 \\ 0 & -4 & 8 \end{vmatrix} \begin{matrix} I_1 \\ I_2 \\ I_3 \end{matrix}.$$

The determinant of resistance may be written as

$$\Delta_R = [R] = 11\begin{vmatrix} 27 & -4 \\ -4 & 8 \end{vmatrix} + 5\begin{vmatrix} -5 & -4 \\ 0 & 8 \end{vmatrix}$$

$$= 11(27\times 8 - 4\times 4) + 5(-40) = 2000.$$

$$\Delta_1 = \begin{vmatrix} 50 & -5 & 0 \\ 0 & 27 & -4 \\ 0 & -4 & 8 \end{vmatrix} = 50\begin{vmatrix} 27 & -4 \\ -4 & 8 \end{vmatrix} = 10000.$$

$$\Delta_2 = \begin{vmatrix} 11 & 50 & 0 \\ -5 & 0 & -4 \\ 0 & 0 & 8 \end{vmatrix} = -50\begin{vmatrix} -5 & -4 \\ 0 & 8 \end{vmatrix} = 2000.$$

$$\Delta_3 = \begin{vmatrix} 11 & -5 & 50 \\ -5 & 27 & 0 \\ 0 & -4 & 0 \end{vmatrix} = 50\begin{vmatrix} -5 & 27 \\ 0 & -4 \end{vmatrix} = 1000.$$

Thus the currents are given by

$$I_1 = \frac{\Delta_1}{\Delta_R} = \frac{10000}{2000} = 5 \text{ amp.}$$

$$I_2 = \frac{\Delta_2}{\Delta_R} = \frac{2000}{2000} = 1 \text{ amp.}$$

and

$$I_3 = \frac{\Delta_3}{\Delta_R} = \frac{1000}{2000} = 0.5 \text{ amp,}$$

The voltage $V_1=(I_1-I_2)\ 5=20$ volts

$V_2=(I_2-I_3)\ 4=2$ volts

and voltage across $6\Omega=6I_1=30$ volts.

Example 5. *Calculate the current in 6Ω resistance in the Fig. 13.15.*

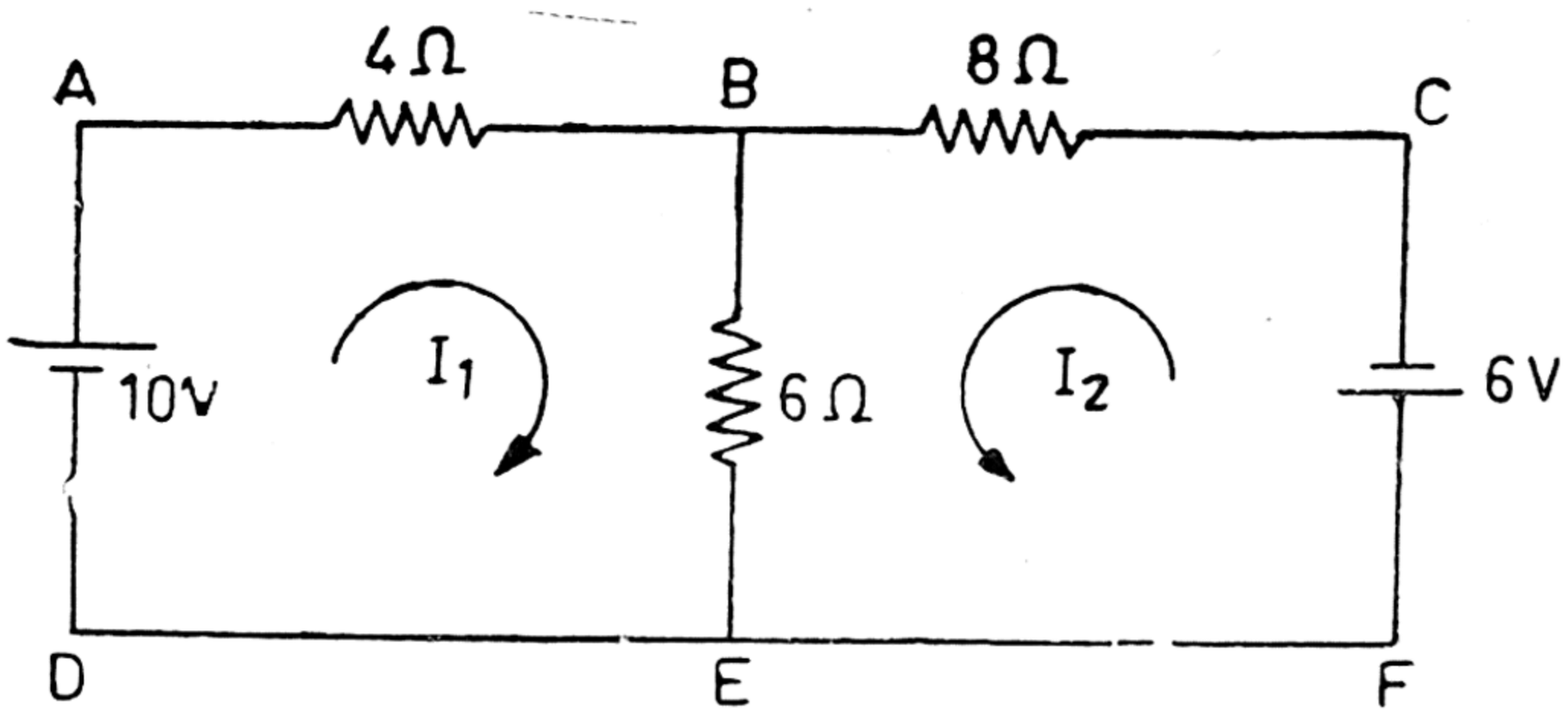


Fig. 13.15.

Let I_1 and I_2 be the currents in the meshes. *ABEDA* and *BCFEB* respectively.

There are two methods of finding current in 6Ω resistance.

(1) *Kirchhoff's Laws.* Applying Kirchhoff's first law at point *E*, we get current along $BE=I=I_1+I_2$. Now applying Kirchhoff's second law to the mesh *ABEDA*

$$I_1\times 4+(I_1+I_2)\times 6=10,$$

or $$10I_1+6I_2=10 \qquad ...(i)$$

and to the mesh *BCFEB*

$$I_2\times 8+(I_1+I_2)\times 6=-6.$$

or $$6I_1+14I_2=-6. \qquad ...(ii)$$

Solving equations (*i*) and (*ii*) we get

$$I_1=22/13 \text{ and } I_2=-15/13 \text{ amp.}$$

$\therefore$ Current in the *BE* arm $=I_1+I_2=7/13$ amp.

(*ii*) *Superposition Theorem.* If the second source is replaced by its internal resistance (*i.e.*, zero) and *I* is the current in *AB* branch, then the external resistance of the whole circuit will be

$$R=4+6\times 8/(6+8)=52/7.$$

$$\therefore \quad \text{Current } I=\frac{V}{R}=\frac{10}{52/7}=\frac{35}{26}.$$

Its component along 6Ω resistance

$$I_1'=\frac{8}{8+6}\ I=\frac{8}{14}\times\frac{35}{26}=\frac{10}{13} \text{ amp.}$$

If the first source is replaced by its internal resistance zero, and the current in the BC arm is I, then the external resistance of the current

$$R=8+4\times 6/(4+6)=52/5 \text{ amp.}$$

$$\therefore \quad \text{Current } I=\frac{V}{R}=\frac{-6}{52/5}=\frac{-15}{26} \text{ amp.}$$

Its component along the 6Ω resistance

$$I_2'=\frac{4}{10}\ I=\frac{4}{10}\times\left(\frac{-15}{26}\right)=-\frac{3}{13} \text{ amp.}$$

The superposition theorem gives the total current in 6Ω resistance as

$$I'=I_1'+I_2'=\frac{10}{13}-\frac{3}{13}=\frac{7}{13} \text{ amp.}$$

Example 6. *Calculate the current in branch BC for the network shown in Fig. 13.16, by using superposition theorem.*

We know that the impedance of capacitor $=-j/\omega C$ and of inductance $=j\omega L$. In our present problem

$$Z_{11}=1+j2+1-j4+3+j6=5+j4,$$
$$Z_{22}=2-j5+2+j4+3+j6=7+j5,$$

and
$$Z_{12}=3+j6.$$

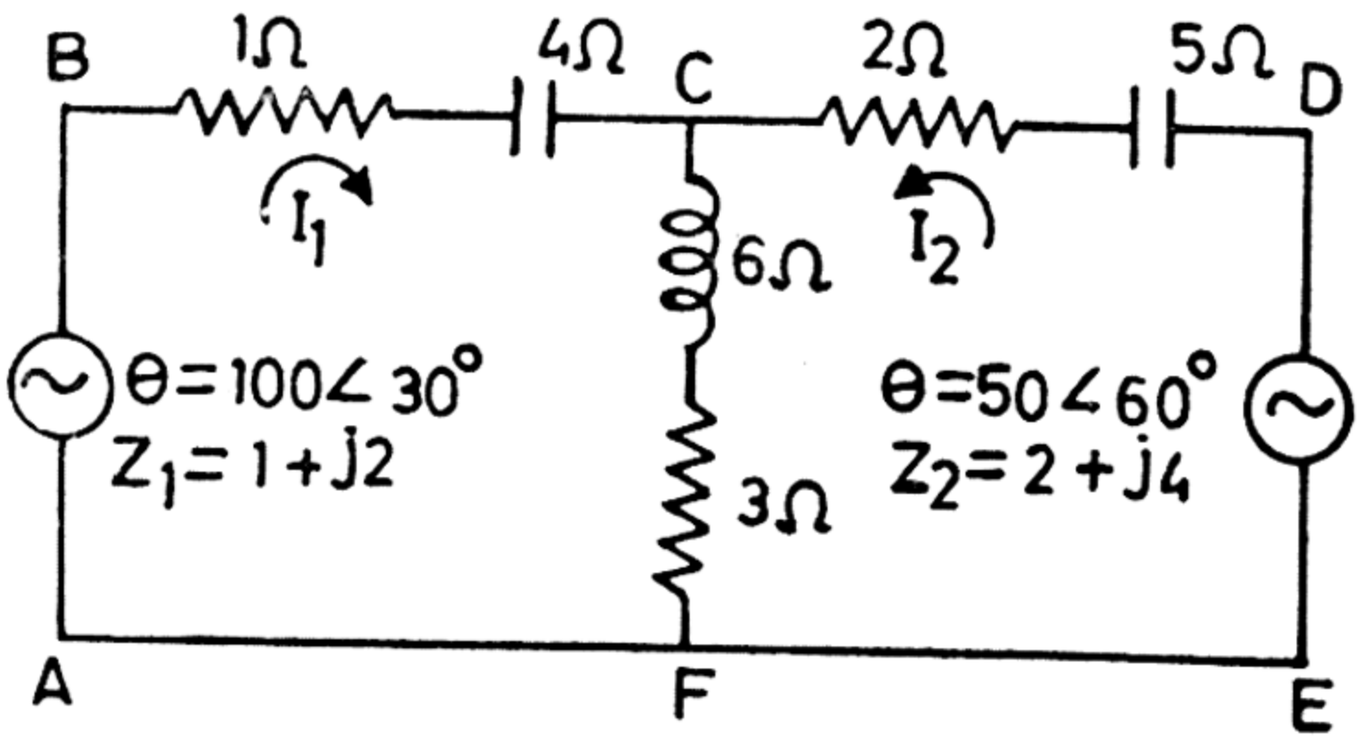


Fig. 13.16.

If I_1' and I_2' be the currents in the left and right meshes respectively when the source from the second mesh is replaced by its impedance. Then we have

$$e_1=Z_{11}I_1'+Z_{12}I_2'$$
$$0=Z_{12}I_1'+Z_{22}I_2'.$$

or
$$I_1'=\frac{e_1Z_{22}}{Z_{11}Z_{22}-Z_{12}^2}=\frac{100\ (\cos 30°+j\sin 30°)(7+j5)}{(5+j4)(7+j5)-(3+j6)^2}$$

$$=\frac{50(\sqrt{3}+j)(7+j5)}{(15+j53)-(-27+j36)}=\frac{50[7\sqrt{3}-5+j(7+5\sqrt{3})}{42+j17}$$

Similarly the current I_1'' in the left hand mesh, when the source from this mesh is replaced by its impedance, is given by

$$I_1''=\frac{-e_2Z_{12}}{Z_{11}Z_{22}-Z_{12}^2}=\frac{-50(\cos 60°+j\sin 60°)(3+j6)}{42+j17}$$

$$=-\frac{25(1+j\sqrt{3})(3+j6)}{42+j17}=-\frac{25[3-6\sqrt{3}+j(3\sqrt{3}+6)]}{42+j17}.$$

By the *superposition theorem*, the current

$$I_1=I_1'+I_2''=\frac{500\sqrt{3}-325+j(200+175\sqrt{3})}{42+j17}$$

$$=15.3+j5.8.$$

Example 7. *Make use of the network shown in Fig. 13.17, calculate the current in branch EF and prove reciprocity theorem by actually calculating the current in AB when the same potential is removed from branch AB and inserted in branch EF.*

Mesh equations for currents are given as

$$e=Z_{11}I_1+Z_{12}I_2$$

$$0=Z_{12}I_1+Z_{22}I_2,$$

where $Z_{11}=2-j2+2-j8=4-j10.$

$$Z_{22}=2+j4+1-j4+2-j8=5-j8$$

and $Z_{12}=2-j8.$

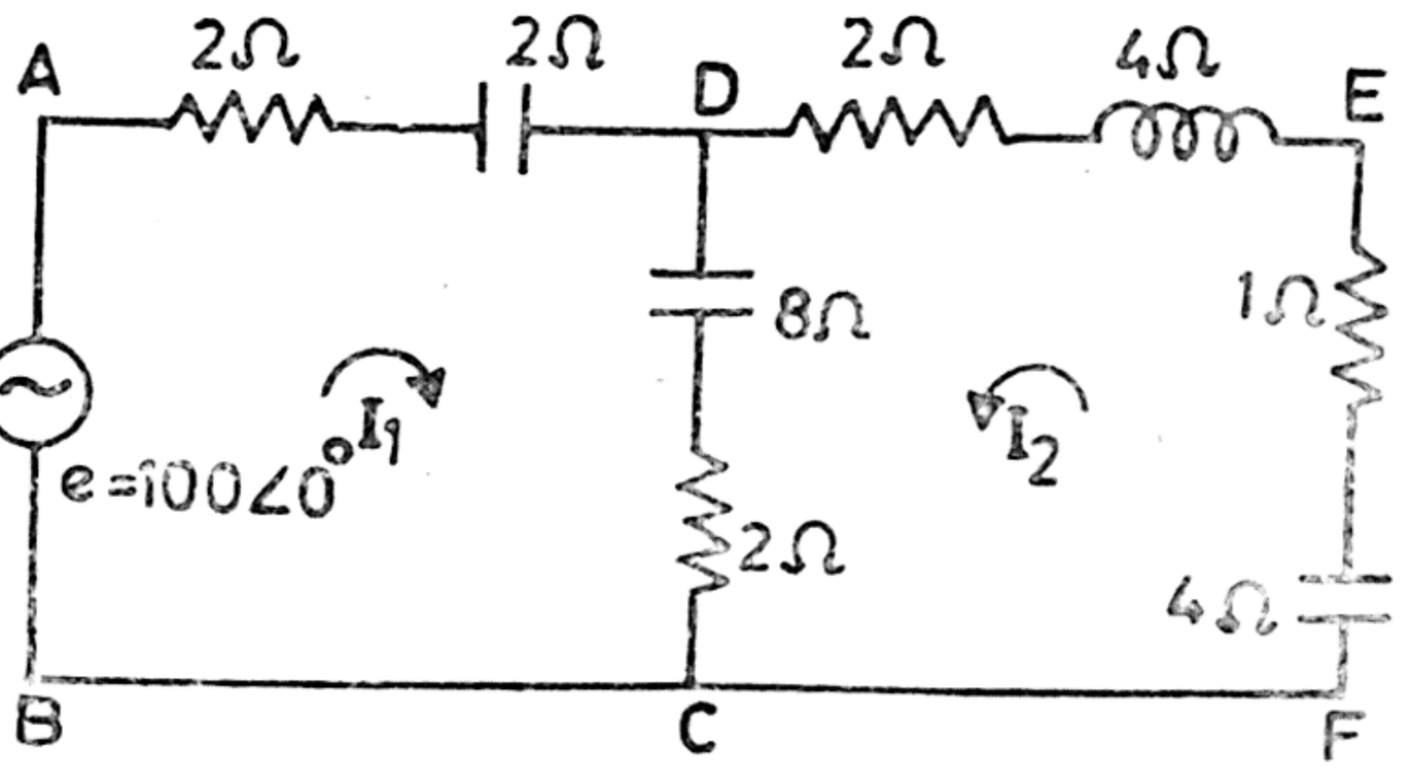


Fig. 13.17.

The current in second mesh is given by

$$I_2=\frac{-eZ_{12}}{Z_{11}Z_{22}-Z_{12}^2}=\frac{-100(2-j8)}{(4-j10)(5-j8)-(2-j8)^2}$$

$$=-100(2-j8)/(-50j)=-16-j4.$$

Similarly the current in first mesh, when its source is introduced in the second mesh, is given by

$$I_1=\frac{-eZ_{12}}{Z_{11}Z_{22}-Z_{12}^2}=-16-j4.$$

Hence the *reciprocity theorem is proved.*

Example 8. *Calculate open circuit voltage and impedance between two terminals E and F in the network shown in Fig. 13.18. Calculate also the load current flowing through a load impedance* $Z_L = 10 - j7.5$.

For calculating the impedance Z_0, between terminals E and F let us remove potential source e and short circuit this branch.

Thus we have

$$Z_0 = \frac{(j5 - j20)(j5)}{(j5 - j20) + (j5)}$$

$$= j7.5.$$

Total impedance across the source e is

$$Z = \frac{Z_{CD} Z_{CGH}}{Z_{CD} + Z_{CGH}}$$

$$= \frac{(-j20)(j5 + j5 - j20)}{-j20 + (j5 + j5 - j20)}$$

$$= -j6.66.$$

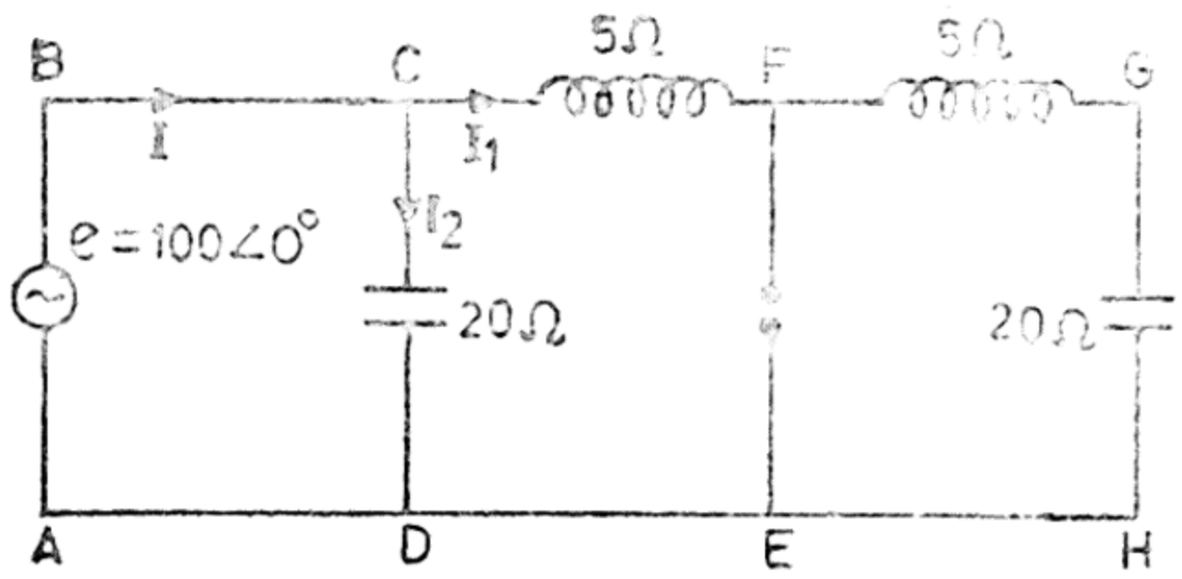


Fig. 13.18.

$\therefore$ Current in BC branch, $I = \frac{e}{Z} = \frac{100/\underline{0°}}{-j6.66} = \frac{100/\underline{0°}}{6.66/\underline{-90°}}$

$$= 15/\underline{90°} = j15.$$

Current in CGH branch, $I_1 = \frac{-j20}{-j30} \times j15 = j10 = 10/\underline{90°}$.

Hence pd across E and F terminals $= Z_{FGH} \times I_1 = -j15 \times j10$

$$= 150/\underline{0°}.$$

$\therefore$ Load current $I_L = \frac{150/0°}{j7.5 + 10 - j7.5} = 15/\underline{0°}$.

Example 9. *Determine the current in the load resistance* R_L *of the ladder-network* [*Fig. 13.19 (a)*].

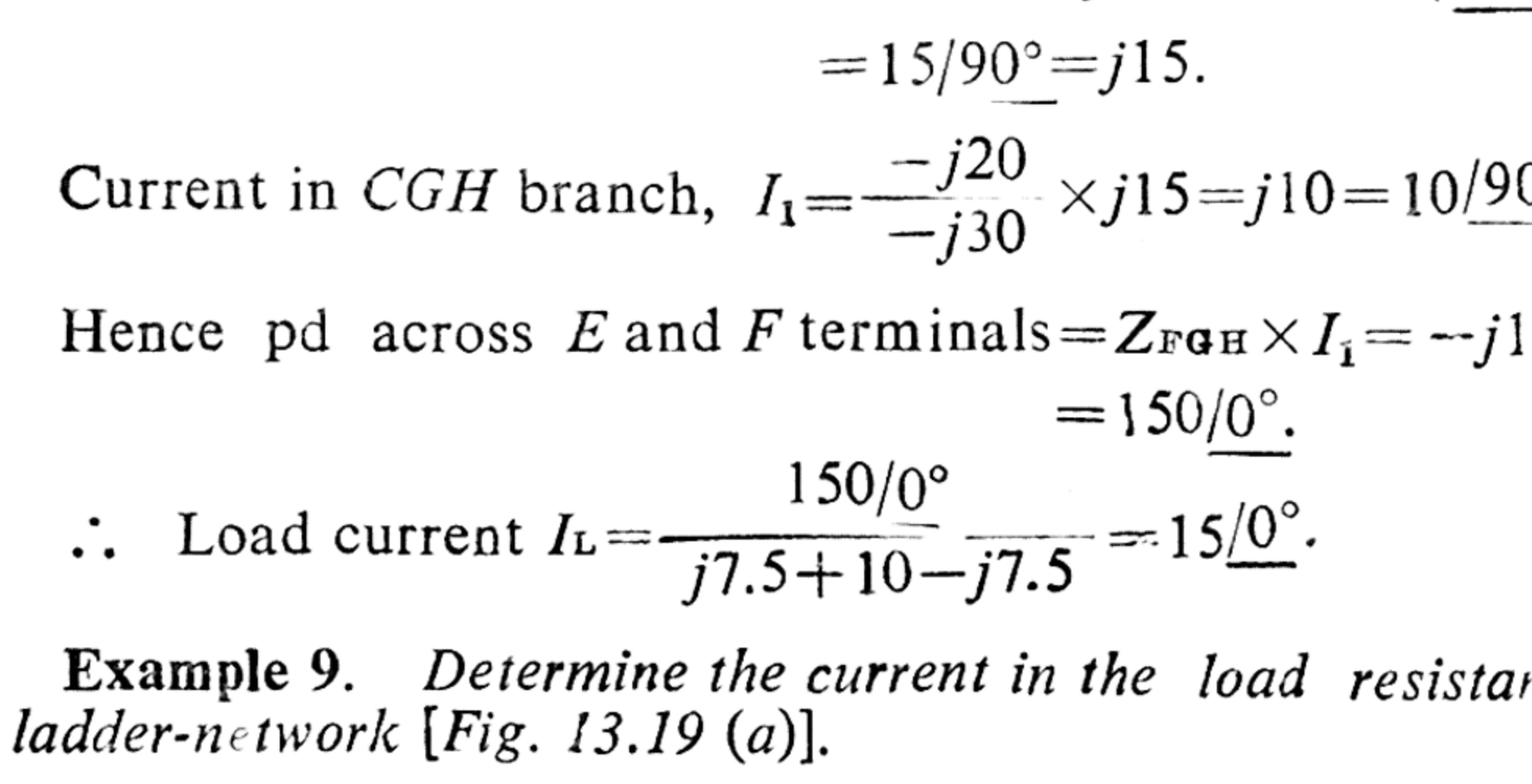


Fig. 13.19 (a).

This problem may be solved by a repeated application of Thevenin's theorem. As a first step, let us treat the network to the right of R_2 as the load resistance. The comparison of the left side of the network with the Fig. 13.15 gives us

Open circuit voltage across $AB = ER_3/(R_1+R_3) = E'$ (say)

and the total resistance between terminals A and B when the generator is replaced by its internal resistance (which is zero in this case) $= R_2 + R_3R_1/(R_1+R_3) = R'$ (say).

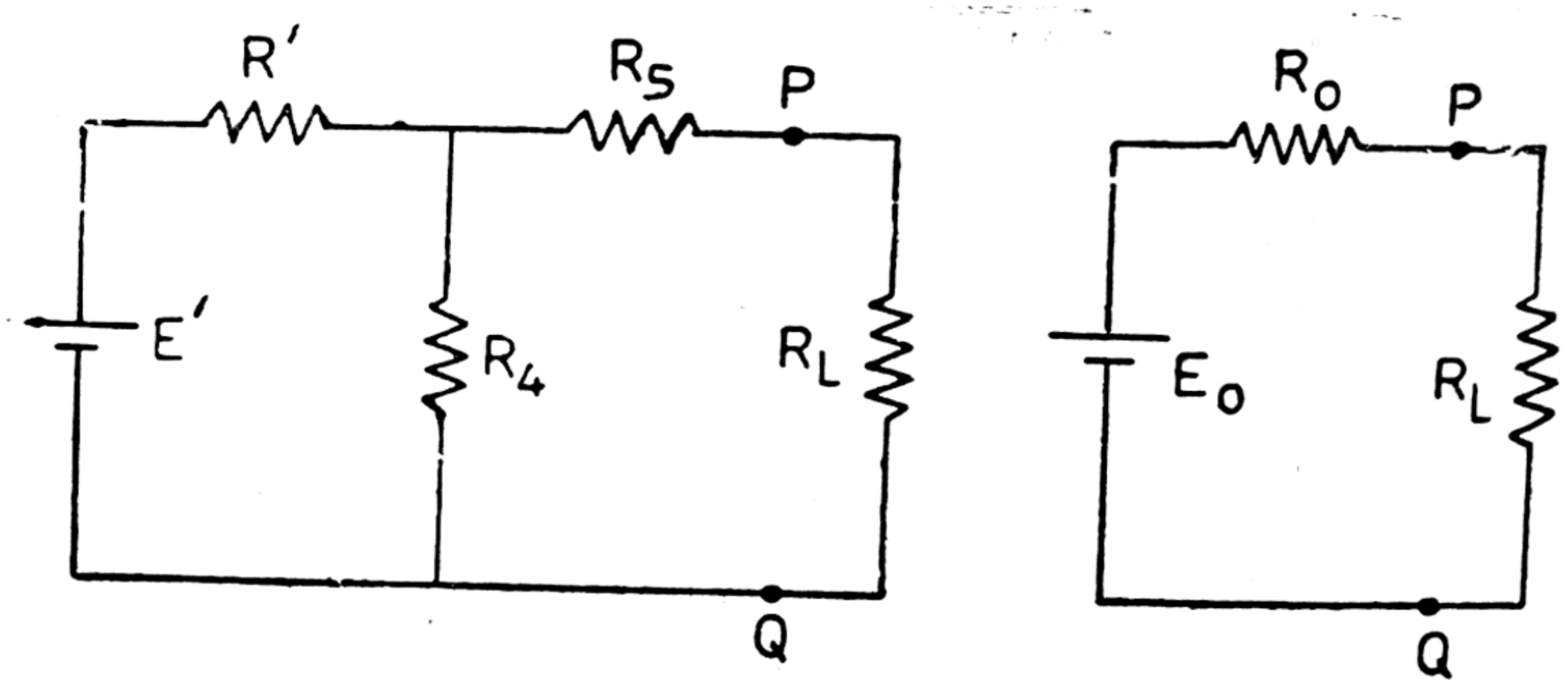


Fig. 13.19 (b).

The use of Thevenin's theorem thus reduces the original circuit to that of Fig. 13.19 (*b*). To solve the problem, let us again apply Thevenin's theorem to this circuit. For this the open circuit voltage across PQ is given by

$$E_0 = \frac{E'R_4}{R'+R_4} = \frac{E\,R_3R_4/(R_1+R_3)}{(R_2+R_4)+R_1R_3/(R_1+R_3)}$$

$$= \frac{E\,R_3R_4}{(R_1+R_3)(R_2+R_4)+R_1R_3}$$

and the total resistance between the terminals P and Q when the generator is replaced by its internal resistance

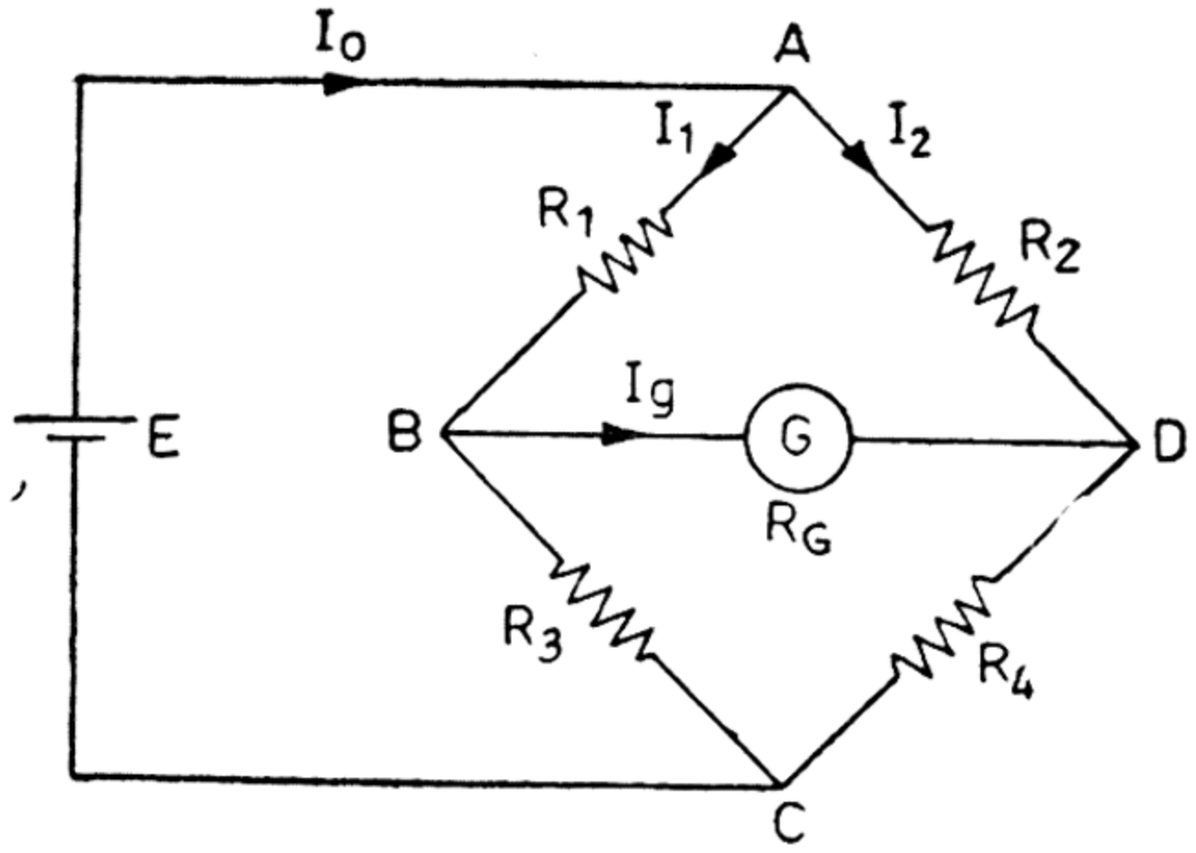
$$R_0 = R_5 + \frac{R'R_4}{R'+R_4} = R_5 + \frac{R_4[R_1R_2+R_1R_3+R_2R_3]}{R_1R_2+R_1R_3+R_2R_3+(R_1+R_3)\,R_4}.$$

Here E_0 and R_0 are called Thevenin's components. Thus the current in the load resistance

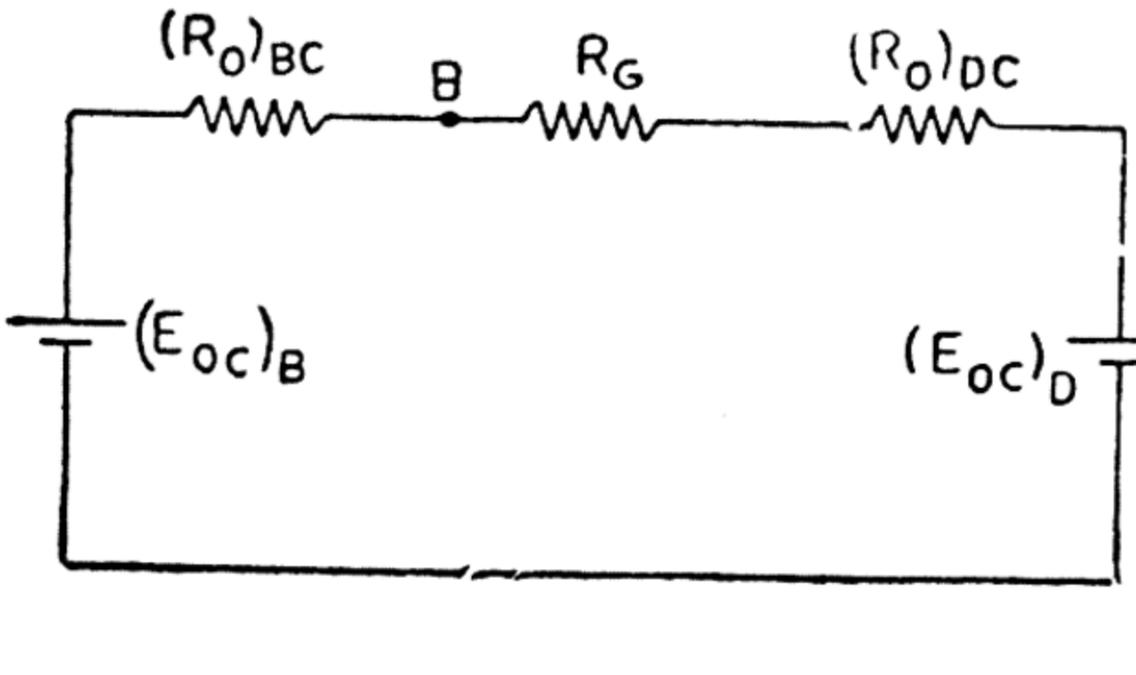
$$I_L = \frac{E_0}{R_0+R_L}$$

$$= \frac{E\,R_3R_4}{(R_5+R_L)[(R_1+R_3)(R_2+R_4)+R_1R_3]+R_4(R_1R_2+R_1R_3+R_2R_3)}.$$

Example 10. *Find the current in the galvanometer arm of a Wheatstones bridge using the Thevenin's theorem.*



(a)



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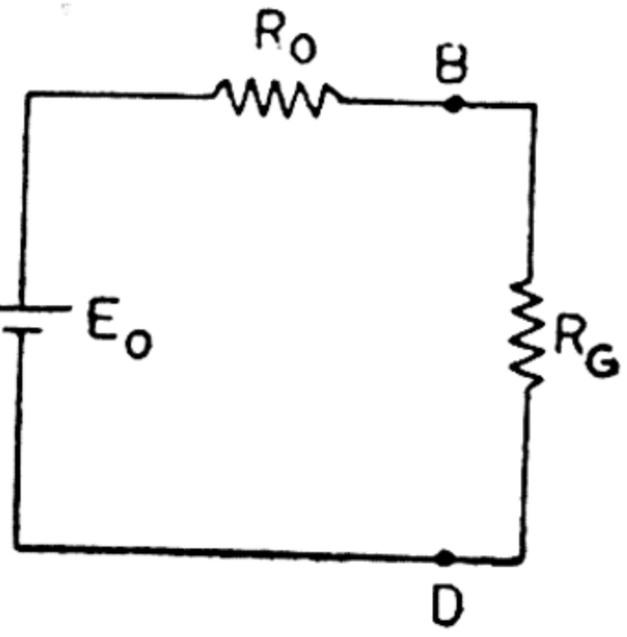


(c)

Fig. 13.20.

Since the Thevenin's theorem can determine the current flowing through the load resistance connected between the two output terminals of a network. Therefore we assume the galvanometer as a load of resistance $R_L = R_G$.

Let us first apply Thevenin's theorem to the point B on the bridge. The open circuit voltage at point B, *i.e.*, when R_G is removed

$$(E_{oo})_B = ER_3/(R_1+R_3)$$

and the resistance across B and C, when the source is short circuited, is

$$(R_0)_{BO} = R_1R_3/(R_1+R_3).$$

Similarly the application of Thevenin's theorem to the point D gives

$$(E_{oc})_D = ER_4/(R_2+R_4)$$

and $$(R_0)_{DO} = R_2R_4/(R_2+R_4).$$

Thus the bridge is reduced to a circuit shown in Fig. 13.20 (*b*). If the current passes in the galvanometer arm from point B to D,

then this circuit will further reduce to the form shown in Fig. 13.20 (*c*), where

$$E_0=(E_{OC})_B-(E_{OC})_D$$

and $$R_0=(R_0)_{BC}+(R_0)_{DO}$$

$\therefore$ Current $I_g=E_0/R_0$.

If $(E_{OC})_D>(E_{OC})_B$, the current I_g will flow in opposite direction. If $(E_{OC})_D=(E_{OC})_B$ the current will be zero and the bridge will be balanced.

Example 11. *Convert linear network Fig. 13.5 (left) with* $Z_1=2$, $Z_2=4$, $Z_3=3$, $Z_L=5$ *and* $e=10$ *volts into Thevenin's equivalent network and then into Norton's equivalent circuit. Show that the power delivered to the load in both the cases is same.*

For converting this network in Thevenin's equivalent, let us calculate Thevenin's components e_0 and Z_0 as

$$e_0=\frac{eZ_3}{Z_1+Z_3}=\frac{10\times 3}{2+3}=6 \text{ volts.}$$

and $$Z_0=Z_2+\frac{Z_1Z_3}{Z_1+Z_3}=4+\frac{2\times 3}{2+3}=5.2 \text{ ohms.}$$

$\therefore$ Power delivered to the load

$$Z_L=I_L^2Z_L=\left(\frac{6}{5.2+5}\right)^2\times 5=1.73 \text{ watts.}$$

The Thevenin's equivalent network is shown in right part of the Fig. 13.5. Now applying Norton's theorem, we have

$$I_0=e_0/Z_0=6/5.2=1.15.$$

The current through the load can be obtained by applying division law in network as

$$I_L=\frac{I_0Z_0}{Z_0+Z_L}=\frac{6}{5.2+5}=0.588.$$

$\therefore$ Power delivered to the load $=(6/10.2)^2\times 5=1.73$ watts. *which is same as that in the Thevenin's equivalent network.*

Example 12. *Calculate the load current in the following network, using Norton's theorem.*

First of all let us calculate Thevenin's components,

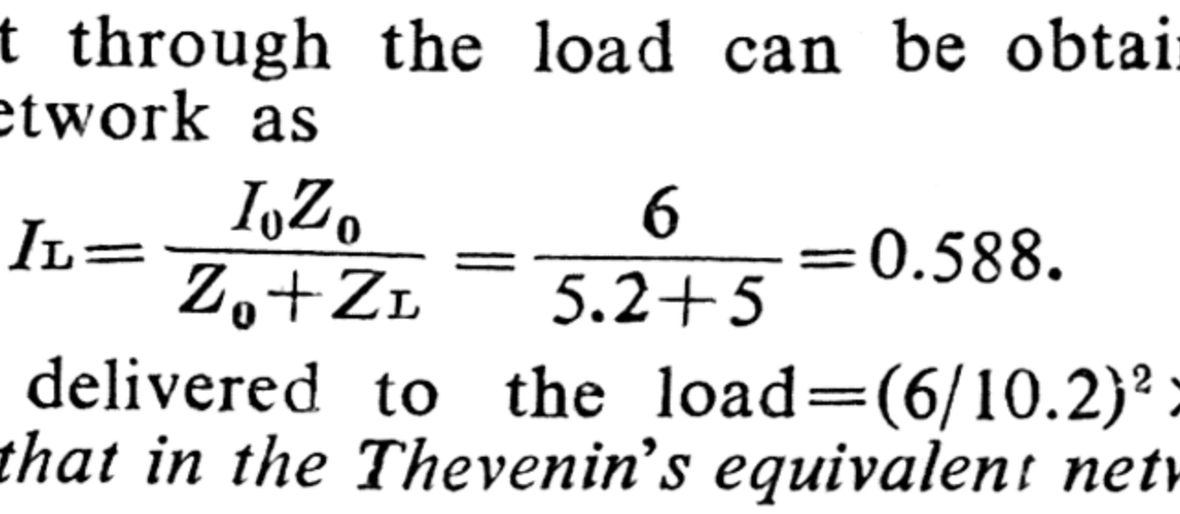


Fig. 13.21.

$$e_0=\frac{eZ_3}{Z_1+Z_3}=\frac{100/\underline{30^\circ}(-j20)}{j10-j20}=200/\underline{30^\circ},$$

and $$Z_0=Z_2+\frac{Z_1Z_3}{Z_1+Z_3}=j10+\frac{j10\times(-j20)}{j10-j20}=j30.$$

Using *Norton's theorem* we have load current

$$I_L=\frac{e_0}{Z_0+Z_L}=\frac{200/30^\circ}{j30+30}=\frac{200/\underline{30^\circ}}{30\sqrt{2}/\underline{45^\circ}}.$$

{As $30+j30=[30^2+30^2]^{1/2}\tan^{-1}(30/30)=30\sqrt{2}/\underline{45^\circ}$}

or $$I_L=4.71/\underline{-15^\circ}.$$

Example 13. *A d.c. generator (emf=40 volts) delivers a maximum power of 10 W to an external load resistance. Calculate the internal resistance of the source and it short-circuit current. How much power will it dissipate when its terminals are shorted ?*

From maximum power theorem, we know that the maximum power will be delivered to the load if the impedance of the network is conjugate to the impedance of the receiving network. In the present problem we haye given only pure resistances, hence

Source resistance R_S=External load resistance R_L and the maximum power delivered to the load

$$P_{L,max}=\frac{E^2}{4R_L}=\frac{E^2}{4R_s} \quad \text{or} \quad R_s=\frac{E^2}{4P_{L,max}}$$

In the present problem, $E=40$ volts and $P_{L,max}=10$ watts

$\therefore \quad R_s=40^2/4\times 10=40\ \Omega.$

The short circuit current of the source $=E/R_s=40/40=1$ amp.

When the source terminals are shorted, *i.e.*, $R_L=0$, the power dissipated will be

$$P=E^2/R_s=40^2/40=40 \text{ watts.}$$

Example 14. *Equivalent circuit of the operation of the valve 6J5 is shown in Fig. 13.22. If $Y_g=20$, $Y_{gk}=j0.8$, $Y_{gf}=j0.7$, $Y_p=140$, $Y_{pk}=j0.9$, $Y_L=8$. The admittances are in micro mhos and $\mu=20$. Calculate the output potential.*

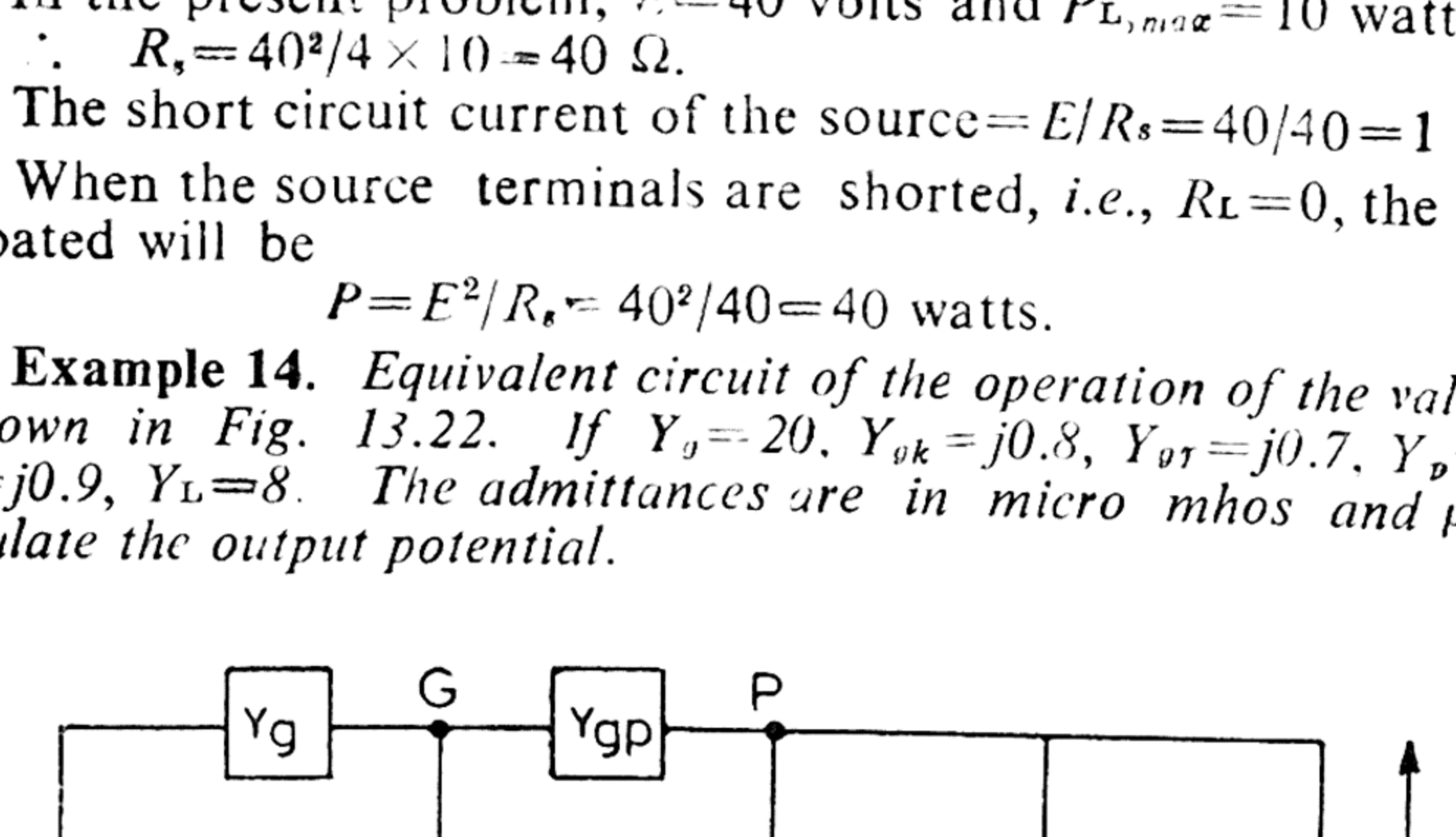


Fig. 13.22.

Applying the *Millman theorem* to node P and taking K as the reference node, we have

$$E_L = \frac{-\mu E_g Y_p + E_g Y_{gp}}{Y_p + Y_L + Y_{gp} + Y_{pk}} = \frac{[-20 \times 140 + j0.7]E_g}{140 + 8 + j0.7 + j0.9}$$

or
$$\frac{E_L}{E_g} = \frac{-2800 + j0.7}{148 + j1.6} \quad \ldots(i)$$

Now applying the theorem to node G, we get

$$E_g = \frac{e_s Y_g + E_L Y_{gp}}{Y_g + Y_{gp} + Y_{gk}} = \frac{e_s 20 + E_L j0.7}{20 + j0.7 + j0.8} \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$E_L = \frac{-2800 + j0.7}{148 + j1.6} \times \frac{e_s 20 + jE_L 0.7}{20 + j1.5}$$

$$= (-12.14 + j9.09)e_s = (15.49/143°8')\ e_s.$$

Oral Questions

1. What do you mean by active and passive elements ?
2. What will be the current if the output terminals to the voltage source are short-circuited ?
3. What are the differences between the network, loop and mesh ?
4. What do you mean by Γ and Π-networks. How are they also called star and delta networks ?
5. What are the uses of star and delta connections ?
6. What is the nodal method for solving complicated networks ?
7. What do you mean by self and mutual resistances in network problems?
8. Whether the reciprocity theorem is correct for ac generators in circuits ?
9. What are the main differences between Thevenin and Norton's equivalent circuits.
10. If the power lost in the internal generator is maximum what will be the power deliver to the receiving generator ?

Problems

1. Calculate the currents in the battery and galvanometer arms, of a Wheatstone's bridge network having values $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$ $R_3 = 30\ \Omega$ $R_4 = 40\ \Omega$, and $R_g = 50\ \Omega$ with the delta-star transformation method. The d.c. source is a battery of emf 2 volts and internal resistance 1 Ω.

(91.2 mA, 2.5 mA)

2. Calculate the resistance between the points A and B of a triangular pyramid $ABCD$, built up of six wires, whose resistances are as follows : $AB = AC = DC = DB = 1\ \Omega$ and $CB = AD = 2\ \Omega$. *(7/12 Ω)*

3. Two batteries ($E_1 = 2.05$ V, $r_1 = 0.05$; $E_2 = 2.15$ V, $r_2 = 0.04\Omega$, are connected in parallel and the combined network is again in parallel with a of 1 Ω. Calculate the current in first battery branch by the principle of resistance superposition. (0.18 amp)

4 Determine the currents in the various meshes of a ladder network (Fig 13.14), by the method of determinants. Given that $E = 30$ V, $R_1 = R_2 = R_3 = 4\ \Omega$ and $R_4 = R_5 = 5\ \Omega$ and $R_L = 10\ \Omega$. **(4.52, 1.54, 0.38 amp)**

5. Calculate the currents in the branches AB ($=4\ \Omega$). BC ($=10\Omega$) and BE ($=6\ \Omega$) by using the superposition theorem, in a network in which the positive terminal of a battery of 20 volts and internal resistance of $2\ \Omega$ is connected through the resistances AB and BC with the positive terminal of the another battery ($E=1$ V and $r=2\ \Omega$). The negative terminals of these batteries are joined with the end E of the resistance BE.

(1.50, 0.33, 1.83 amp)

6. Calculate the current in branch BC for the network shown in Fig 13.16, if $e_1=100\ \underline{/0°}$, $Z_1=1+j3$. $e_2=50\ \underline{/30°}$, $Z_2=1+j5$, in BC arm $R=1\ \Omega$. $C=4\ \Omega$, in CD arm $R=20\ \Omega$, $C=8\ \Omega$ and in CF arm $R=2\ \Omega$ and $L=4\ \Omega$.

(15.77—j8.92)

7. How the Thevenin's theorem can be used to study the charging of a capacitor when a voltage divider arrangement is used and the capacitor itself is used as a load. Determine the time constant and the voltage across the capacitor ($C=1000\ \mu$f) at the end of the period. Also calculate the currents in R_1 and R_2 at this stage. Given that $E=25$ volts, $R_1=10\Omega$ and $R_2=40\ \text{k}\Omega$. The capacitor is charged across the resistance R_2.

(8 sec., 12.6 V, 1.24mA, 0.315 mA)

8. Calculate the open circuit voltage and Thevenin resistance of a simple two terminal network. In it two resistors 20 and 30 Ω and a source of 100 volts are connected in series and the terminals are the ends of resistance 30 Ω.

(60V, 12 Ω)

9. For the network shown in Fig 13.21, calculate the current through load impedance Z_L using Norton's theorem if $e_1=100\underline{/0°}$ instead of $100\ \underline{/30°}$.

(4.71/—45)

10. A Π-network is composed of resistances, each of $100\ \Omega$. It has a 10 volt source at one pair of terminals and a varaible resistance R_L at the other pair, Find the value of R_L coresponding to the maximum transfer of power and the magnitude of the power.

(50 Ω, *125 mW)*

14

Units and Their Absolute Measurements

14.1. SYSTEM OF UNITS

Some thousands of years ago when man wandered from place to place in groups, living on wild grains he gathered and the animals he hunted, there was no need for any system of measurement for trading with his follows. The beginning of civilisation came and the systems of measurement developed. With the advance of science during the 17th and 18th century scientists began more and more to rail against the confusion presented by multiplicities of systems of measurement. The units generally discussed at school level are *C.G.S.*, *F.P.S.* and *M.K.S.*

Unlike the other branches of Physics, the study of units in electromagnetism is confusing on account of various systems of electrical units used at various times. As the quantities used in electricity and magnetism can not be defined in terms of the fundamental units of mass, length and time and the fourth fundamental unit is required. In the early works, this fourth unit was introduced in two different ways. Consequently two different systems of electrical units : (*i*) electrostatic system of units (*e.s.u.*) and (*ii*) electromagnetic system of units (*e.m.u.*) were developed. The former was based upon theoretical considerations while the later on the experimental facts. The permittivity of the medium was assumed as the fourth fundamental unit in the former system of units and the permeability in the later. Let us derive other electrical quantities in terms of fundamental units in these two cases.

14.2. ELECTROSTATIC SYSTEM OF UNITS

The force between two charges q_1 and q_2 placed at a distance r apart is given by the relation

$$F = q_1 q_2 / 4\pi\varepsilon r^2, \qquad \ldots (1)$$

where ε is the permittivity or dielectric constant of the medium in which charges are situated. In the electrostatic system of units ε is taken as the fourth dimension. ε is usually written as ε_0 for empty space. The dimensions of charge can be obtained as

(*i*) $[q]\,[q]=[F]\,[\epsilon_0]\,[L^2]=[MLT^{-2}]\,[\epsilon_0 L^2]$

or $[q]=[\varepsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-1}]$.

The dimensions of the other electrical units can be found by using appropriate relations as :

(*ii*) Work done W=charge $q\times$potential difference V.

$$\therefore\quad [V]=\frac{[W]}{[q]}=\frac{[ML^2T^{-2}]}{[\epsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-1}]}=[\epsilon_0{}^{-1/2}M^{1/2}L^{1/2}T^{-1}].$$

The same will be the dimensions for *electromotive force*.

(*iii*) Electric intensity E=Force experienced by a unit charge $=F/q$.

$$\therefore\quad [E]=\frac{[F]}{[q]}=\frac{[MLT^{-2}]}{[\varepsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-1}]}=[\varepsilon_0{}^{-1/2}M^{1/2}L^{-1/2}T^{-1}].$$

(*iv*) Electric induction or displacement $D=\varepsilon_0 E$.

$\therefore\quad [D]=[\varepsilon_0]\,[E]=[\epsilon_0{}^{1/2}M^{1/2}L^{-1/2}T^{-1}]$.

(*v*) Electric flux Φ=Electric field$\times$Area

$[\Phi]=[\varepsilon_0{}^{-1/2}M^{1/2}L^{-1/2}T^{-1}]\,[L^2]=[\varepsilon_0{}^{-1/2}M^{1/2}L^{3/2}T^{-1}]$.

(*vi*) Electric current i=Rate of flow of charge$=q/t$

$\therefore\quad [i]=[q]/[T]=[\varepsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-2}]$.

(*vii*) Resistance R=Emf/current$=V/i$.

$$\therefore\quad [R]=\frac{[V]}{[i]}=\frac{[\varepsilon_0{}^{-1/2}M^{1/2}L^{1/2}T^{-1}]}{[\varepsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-2}]}=[\varepsilon_0{}^{-1}L^{-1}T].$$

(*viii*) Capacitance C=charge q/potential difference V

$$\therefore\quad [C]=\frac{[q]}{[V]}=\frac{[\epsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-1}]}{[\varepsilon_0{}^{-1/2}M^{1/2}L^{1/2}T^{-1}]}=[\varepsilon_0 L].$$

(*ix*) Inductance L=Emf V/Rate of change of current (di/dt) :

$$\therefore\quad [L]=\frac{[V]}{[i]\,[T^{-1}]}=\frac{[\varepsilon_0{}^{-1/2}M^{1/2}L^{1/2}T^{-1}]}{[\varepsilon_0{}^{1/2}M^{1/2}L^{3/2}T^{-2}]\,[T^{-1}]}=[\varepsilon_0{}^{-1}L^{-1}T^2].$$

(*x*) Magnetic field intensity $H=i\,dl\sin\theta/r^2$.

$\therefore\quad [H]=[i]\,[L]/[L]^2=[\epsilon_0{}^{1/2}M^{1/2}L^{1/2}T^{-2}]$

(*xi*) Magnetic Induction B=Induced emf$/lv$

$\therefore\quad [B]=[\epsilon_0{}^{-1/2}M^{1/2}L^{1/2}T^{-1}]/[L]\,[LT^{-1}]=[\varepsilon_0{}^{-1/2}M^{1/2}L^{-3/2}]$.

(*xii*) Magnetic moment M=couple$/B\sin\theta$

$\therefore\quad [M]=[ML^2T^{-2}]/[\epsilon_0{}^{-1/2}M^{1/2}L^{-3/2}]=[\varepsilon_0{}^{1/2}M^{1/2}L^{7/2}T^{-2}]$.

(*xiii*) Magnetic pole strength m=Magnetic moment/length of tbe magnet

$$[m]=[M]/[L]=[\varepsilon_0{}^{1/2}M^{1/2}L^{5/2}T^{-2}].$$

14.3. ELECTROMAGNETIC SYSTEM OF UNITS

The force between two magnetic poles m_1 and m_2, situated at a distance x apart is given by the relation

$$F=\mu m_1 m_2/4\pi r^2, \qquad \ldots(2)$$

where μ is the magnetic permeability of the medium. For the free space $\mu=\mu_0$. As neither μ nor m can be expressed in $[M]$, $[L]$ and $[T]$, hence to represent m, one can assume μ as a fourth dimension. Thus for free space.

(*i*) $[F]=[\mu_0]\,[m]\,[m]/[L^2]$

$\therefore\ [m]=[\mu_0^{-1}FL^2]^{1/2}=[\mu_0^{-1}MLT^{-2}L^2]^{1/2}=[\mu_0^{-1/2}M^{1/2}L^{3/2}T^{-1}]$.

The dimensions of other electric quantities can be obtained from fundamental formula as follows :

(*ii*) Magnetic induction $B=$Force experienced by a unit pole F/m.

$$\therefore\quad [B]=\frac{[F]}{[m]}=\frac{[MLT^{-2}]}{[\mu_0^{-1/2}M^{1/2}L^{3/2}T^{-1}]}=[\mu_0^{1/2}M^{1/2}L^{-1/2}T^{-1}].$$

(*iii*) Magnetic field intensity $H=B/\mu_0$

$\therefore\ [H]=[B]/[\mu_0]=[\mu_0^{-1/2}M^{1/2}L^{-1/2}T^{-1}]$.

(*iv*) Magnetic flux $\Phi=$Magnetic induction $B\times$Area A

$\therefore\ [\Phi]=[B]\,[A]=[\mu_0^{1/2}M^{1/2}L^{-1/3}T^{-1}]\,[L^2]=[\mu_0^{1/2}M^{1/2}L^{3/2}T^{-1}]$.

(*v*) Magnetic moment $M=$Pole strength $m\times$length of the magnet $2l$.

$$\therefore\quad [M]=[m]\,[l]=[\mu_0^{-1/2}M^{1/2}L^{3/2}T^{-1}]\,[L]=[\mu_0^{-1/2}M^{1/2}L^{5/2}T^{-1}].$$

(*vi*) Intensity of magnetisation $I=$Magnetic moment/volume.

$$[I]=[M]/[\tau]=[\mu_0^{-1/2}M^{1/2}L^{5/2}T^{-1}]/[L^3]=[\mu_0^{-1/2}M^{1/2}L^{-1/2}T^{-1}].$$

(*vii*) Electric current is related with magnetic field as

$$H=i\,dl\sin\theta/r^2.$$

$$\therefore\quad [i]=[H]\,[r^2]/[dl]=[\mu_0^{-1/2}M^{1/2}L^{-1/2}T^{-1}]\,[L^2]/[L]$$
$$=[\mu_0^{1/2}M^{1/2}L^{1/2}T^{-1}].$$

(*viii*) Electric charge $q=$current $i\times$time t.

$$\therefore\quad [q]=[i]\,[t]=[\mu_0^{-1/2}M^{1/2}L^{1/2}T^{-1}]\,[T]=[\mu_0^{-1/2}M^{1/2}L^{1/2}].$$

(*ix*) Electric field intensity $E=$Force experienced by a unit charge $=F/q$.

$$\therefore\quad [E]=\frac{[F]}{[q]}=\frac{[MLT^{-2}]}{[\mu_0^{-1/2}M^{1/2}L^{1/2}]}=[\mu_0^{1/2}M^{1/2}L^{1/2}T^{-2}].$$

(*x*) Potential difference $V=$Work done by a unit charge $=W/q$

$$\therefore\quad [V]=\frac{[W]}{[q]}=\frac{[ML^2T^{-2}]}{[\mu^{-1/2}M^{1/2}L^{1/2}]}==[\mu_0^{1/2}M^{1/2}L^{3/2}T^{-2}].$$

(*xi*) Resistance R = Potential difference V/Current i.

$$\therefore \quad [R] = \frac{[V]}{[i]} = \frac{[\mu_0^{1/2} M^{1/2} L^{3/2} T^{-2}]}{[\mu_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]} = [\mu_0 L T^{-1}]$$

(*xii*) Capacitance $C = q/V$.

$$[C] = [\mu_0^{-1/2} M^{1/2} L^{1/2}]/[\mu_0^{1/2} M^{1/2} L^{3/2} T^{-2}] = [\mu_0^{-1} L^{-1} T^2]$$

(*xiii*) Inductance L = Emf V/Rate of change of current di/dt.

$$\therefore \quad [L] = \frac{[V]}{[i]/[t]} = \frac{[\mu_0^{1/2} M^{1/2} L^{3/2} T^{-2}]}{[\mu_0^{-1/2} M^{1/2} L^{1/2} T^{-2}]} = [\mu_0 L].$$

14.4. COMPARISION BETWEEN THE ELECTROSTATIC AND ELECTROMAGNETIC SYSTEMS OF UNITS

On comparing the dimensions of any electric quantity in both the systems of units, it is seen that they differ. It is impossible to have different dimensions for the same electric quantity. Thus one is bound to consider dimensions of μ_0 and ϵ_0.

If we compare dimensions of charge q obtained in both the systems of units, then

$$\text{In } e.s.u.\ [\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-1}] = \text{In } e.m.u.\ [\mu_0^{-1/2} M^{1/2} L^{3/2}].$$

$$\therefore \quad \mu_0^{-1/2} \varepsilon_0^{-1/2} = [L T^{-1}], \qquad \ldots (3)$$

which are the dimensions of velocity. The magnitude of this velocity can be found by measuring a given charge first in *e.s.u.* and then in *e m.u.* The ratio of these values is found nearly equal to the velocity of light. Thus we see that

$$[q]_{emu}/[q]_{esu} = \text{velocity of light } c.$$

Similarly we can obtain other ratios as :

(*i*) $$\frac{C_{emu}}{C_{esu}} = \frac{[\mu_0^{-1} L^{-1} T]}{[\varepsilon_0 L]} = \mu_0^{-1} \epsilon_0^{-1} [L^{-2} T^2] = c^2.$$

(*ii*) $$\frac{i_{emu}}{i_{esu}} = \frac{[\mu_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-2}]} = \mu_0^{-1/2} \varepsilon_0^{-1/2} [L T^{-1}]^{-1} = c.$$

(*iii*) $$\frac{V_{emu}}{V_{esu}} = \frac{[\mu_0^{1/2} M^{1/2} L^{3/2} T^{-2}]}{[\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]} = \mu_0^{1/2} \epsilon_0^{1/2} [L T^{-1}] = c^{-1}.$$

(*iv*) $$\frac{R_{emu}}{R_{esu}} = \frac{[\mu_0 L T^{-1}]}{[\varepsilon_0^{-1} L^{-1} T]} = \mu_0 \varepsilon_0 [L T^{-1}]^2 = c^{-2}.$$

(*v*) $$\frac{L_{emu}}{L_{esu}} = \frac{[\mu_0 L]}{[\varepsilon_0^{-1} L^{-1} T^2]} = \mu_0 \varepsilon_0 [L T^{-1}]^2 = c^{-2}.$$

(*vi*) $$\frac{H_{emu}}{H_{esu}} = \frac{[\mu_0^{-1/2} M^{1/2} L^{-1/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{1/2} T^{-2}]} = \mu_0^{-1/2} \varepsilon_0^{-1/2} [L T^{-1}]^{-1} = c.$$

$$(vii)\qquad \frac{M_{emu}}{M_{esu}}=\frac{[\mu_0^{-1/2}M^{1/2}L^{5/2}T^{-1}]}{[\varepsilon_0^{1/2}M^{1/2}L^{7/2}T^{-2}]}=\mu_0^{-1/2}\varepsilon_0^{-1/2}\,[LT^{-1}]^{-1}=c.$$

$$(viii)\qquad \frac{E_{emu}}{E_{esu}}=\frac{[\mu_0^{1/2}M^{1/2}L^{1/2}T^{-2}]}{[\varepsilon_0^{-1/2}M^{1/2}L^{-1/2}T^{-1}]}=\mu_0^{1/2}\varepsilon_0^{1/2}\,[LT^{-1}]=c^{-1}.$$

Similarly we can compare the values in *emu* and *esu* of other electrical quantities.

14.5. PRACTICAL AND INTERNATIONAL UNITS

The electrostatic units of charge and current and the electromagnetic unit of potential are inconveniently small for practical purpose. The practical units are coulomb for charge, ampere for current and volt for potential. From these are derived the ohm for resistance, watt for power, farad for capacity and henery for inductance.

International conference on Electrical Standards (London, 1908) defined these practical units as given below and thus named as '*International units*'.

One international *ampere* is that current which will deposit 0.001118 gm. of silver in one sec. in the electrolysis of $AgNO_3$.

One international ohm is the resistance of a column of mercury of length 106.3 cm. at 0°C, of a uniform cross sectional area and 14.452 gm. in mass.

One international volt is defined as the potential difference across 1 international ohm carrying 1 international ampere. It was also defined in terms of *emf* of the Weston cell, assuming it to be 1.0183 volts.

Absolute measurements have now been made by different methods. Practical units are therefore based on the absolute system of units directly. The absolute units are related by powers of 10 to the electromagnetic units and differ slightly from the international units. Modern instruments are calibrated in absolute practical units. The relationship between the international units as practical units are

1 international ampere = 0.999835 amp.
1 international ohm = 1.000425 ohms.
1 international volt = 1.00033 volts.

14.6. MEASUREMENT OF ABSOLUTE QUANTITIES

1. Determination of resistance in absolute measure by Lorenz Method. Lorenz method is based on the principle of measuring the inductance and time as inductance has dimensions of length in *emu*, and the resistance has dimension $\mu_0 LT^{-1}$ in *emu*.

In this method, a solenoid of n turns per unit length is placed in east-west direction. A circular disc D is mounted concentrically such that its plane is perpendicular to the magnetic lines of forces. The disc is driven with uniform speed by means of an electric motor. Thus an *emf* is set up between the central axis (axle) and the circumference of the disc. To measure this *emf* two small metal brushes B_1

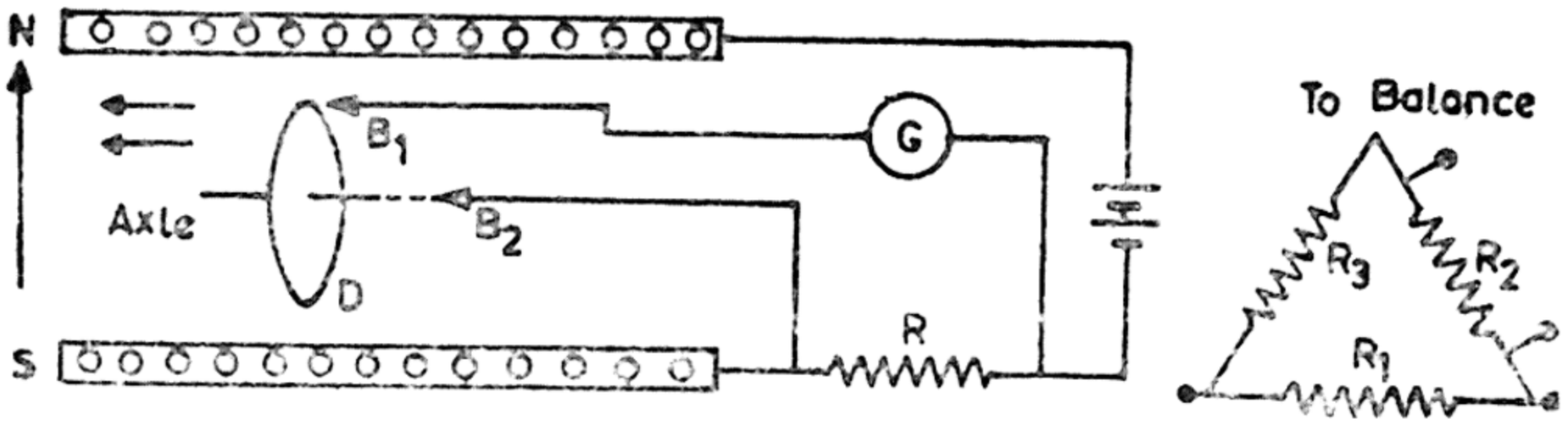


Fig. 14.1.

and B_2 are connected with the two terminals of the experimental resistance R. A battery and a galvanometer are connected as shown in Fig 14.1.

Let the disc of radius r be rotated through an angle θ, then the area swept out is $\frac{1}{2}\theta r^2$. If B is the flux density, then the flux cutting the disc perpendicularly is $\frac{1}{2}\theta r^2 B$.

If the disc is having m revolutions per sec, the time taken to turn through angle θ is $\theta/2\pi m$ sec.

$\therefore$ *emf* between the centre and circumference

$$= \frac{d\Phi}{dt} = \frac{\frac{1}{2}\theta r^2 B}{\theta/2\pi m} = \pi r^2 m B. \qquad \ldots (4)$$

This is in *emu*, if B is in gauss and r in centimeters.

If i *emu* current is passing through the solenoid, the magnetic flux

$$B = \mu_0 4\pi n i. \qquad \ldots (5)$$

The solenoid should be at right angles to the magnetic meridian to eliminate the effect of the earth's magnetic field. Hence

$$\text{Induced } emf\ \mathcal{E} = \pi r^2 m \mu_0 4\pi n i = 4\mu_0 \pi^2 r^2 n m i \ emu.$$

This *emf* is balanced against the p.d. across a resistor R which is in series with the solenoid. Hence for no deflection in galvanometer

$$\text{Induced } emf\ 4\mu_0 \pi^2 r^2 n m i = \text{p.d. across the resistance } iR$$

$$R = 4\mu_0 \pi^2 r^2 n m. \qquad \ldots (6)$$

This equation is dependent on n, r and m, which are measured in absolute units of length and time.

The method suffers errors due to the following causes : (*i*) the earth magnetic field, (*ii*) the significant value of the thermo-*emf*., (*iii*) the requirement of the measurement of disc radius whilst it is moving with a very high speed, and (*iv*) the difficulty in electrical connections to the centre and cirumference of the rotating disc.

Instead of a long solenoid a coil system is more useful. If M be the mutual inductance between the disc and the coil system, then

$$emf = \text{Rate of change of } Mi = Mim$$

$$= \text{p.d. across the resistance at the time of balance}$$

$$\therefore \quad R = mM, \qquad \ldots(7)$$

where M the mutual inductance is dependent on the geometrical dimensions. As the resistance in the coil circuit is small, hence Lord Rayleigh and Mrs Sidgwick used an arrangement of three resistors. R_2 was the smallest, whilst R_3 was large compared with R_1. The equivalent resistance was thus $R_1R_2/(R_1+R_2+R_3)$.

This method was much improved by F.E. Smith. After a series of experiments the result was stated in the form that the ohm was represented by a column of mercury at 0°C of 14.4446 ± 0.0006 gm in mass of 1 sq mm cross section area and of length 106.245 ± 0.004 cm.

2. Determination of current in Absolute Units by Rayleigh's Current balance method. If two current carrying coils are placed in such a way that they are parallel and their axes are coincident, a force will exist between the coils. This force is proportional to the product of the two currents in the coils and can be measured with a balance.

Lord Rayleigh was the first who could measure the current with a highest accuracy. The Rayleigh's balance consists of four stationary coils and two smaller coils which form part of the beam of a balance. The currents through the larger coils are arranged such that the magnetic lines of induction enter the smaller coils at the top and leave at the bottom. Thus the lines of induction are every where at right angles to the current and the force will act on the smaller coils (E and F) in vertical direction.

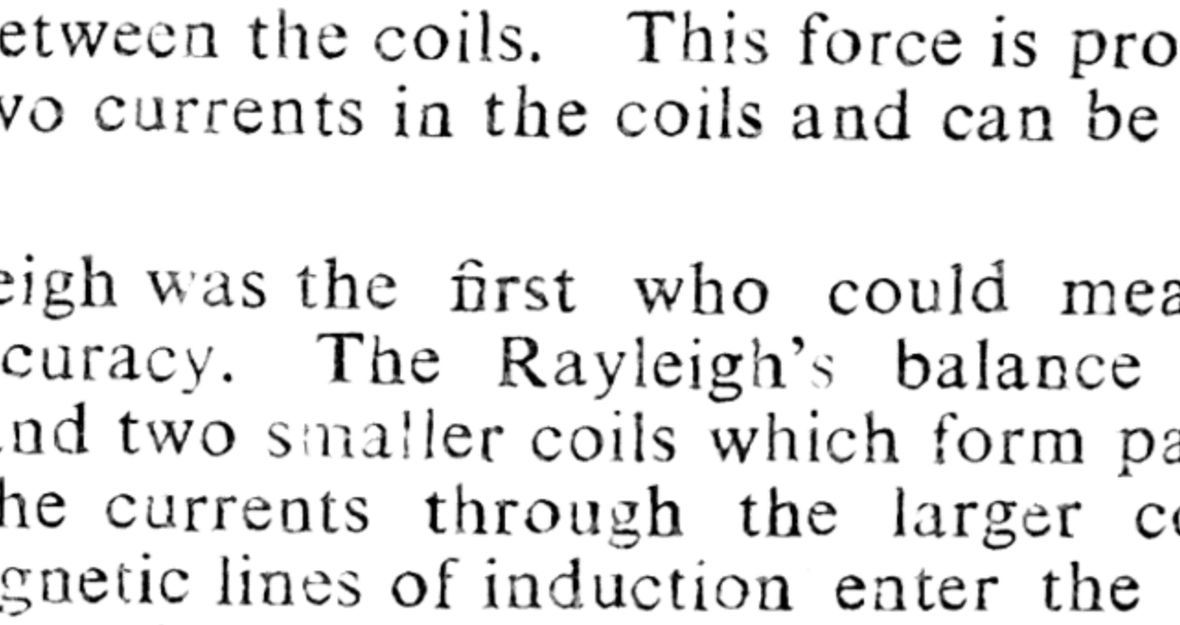


Fig. 14.2.

The balance beam is suspended at its mid point by two ribbons of very fine flexible copper wires, one on each side of the beam.

The machanical force on the small coils is measured by sliding a rider of known weight on the beam. The weight can be calibrated as the square of the current.

The theory of the force exerted is best presented after that of Ayrton Mather and Smith. If a current i is passing through the coils, then

The force in dynes on the small coil $=i\times$ number of truns per cm of small coil $n\times$ number of lines of induction entering the coil, *i.e.*,

$$F = ni[\pi r_2^2(dB/dx)\times \text{length of the small coil}]$$

$$=i\pi r_2^2\times(dB/dx)\times \text{totai number of the turns on smal coil } n_2,$$

where r_2 is the radius of the smaller coil and dB/dx is the average rate of change of the field due to one of the big stationary coils along the line joining the centres.

We know that the magnetic induction B on the axis of a coil of n_1 turns, radius r_1, at a distance x from its centre is

$$B=2\pi n_1 i r_1^2/(r_1^2+x^2)^{3/2}.$$

$$\therefore \quad dB/dx=2\pi n_1 i r_1^2(-3x)/(r_1^2+x^2)^{5/2}.$$

We know that dB/dx is maximum when $x=\frac{1}{2}r_1$. Hence at this position

$$dB/dx=96\pi n_1 i/25\sqrt{5}r_1^2.$$

Hence the total force on the small coil is

$$F=\pi r_2^2 n_2 i\times 96\pi n_1 i/25\sqrt{5}r_1^2=96\times 5^{-5/2}\pi^2 n_1 n_2 i^2(r_2^2/r_1^2) \quad ...(8)$$

The value of this force will remain almost independent of the small variations in the vertical position of the small coil. This force also depends on direct counts of the number of turns and on the ratio of the radii of the two coils. In this way current can be measured in terms of force, *i.e.*, the weight placed on the beam and the displacement from the point of suspension. With the improved Rayleigh balance it is possible to compare the currents to an accuracy of two in a million.

14.7. RATIONALISED MKS SYSTEM OF UNITS

In 1901 Giorgi, pursuing an idea originally due to Maxwell, considered that a great simplification of the units in electricity could be brought about by adopting meter and kilogram as the units of length and mass. He pointed out that if a unit of electrical measurement was added as a fourth basic unit, a system could be constructed which would be satisfactory for uniform application in most sciences and technologies. The International Electrotechnical Commission in 1950 recommended ampere as the fourth unit. The important features of this system are as follows :

1. The dyne is so small that 981 dynes act on our hand when we hold one gramme. A unit of force which makes one kilogram to accelerate by one meter/sec^2 is therefore much more reasonable in size. This new unit is called '*newton*'. Similarly unit of work done, joule, is somewhat practicable.

2. The outstanding advantage of this system lies in the sphere of electrical units, as it ends the use of *emu* and *esu*. All experiments, whether purely electric, purely magnetic, or a combination use the same units.

3. In most cases the quantities used in the *mks* system are identical with the practical quantities of electricity and magnetism, such as amperes, volts, ohms and farads which are generally used as practical units since a long time.

4. The introduction of a factor 4π into the fundamental equations converts the system into rationalized *mks* system. It makes simpler form to the frequently used derived equations.

The equations expressing Coulomb's law in electricity and magnetism are

$$F = q_1 q_2 / 4\pi\epsilon_0 r^2$$

and

$$F = \mu_0 m_1 m_2 / 4\pi r^2.$$

The ampere's law of force between two current elements is

$$F = \mu_0 i_1 i_2 \, dl_1 dl_2 / 4\pi r^2. \qquad \ldots(9)$$

These equations show that the introduction of a factor 4π constitutes the process of '*rationalization*'. It is obvious that we can not just put the factor 4π into the constants ϵ_0, μ_0, as we also have relations without any factor 4π, such as

$$\mathbf{D} = \epsilon_0 \mathbf{E} \text{ and } \mathbf{B} = \mu_0 \mathbf{H}. \qquad \ldots(10)$$

International system of Units (SI). In 1960, International general conference of weights and measures recommended that if the MKSA system were enlarged by the addition of basic units *candela* for light measurement and kelvin for temperature, then this would be acceptable for all branches of science. This new system was named as International system of units, adbreviated as SI. Instead of six basic units (meter, kilogramme, second, ampere, candela, kelvin), two supplementary units are also used

radian (rad) for plane angle, steradian (sr) for solid angle.

SI is rational and coherent. It is rational in that it admits only one measurement unit for any one physical quantity. It is coherent in that every unit used in science can be derived from six basic units and two supplementary units.

Dimensions of Electrical Units in the MKSA system

Charge : $-q$ = current × time = ampere sec = coulomb

$$[q] = [I] \times [T] = [IT].$$

Potential Difference : —

V = work done per unit charge = joule/coul = volt.

$$[V] = \frac{[W]}{[q]} = \frac{[ML^2T^{-2}]}{[IT]} = [I^{-1}ML^2T^{-3}].$$

Electric field strength :— $E = -dV/dx$ = volt/meter.

$$[E] = [V]/[L] = [I^{-1} M L T^{-3}]$$

Electric diplacement :— $\int \mathbf{D}.\, d\mathbf{S} = q$ or D = coulomb/meter^2.

$$[D] = [q]/[L^2] = [IL^{-2}T].$$

Permittivity :— $\varepsilon_0 = \dfrac{D}{E} = \dfrac{\text{coulomb/meter}^2}{\text{volt/meter}} = \dfrac{\text{coulomb}}{\text{volt.meter}}$.

$$[\epsilon_0] = \frac{[D]}{[E]} = \frac{[IL^{-2}T]}{[I^{-1}MLT^{-3}]} = [I^2 M^{-1} L^{-3} T^4].$$

Resistance :— R = Potential/current = volt/ampere = ohm.

$$[R] = [V]/[I] = [I^{-2} M L^2 T^{-3}].$$

Capacitance :— C = Charge/Potential = coulomb/volt = farad.

$$[C] = \frac{[q]}{[V]} = \frac{[IT]}{[I^{-1} M L^2 T^{-3}]} = [I^2 M^{-1} L^{-2} T^4].$$

Inductance : $-L$ = — Induced emf/(di/dt) = volt sec/amp = henry.

$$[L] = [V]/[IT^{-1}] = [I^{-2} M L^2 T^{-2}].$$

Magnetic field strength :— $\int \mathbf{H}.\mathbf{dl} = i$ or H = ampere turn/meter.

$$[H] = [I]/[L] = [IL^{-1}].$$

Magnetic flux :—Induced emf $\mathcal{E}$ = — Rate of change of flux $(d\Phi/dt)$.

$\therefore$ Φ = volt. sec per turn = weber.

$$[\Phi] = [V]\,[T] = [I^{-1} M L^2 T^{-2}].$$

Magnetic Induction B = magnetic flux/area = weber/meter^2.

$$[B] = [\Phi]/[L^2] = [I^{-1} M T^{-2}].$$

Permeability $\mu_0 = \dfrac{B}{H} = \dfrac{\text{volt sec/meter}^2}{\text{ampere/meter}} = \dfrac{\text{volt. sec}}{\text{amp. meter}} = \dfrac{\text{henry}}{\text{meter}}$.

$$[\mu_0] = \frac{[B]}{[H]} = \frac{[I^{-1} M T^{-2}]}{[IL^{-1}]} = [I^{-2} M L T^{-2}].$$

$$[\varepsilon_0 \mu_0]^{1/2} = [I^2 M^{-1} L^{-3} T^4\ I^{-2} M L T^{-2}]^{1/2} = [L^{-2} T^2]^{1/2} = [L T^{-1}]^{-1}$$

TABLE 14.1

Electromagnetic Quantities and their units

Physical concept	*S.I. units*	*Gassian units*	*S.I./Gaussian*
Charge q	Coulomb	Statcoulomb	3×10^9
Current i	Ampere	Statampere	3×10^9
Electric field intensity E	Newton/coulomb =volt/meter	Dyne/statcoulomb =Stat volt/cm.	$1/3\times10^4$
Potential V, emf $\mathcal{E}$	Volt=joule/coul	Statvolt =erg/stat coul.	1/300
Capacitance C	Farad=coul/volt	Statfarad, cm	9×10^{11}
Electric dipole moment p	Meter-coulomb	Cm-stat coul	3×10^{11}
Electric polarization P	Coul./m^2	Stat coul./cm.^{2}	3×10^5
Permittivity ε, ε_0	Coul^2/newton m^2 =farad/meter	Not used	
Displacement D	Coul./m^2.	Dyne/stat coul.	$12\pi\times10^5$
Resistance R	Ohm=volt/amp	Stat ohm=stat volt/stat amp.	$1/9\times10^{11}$
Resistivity ρ	Meter-ohm	Stat ohm-cm	$1/9\times10^9$
Magnetic induction B	Weber/m^2=Tesla =newton/amp. m	Gauss=μ_0(oersted)	10^4
Permeability μ, μ_0	Weber/m. amp =henry/m. =newton/amp^2.	Gauss/oersted	$10^7/4\pi$
Inductance L	Henry =web./amp.	Erg/stat amp^2	9×10^{11}
Magnetic dipole moment p_m	Amp-meter^2= Weber-meter	Gauss cm^2=erg/ Oersted=erg/gauss	10^3
Magnetic intensity H	Amp/meter	Oersted	$4\pi\times10^{-3}$
Pole strength g_m	Amp. meter	Gauss cm^2	$(10^8/4\pi)$
Conductance Y	Mho=amp/volt.	Stat mho	9×10^{11}
Magnetic flux Φ	Weber	Maxwell, gauss-cm^2	10^8
Magnetic vector potential A	Weber/m	Gauss cm	10^6
Conductivity	Mho/meter	Stat mho/cm	9×10^{-9}

TABLE 14.2

Equations of electromagnetism for SI and Gaussian units—

Equations Content	*Equation for SI Units*	*Equation for Gaussian Units*
Coulomb's law	$F=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}\hat{r}$	$F=\frac{q_1q_2}{r^2}\hat{r}$
Gauss's law	$\oint \mathbf{E}.\,d\mathbf{S}=q/\varepsilon_0$ or $\nabla\times\mathbf{E}=\rho/\varepsilon_0$	$\oint \mathbf{E}.\,d\mathbf{S}=4\pi q$ $\nabla\times\mathbf{E}=4\pi\rho$
Electric field near a conductor	$E=\sigma/\varepsilon_0$	$E=4\pi\sigma$
Potential of point charge	$V=q/4\pi\epsilon_0 r$	$V=q/r$
Magnetic force on a moving charge	$\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$	$\mathbf{F}=\frac{q}{c}(\mathbf{v}\times\mathbf{B})$
Magnetic force on a current element	$d\mathbf{F}=i(d\mathbf{l}\times\mathbf{B})$	$d\mathbf{F}=\frac{i}{c}(d\mathbf{l}\times\mathbf{B})$
Force per unit length on parallel currrent	$\frac{dF}{dl}=\frac{\mu_0}{2\pi}\frac{i_1i_2}{d}$	$\frac{dF}{dl}=\frac{2i_1i_2}{c^2d}$
Magnetic field created by a moving charge	$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{q}{r^3}(\mathbf{v}\times\mathbf{r})$	$\mathbf{B}=\frac{q}{cr^3}(\mathbf{v}\times\mathbf{r})$
Magnetic field created by a current element	$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{i}{r^3}(d\mathbf{l}\times\mathbf{r})$	$d\mathbf{B}=\frac{i}{cr^3}(d\mathbf{l}\times\mathbf{r})$
Magnetic field of long wire	$B=\mu_0 i/2\pi r$	$B=2i/cr$
Magnetic field within a long solenoid	$B=\mu_0 ni$	$B=4\pi ni/c$
Faraday's law of electro-magnetic induction	$\int \mathbf{E}\cdot d\mathbf{l}=\frac{d\Phi}{dt}$ or $\nabla\times\mathbf{E}=\frac{\partial\mathbf{B}}{\partial t}$	$\int \mathbf{E}\cdot d\mathbf{l}=\frac{1}{c}\frac{d\Phi}{dt}$ $\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$
Energy density of electro-magnetic force	$u=\frac{1}{2}\varepsilon_0E^2+\frac{B^2}{2\mu_0}$	$u=\frac{1}{8\pi}(E^2+B^2)$
Magnetic moment of circling particle	$\mathbf{m}=\frac{q}{2m}\mathbf{L}$	$\mathbf{m}=\frac{q}{2mc}\mathbf{L}$
Magnetic moment of current loop	$m=iA$	$m=iA/c.$

14.8. FUNDAMENTAL CONSTANTS

In the following table the unified atomic mass scale ($^{12}C \equiv 12$) has been used and the values are given for SI units with the following abbreviations.

C=coulomb
J=joule
K=degree kelvin
$k(g)$=kilogram
m=meter
mol=molecules
N=newton
s=second
T=tesla
u=atomic mass unit

Constant	Symbol	Value in SI units
Avogadro's number	N_A	6.02205×10^{23} mol^{-1}
Bohr magneton	$\mu_B = e\hbar/2m_e$	9.27408×10^{-24} JT^{-1}
Boltzmann constant	k	1.38066×10^{-23} JK^{-1}
Electron rest mass	m_e	5.48586×10^{-4} u
Elementary charge	e	1.60219×10^{-19} C
Fine structure constant	$1/\alpha = 4\pi\varepsilon_0\hbar c/e^2$	137.03604
First Bohr Orbit	$a_0 = 4\pi\varepsilon_0\hbar^2/m_e e^2$	5.29177×10^{-11} m
Mass unit (unified mass scale)	u	1.66057×10^{-27} kg
Neutron rest mass	m_n	1.00867 u
Nuclear magneton	μ_N	5.05082×10^{-27} JT^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ NA^{-2}
Permittivity of free space	ε_0	8.85419×10^{-12} $C^2 N^{-1} m^{-2}$
Planck constant	h	6.62618×10^{-34} Js
Proton rest mass	m_p	1.00728 u
Rydberg constant	R_∞	1.09737×10^7 m^{-1}
Speed of light in vacuum	c	2.99792×10^8 ms^{-1}

Energy Convesion Factors

1 joule $= 10^7$ ergs $= 6.70043 \times 10^9$ $u = 6.24146 \times 10^{18}$ eV
1 kg $= 6.02205 \times 10^{26}$ $u = 5.60955 \times 10^{35}$ eV
1 °$K = 9.25103 \times 10^{-14}$ $u = 8.61735 \times 10^{-5}$ eV
1 cal $= 2.803460 \times 10^{10}$ $u = 2.61143 \times 10^{19}$ eV

Part II

1

Thermionic Tubes

1.1. ELECTRON EMISSION

A metal consists of atoms, due to mutual forces, these atoms are held in a regular pattern called *lattice*. Each atom consists of a nucleus surrounded by a number of electrons which move in various orbits. The electrons in outer most orbit can move freely from one atom to another in all possible directions within entire region of metal. The free electrons can not leave the metal surface since, as soon as, any electron leaves the surface, it will develop one positive charge on the surface which will attract the electron. This attracting or restraining force acts like *surface barrier of electrons*. So electrons can not leave the surface unless their kinetic energy is very large, *i e.*, sufficient to overcome the

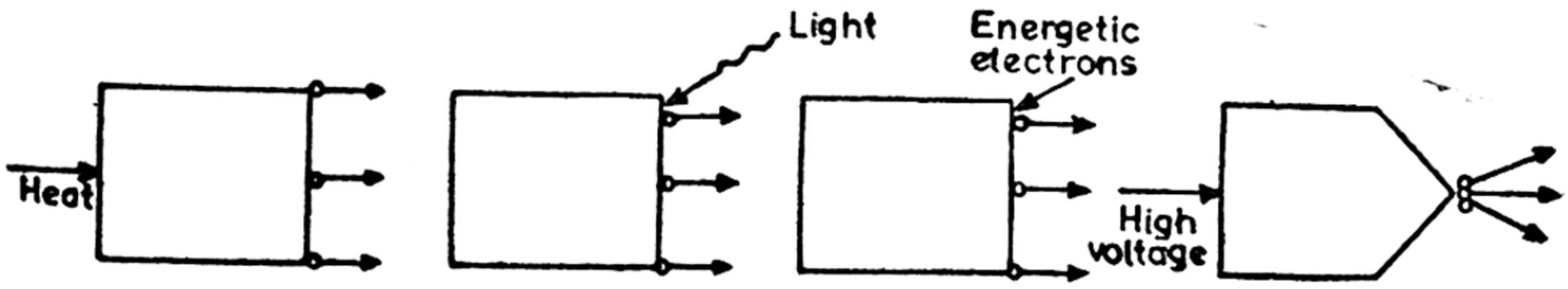


Fig. 1.1.

barrier. This additional energy necessary for emission of free electrons can be supplied in following ways :—

(*i*) *Thermionic emission.* When a metal is heated, due to heat energy, the emission of electrons takes place. The phenomenon is called *thermionic emission* [*Fig. 1.1*(*a*)].

(*ii*) *Photo-electric emission.* When a metal is exposed to e.m. radiations, *e.g.* light radiations or photons, the emission of electrons takes place. This phenomenon is called *photo-electric emission* [*Fig. 1.1.* (*b*)].

(*iii*) *Secondary emission.* When highly energetic electrons collide with the surface of metal, the electrons are emitted from the metal. This is called *secondary emission* [Fig. 1.1 (*c*)].

(*iv*) *Field emission.* When a high electric field ($\sim 10^8$ volts/meter) is established at the surface, the electrons are emitted.

This phenomenon of emission of electrons is called *field emission* [Fig. 1.1 (*d*)].

Out of the above processes, thermionic emission process is most frequently used. We will discuss here first two processes only.

1.2. WORK FUNCTION

The emission of electrons into the surrounding space is analogous to the evaporation of liquid. In both the cases the pressure of the surrounding electron (or gas) cloud is a constant at equilibrium positions and depending on the temperature and the metal (or liquid). Work is to be done in both the cases. In the latter case this work is the latent heat of vaporization which is required against the mutual cohesive forces (surface tension forces, etc). The forces and work in the former case can be explained as under : *when the electron leaves the surface, a positive charge is induced on the surface. As the metal surface must be an equipotential, the method of electrical images dictates that the field distribution will remain unchanged if the surface charge is assumed to be concentrated at the image.* Thus for an electron ($-e$) at a distance x from the surface, the force

$$f=(+e)(-e)/4\pi\epsilon_0\ (2x)^2=-e^2/16\pi\epsilon_0 x^2. \qquad ...(1)$$

$-$ve sign represents that the electron is attracted toward the surface of the metal. It is to be noted that the surface is assumed plane, which is possible if the electron is at a distance greater than few atomic diameters.

If the electron is removed to an infinite distance, work done

$$U=-\int_{\infty}^{x} f(x)\ dx = \frac{e^2}{16\pi\epsilon_0 x}. \qquad ...(2)$$

This work done by the electron, or the energy required for the electron to pass into space is called the *surface barrier energy*. If it is in electron volts and represented by E_b, then

$$U=eE_b. \qquad ...(3)$$

The distribution of energy for free electrons inside the solid is given by *Fermi-Dirac statistics*. According to this theory even at $0°K$, there are sufficient number of electrons which have considerable energy up to the maximum value E_f (in electron volts), called *Fermi energy*. But for just emission of electrons the minimum energy required is surface binding energy (E_b) which is more than Fermi energy. Hence the additional energy (even at $0°K$) required to emit electrons from the metal surface will be the difference between the two energies, *i.e.* (E_b-E_f). This *minimum energy required to emit electrons at absolute zero temperature is known as work-function or threshold energy* (E_w) of the metal.

$$\therefore \qquad E_w=E_b-E_f. \qquad ...(4)$$

The work function for a metal will depend on the nature of the metal, condition of surface, impurities, etc. When the electron of charge e is accelerated through a potential difference of ϕ volts such that

$$e\phi=(E_b-E_f)=E_w, \qquad \ldots(5)$$

the ϕ is known as the *voltage equivalent of the work function*. In this expression E_w is in the units of electron volts.

Thus those electrons whose energy is more than the work function are emitted. This phenomenon is known as *thermionics* and electrons emitted are called as therm-electrons or *thermions*.

1.3. PHOTOELECTRIC EMISSION

The energy associated with a quantum of radiation, $h\nu$, may be transferred to the electron of a metal. If $h\nu > E_w$, the work function of the metal (measured in eV), the electron may be ejected from a metallic surface with a maximum kinetic energy

$$K.E.=h\nu-E_w. \qquad \ldots(6)$$

This equation is known as *photoelectric equation* and the effect as *photoelectric effect*. The threshold value of frequency beyond which photoelectric process can take place is given by

$$\nu_c=E_w/h \qquad \ldots(7)$$

and the corresponding wavelength beyond which this effect does not take place is given by

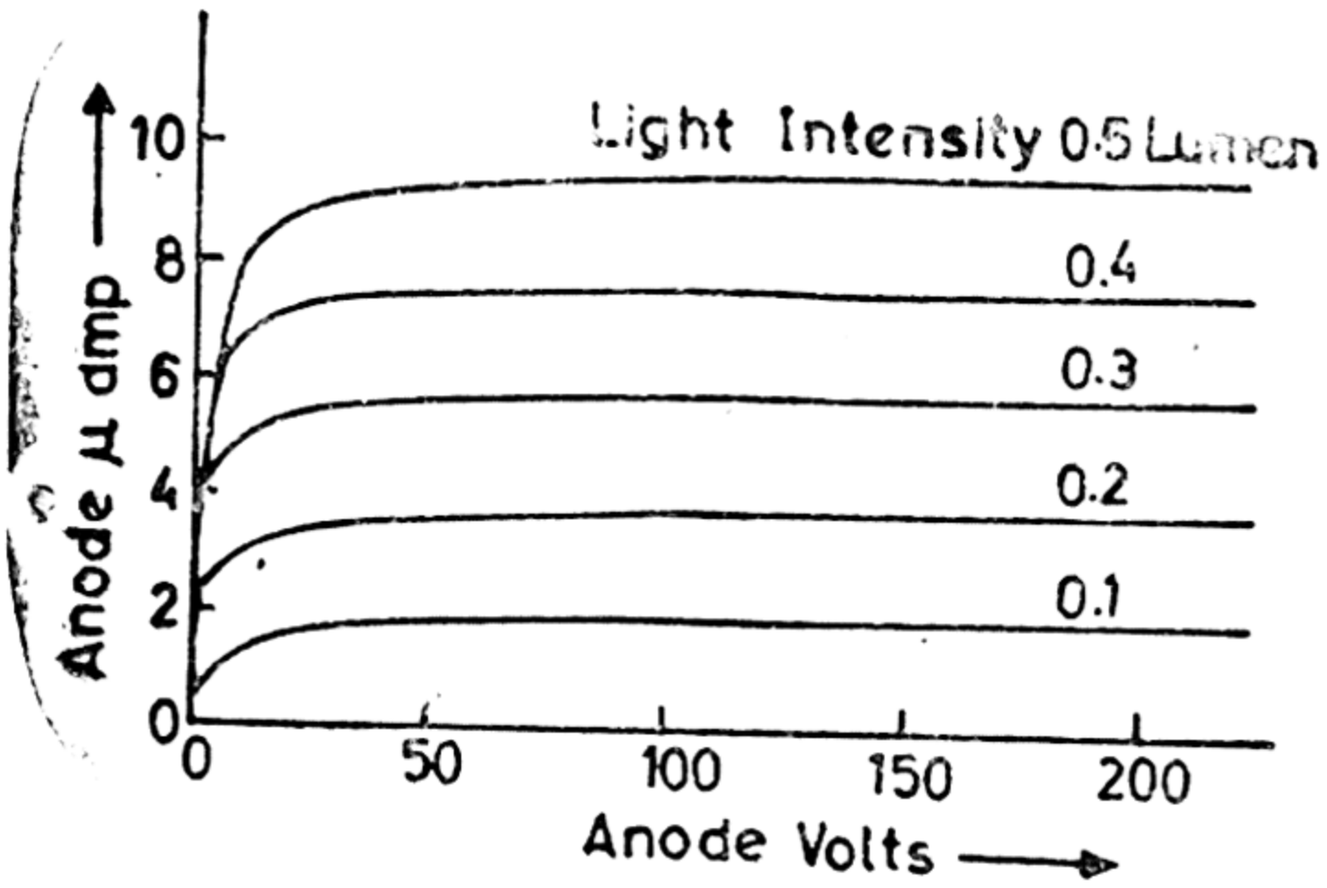
$$\lambda_c(\text{cut-off wavelength})=ch/E_w. \qquad \ldots(8)$$

Substitution of numerical values shows that λ_c lies in the visible region only for the alkali metals and barium and strontium, which have low values of E_w and in the ultraviolet region for other metals.

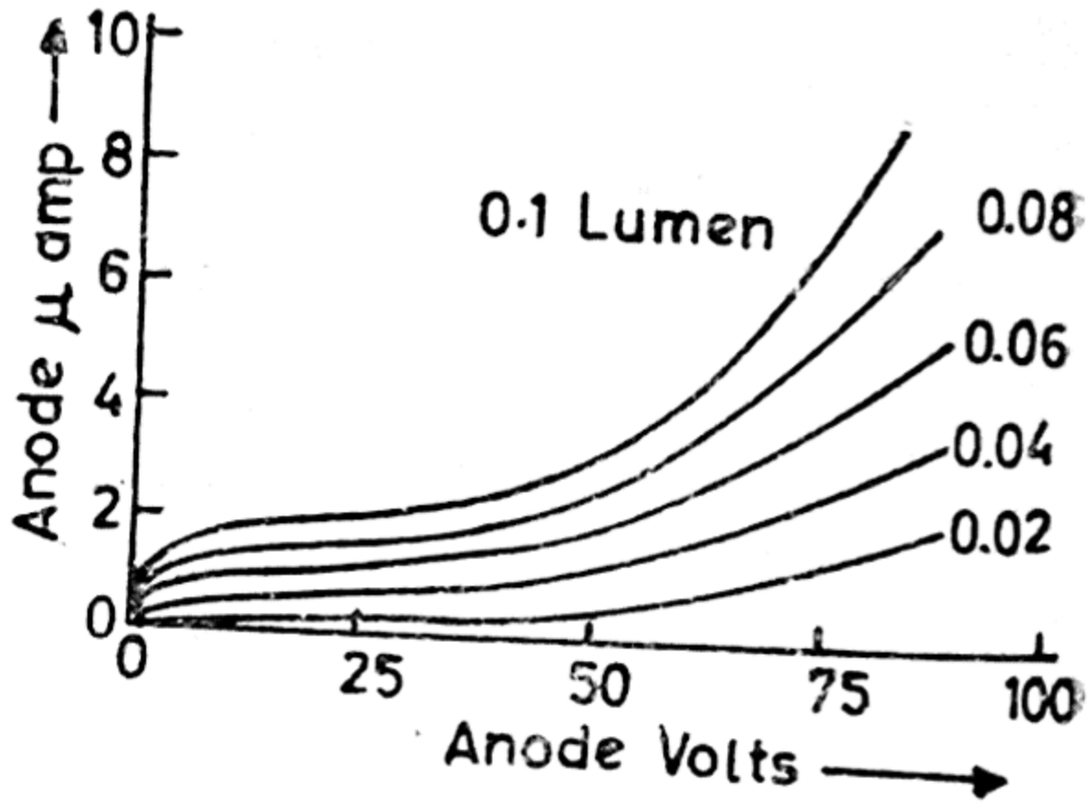
To achieve a *photo-cathode* sensitive to light in the visible region of the spectrum, composite materials are used. Caesium on silver oxide ($Ag-O-Cs$) has peak sensitivity in the U.V. and a second peak in the infrared with a threshold wavelength 12000A°. Antimony-caesium alloy ($Sb-Cs$) has peak sensitivity to blue light but smaller red sensitivity. The combination of bismuth, silver, oxygen and caesium ($Bi-Ag-O-Cs$) has high sensitivity in the centre of the visible region of the spectrum.

A *photo-cell* consists of the prepared photo-cathode on a semi-cylindrical metal plate. This cathode when illuminated releases electrons which are collected by an anode in the form of a straight wire or a wire frame. It is so mounted as to not cast a shadow on the cathode. The whole is assembled in a cylindrical glass bulb, that either is evacuated or consists of inert gas at low pressure.

Vacuum photo-cell. With light of appropriate wavelength striking the cathode, a small voltage causes a small current. As the voltage is increased, the current rises rapidly and reaches to the saturation value. This saturated value of current increases as we increase the light intensity. The volt-ampere curves with light intensity (*in lumens*) as a parameter are shown in Fig. 1.2 (*a*). This figure shows that the current reaches near saturation values for very low values of applied potential.



(*a*)



(*b*)

Fig. 1.2.

Gas filled photo-cell. The presence of an inert gas increases the current output. The emitted electrons, in moving to the anode, collide with gas atoms and produce ionization. The electrons freed from the atoms and the secondary electrons emitted by the positive ions from the cathode surface increase the original anode current to four to seven times. The characteristic curves are of the same shape as for the vacuum cell up to the ionization potential of the gas. The current for the gas tube rises above ionization potential of the gas and the gas-filled cell is seen to have sensitivity exceeding that of the vacuum cell. The main disadvantage of the gas-filled cell is the nonlinear variation of current output with light intensity, Fig. 1.2 (*b*). This is not serious for small variations in light intensity.

1.4. RICHARDSON-DUSHMAN EQUATION

Richardson gave a quantitative treatment for the thermionic emission from the metallic emitters. He assumed the emitted electrons from the metal surface in dynamical equilibrium with the electrons in the metal and therefore used Clapeyron's latent heat equation. The relation so obtained was

$$\text{Current density } j = A_0 T^{1/2} e^{-\phi/kT}, \qquad \ldots(9)$$

where $A_0 = Ae$, a constant and ϕ = work function of the metal.

To have the values of A_0 and ϕ in agreement with the experimental values, Richardson later suggested $\phi = \phi_0 + \frac{3}{2} kT$ and found

$$j = A_0 T^2 e^{-\phi_0/kT}, \qquad \ldots(10)$$

where ϕ_0 = work function of the metal at $0°K$.

The above relation was also derived by Dushman, using quantum mechanical process, hence named as *Richardson-Dushman equation*.

The process of thermionic emission was best explained by using Fermi-Dirac statistics. In this case the number of electrons lying between momentum space p and $p+dp$ is given as

$$N = \frac{8\pi p^2\, dp}{h^3[e^{(E_a - E_f)/kT} + 1]}, \qquad \ldots(11)$$

where E_f is the Fermi energy and E_a is the surface energy and the difference

$$E_a - E_f = e\phi = e \times \textit{work function} = E_w. \qquad \ldots(12)$$

Here E_w is the *work function* in electron volts.

Since $p = mv$, $dp = mdv$, hence Eq. (11) can be written as

$$N = \frac{2(m/h)^3\, dvx\, dv_y\, dv_z}{e^{(E_a - E_f)/kT} + 1},$$

where vx, v_y and v_z be the velocity components of the electron along the x, y and z-directions respectively ($\because\ 4\pi v^2\, dv = dvx\, dv_y\, dv_z$).

Since $(E_a - E_f) \gg kT$, hence we can write

$$N = 2(m/h)^3 \left[e^{(E_a - E_f)/kT}\right]^{-1} dvx\, dv_y\, dvz$$

$$= 2(m/h)^3 e^{E_f/kT}\, e^{-\frac{1}{2}m(v^2x + v^2_y + v^2_z)}\, dvx\, dv_y\, dv_z$$

$$= 2(m/h)^3 e^{E_f/kT}\, e^{-\frac{1}{2}mv^2x}\, e^{-\frac{1}{2}mv^2_y}\, e^{-\frac{1}{2}mv^2z}\, dvx\, dv_y\, dv_z$$

The number of electrons arriving per unit area in $y-z$ plane is given as

$$N' = Nv_x = 2(m/h)^3 e^{Ef/kT} \int_{\sqrt{2E_a/m}}^{\infty} vx\, e^{-\frac{1}{2}mv^2x/kT}\, dv_x$$

$$\times \int_{-\infty}^{\infty} e^{-\frac{1}{2}mv^2y/kT}\, dv_y \int_{-\infty}^{\infty} e^{-\frac{1}{2}mv^2z/kT}\, dv_z$$

$$= 4\pi mk^2\, h^{-3}\, T^2 e^{-(Ea - Ef)/kT}.$$

Therefore the saturation emission current density per unit area

$$jx = eNv_x = 4\pi\, mek^2\, h^{-3}\, T^2\, e^{-(Ea - Ef)/kT}$$

$$= AT^2\, e^{-e\phi/kT}. \qquad \ldots(13)$$

This equation is known as Richardson-Dushman equation. Here

$A = 4\pi mek^2/h^3$ = constant for a particular material.

The constant A and $b_0(=e\phi/k)$ can be calculated by putting the above equation as

$$\log_e jx/T^2=\log_e A-e\phi/kT=\log_e A-b_o/T.$$

The graph between $\log_e jx/T^2$ and $1/T$ is a straight line. The slope gives b_0 and the intersept on y-axis ($\log_e jx/T^2$) gives the value of $\log_e A$. Therefore A and b_0 can be calculated for different emitters. The values of certain metals are given below in Table 1.1.

Table 1.1. Values of the emission constants for certain metals :

Material	A_0 ($Amp/m^2\ °K^2$)	$b(°K)$	E_w (eV)	*Melting point °K*
Carbon	60.2×10^4	46500	4.00	3773
Caesium	16.2×10^4	21000	1.81	302
Molybdenum	60.2×10^4	51500	4.44	2893
Nickel	26.8×10^4	32100	2.77	1725
Platinum	60.2×10^4	59000	5.08	2028
Thorium	60.2×10^4	38900	3.35	2118
Tungsten	60.2×10^4	52400	4.52	3643
Oxide coated	0.01×10^4	11600	1.00	2200

Schottky effect and field emission. The electron emission at very high fields is almost independent of temperature. It depends on the external electric field E and the work function ϕ. This field alters the effective surface barrier (which is due to the electrostatic force between the outgoing electron and the image charge on the emitter). There is a reduction in the barrier height and the thickness of the barrier. The reduction in height of the barrier is called *Schottky effect*. It is proportional to square root of electric field. It increases the thermionic emission at high fields in vacuum tubes. The reduction in thickness of barrier leads to the possibility of quantum mechanical tunnelling.

1.5. MATERIALS FOR THERMIONIC EMISSION

The Richardson's equation suggests that for emission of electrons a metal with high melting temperature and low work function should be used, so as not to be melted at operating temperatures but should give copious emission of electrons. Following types of materials are used as electron emitter :—

(*i*) **Tungsten.** The pure metal has high melting point (3643°*K*) and high work function (4.52 *eV*). It can be operated at tempera-

tures between 2500 and 2800°*K* to provide adequate emission of electrons. At 2500°*K* it has a life of 2000 hrs. It can be successfully used as the filament in high potential plates (even more than 10,000 volts). It has a poor *emission efficiency* (the ratio of emission current to the power consumed by heating), nearly 80 $A/m^2 \times$ watt at the temperature 2600°*K*. *It is used in high potential rectifier tubes, large power amplifier tubes and in X-ray tubes, etc.*

(*ii*) **Thoriated Tungsten Emitters.** In the year 1914 Langmuir and Roger developed an excellent electron emitter, thoriated tungsten with work function 2.6 *eV*. It operates at temperature 1900°*K*. It has high electron emission efficiency of the order of 50 to 1000 A/m^2 watt.

Before drawing tungsten into a wire a small quantity (1 or 2 percent) of thorium oxide (ThO_2 or thoria) is added to the tungsten powder. During the evacuation process of the tube, the oxide is converted into pure thorium and a portion of it diffuses through the wire to form a mono-atomic layer of thorium on the surface. An intense electric field is developed between absorbed thorium atom and base tungsten which reduces the work function, both in comparison to tungsten (4.52 *eV*) and thorium (3.35 *eV*). In order to reduce the evaporation of thorium layer from the filament it is carbonized by heating at temperature about 1600°*K* in the presence of some suitable hydrocarbon (*e.g., nephthalene*). The applied potential for these filaments should be less than 10,000 volts. These are used for large tubes transmitting power. These filaments are also susceptible to damage and decrease efficiency due to the bombardment of gas ions. *These filaments are used under high vacuum conditions.*

(*iii*) **Oxide Coated Emitters.** In the year 1903, Wehnelt developed oxide coated emitters with quite low work function 1.0 *eV*. But these came in frequent practical use after the year 1922. The oxide coated emitter operates at the relatively low temperature of 1000°*K*. It has very high electron emission efficiency of the order of 100 to 10000 A/m^2 watts. The filament has the base of nickel or konel (alloy of nickel, cobalt and titanium) or some other alloy, coated with the oxide of barium and strontium. During the evacuation process some of the oxide is converted to pure metal and this forms an absorbed layer which is electropositive to the base and reduces the work function. These are most common filaments used in the tubes. *These are used in small tubes, also in power tubes with sufficiently high anode voltages* (up to 1000 volts). These are also susceptible to oxygen poisoning and damage due to gas ion bombardment.

Directly and Indirectly Heated Emitters. The thermionic emitters are of two types in their practical form.

(1) *Directly heated or filament type.* The directly heated filaments are taken in the form of a *V*, *W* or a straight wire (Fig. 1.3).

The electron emission takes place directly from these wires when heated by allowing a current through them. Tungsten and thoriated tungsten emitters are used for this purpose. These are used for very high voltage systems. These suffer with following disadvantages : (*a*) When the wire is connected to a dc source, the potential difference is distributed along the wire, so various points of the wire (emitter) will be at different potentials. (*b*) When it is connected to an ac source, it will result in an alternating electrostatic field near emitter, which will cause noise in the output.

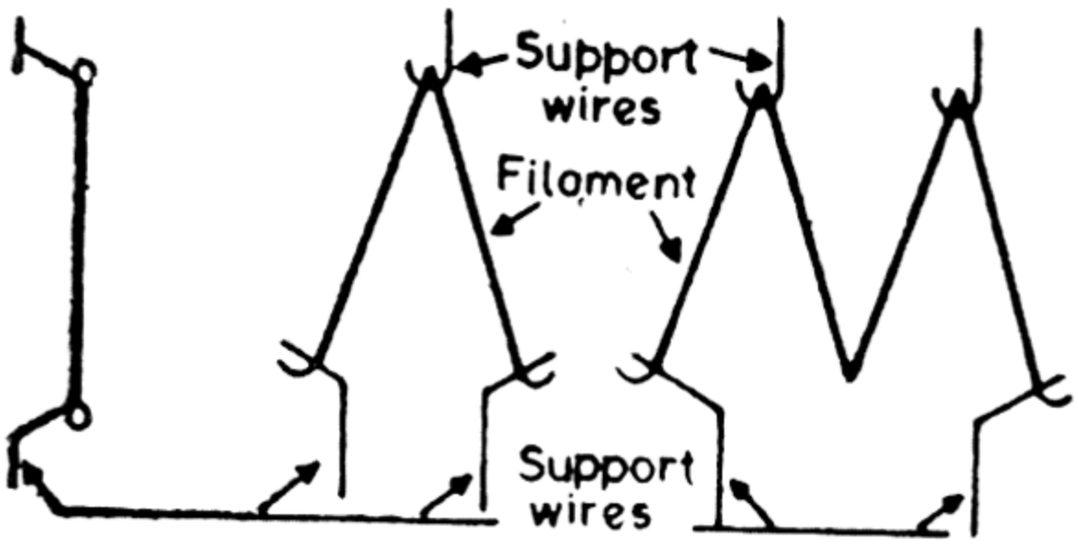


Fig. 1.3.

These emitters were used in early stages. The disadvantages are avoided in the indirectly heated emitters.

(2) *Indirectly heated type.* These emitters are commonly used due to their high emission efficiency. These consist of cylindrical sleeve of nickel, coated by oxides of barium or strontium on the external surface. These are heated by an insulated heater coil of tungsten (Fig. 1.4). The current is passed in heater coil and not through the cylinder, so all its parts remain at the same potential. It has number of advantages : (*a*) *It is more stable than directly heated,* (*b*) *It has a very high emission efficiency due to the use of oxide-coated emitting surface,* (*c*) *The emitter surface is at same potential and* (*d*) *It avoids the objectional hum, noise, etc.*

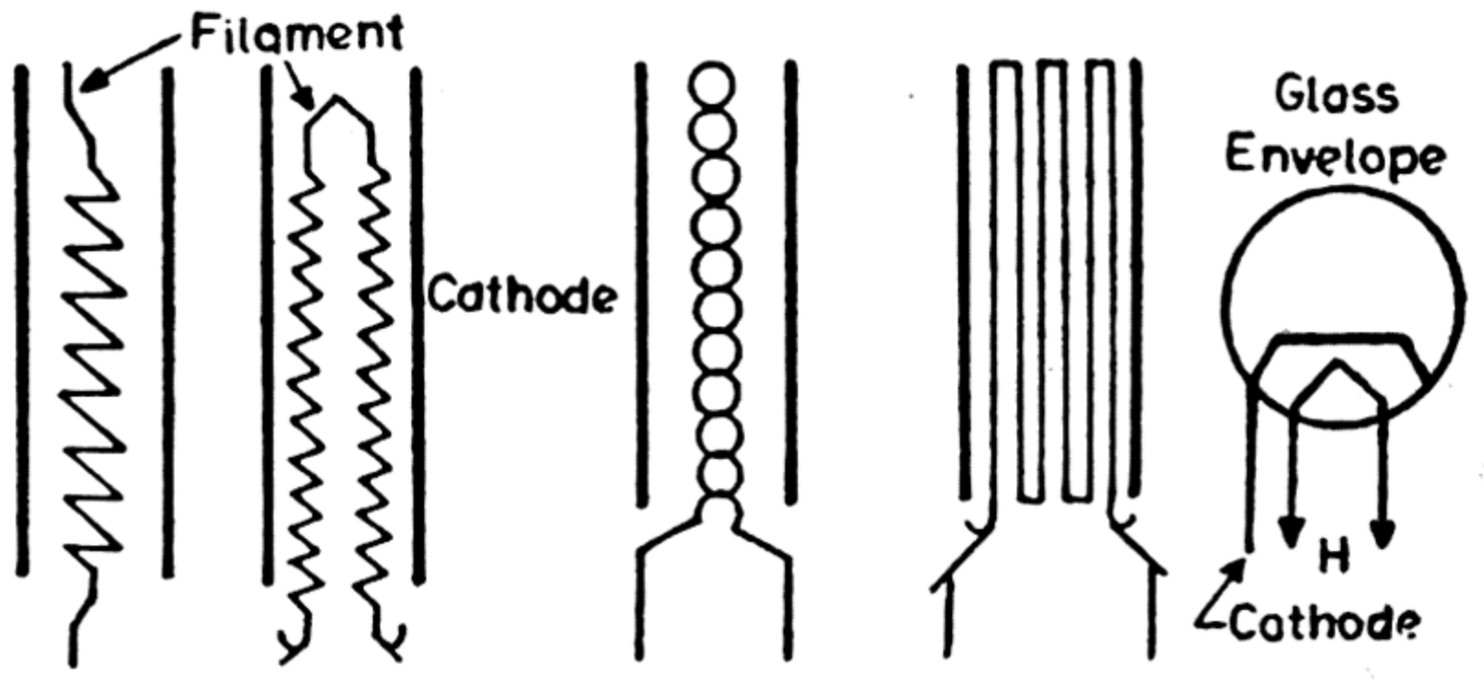


Fig. 1.4.

1.6. VACUUM DEVICES

Thomas Alwa Edison while experimenting with incandescent lamps introduced a positive electrode in the lamp and found a current flow through the circuit. The ages of electronics began with this discovery in 1883. The vacuum tubes were used as a rectifier and detector of electromagnetic waves of early radio. The introduction of control grid to the vacuum tube enabled amplification of signals. The vacuum tubes have been improved for various purposes. The developments of solid state devices

reduced the application of vacuum devices. However, the vacuum devices are still used in certain areas like high power and microwave applications and cathode ray tube applications.

1.7. DIODE VALVE

A diode valve consists of two electrodes in an evacuated bulb of glass or metal. The two electrodes are cathode (K) and anode or plate (P). The cathode may be either directly heated or indirectly heated. In case of indirectly heated cathode, the filament/heater (H) is heated by a low tension source either of dc or ac, which heats the cathode but does not emit electrons. A hollow, coaxial metallic cylinder serves the purpose of anode or plate. The plate attracts the electrons emitted by the cathode since it is kept at positive potential. When the plate and cathode are connected by some external circuit, then in the circuit, the electron will flow from plate to cathode and the conventional current will be from cathode to plate. If the plate is connected with the —ve potential, the electrons will be forced back to the cathode and no current will flow in the circuit. Thus we see that current is *only from plate to cathode inside the diode. On account of this unidirectional flow of current in diode, it is called diode valve.*

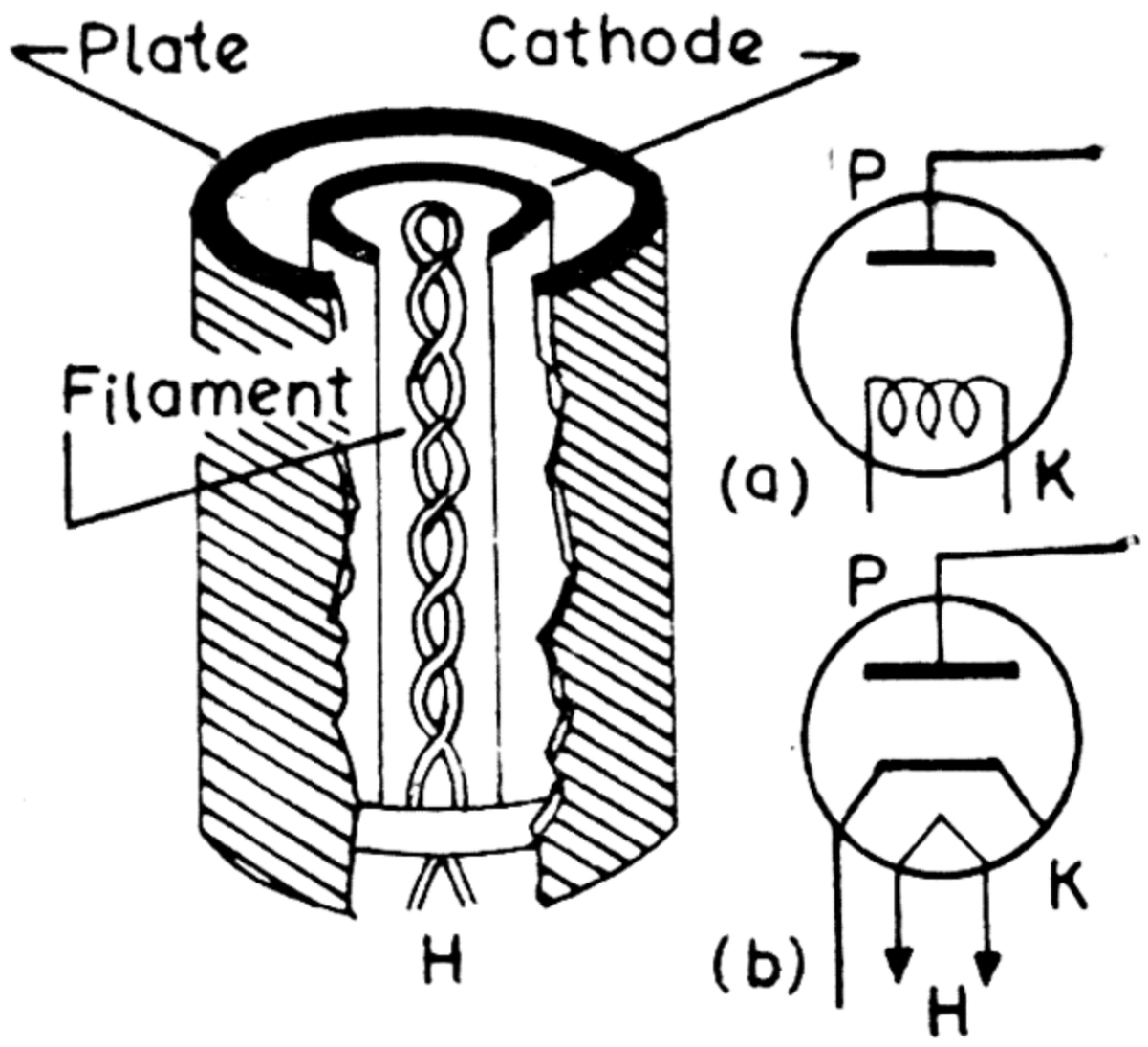


Fig. 1.5.

Since the electrons falling on the plate are having high kinetic energy, hence the plate should be of a metal having high work function, otherwise a secondary emission from the plate will take place. Generally the plate is of nickel or some alloy.

Diode Valve Characteristics. The circuit to study the variation of plate current (I_p) with plate voltage (V_p) is shown in Fig. 1.6. The necessary plate voltage is given from *dc* source of high tension supply (H.T.), which is adjustable with the help of a potential divider. V is a voltmeter to measure the potential difference (V_p) between the plate P and cathode K and mA is milliam meter to measure plate current

Fig. 1.6.

(I_p). To heat cathode at a particular temperature (T_1), terminals of heater H are connected to a low tension (L.T.) source. The filament current (I_f) is adjusted with a rheostat and is measured with an another milliammeter. *The study of change in plate current due to change in plate voltage for a particular value of filament current and therefore at a fixed temperature of cathode gives the characteristic curves of diode valve, which are shown is Fig. 1.7.*

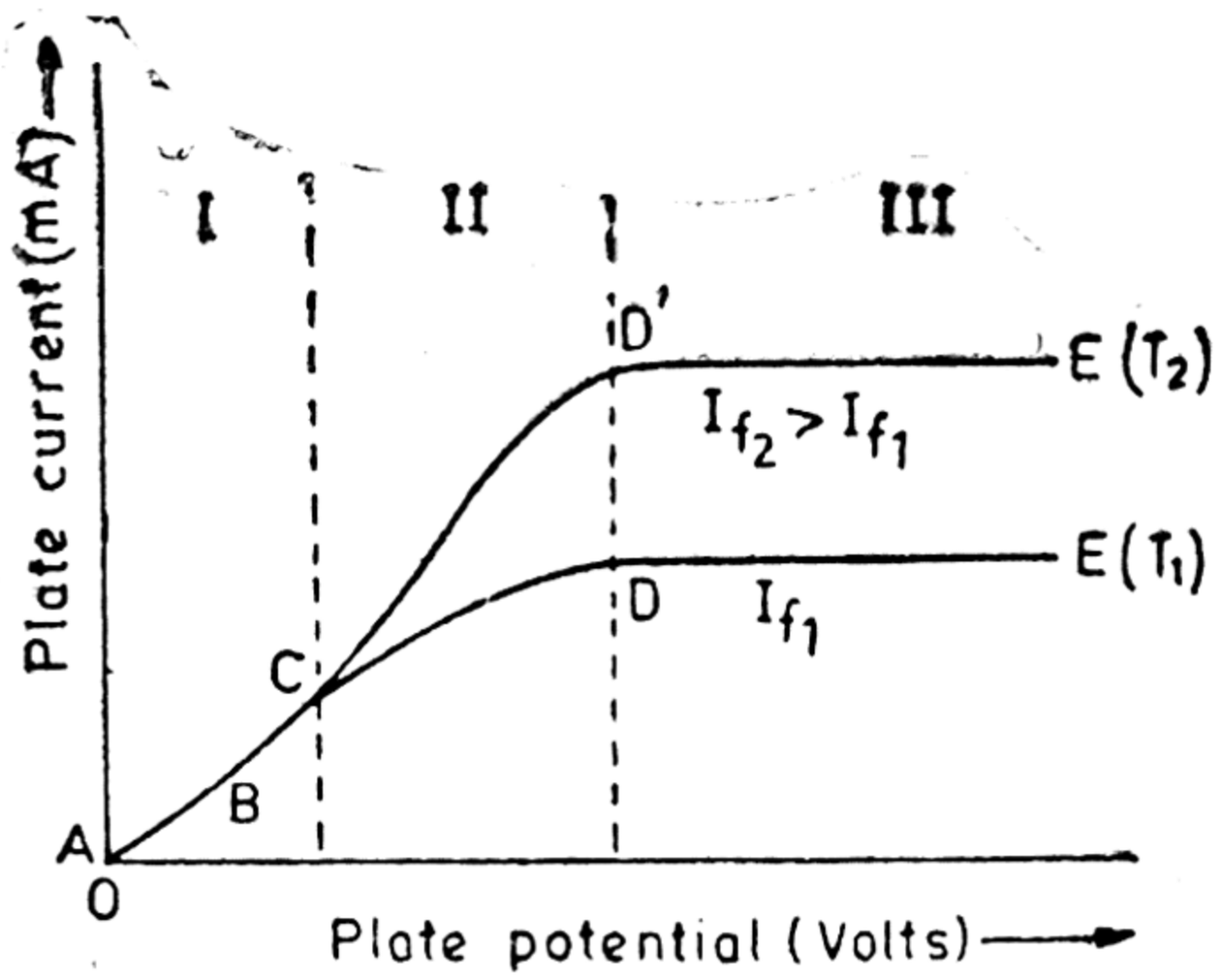


Fig. 1.7.

At any temperature of cathode, Richardson equation gives the number of electrons emitted from cathode which are not affected by the plate potential. But the number of electrons attracted by plate depends upon the potential difference V_p. If the positive potential of the plate is very low so as it can not attract the electrons emitted from cathode, the electrons will be collected round the cathode and will form a negatively charged electron cloud. The accumulation of electrons between plate and cathode is called *space charge*. This negative cloud will repel the incoming electrons from cathode. A situation will reach when the repelling force due to space charge becomes sufficient to stop the further emission of electrons. When the plate potential is low up to C, the space charge repells the electron coming out of the cathode. Thus a very small current flows through the anode since few of the fast moving electrons (high kinetic energy) can come out of the space charge.

With the increase of the positive plate potential the more electrons will be attracted and collected by the plate. The plate current increases up to D and the space charge reduces (*also the repelling electric field on cathode surface reduces*). On further increase in the plate potential equilibrium can be attained such as all the electrons emitted by cathode are being attracted by the plate. This is called as *saturation current*. If the plate potential is increased beyond this value the plate current does not increase further, as shown by the curve DE. In this condition there exists no space-charge. Thus the characteristic curve can be divided into three regions : (1) *space charge limited current region I*, up to C ; (2) *transition region II*, from C to D and (3) *saturation current limited region* or temperature limited current region, from D to E.

If now filament current is changed (suppose increased) the temperature of cathode will rise (say $T_2 > T_1$). According to Richard-

son-Dushman equation the electron emission increases with the temperature. So similar curve $BCD'E'$ will be obtained giving more saturation plate current. There is no change in region I. Eq. (29) also shows that in space charge limited region the current is independent of cathode temperature and the work function. It varies as the three halves power of the plate potential.

The characteristic curves in Fig. 1.8 show that the variation of plate current is not linear with plate potential, *i.e., diode does not obey Ohm's law.* It is called as non-linear resistive element.

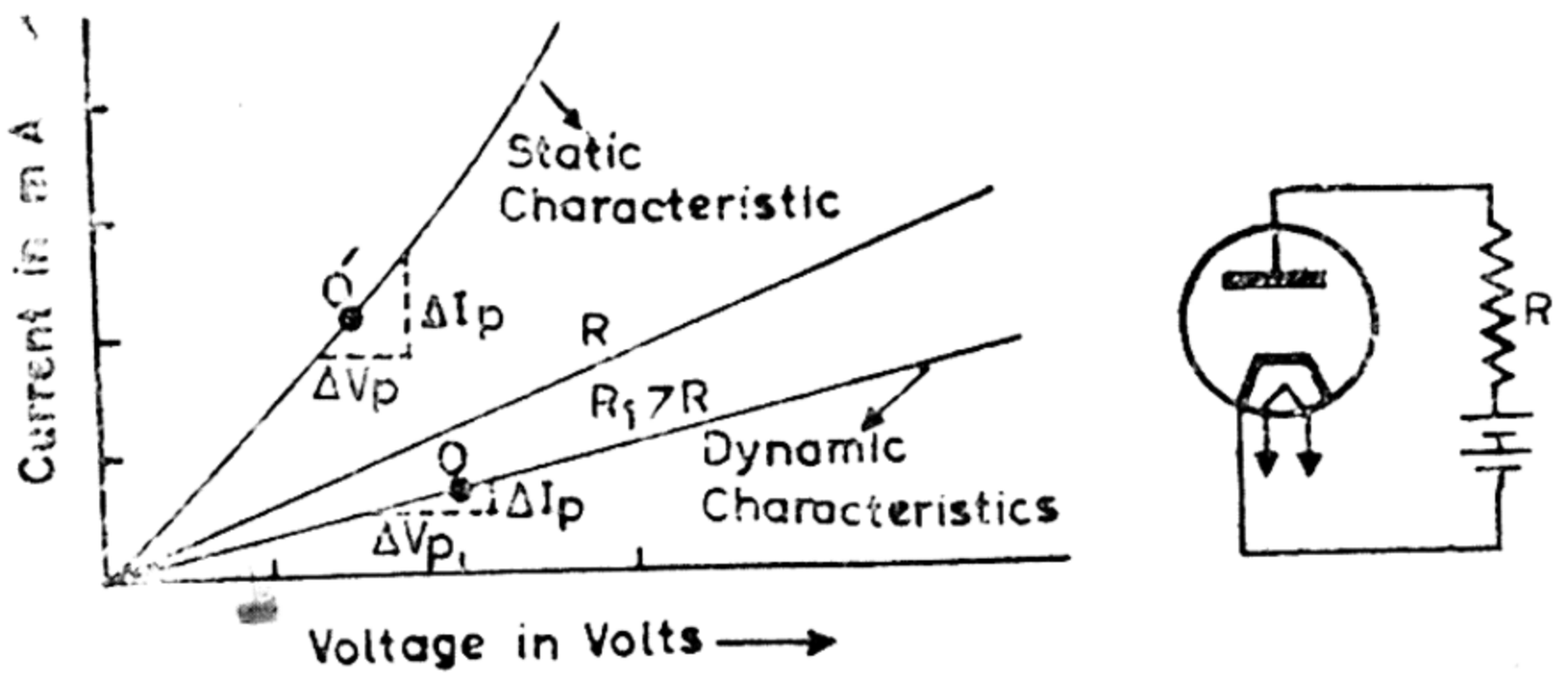


Fig. 1.8.

The characteristic curves will be different when these are drawn with a resistance in the circuit. The characteristic curves without any external resistance in the circuit are known as *static characteristic* curves and with some resistance are known as dynamic *characteristic curves* (Fig. 1.8). The dynamic characteristic curves are more straight, *i.e.*, their variation of plate current with plate potential is more linear.

The internal resistance of diode at a point, also known as *anode resistance*, is the ratio of change in plate potential to the corresponding change in plate current at that point. This will be different for a point on static and dynamic characteristics. We can calculate by evaluating the slope at a point and is expressed as

$$R_p = \triangle V_p / \triangle I_p, \qquad \ldots(20)$$

the *static internal resistance* at O' and

$$r_p = \triangle V_p / \triangle I_p, \qquad \ldots(21)$$

the *dynamic internal resistance* at O.

Since the characteristic curve is not in the form of a straight line at all points, so this ratio is not constant. The variation is more in the case of static internal resistance than dynamic internal resistance. When $\triangle V_p$ is taken in volts and $\triangle I_p$ in amperes, the internal resistance will be in ohms.

1.8. CHILD-LANGMUIR OR THREE HALVES POWER LAW

Let us consider a simple diode consisting of a plane parallel cathode and anode surfaces lying in the $x-z$ plane. A pd is applied

between the plates. To find the diode current and to make the mathematics simple, let us assume that—

(*i*) *The spacing between the electrodes is small so that the field can be assumed uniform and normal to the surfaces of the electrodes.*

(*ii*) *The electrodes are equipotential, i.e., there is no potential variation in $x-z$ plane.*

(*iii*) *The calhode is a source of infinite electrons supply and is at the plane $y=0$.*

(*iv*) *The electrons are emitted with zero 'initial' velocity under space charge limited region olong the y-direction.*

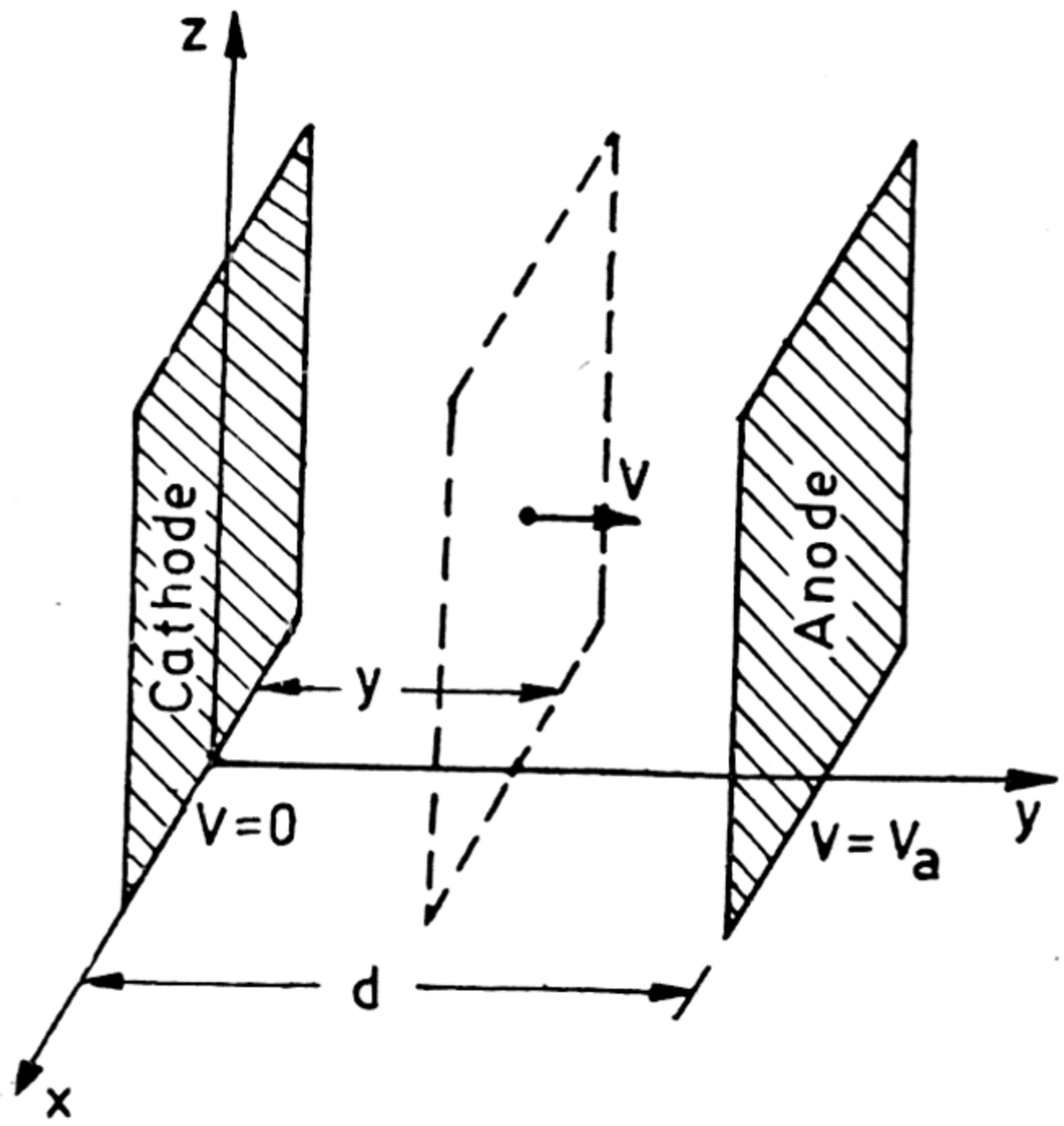


Fig. 1.9.

According to the Poission equation, the potential V at any point in the space between the electrodes is given by

$$\partial^2V/\partial x^2+\partial^2V/\partial y^2+\partial^2V/\partial z^2 = -\rho/\epsilon_0, \quad \ldots(22)$$

where ρ is the charge density at this point. As potential varies along y-axis and is constant in $x-z$ plane, hence

$$\partial^2V/\partial x^2=\partial^2V/\partial z^2=0$$

$$\text{and } d^2V/dy^2=-\rho/\varepsilon_0 \quad \ldots(23)$$

Assuming the initial velocity of electron as zero. Its velocity v at any point is related with potential V through which electron has traversed and is given by the relation

$$\tfrac{1}{2}mv^2=eV \text{ or } v=(2eV/m)^{1/2}. \quad \ldots(24)$$

Current density at this point $j=-nev=\rho v$, .. (25)
where n is the number of electrons per unit volume at that point.

Using relations (24) and (25), Eq. (23) reduces to

$$d^2V/dy^2=j/v\epsilon_0=j/\varepsilon_0\ (2eV/m)^{1/2}. \quad \ldots(26)$$

On multiplying both sides by $2dV/dy$ and integrating, we have

$$\left(\frac{dV}{dy}\right)^2=4\ \frac{j}{\epsilon_0}\left(\frac{m}{2e}\right)^{1/2} V^{1/2}+C.$$

The constant of integration $C=0$ as at $y=0$, V and dV/dy are both zero.

$$\therefore \qquad \frac{dV}{dy}=2\left[\frac{j}{\epsilon_0}\left(\frac{m}{2e}\right)^{1/2}\right]^{1/2} V^{1/4}$$

or $$V^{-1/4}dV=2\left[\frac{j}{\epsilon_0}\left(\frac{m}{2e}\right)^{1/2}\right]^{1/2} dy.$$

On integrating, we have

$$\frac{4}{3}V^{3/4}=2\left[\frac{j}{\epsilon_0}\left(\frac{m}{2e}\right)^{1/2}\right]^{1/2} y+C'.$$

Again constant of integration $C'=0$ as at $y=0,\ V=0$.

$$\therefore \quad V^{3/2}=\frac{9}{4}\cdot\frac{j}{\epsilon_0}\left(\frac{m}{2e}\right)^{1/2} y^2.$$

At the anode $$V_a^{3/2}=\frac{9j_a}{4\epsilon_0}\left(\frac{m}{2e}\right)^{1/2} d^2. \qquad \text{..(27)}$$

or current density at anode $$j_a=\frac{4}{9}\epsilon_0\left(\frac{2e}{m}\right)^{1/2}\frac{V_a^{3/2}}{d^2}$$

$$=2.33\times10^{-6}\ V_a^{3/2}/d^2. \qquad \text{...(28)}$$

This is the famous *Child-Langmuir or three halves power law.* It shows that the space charge current density is independent of the cathode temperature T and work function ϕ.

If the plates are concentric cylinders, instead of parallel plane plates, the similar results are obtained. The space charge limited current for cylindrical electrodes is given by the relation

$$i_a=2.33\times10^{-6}.2\pi r_a l_a\ V_a^{3/2}/r_a^2.$$

This is an approximate solution. For an exact solution a factor β^2 which is a function of the ratio of the radius at any point to the cathode radius, *i.e.*, r_a/r_k is introduced and we have

$$i_a=14.6\times10^{-6}\frac{l_a}{r_a}\cdot\frac{V_a^{3/2}}{\beta^2}. \qquad \text{...(29)}$$

For $r_a/r_k>7$, the value of β^2 differs from 1 by less than 10%.

1.9. TRIODE VALVE

As one can not control diode on account of space charge, hence a third electrode, known as *control grid,* was introduced by Lee de Forest. It consists of a spiral of fine wire and is placed between the cathode and the anode, usually near the cathode. The tube or valve thus consists of three electrodes, hence is called *triode.*

The construction of a typical triode is shown in Fig. 1.10 with its symbolic representation. The third electrode is in the

form of fine, uniform metallic (nichrom, molybdenum, iron, nickel, tungsten, or other alloys) mesh, screen or spiral winding, which is well supported and surrounded the cathode K completely. The various turns of mesh are equidistant with large separation between them in comparison to the diameter of wire used. The grid is placed nearer to cathode, so as to be more effective. The electrons from the cathode reach the plate from the open space without hindrance. Actually some potential is applied on grid, the electric field between grid and cathode affects (or controls) the electron flow. Depending upon the field configuration and design of the tube the shape (may be circular, elliptical or flat helix, ladder type), size and spacing of the grid wire are selected.

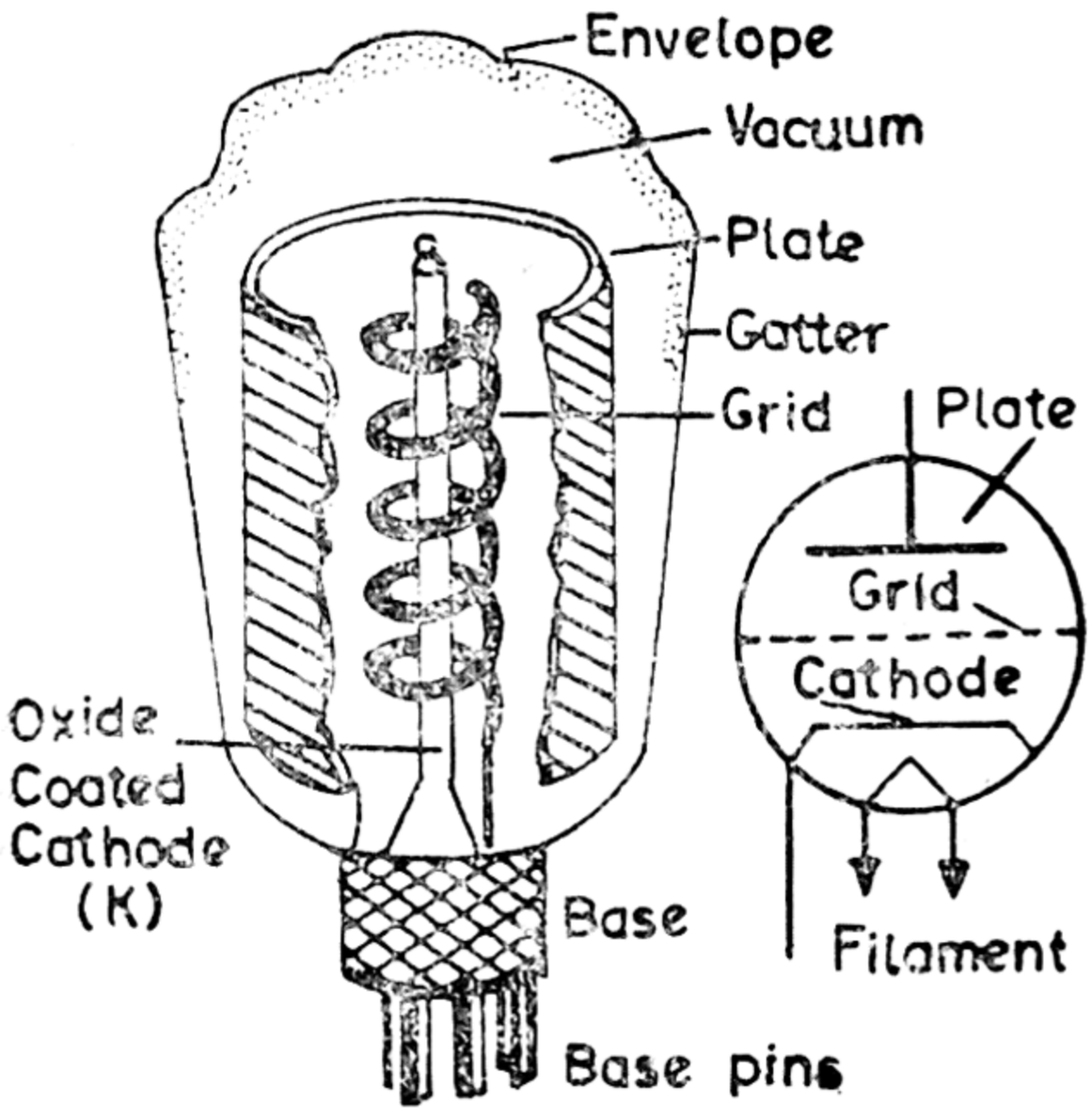


Fig. 1.10.

All the three electrodes, plate (anode), grid and cathode are mounted inside a highly evacuated ($\sim 10^{-8}$ meter of *Hg*) metallic or glass bulb. Some triode valves contain traces of inert gas. Some chemically active substance, like barium or magnesium is deposited by the process of vaporisation on the inner side of the bulb. It will absorb the residual gases by chemical action, and will keep high degree vacuum inside the bulb for the successful operation of the tube. These substances are called *getters*. The properties and construction of plate (anode) and cathode (directly or indirectly heated) are similar to as in diode valve. Since the indirectly heated valves have many advantages, a coated tungsten wire is used as heater which is very close to cathode. Since the cathode is oxide coated so at low temperature copious electron emission takes place. The plate is in the form of flat metallic plate if the grid is flat or a *Cu* or *Ni* cylinder surrounding the grid if the grid is spiral. It should be noted that the plate, grid and cathode are symmetrically placed always. The plate collects the electrons, emitted from the cathode as it is usually connected to the positive potential. The terminals of all these electrodes and heater are brought outside through the backelite or ebonite base, in the form of base pins.

To bring the cathode to its proper temperature, for normal emission of electrons, heater or filament is connected to a low voltage (ac or dc). The plate or anode is normally connected to a high positive voltage, to attract the stream of electrons from cathode. To control the electron flow the third electrode, control-grid is connec-

ted with a fixed dc voltage, called the *grid bias*, which is normally a few volts negative than cathode. On this bias-voltage is superimposed. This will vary the effective grid-bias and thus the flow of electrons, which changes the plate current in plate circuit.

The electrons emitted from cathode are collected in its immediate vicinity and form electron cloud, called *space charge* and repel the other similar charged electrons. The plate is at positive potential, with respect to cathode. It attracts electrons which flow from cathode to plate through the spacing of the grid. With the increase in positive potential of plate more number of electrons will be attracted, thus the space charge will decrease. Now suppose that the grid is at negative potential with respect to cathode, it will exert a repelling force on the electrons coming towards the plate. This will increase the space charge between the grid and cathode and will decrease the plate current. Thus if the grid bias is further decreased, the plate current will further decrease and the bias voltage that is just sufficient to cut off the plate current is obtained. It is called the *cut off bias. At cut off bias of grid the electric field due to grid neutralizes the attractive field of plate.*

When the grid is positive with respect to cathode, electrons will be accelerated under the combined effect of grid and plate potentials. This will decrease the space charge and increase the plate current. If the grid bias is sufficiently positive a stage will be attained when all the electrons emitted from the cathode are attracted by the plate. Under these conditions there exists no accumulation of electrons (*i.e.*, no space charge) and the current is called *saturation current.*

Since the grid wire is positive, a part of electrons is attracted by the grid and causes a comparatively small grid current in grid-cathode circuit and some power is dissipated. As a large saturation current in plate cathode circuit can damage the tube, *so the electron tubes are generally operated under negative grid bias conditions.* Thus the variation of plate current is possible with grid bias. Since the grid is nearer to the cathode, so grid bias is more effective in controlling the electrons or the plate current in the circuit. Due to these reasons the grid is known as control grid.

Static Characteristic curves of a triode valve. Fig. 1.11 shows the circuit arrangement for the characteristic curves of a triode valve. The filament or heater is connected to ac or dc supply (generally 6 volts), through a rheostat (not shown in figure), to give necessary heater current to heat the cathode to its proper temperature for supply of electrons. The cathode is connected to negative terminal of a high tension (H.T.) battery E_{bb} of high potential of nearly 250 volts. This gives necessary positive plate potential through a rheostat R_1, used as potential divider. To

record the plate current and plate potential at any instant a milliammeter *mA* and voltmeter V_1 are connected in the plate cathode circuit. A low tension (L.T.) battery E_{cc} of about 15 to 25 volts is used in the grid cathode circuit as the grid bias. The grid bias can be negative or positive. This is also adjustable with the help of

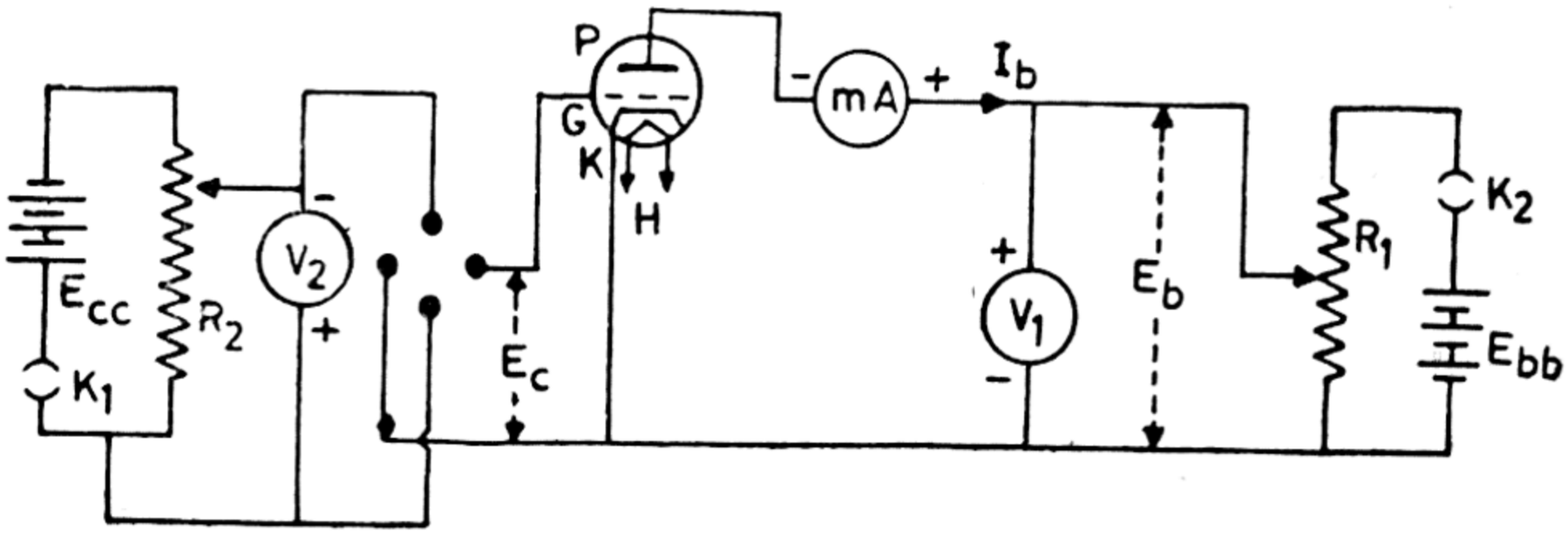


Fig. 1.11.

rheostat R_2 used as potential divider. The value of grid bias at any instant is recorded with another voltmeter V_2.

As there is no load in the circuit, hence the characteristic curves so obtained are called *static characteristics.* Since the plate current I_b of the triode valve depends upon the plate potential E_b and the grid potential E_c, hence we can write

$$I_b = f(E_b, E_c).$$

By plotting these three variables simultaneously, we obtain following three families of characteristics :

(*a*) *Plate or Anode Characteristic Curves,* The grid bias is adjusted with rheostat R_2 to any particular value E_c (−ve or +ve) recorded at V_2. Now the plate potential E_b in increased slowly in steps with R_1, and recorded at V_1 and plate current I_b for each V_1 is recorded at *mA*. This will give one anode characteristic curve. By keeping other fixed values of *Ec*, similar sets of readings are plotted which give the characteristic curves in each case as shown in Fig. 1.12. The anode characteristics are drawn usually for −ve values of grid bias. As the grid is more negative, the curves are shifted towards the higher plate potential side, without much change in the shape of the characteristics.

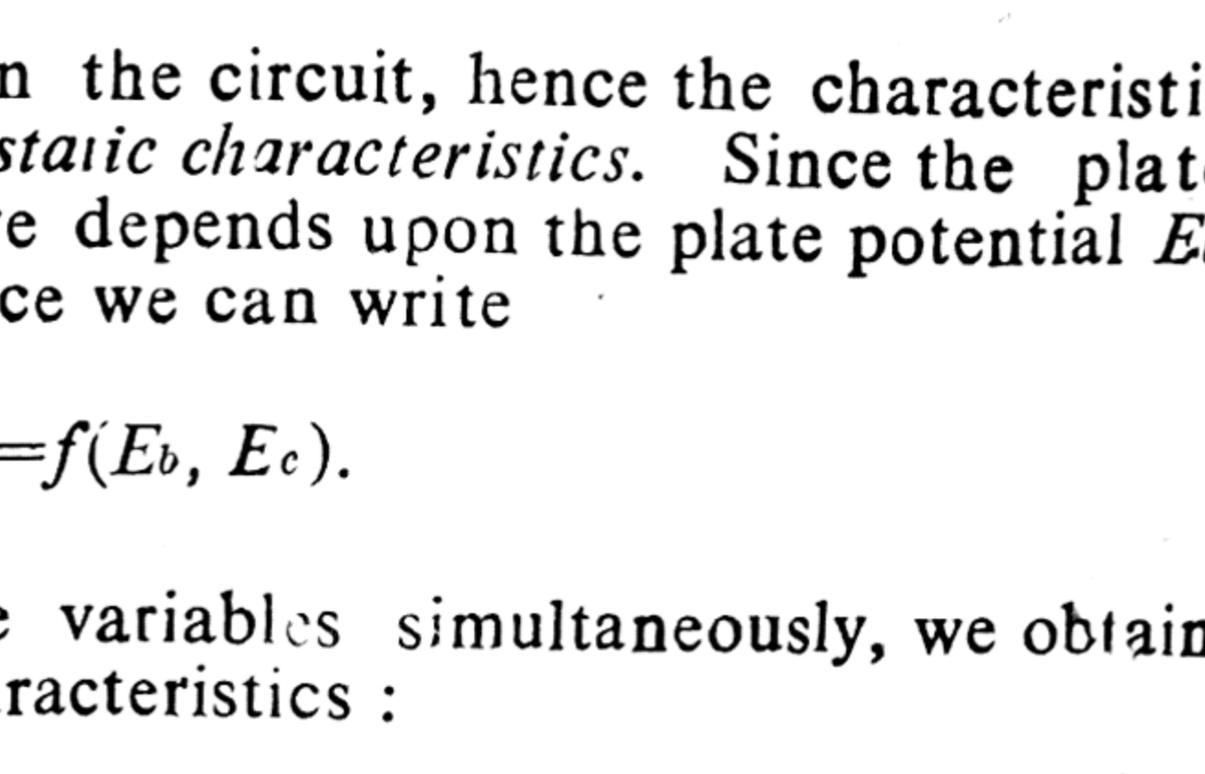


Fig. 1.12.

(*b*) ***Mutual or Transfer Characteristic Curves.*** The plate potential is kept at any constant value E_b with R_1, recorded at V_1. Now the grid bias is increased in steps by adjusting slowly R_2 (from −ve to +ve value). This grid bias is recorded at V_2 and the corresponding plate current i_b is recorded at *mA*. By plotting these observations we have *one mutual characteristic curve.* Different sets of observations with other fixed values of E_b are similarly recorded and plotted. The curves obtained in each cases are shown in Fig. 1.13. These are found similar to each other but are displaced with the different plate potentials. For each plate voltage there is a particular grid voltage at which the plate current is zero, the *cut off-potential.* Also there is a saturation current, when the variation of grid potential has no effect on the plate current.

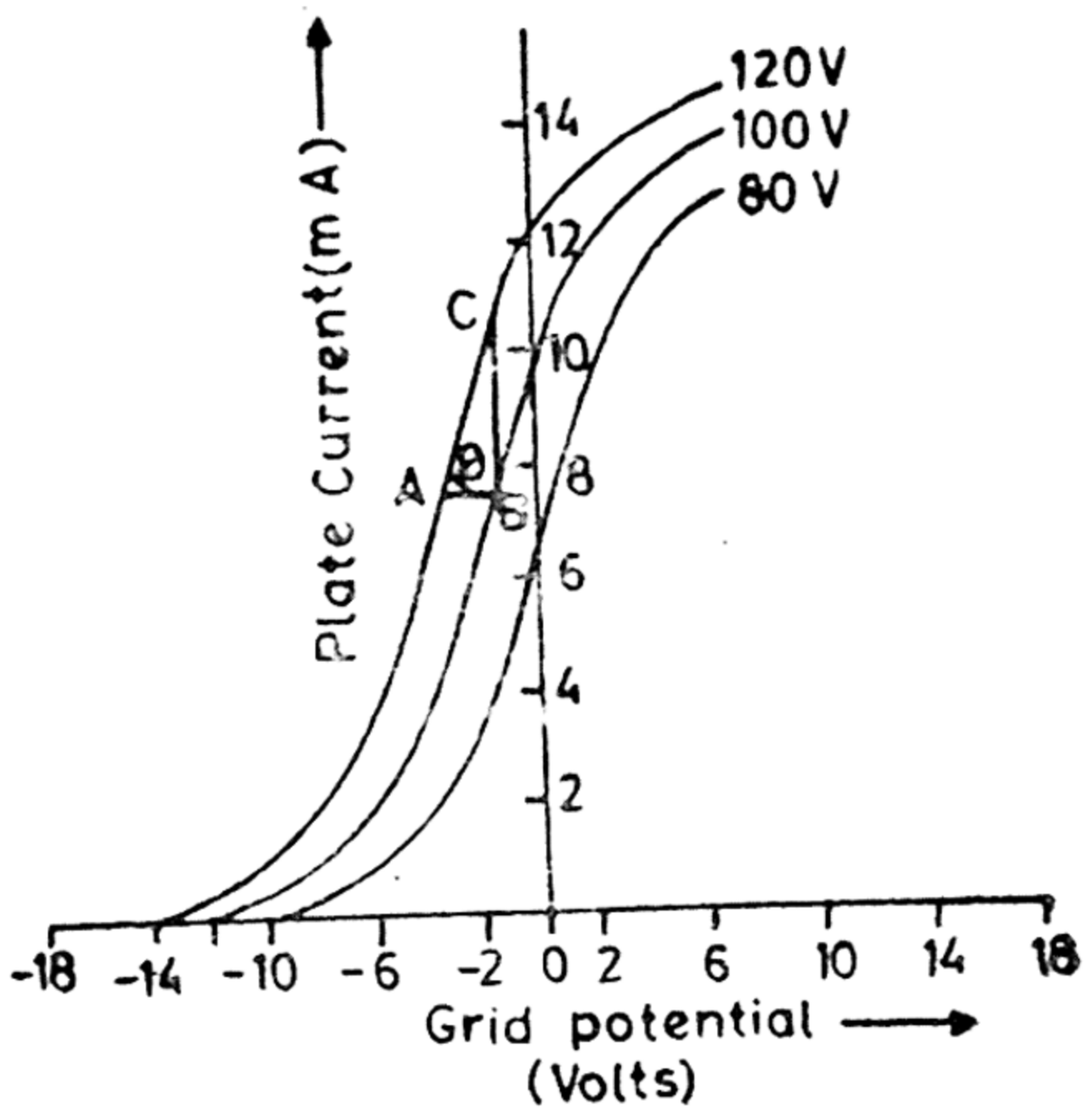


Fig. 1.13.

(*c*) ***Amplification or Constant Current Characteristic Curves.*** The plate and grid potentials (E_b and E_c) are adjusted simultaneously to have a same plate current (I_b). For a particular value of I_b, we can record grid voltages corresponding to different values of plate voltages. By plotting these values we will obtain one curve. Different sets and hence curves can be obtained with the different values of I_b. These curves Fig. (1.14) show the effects of the plate and grid voltages on the plate current of the tube.

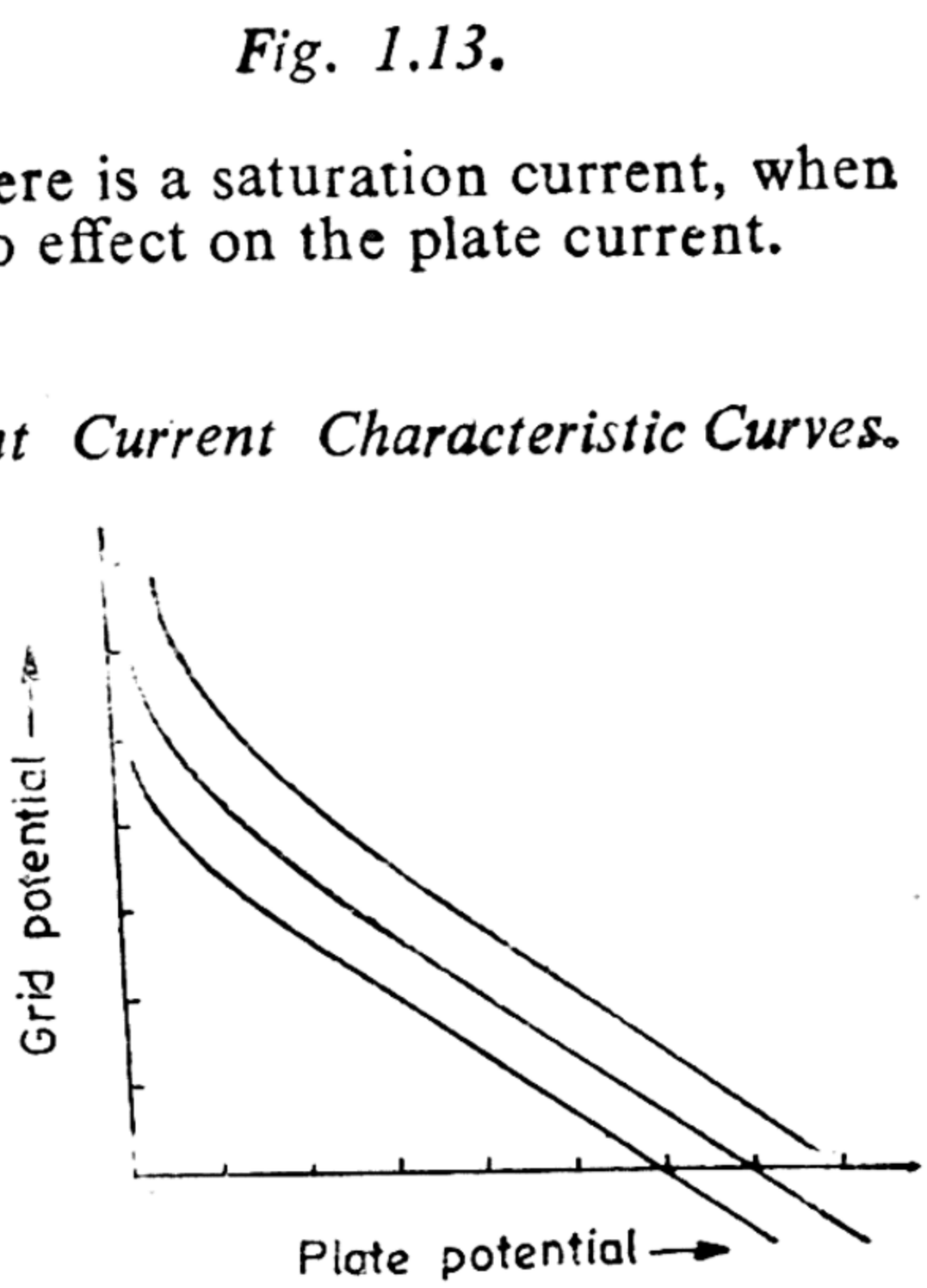


Fig. 1.14.

Coefficient or Parameters of a Triode Valve. These characteristic curves show that the effect of grid potential is more on plate current in comparison to plate potential. It is because the grid is nearer to cathode. The set of characteristics for a tube depends on geometric configuration of electrodes, the spacing between them, the power dissipation capabilities and other factors.

The above design factors of the tube are summarized by three major tube constants or parameters.

(*i*) *Amplification factor* (μ). The slope of the constant current characteristic gives the amplification factor μ. We can define it as the ratio of the change in plate voltage $\triangle E_b$ to the change in grid voltage $\triangle E_c$ for a constant plate current I_b, *i.e.*,

$$\mu = -(\triangle E_b / \triangle E_c)_{I_b = \text{constant}}. \qquad \ldots(30)$$

Negative sign is necessary to have μ positive as the effect on the anode current of an increase of the anode potential is compensated for by decrease of the grid potential. μ is directly obtained from the slope of the constant current characteristic, but can also be calculated from remaining two characteristics. For example with mutual characteristic Fig. 1.13,

$$\mu = 20/AB. \qquad \ldots(31)$$

As the curves are nonlinear, hence we should write

$$\mu = -(\partial E_b / \partial E_c)_{I_b = \text{constant}}. \qquad \ldots(32)$$

It varies from 2 to 100, with most common value of 20. The reciprccal of μ is called the *penetration factor.*

(*ii*) *Plate resistance* (r_p). It is the measure of the resistance offered by the valve to the variation of current in the plate circuit. Keeping the grid voltage E_b to be constant, if there is a small increase of plate current $\triangle I_b$ due to increase in plate potential $\triangle E_b$, the plate resistance is given by

$$\text{Plate resistance } (r_p) = \frac{\text{Change in plate voltage } (\triangle E_b)}{\text{Change in plate current } (\triangle I_b)}$$

or

$$r_p = (\partial E_b / \partial I_b)_{E_c = \text{constant}}. \qquad \ldots(33)$$

To evaluate the plate resistance a small change in plate voltage and corresponding change in plate current is recorded, keeping the grid voltage constant. This can be measured as the reciprocal of the slope at any point of the anode characteristic. One can also calculate r_p from other two characteristic curves, *e.g.*, $r_p = 20/CB$ from mutual characteristic curve. As it is the ratio of volts to amp, so its unit is ohms. Its value varies from 500 ohms to 100 kilo ohms. The reciprocal of r_p is called the *plate conductance*, g_p.

(*ii*) *Trans-* (*or Mutual*) *Conductance.* It is a measure of the mutual effect of the grid and plate circuits and is defined as the ratio of the change in the plate current ($\triangle I_b$) to a change in the control grid voltage ($\triangle E_c$) such that the plate voltage E_b is kept constant.

$$\text{Mutual conductance } (g_m) = \frac{\text{Change in plate current } (\triangle I_b)}{\text{Change in grid potential } (\triangle E_c)}$$

or $$g_m=(\partial I_b/\partial E_c)_{E_b}=\text{constant.} \qquad \ldots(34)$$

Mutual conductance, also known as *plate-grid transconductance* is the slope of the mutual characteristic. As the mutual characteristic is nearly a straight line for a certain region of values of V_g, hence g_m is nearly constant for this region.

This is the ratio of amp and volts, measured in units of mhos. For triode the value of g_m lies between 0.5 and 5 millimhos.

Relation between the Parameters of Triode. As we know that

$$I_b=f(E_b, E_c),$$

or $$dI_b=\left(\frac{\partial I_b}{\partial E_b}\right)_{E_c} dE_b+\left(\frac{\partial I_b}{\partial E_c}\right)_{E_b} dE_c.$$

If the small changes in plate voltage and grid voltage are such that the net change in current is zero, then

$$(\partial I_b/\partial E_b)_{Ec}\ dE_b+(\partial I_b/\partial E_c)_{Eb}\ dE_c=0.$$

From Eqs. (33) and (34), we have

$$(1/r_p).dE_b+g_m dE_c=0 \text{ or } g_m r_p=-(dE_b/dE_c).$$

This is under the condition that $dI_b=0$, or I_b is constant. Hence from Eq. (32), we have

$$g_m r_p=\mu. \qquad \ldots(35)$$

This equation relates the three parameters of the triode valve.

Variation of the Triode Parameters. The values of the triode parameters μ, r_p and g_m are not constants but vary with the change in plate current. The Fig. 1.15 shows the variation for any fixed value of plate potential. This is also because of the variation in slope of the characteristics at different points. The amplification factor (μ) is almost constant over a quite wide range of operation. The plate resistance (r_p) varies with different values of plate current. It is high for low values of current and attains a constant value at higher plate current. The variation in transconductance (g_m) is reciprocal of the variation of r_p and also attains a constant value at higher plate current. So *these are not constants of a valve but are coefficients or parameters of the valve.*

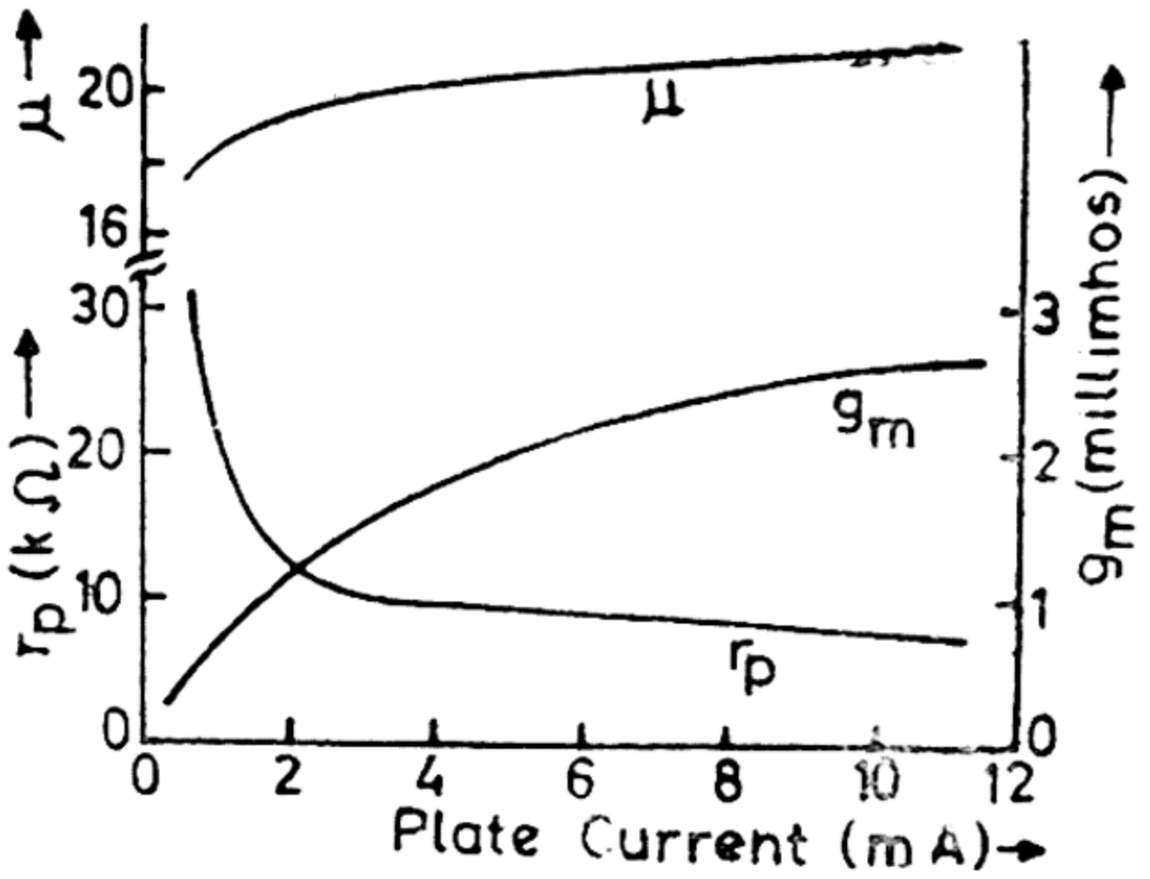


Fig. 1.15.

Dynamic Characteristic Curve.

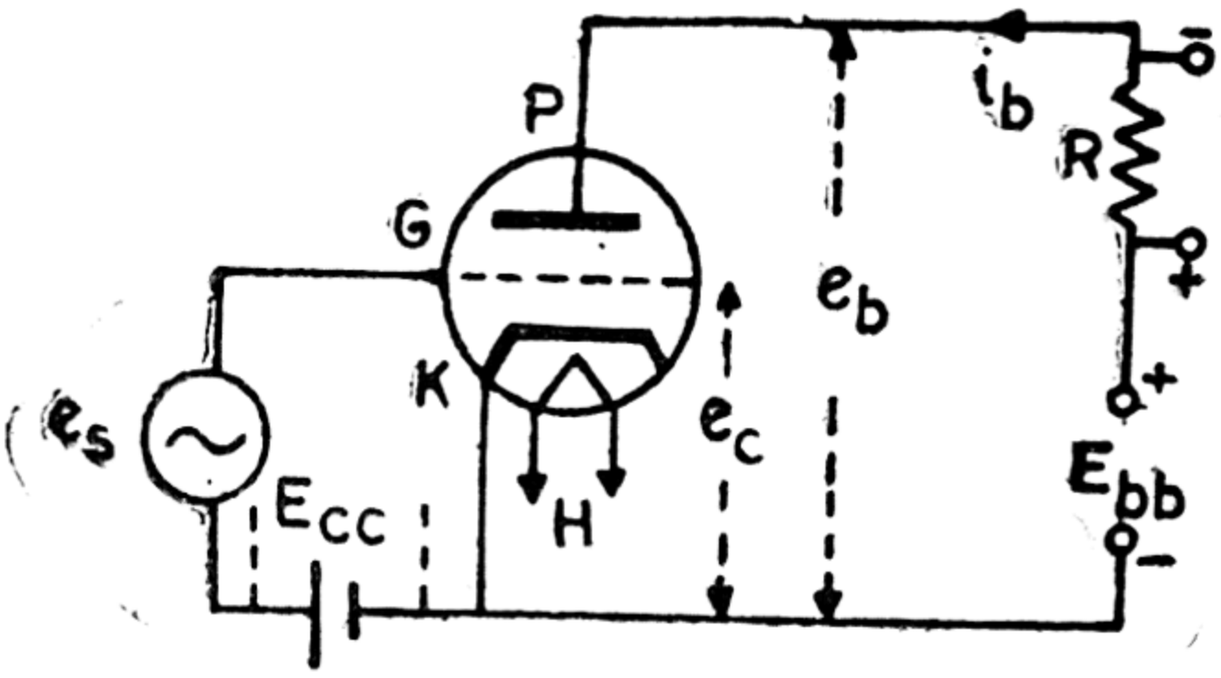


Fig. 1.16.

The characteristic curves so far studied are static characteristic as there was no load in the circuit. But when tube is subject to active operation (as an amplifier, etc.) load is always used in the plate circuit. The output is obtained across this load. The characteristic curve now obtained is called *dynamic characteristic.*

The Fig. 1.16 shows a simple circuit diagram for a triode with a load in plate circuit. Let e_s be the instantaneous *ac* component of input supply at grid bias E_{cc} as such that the instantaneous value of the grid voltage is e_c. Due to this grid voltage the total anode current is i_b at any instant in the direction shown by the arrow. As the anode voltage is E_{bb} from H.T., the anode voltage e_b, at any instant will be given by

$$e_b = E_{bb} - i_b R, \qquad \ldots(36)$$

where $i_b R$ is the drop in potential across the load resistance. This potential and hence e_b will vary with the plate current and also with load resistance R. The plate current i_b is given by

$$i_b = -e_b/R + E_{bb}/R. \qquad \ldots(37)$$

Dynamic Transfer (Mutual) Characteristic. As in the case of diode, the dynamic transfer characteristic is the curve between e_c and i_b with both the plate supply E_{bb} and resistance R as constant. These characteristics (Fig. 1.17) are quite straight lines (continuous

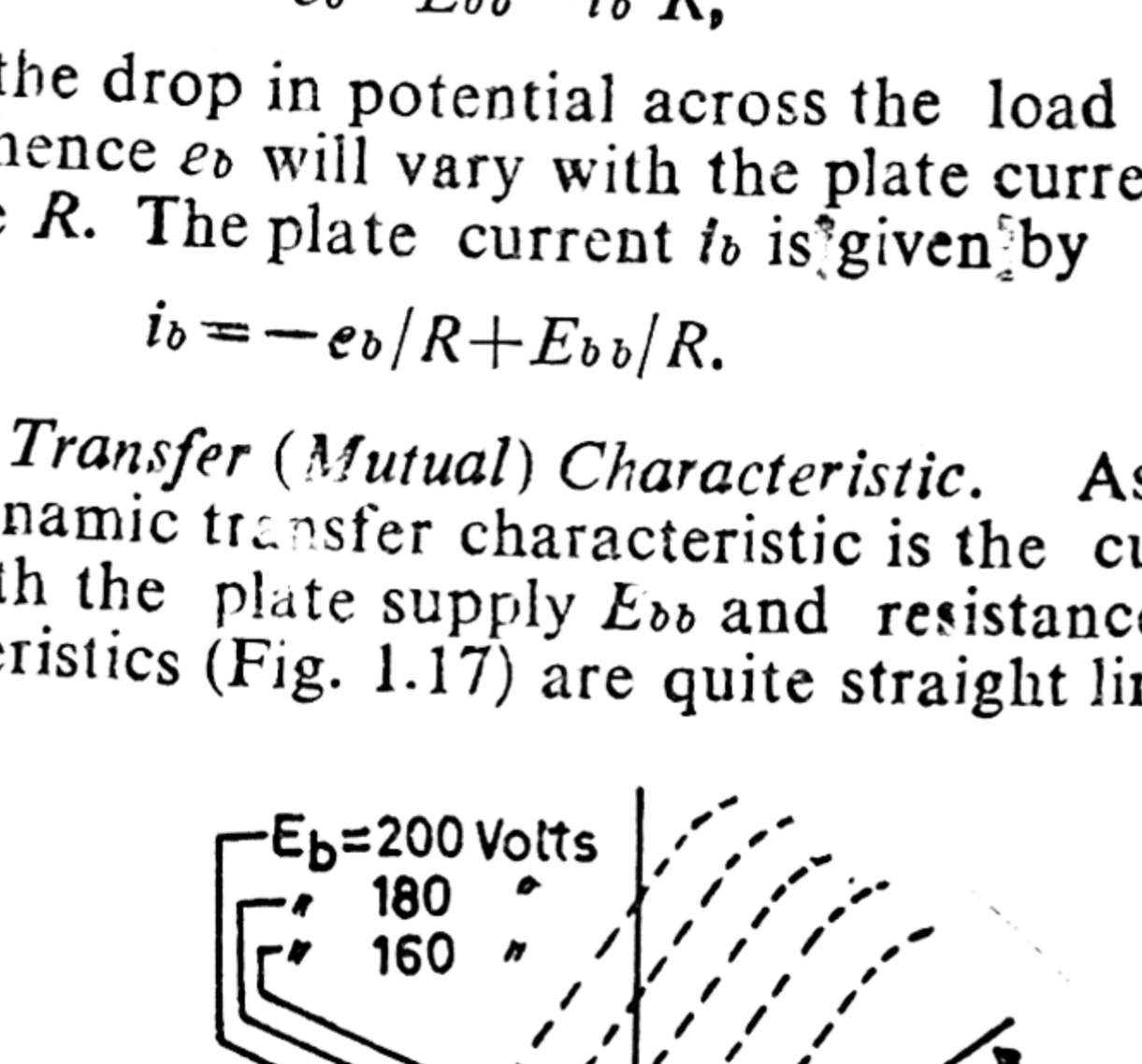


Fig. 1.17.

curves) and have different slopes than static (shown dotted) characteristics. The dynamic characteristic depends on the valve design and circuit elements both, while the static characteristic depends on the valve configuration only.

Dynamic curves can be drawn graphically also from Eq. (36) by subtracting the quantity $i_b R$ from each ordinate value of static curve. Let $E_{bb}=200V$ and $R=10{,}000$ ohms. At cut off i_b is zero, so $e_b=200$ volts, the full voltage is applied at the plate, we thus have point A. If now grid is less negative i_b will rise. Let $i_b=2mA$, then $i_b R=2\times10^{-3}\times10\times10^3=20$ volts, so $e_b=200-20=180$ volts and we have point B (at $E_b=180$ volts of static characteristic). Similarly further if $i_b=4mA$, $e_b=160$ volts, we have point C and so on. The line joining the points A, B, C,...gives the dynamic characteristic for a particular load R. Similarly other dynamic characteristics can be drawn for other loads. With the increase in load, the slope of the characteristic decreases. Dynamic curves are more linear and have less slope than static curves. Due to the linearity of the dynamic curves the output wave form is not distorted.

Dynamic Anode Characteristic. It is the curve between e_b and i_b with e_c as constant. Similar to the graphical drawing of dynamic mutual characteristics the dynamic anode characteristics can also be graphically drawn from static anode characteristics with the help of Eq. (37). If this Eqn is plotted on the i_b, e_b plotting, then the new characteristic obtained will be a straight line with the slope $(-1/R)$ as shown in Fig. 1.18. This lind AB is called *load line.* It cuts the i_b axis at E_{bb}/R, point A and the e_b axis at $e_b=E_{bb}$, point B. So the Eq. (37) indicates the equation of load line for a given value of E_{bb} and R. *The load line is the characteristic of the external circuit elements and not the valve characteristic.* The intersection of the line with the static anode characteristic gives information about the external condition of the plate current and voltage. If the valve is operated at particular grid voltage $e_c=-2$ volts (say) the load line cuts the characteristic at point Q, known as *operating point.* The power output can be calculated from the expression

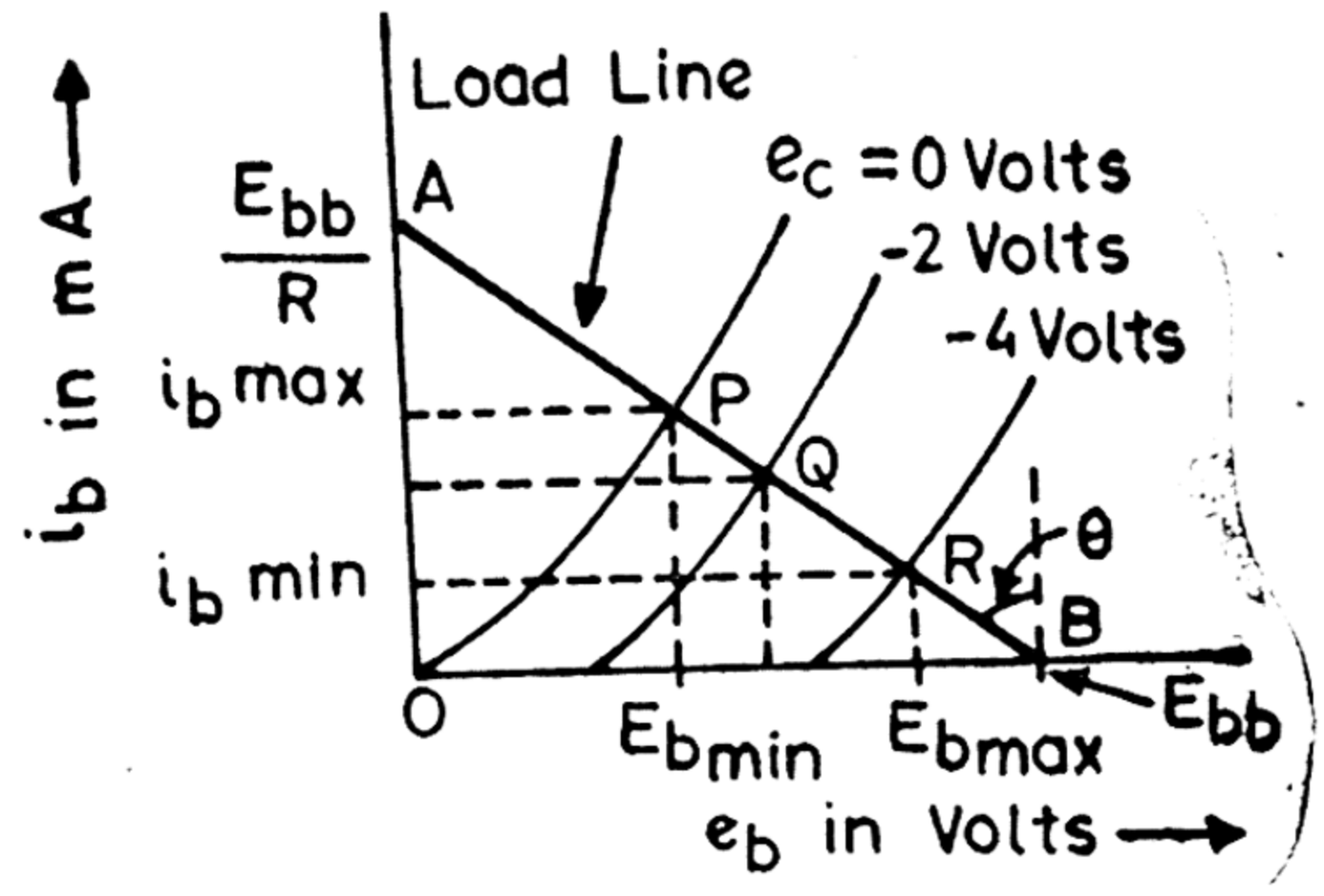


Fig. 1.18.

$$\text{Power output}=\tfrac{1}{2}\,(E_{b\,max}-E_{b\,min})\,(I_{b\,max}-I_{b\,min}). \qquad \ldots(38)$$

Thus a load line plotted over the tube anode characteristic is a very useful tool in the graphical analysis and for the complete behaviour or response of the tube.

1.10. TETRODES

A tetrode valve, also referred to as a *screen-grid tube or four electrodes tube*, consists of two grids in addition to cathode and anode. The first grid called *control grid* (G_1), is kept close to the cathode, and the second grid called *screen grid* (G_2), is placed in between the control grid and the anode. This screen grid is usually kept at positive potential. The tetrode valve and its symbol are shown in Fig. 1.19.

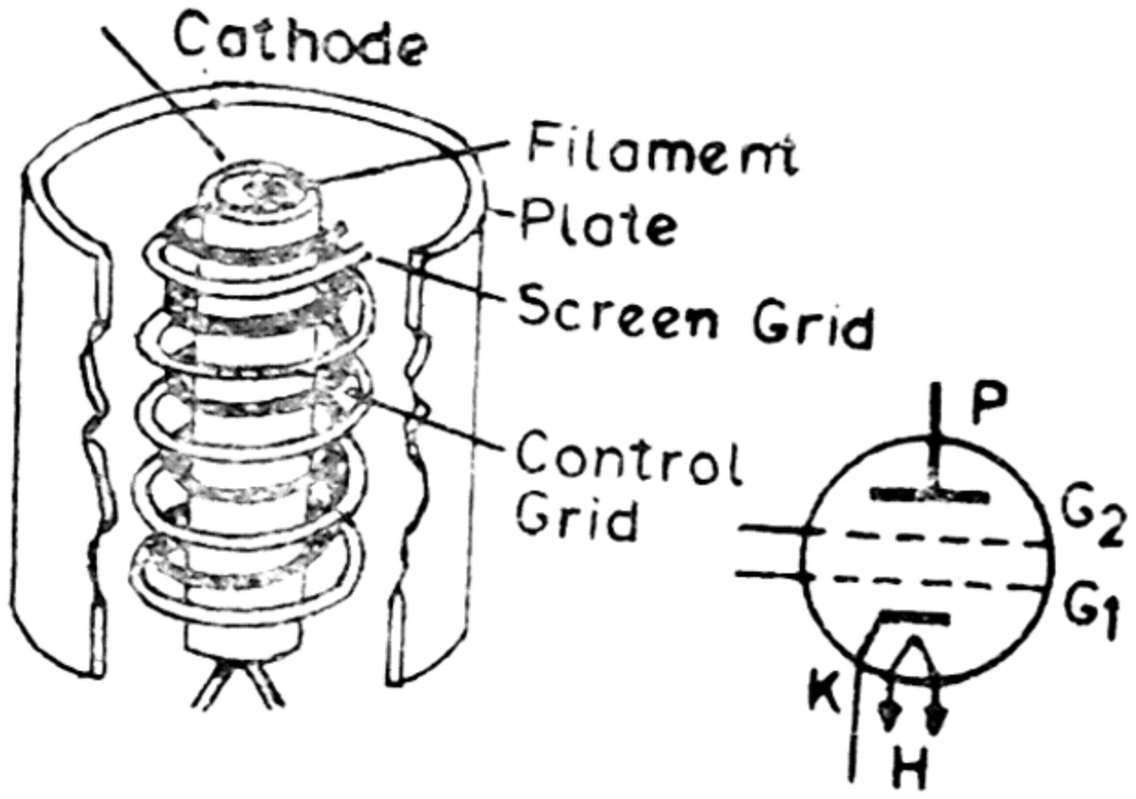


Fig. 1.19.

There exists capacitance effect between the cathode, plate and grid of triode valve, because of the electric field present between these charged electrodes especially at high frequencies. This is called as *interelectrode capacitance*. These are designated as grid-cathode capacitance C_{gk}, the grid-plate capacitance C_{gp}, and the plate-cathode capacitance C_{pk} as shown in Fig. 1.20 Due to this capacitance effect (mainly due to grid-plate capacitance), some energy is feed back from the plate circuit (output) to the grid circuit (input) which leads to instability and undesirable oscillations at radio-frequency.

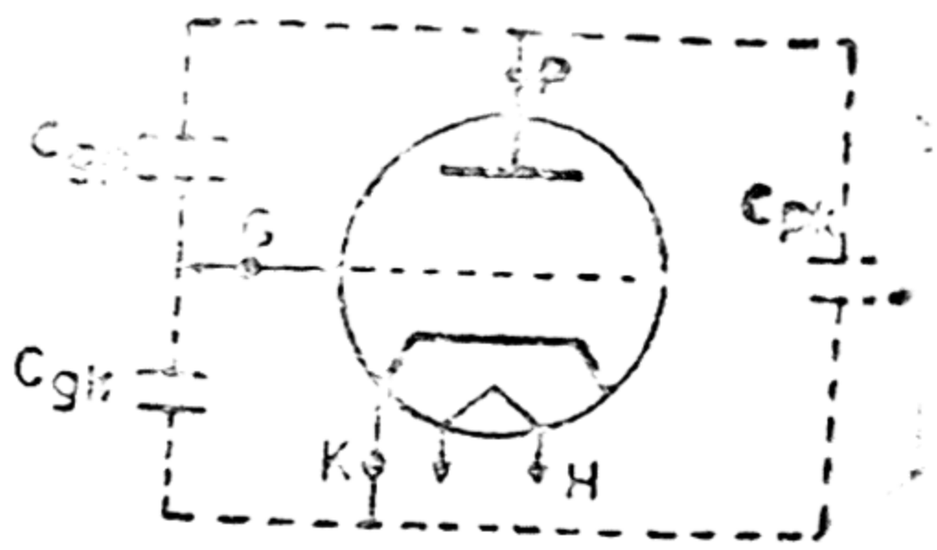


Fig. 1.20.

This feedback effect increases also the effective input capacitance (*i.e.*, the ratio of the change in charge on the grid to the change in the grid potential). When charge is added to the grid to remain the anode potential constant the input capacitance would be the sum of the grid-cathode capacitance C_{gk} and the grid-plate capacitance C_{gp}. In an amplifier circuit, the anode voltage decreases by $|A|$ or $|K|$ (the voltage gain) times the change in grid voltage, which tends to decrease the grid voltage by induction. Thus an additional charge ($C_{gp}\ \Delta V_a = |A|\ C_{gp}\ \Delta V_g$) must be put on to the grid to counteract this effect. Hence, when the plate voltage changes with grid voltage, the input capacitance is increased by the term $|A|\ C_{gp}$ and becomes

$$C_{in} = C_{gk} + C_{gp} + |A|\ C_{gp} = C_{gk} + (1 - A)\ C_{gp}.$$

As the voltage gain $|A|$ may be fairly large, a small capacitance C_{gp} may result in appreciable *a-c* grid current. This is called *Miller effect*. The gain dependent term in the input capacitance is made negligible by inserting a screen grid between the control grid and the anode. In such a case the input capacitance

$$C_{in} = C_{gk} + C_{gp},$$

where C_{gp} is the control grid-screen grid capacitance.

Function of Screen Grid. (*i*) It reduces the capacitance between the control grid and the plate of a triode to a negligible amount, few thousandths of that of a triode. (*ii*) It thus acts as an electrostatic shield between the plate and the control grid. This shielding action can further be increased by connecting a bypass capacitor between the screen grid and the eathode. (*iii*) It reduces the space charge, as the electrons coming from the cathode receive added acceleration on their way to the plate.

Comparison with the Triode. (*i*) The plate in a triode controls the space and collects the plate current. In a tetrode the plate only collects the electrons which succeed in passing through the screen. (*ii*) The passive character of the plate makes the tetrode a much better voltage amplifier than the triode. (*iii*) The plate-grid capacitance is only a few thousandths of that of a triode. (*iv*) The plate-grid transconductance is roughly the same as that of a triode. (*v*) The amplification factor and plate resistance are about ten or one hundred times that of a triode.

Tetrode Characteristics. The tetrode operation is just similar to that of triode, *i.e.*, cathode at ground potential, the control grid at a small negative potential and plate at high positive potential. The screen grid is also at positive potential but less than the plate potential (Fig. 1.21). The positive screen accelerates the electrons and adds to the electrostatic field of the plate. There is

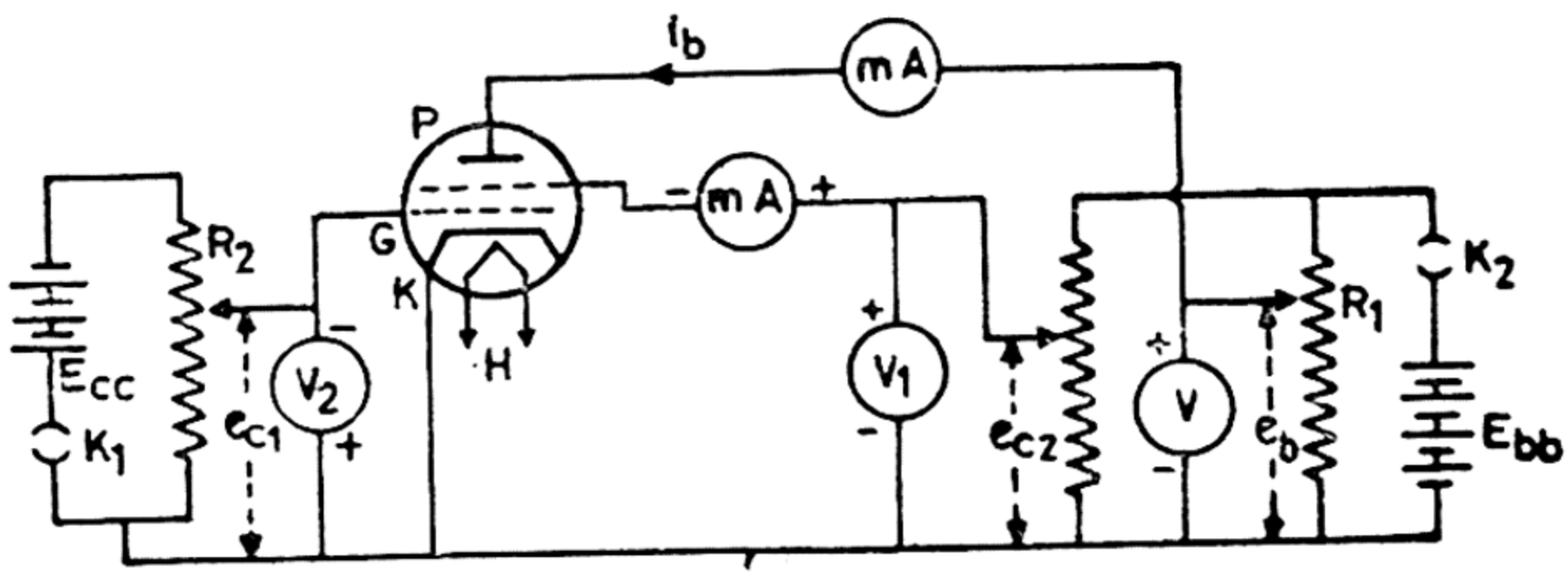


Fig. 1.21.

also small screen current but most of the electrons pass through the open space of the screen grid towards the plate and produce the plate current. Because of the shielding effect of the screen grid the electrostatic field of the plate has a little effect on the space charge near cathode. So, with the variation of plate potential the plate current is not much affected. Hence plate resistance ($\partial V_p/\partial I_p$) of tetrode is very high, of the order of 0.5 to 1 mega ohm. But the control grid is quite effective in governing the flow of electrons, *i.e.*, plate current. Since the amplification factor (μ) is the ratio of the change in plate potential to the change in grid potential for same change of current hence μ will also be higher than that of triode nearly 400 to 800 times. Generally the transconductance of tetrode is same as triode (nearly 1 to 1.5 milli mhos).

The typical static *plate characteristic curves* for a tetrode are shown in Fig. 1.22. These curves show the relation between the plate current (i_b) and the plate voltage (e_b) for the values of constant grid voltage (e_{c_1}), the screen grid potential (e_{c_2}) is kept constant (125 *V*). One curve for screen grid current, (i_{c_2}) with plate voltage (e_b) for the control grid voltage of zero volt is also shown. In the initial part of this characteristic, the i_{c_2} is highly positive while $e_b = 0$ volts, most of the electrons are attracted by the screen and i_b is zero. As e_b increases, some electrons are diverted to the plate thus i_b increases and i_{c_2} decreases rapidly. The total current due to both i_b and i_{c_2} remains almost unaltered. With the further rise in e_b, i_b falls suddenly while i_{c_2} increases correspondingly (part xy). At this region of e_b, electrons have sufficient high velocities to drive out loose electrons from the plate material and project them between the region of plate and screen grid. These electrons from anode (known as secondary electrons)

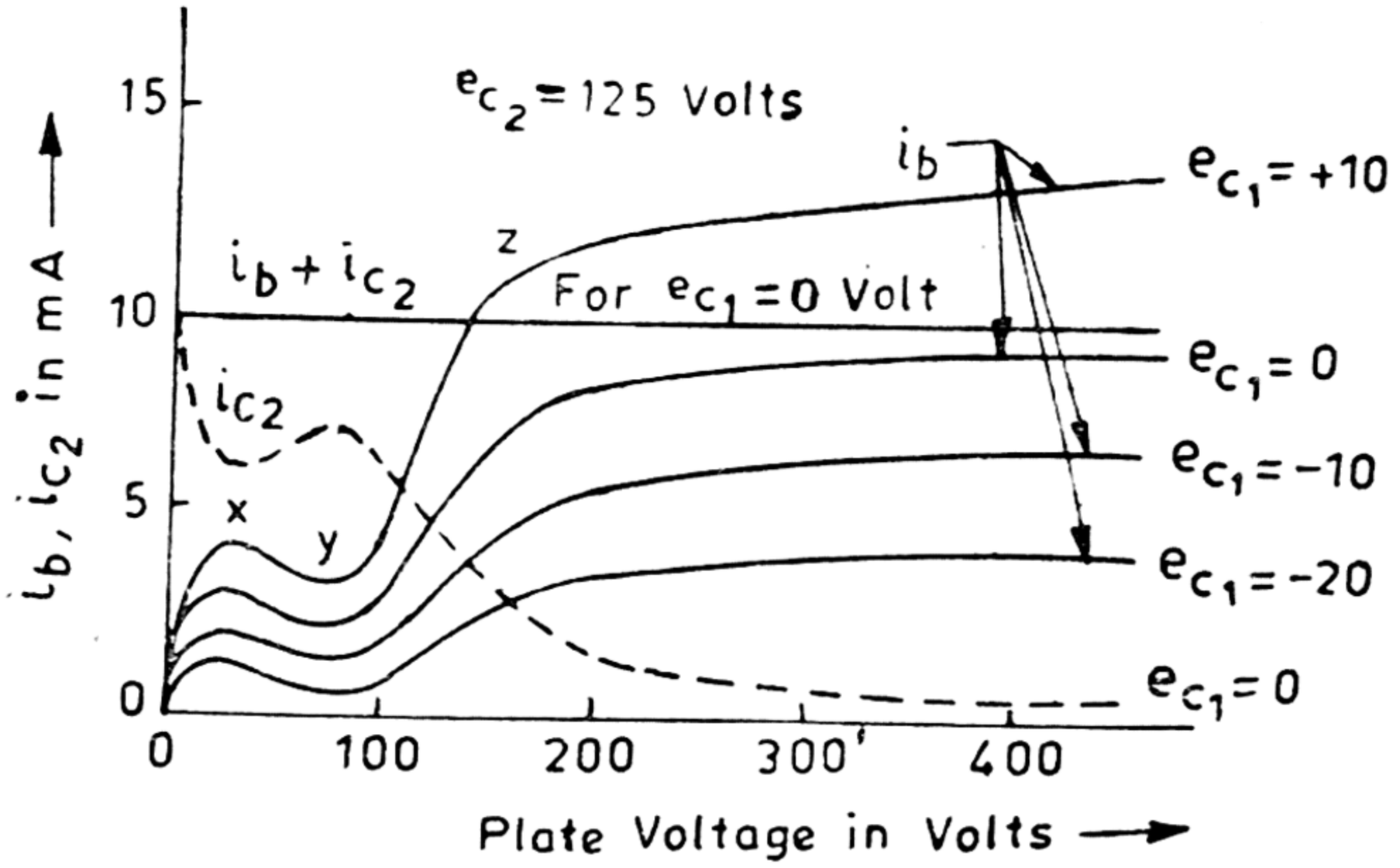


Fig. 1.22.

are collected by screen grid (since $e_{c_2} > e_b$ at this stage). Now the number of electrons with plate has decreased, hence the plate current i_b decreases and there is a negative slope in e_b—i_b curve. When e_b is further increased, becomes more than e_{c_2}, the force of attraction due to plate is increased, the secondary electrons will return to plate, i_b will rise and i_{c_2} will fall sharply. When e_b is sufficiently high than e_{c_2}, it collects all the electrons and conduction of saturation current is achieved (only those electrons which are intercepted by the screen will constitute a low current i_{c_2}).

Transfer-characteristics. We know that the plate of the tetrode has very little influence on the space current. The curves for fixed screen grid potential e_{c_2} show the effect of variation of the plate potentials, *i.e.*, the variation of i_b with e_{b_1} corresponding to a particular value of e_b. Because of the slight influence of the plate, the transfer curves are much more bunched together than the transfer curves of triodes for different values of e_b, provided $e_b > e_{c_2}$.

Tube Parameters. The plate current of a tetrode may be expressed as a function of potentials e_b, e_{c_1} and e_{c_2} as

$$ib = f(e_b, e_{c_1}, e_{c_2}) \qquad \ldots(39)$$

$$\therefore \quad \Delta ib = \left(\frac{\partial ib}{\partial \epsilon b}\right) \Delta \epsilon b + \left(\frac{\partial ib}{\partial e_{c_1}}\right) \Delta e_{c_1} + \left(\frac{\partial ib}{\partial e_{c_2}}\right) \Delta e_{c_2}. \qquad \ldots(40)$$

Generally the screen potential is maintained constant at some appropriate value. Hence using $\Delta e_{c_2} = 0$ and proceeding as for the case of tricdes, tube parameters can be defined by partial differential coefficients as :

plate resistance $r_p \equiv (\partial eb/\partial ib)_{cc_1,\ e_{e_2}}$

plate control grid transconductance $g_m \equiv (\partial ib/\partial e_{c_1})_{eb,\ e_{c_2}}$

and amplification factor $\mu \equiv -(\partial eb/\partial e_{c_1})_{ib,\ e_{b_2}}$. $\qquad \ldots(41)$

It is clear that the relation $\mu = r_p g_m$ also applies in this case.

These parameters are found by constructing a triangle on the characteristic curve about the operating point.

Negative Resistance. Over the region xy of the tetrode plate characteristic, the anode current decreases with the anode potential increasing. The anode slope resistance is negative. This region is called the region of *negative resistance*, or *dynatron region*. This limits the range of operation of tetrode and producing instability. But this part is utilized as an oscillator under certain conditions. This defect of tetrode is overcome in pentode valve.

1.11. PENTODE

The pentode valve consists of three grids in addition to cathode and anode. The more grid than of tetrode is in between plate and screen grid, called *suppressor grid*. The suppressor grid (G_3) consists of a spiral wound wire or a coarse-mesh screen and

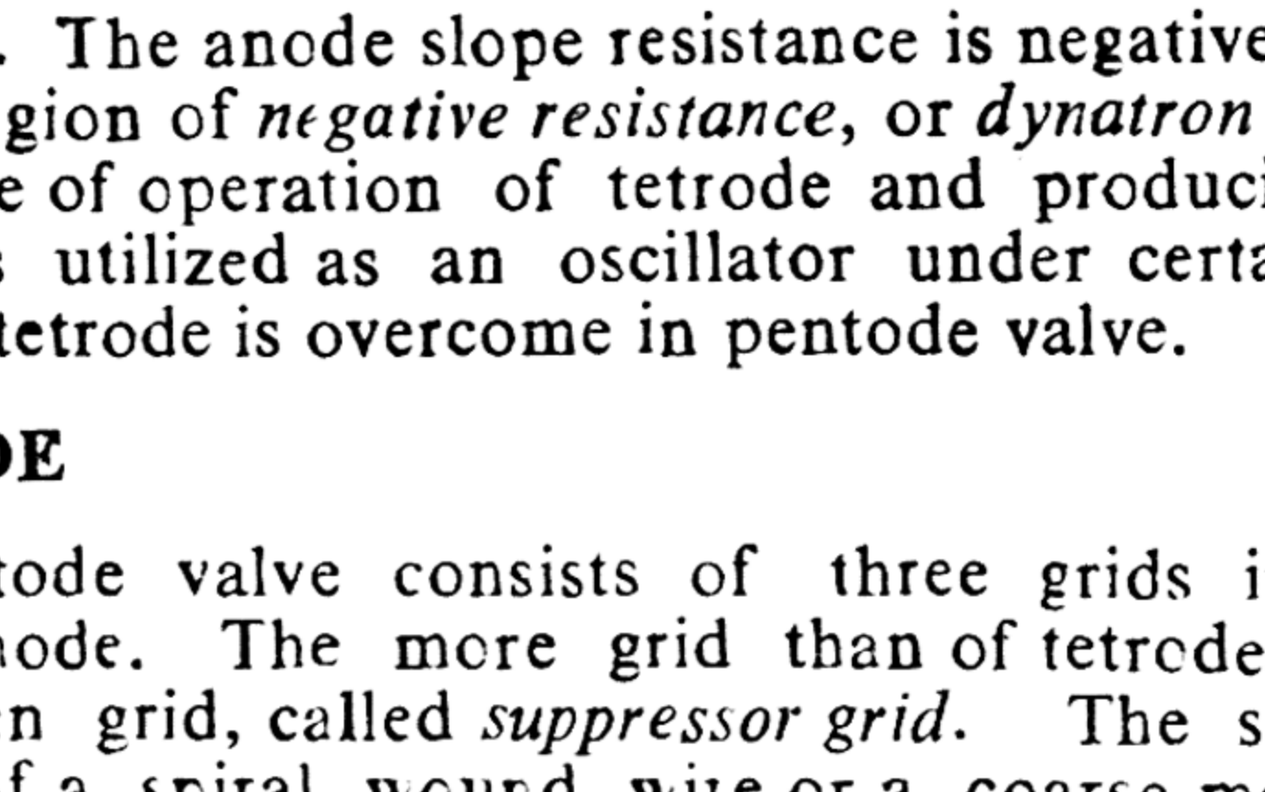


Fig. 1.23.

is maintained at cathode potential. The pentode valve and its symbol are shown in Fig. 1.23. The pentode has all the advantages of tetrode as : low grid-plate capacitance, high amplification factor

and high power output. The usually undesirable feature of secondary emission of negative resistance is overcome by the use of additional electrode in the form of suppressor grid.

Function of Suppressor Grid. G_3 is usually connected directly to the cathode and is thus at a negative potential with respect to plate. The secondary electrons from plate are repelled back to plate by the negative field of suppressor grid. Thus the secondary electrons can not reach the screen grid (even it is positive than plate). Thus the plate current rises smoothly without any fall (in negative resistance region) up to the saturation condition. The suppressor increases the shielding effect between plate and screen grid, which further reduces the grid-plate inter-electrode capacitance.

Pentode Characteristics. The operation of pentode is exactly same as that of tetrode. The basic circuit for obtaining the static characteristics of pentode is that for a tetrode, as the fifth electrode, *i.e.*, the suppressor grid is connected to the cathode. Starting with zero voltage on the control grid (e_{c_1}) and screen grid (e_{c_2}) at 100 volts, values of plate and screen currents (ib and i_{c_2} respectively) are measured by increasing the plate voltage (eb) from zero to maximum. The observations are repeated for various negative values of control grid potential. The curves between ib and e_b are called anode characteristic curves (Fig. 1.24). These curves are free from any kinks, which increase the range of the operation of pentode. These undistorted characteristic curves give a large output power without distortion. Hence the pentode valve is frequently used in radio frequency amplifiers.

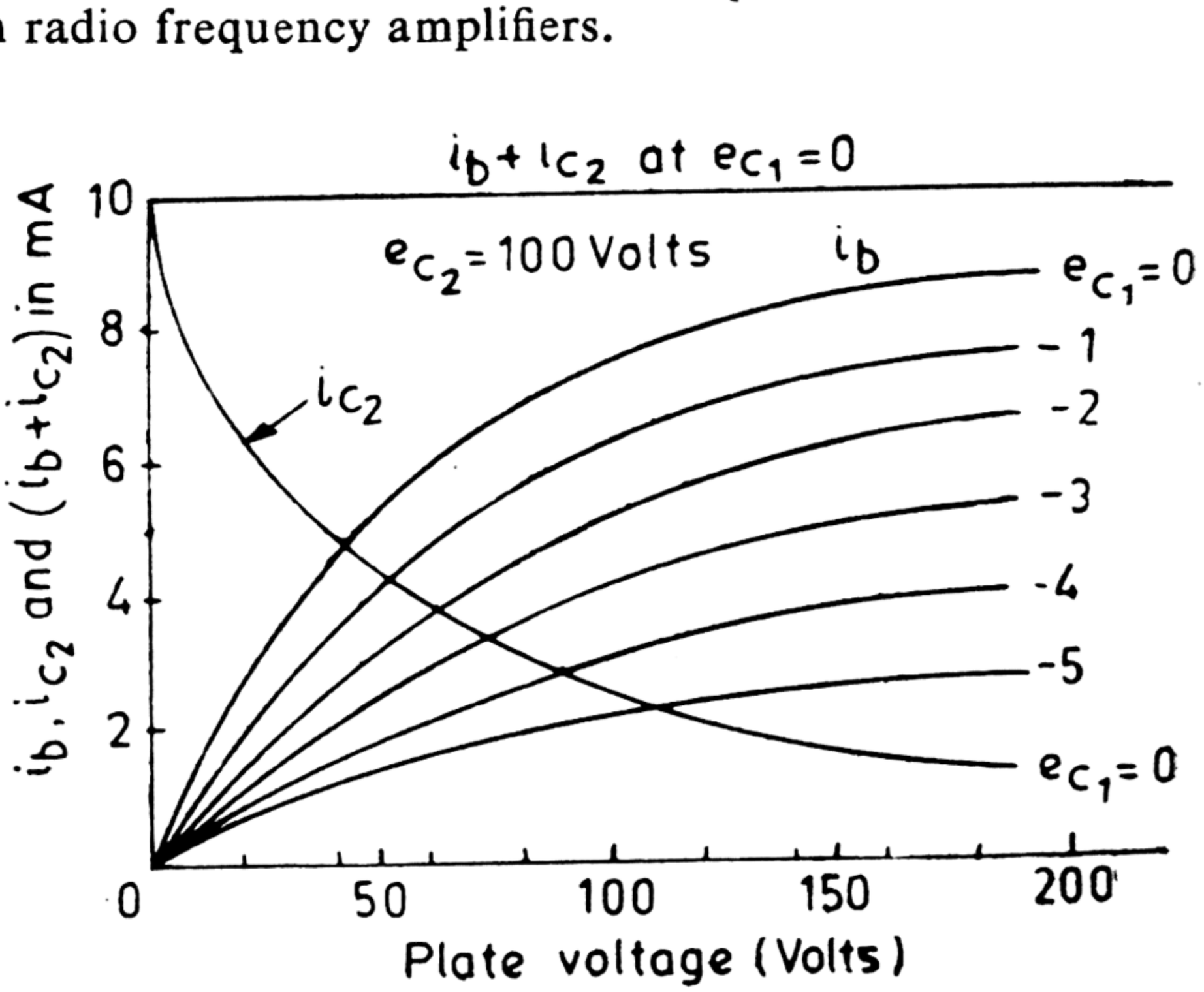


Fig. 1.24.

Constants of the Pentode. The tube constants of the pentode are determined in the same manner as for the triode and tetrode, by constructing triangles about the operating point. As the change in plate voltage is still less than with the tetrode, the amplification factor μ and the plate resistance r_p are higher for the tetrode. The transconductance is about the same as for a similar triode or tetrode.

The pentode may be used as a tetrode by connecting the suppressor grid externally with the plate. It may be used as a triode if its screen grid is connected with its plate and no connection is made at the suppressor grid.

1.12. MAGNETRON

A vacuum tube in which the electrons move under magnetic and electrostatic fields between the electrodes is called *magnetron*. These fields are usually-mutually perpendicular to each other. Most of the magnetron tubes are designed for use with short pulses. Several other types have also been developed for continuous-wave operation at powers of the order of 50 watts or even higher.

Usually two types of magnetrons are used. (*i*) Split anode magnetron and (*ii*) cavity magnetron. In the former type the anode is split in two segments. In the latter type, the anodes are of the multi-cavity type. There is a filament at the centre which is surrounded by a circular anode block in which a number of cavities are machined. These cavities are hollow chambers. The magnetic and electric fields are developed inside the each chamber. Each cavity acts as resonant tuning circuits as the fields of certain frequencies can be developed for a given cavity size more strongly than fields of other frequencies. The total energy is coupled out of the magnetron by means of a single loop of wire which is in one cavity.

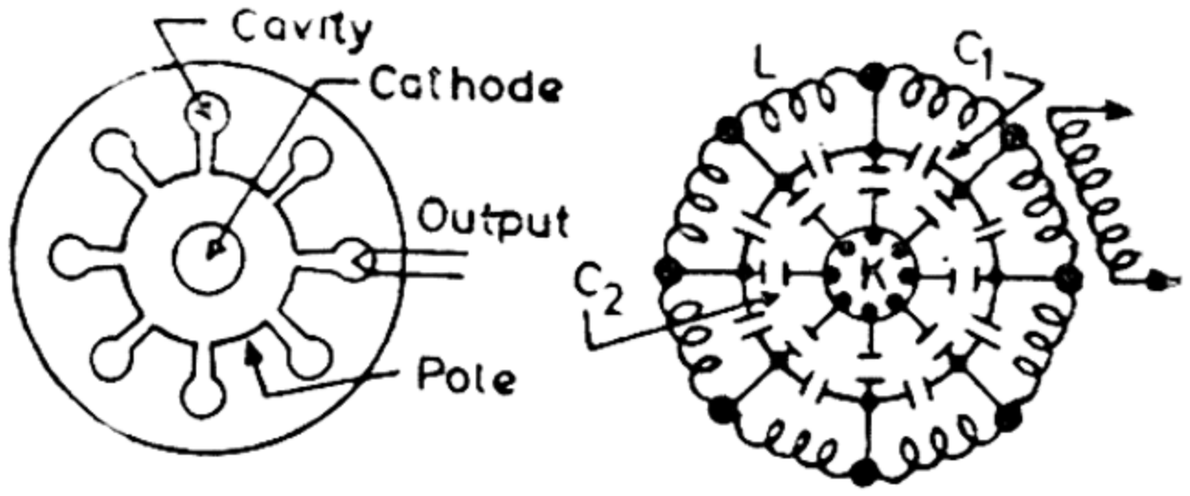


Fig. 1.25.

The anode cavities, together with spaces at the top and the bottom, serve as resonant circuits. The alternate poles become instantaneously +ve and —ve because of high frequency voltages. Although the whole anode structure is at +ve dc potential, but the effect is that of a number of resonant circuits in series. There is a capacity across each gap and an inductance formed by the inner part of the cavity. In the equivalent circuit, C_1 represents the capacity between two adjacent pole faces, C_2 represents the capacity between a pole face and the cathode and L, is the inductance of the inner surface of circular hole.

For a given pd between anode and cathode, the electron path depends upon the magnetic field. As the magnetic field is increased gradually, the electron path will tend to curve more and more until we reach to a value beyond of which electrons will never reach the anode. We can achieve a similar variation in electron motion by altering the anode potential. The increase of potential will increase the electron circle. Thus in magnetron, the problem is to adjust the magnetic field and anode potential to obtain the desired oscillation.

To operate the magnetron, a positive dc voltage is connected between the cathode and anode block and an axial magnetic field is provided by an external magnet. The electrons emitted from the cathode are attracted by the anode. On account of an axial magnetic field the electrons travel in the circular path. While travelling toward the anode the electrons shock excite the cavity resonators. As a result of this shock excitation, the cavity resonators will oscillate at their resonant frequencies.

1.13. GAS-FILLED TUBES

Although the vacuum tubes are very much useful in communication purposes. When a large current is required, gas-filled tubes are used. They depend for their operation on the characteristics of gaseous discharges. The gas tubes may be classified as (1) gas-filled diodes, (2) gas-filled triodes, (3) cold cathode tubes (diodes and tricdes and (4) multi-element pool type tubes.

1. **Gas-Filled Diodes.** When the vacuum diode is conducting, the electrons which are in transit from the cathode to the anode, tend to limit the attractive force of the anode. This effect can be reduced if the tube is filled with mercury vapour or an inert gas like argon, xenon, etc.

In the mercury vapour diode, a directly heated oxide coated robust filament of considerable emission area is used. After exhausting of air, a small amount of mercury is introduced in the tube. This mercury vaporises when the filament is heated. If the cathode to anode voltage exceeds 10.4 volts (the ionization potential of mercury), the emitted electrons have enough energy to ionize the mercury atoms. The liberated electrons are attracted to the anode, while the positive mercury ions move much more slowly, due to their heavy weights, toward the cathode and thus neutralise the negative space charge. Thus with anode potentials somewhat in excess of the ionisation potential, the anode current is equal to almost the entire filament emission. The resistance of the valve is very small in comparison with the vacuum diode. The anode current is limited by the anode load, otherwise valve will damage. Even for very large currents, the anode voltage should not be much greater than the ionization potential. If no provision is made to limit the currents to a safe value, the tube will be destroyed.

The gas pressure corresponds to that of mercury vapour at the temperature of the point of mercury condensation. The pressure depends upon the temperature, hence the characteristics of mercury vapour tubes are functions of temperature. For critical uses, the gas used are argon or xenon, as for these gases the gas pressure changes slightly with temperature.

High pressure gas diodes are available which contain argon or a mixture of argon and mercury at a pressure of about 5 cm. In such tubes cathodes consist of a short, heavy thoriated tungsten or oxide-coated filament and are located close to heavy graphite anodes. The fairly high pressure gas provides the positive ions for reducing the space charge and to prevent the evaporation of the thorium or the coating from the filament. These diodes are limited to low potential operation.

2. **The Thyratron.** The thyratron is a *hot cathode gas triode.* It is filled with mercury vapour or inert gas and employs a robust electrode structure with an indirectly heated cathode. The massive grid structure is so designed as to provide almost complete electrostatic shielding between the cathode and the anode. The grid usually consists of a cylindrical structure, a baffle or a series of baffles containing small holes being inserted between the anode and the cathode. The shielding by the grid is so complete that the application of a small grid potential before conduction is started is adequate to overcome the field at the cathode due to a large anode potential.

Since the grid potential affects the energy of the electrons accelerated to the anode. Suppose the grid is set at a negative potential of $-V_g$, then the anode potential will not cause electrons to traverse the grid aperture until its potential is cV_g, where c is greater than one and depends on the electrode geometry. When the anode potential V_a just exceeds this value, the electrons traverse the grid aperture and are accelerated to the anode If V_a is greater than the ionization potential of the residual gas, the gas ionises and the positive ions thus formed will neutralise the space charge. The more electrons are thus accelerated and an arc discharge forms from cathode to anode. The discharge starts at the critical grid potential which permits the first electron to ionise. Once the arc has been initiated, the grid losses complete control over the arc. The tube can be extinguished only if we reduce the tube current below the minimum value required to maintain ionization. When the gas tube stops conducting the grid regains control and the grid bias determine the amount of plate voltage needed to fire the tube again. The curve that relates the grid ignition potential with the potential of the anode for conduction just to begin is known as the *critical grid curve* and is shown by graph in Fig. 1.26 (*b*). The characteristics in mercury vapour are functions of temperature. Argon or neon types are available for temperature sensitive or *critical application.* The criti-

cal grid characteristics of —ve control thyratron with *Hg* condensation temperature as parameter are shown in Fig. 1.26 (*c*).

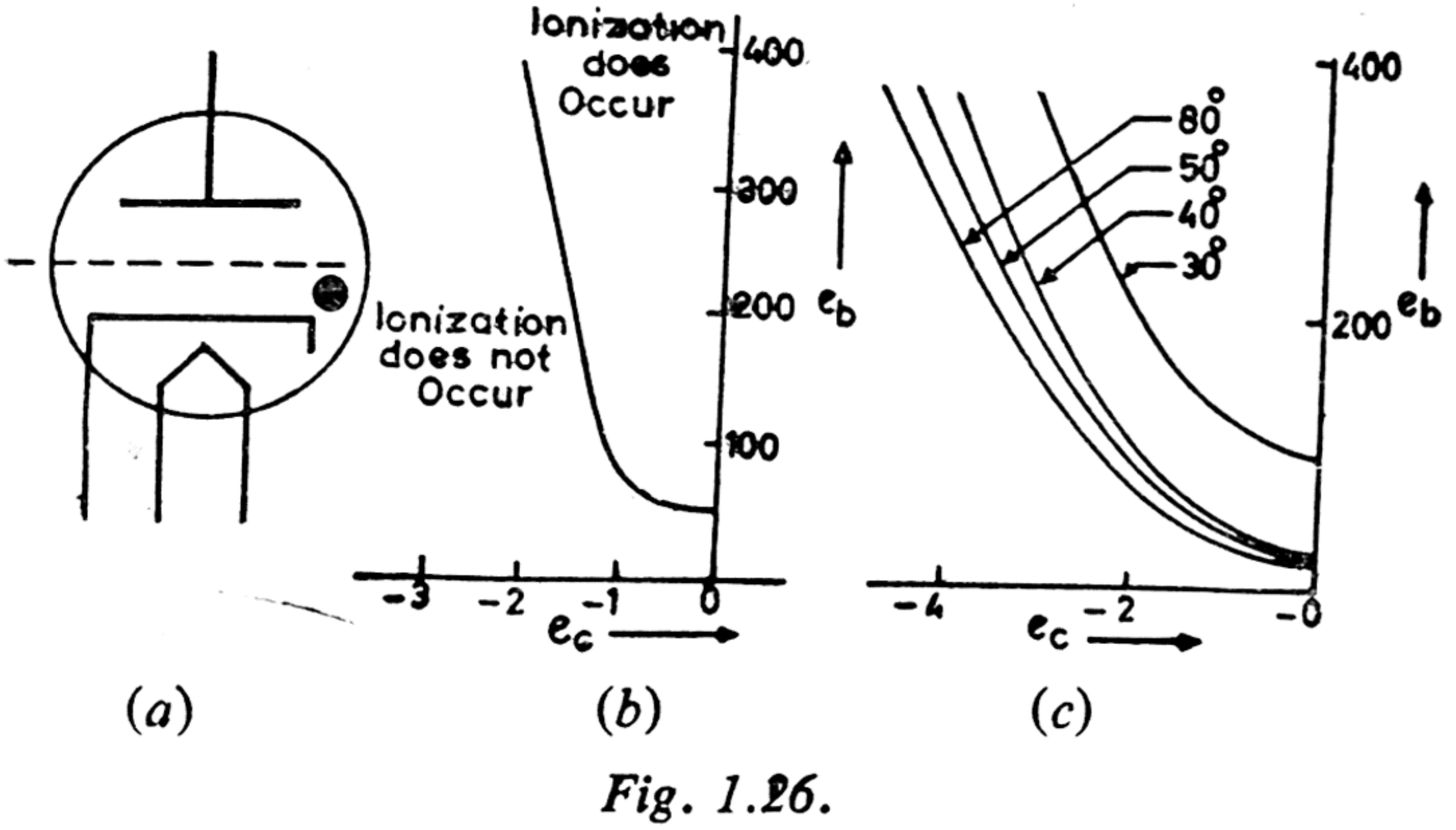


Fig. 1.26.

It is desired that the grid should control the starting of arc in each half cycle of alternating anode voltage. This places an upper frequency limit on thyratron application, since the ionization is established in a few micro-seconds and to quench ionization current must be stopped for a period of the order of 100μ sec. For this reason, thyratrons are often used with alternating anode potential in relay circuits.

The main difference between a vacuum triode and thyratron is that in the former case the plate current is controlled by grid voltage while in the latter case the grid voltage only initiates the arc and having no control on the plate current. The tubes containing mercury vapour carry large currents and are very suitable for industrial purposes. For high speed working any inert gas may be used.

3. **Cold Cathode Gas Diodes (VR-tubes).** Cold cathode gas diode is a common glow tube. It operates in the normal glow discharge region. As the voltage drop across such a tube over the operating range is fairly constant and independent of the load current variation, hence it is also named as *voltage regulator tube* (*VR-tube*). One type of commercial *VR*-tube consists of a central anode wire surrounded by a cylindrical cathode. The electrodes are of nickel and the inner surface of the cathode is oxide coated. The gases commonly used are helium, neon and argon. The tubes containing helium or neon usually contain a small amount (1%) of argon, as the latter lowers the starting voltage. The type of the gas and the cathode materials are the only factors which give a definite voltage. The tubes of the types *VR*-75, *VR*-90, *VR*-105, and *VR*-150 have normal voltages of 75, 90, 105 and 150 volts respectively, with a normal current range of 5 to 40 *mA*. The current rating of these tubes is determined by the area of the cathode and by the pressure of the gas in the tube.

If the potential between these two electrodes is high enough, there will be an exponential increase in current with voltage due to the further ionization by the ions and electrons. This non-self maintaining region of the *V-I* curve is called the *Townsend discharge*. At some high potential the heavy ions cause secondary emission by bombardment of the cathode. The emitted secondary electrons in turn ionize by collision and thus making process self-maintaining. The potential at which this happens is known as *breakdown potential*. Upon breakdown, the potential across the tube is nearly independent of the current and a visible glow (*normal glow*) is appeared in the gas and on the cathode. The tubes in this region are used for voltage regulation.

A circuit for operation of *VR*-tubes is shown in Fig. 1.27. These tubes are rated according to the firing voltage, the voltage drop which will be maintained across it and current range over which voltage will remain constant. The difference between the supply voltage and the maintaining voltage of the *VR*-tube will be developed across *Rs*, which will depend on the changes expected in the supply voltage, the load resistance and on the *VR*-tube characteristics. The current (I_1+I_2) will be sufficient to provide a voltage drop in *Rs* equal to V_1-V_2. To maintain the drop in *Rs* and voltage V_2 constant, a decrease in load current will cause a corresponding increase in tube current.

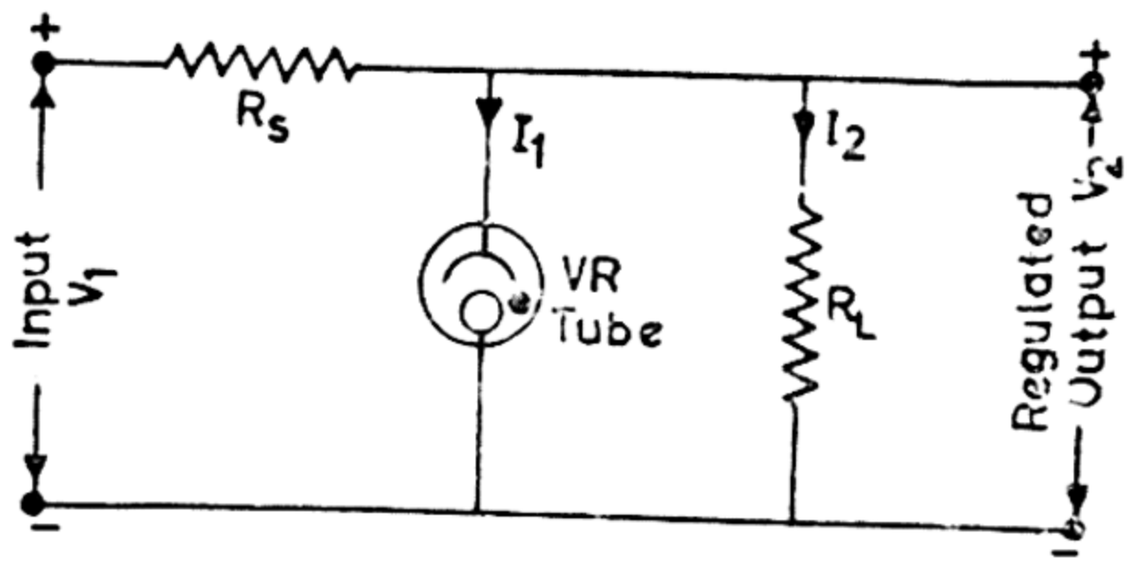


Fig. 1.27.

Resistor $R_S=(V_1-V_2)/I_{max}$. ...(41)

The tube also has some rated minimum current I_{min} and the load may draw any current from zero to $I_{max}-I_{min}$ without effecting the operation of *VR*-tubes.

The *VR*-tubes may be connected in series for higher controlled voltages. Since the action of regulation is rapid, fast voltage changes are wiped out and the tube will act as a filter.

If two *VR*-tubes of same constant voltage say (150 volts) are desired. The tube with widest operating range should be chosen. If a potential higher than that of the glow tube is desired, several tubes must be connected in series. For example, *VR*-150 and *VR*-105 in series will provide a constant 255 volt source.

VR-tubes operate quite satisfactorily but are limited in their usefulness because of their fixed potential ratings and the low current carrying capacity.

4. Cold Cathode Gas Triodes. This tube is referred to as a *grid glow tube*, although the third element, '*grid*' is named as

cal grid characteristics of —ve control thyratron with *Hg* condensation temperature as parameter are shown in Fig. 1.26 (*c*).

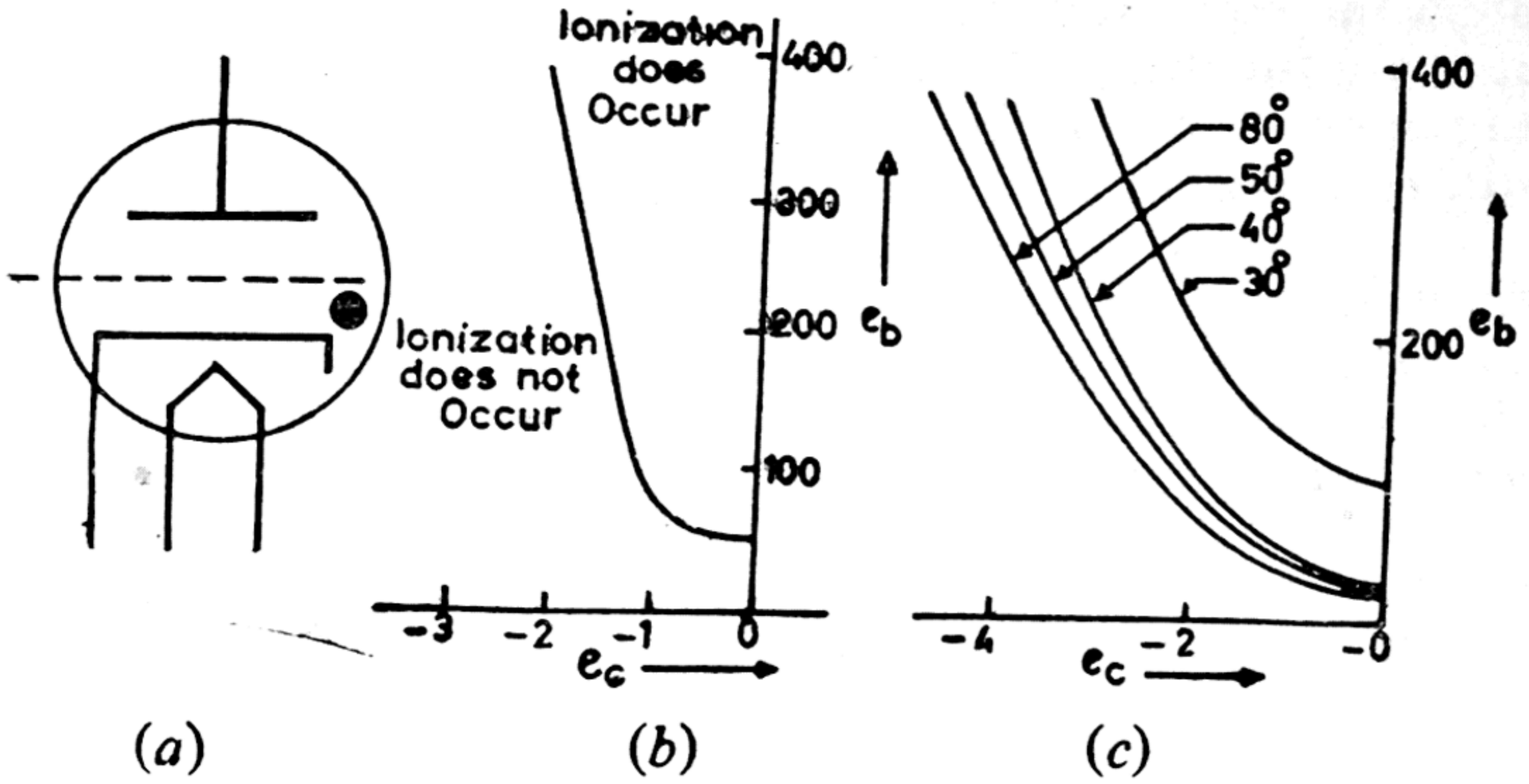


Fig. 1.26.

It is desired that the grid should control the starting of arc in each half cycle of alternating anode voltage. This places an upper frequency limit on thyratron application, since the ionization is established in a few micro-seconds and to quench ionization current must be stopped for a period of the order of 100μ sec. For this reason, thyratrons are often used with alternating anode potential in relay circuits.

The main difference between a vacuum triode and thyratron is that in the former case the plate current is controlled by grid voltage while in the latter case the grid voltage only initiates the arc and having no control on the plate current. The tubes containing mercury vapour carry large currents and are very suitable for industrial purposes. For high speed working any inert gas may be used.

3. **Cold Cathode Gas Diodes (VR-tubes).** Cold cathode gas diode is a common glow tube. It operates in the normal glow discharge region. As the voltage drop across such a tube over the operating range is fairly constant and independent of the load current variation, hence it is also named as *voltage regulator tube* (*VR-tube*). One type of commercial *VR*-tube consists of a central anode wire surrounded by a cylindrical cathode. The electrodes are of nickel and the inner surface of the cathode is oxide coated. The gases commonly used are helium, neon and argon. The tubes containing helium or neon usually contain a small amount (1%) of argon, as the latter lowers the starting voltage. The type of the gas and the cathode materials are the only factors which give a definite voltage. The tubes of the types *VR*-75, *VR*-90, *VR*-105, and *VR*-150 have normal voltages of 75, 90, 105 and 150 volts respectively, with a normal current range of 5 to 40 *mA*. The current rating of these tubes is determined by the area of the cathode and by the pressure of the gas in the tube.

If the potential between these two electrodes is high enough, there will be an exponential increase in current with voltage due to the further ionization by the ions and electrons. This non-self maintaining region of the *V-I* curve is called the *Townsend discharge*. At some high potential the heavy ions cause secondary emission by bombardment of the cathode. The emitted secondary electrons in turn ionize by collision and thus making process self-maintaining. The potential at which this happens is known as *breakdown potential*. Upon breakdown, the potential across the tube is nearly independent of the current and a visible glow (*normal glow*) is appeared in the gas and on the cathode. The tubes in this region are used for voltage regulation.

A circuit for operation of *VR*-tubes is shown in Fig. 1.27. These tubes are rated according to the firing voltage, the voltage drop which will be maintained across it and current range over which voltage will remain constant. The difference between the supply voltage and the maintaining voltage of the *VR*-tube will be developed across *Rs*, which will depend on the changes expected in the supply voltage, the load resistance and on the *VR*-tube characteristics. The current (I_1+I_2) will be sufficient to provide a voltage drop in *Rs* equal to V_1-V_2. To maintain the drop in *Rs* and voltage V_2 constant, a decrease in load current will cause a corresponding increase in tube current.

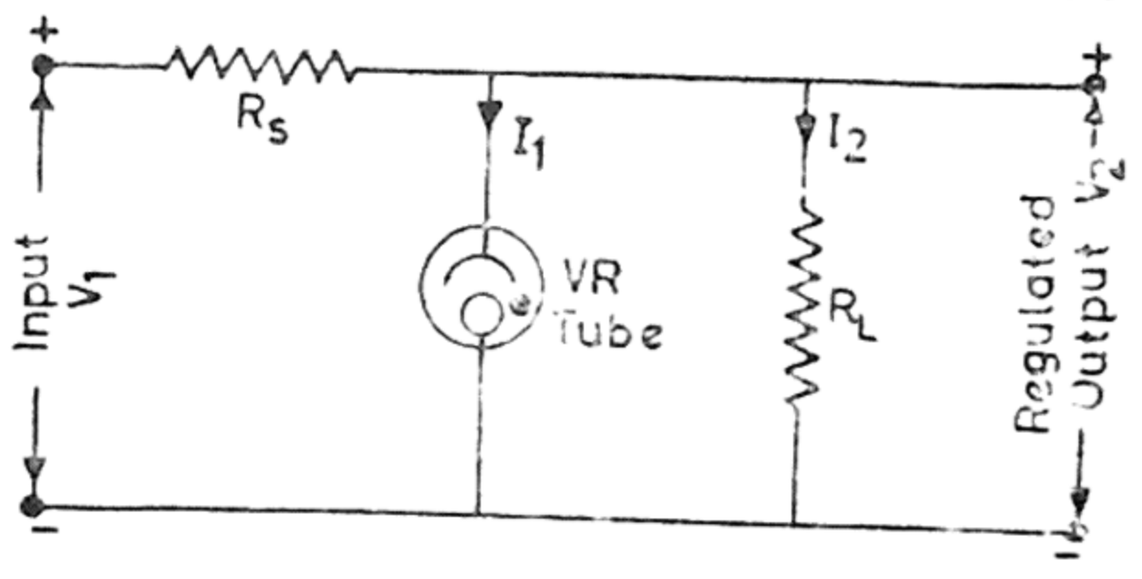


Fig. 1.27.

Resistor $R_S=(V_1-V_2)/I_{max}$. ...(41)

The tube also has some rated minimum current I_{min} and the load may draw any current from zero to $I_{max}-I_{min}$ without effecting the operation of *VR*-tubes.

The *VR*-tubes may be connected in series for higher controlled voltages. Since the action of regulation is rapid, fast voltage changes are wiped out and the tube will act as a filter.

If two *VR*-tubes of same constant voltage say (150 volts) are desired. The tube with widest operating range should be chosen. If a potential higher than that of the glow tube is desired, several tubes must be connected in series. For example, *VR*-150 and *VR*-105 in series will provide a constant 255 volt source.

VR-tubes operate quite satisfactorily but are limited in their usefulness because of their fixed potential ratings and the low current carrying capacity.

4. Cold Cathode Gas Triodes. This tube is referred to as a *grid glow tube*, although the third element, '*grid*' is named as

control electrode or control anode. The tube consists of three electrodes. The *cathode* is a nickel surface coated with a barium and strontium mixture. The *anode* is a pure nickel wire. The *control electrode*, also named as *starter anode*, is placed close to the cathode. It may be a wire loop in some cases and a metallic shield with a hole in other cases. The tube contains a mixture of neon (99%) and argon (1%) at a pressure of 40 mm *Hg*.

The spacing of the electrodes are so adjusted that a discharge takes place from the cathode to the control anode at a lower voltage than is required for a discharge from cathode to the anode. Now it is possible to transfer discharge to the anode. The anode-cathode voltage required for this process is a function of the current in the cathode to control electrode circuit, *i.e.*, the *transfer current*. The curves between anode voltage *v/s* transfer current are called the *transfer* or *transition* characteristics. For zero transfer current the anode voltage is equal to the breakdown voltage between the cathode and the anode. If the transfer current is increased, the required anode-cathode voltage will fall rapidly, but will not go below the maintaining voltage for this gap. The transfer characteristic thus depends upon the type of gas, the pressure and the electrode structure.

The cold cathode gas triodes are very much used as rectifiers in selective ringing telephone circuits, as voltage regulators, as relaxation oscillators, and as relays or switching units. The circuit diagram for its use as relay is shown in Fig. 1.28. The plate supply voltage E_{bb} is greater than the main gap sustaining voltage but less than the main gap breakdown voltage. The voltage E_{cc} is less than the control gap breakdown voltage. Thus a positive pulse will raise the control anode voltage above the breakdown point and a discharge will occur between these electrodes. If the transfer current is sufficiently high so that E_{bb} will cause breakdown, then the discharge will transfer to the anode circuit. In this way a small input current is able to control several milliamperes in the load circuit.

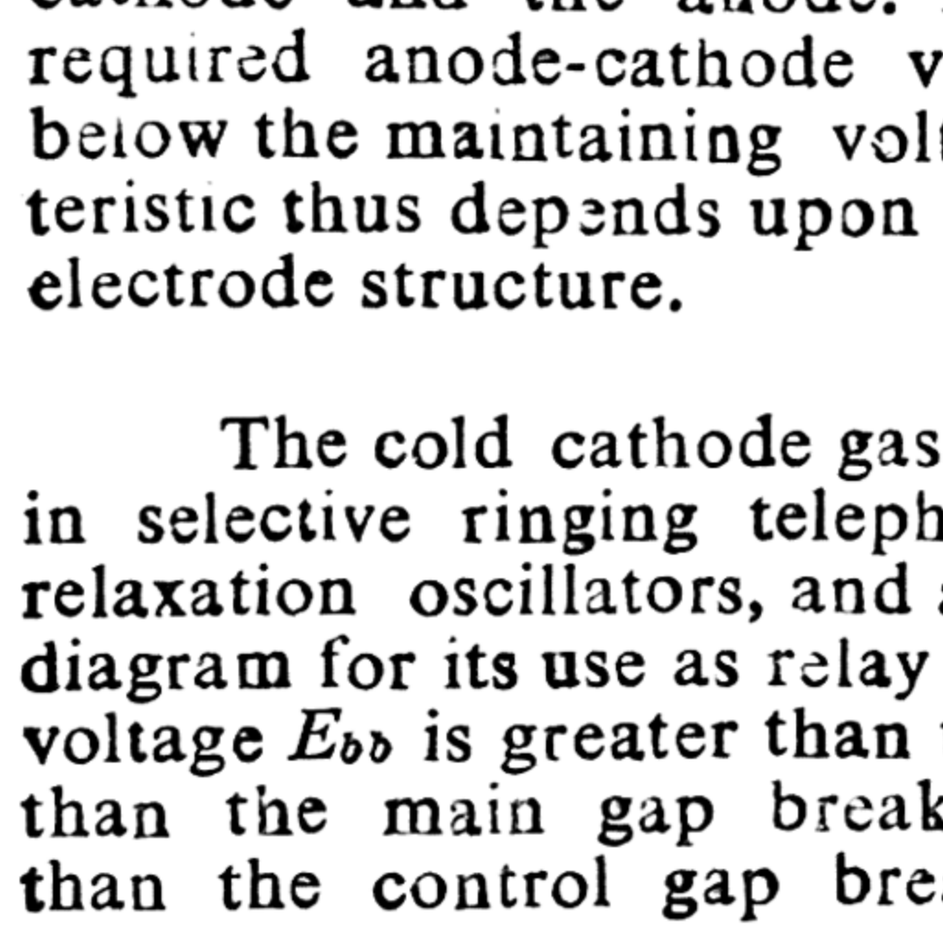


Fig. 1.28.

The cold cathode gas triodes have many features like the thyratron. All these triodes contain control electrodes. The discharge between the cathode and the anode cannot take place unless the control electrode is properly energized. Thus the transfer characteristic of the tube is the counter part of the critical grid breakdown characteristic of the thyratron. Despite the common features, the fields of application are quite different, as the maximum current from a glow tube is $\sim 100 mA$, while from the thyratron is $\sim 100 A$.

1.14. BEAM POWER TUBE

It is the modified form of pentode and contains a *cathode*, a *control grid*, a *screen grid* and an *anode*. It is constructed so that the electrons flowing to the plate are made to travel in concentrated beams. As it is capable of handling larger amounts of power, it is known as *beam power tube*.

In the modified form of the tube the screen grid wires are wound with the same pitch as the control grid. In the tube they are placed so that each turn of the screen grid is shaded from the cathode by a turn of the control grid. The number of electrons captured by the screen is thus reduced and the electrons are confined in thin slices or flat beams in passing between the grids. The electron beams are further concentrated by cathode connected beam forming plates. These plates are included to maintain uniform current density over a cross section of the beam and to prevent the

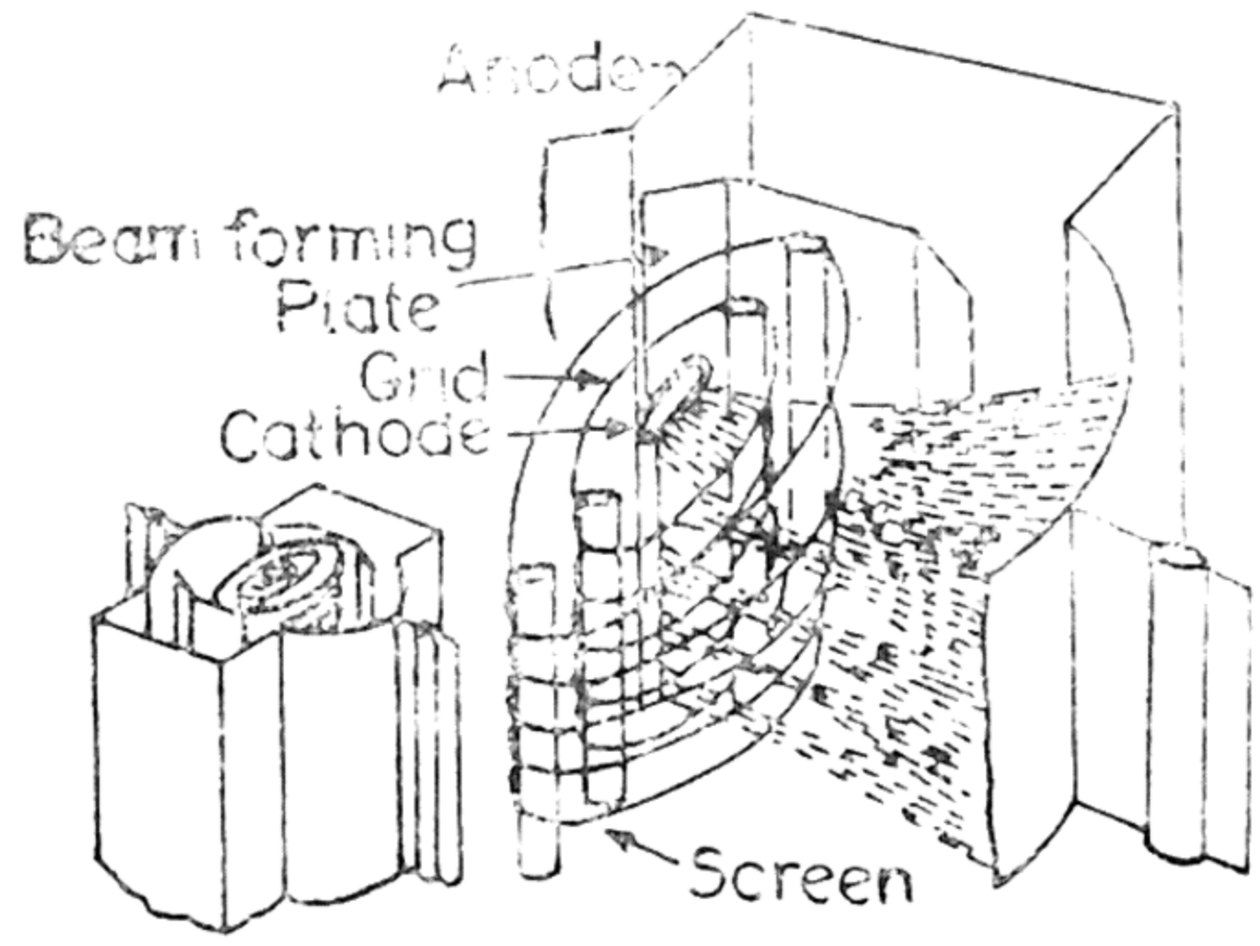


Fig. 1.29.

return of secondary electrons by paths outside the beam. The effect is that of a virtual suppressor grid. In some beam power tubes an actual suppressor grid is used. By increasing the spacing between the screen grid and the plate, a space charge is set up in this area. This space charge repels the secondary electrons emitted from the plate and forces them back into the plate.

Plate characteristics of a beam power tube (6 L 6) is shown in Fig. 1.30. These curves (represented by *B*) are similar to those of the pentode (represented by *P*). The major advantages of the beam power tube over the pentode are that the knees of the curves are sharper, the relatively straight and almost horizontal part of the curves covers a wider range of plate voltage, for a large power outputs

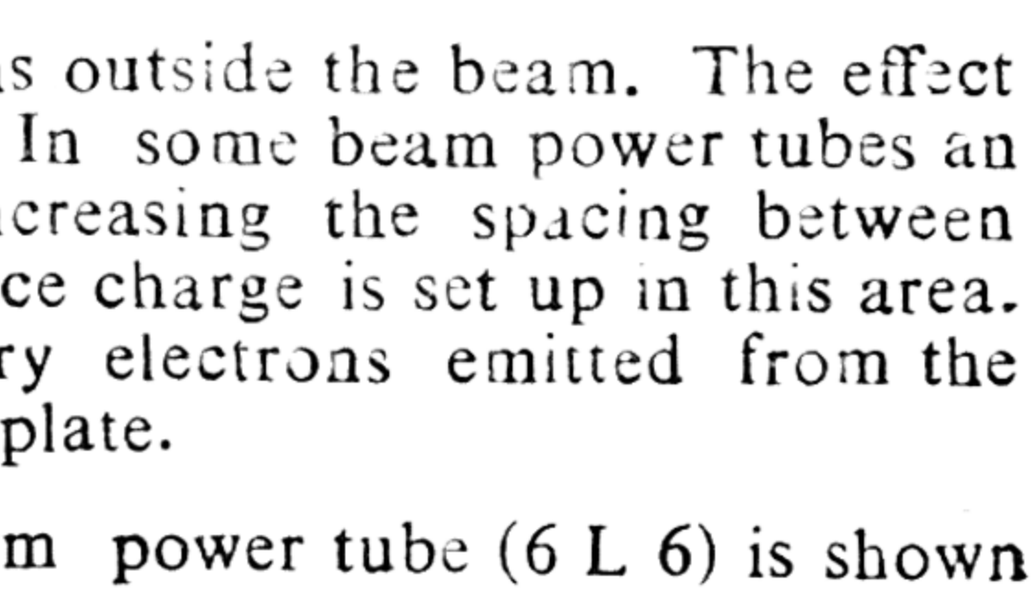


Fig. 1.30.

a lower input voltage is required, g_m is appreciably larger. μ is high when compared with triodes and low when compared with tetrodes and pentodes. The plate resistance is high but not so high as that of pentodes. This tube is having *high output*, *high efficiency* and *high* power sensitivity.

1.15. TECHNICAL DATA FOR THE VACUUM TUBES

The technical descriptions of receiving tubes used in standard broadcast, *FM* and television receivers in radio amplifiers and in many other diverse applications are given in the *RCA*-receiving tube manual, technical series *RC-27*. The tubes are assigned numbers which provide information about the construction and application of the tubes. The tubes generally utilize a glass or metal envelope and a base. Present-day conventional tube designs utilizing glass envelopes and integral glass bases include the seven-pin and nine-pin *miniature* types, the nine-pin *novar* and neonoval types and the twelve pin *duodecar* types. These pins are inserted in the sockets. The required sockets are 7-contact, octal, miniature 9-contact, duodecar 12-contact, etc., types depending upon the tube requirement. The tube number is comprised of at least three units. The first unit is a number representing the rated value of the filament or heater voltage. The second unit has one or more letters. The third unit is again a number representing either the number of elements in the tube or the number of useful leads. The fourth unit consists of one or more letters which indicate the constructional features of the tube, *e.g.*, *G* indicates a glass tube and octal base and *GT* indicates a miniature (or tubular) glass tube octal base.

Exercises

Example 1. *Calculate the work function in electron volts for sodium metal, given that the threshold wavelength is 6800 A°.*

Threshold wavelength is the wavelength of the incident light for which the corresponding incident energy of the protons is just equal to the wavefunction ϕ, *i.e.*,

$$h\nu = \phi \qquad \text{or} \qquad hc/\lambda = \phi.$$

The substitution of numerical values gives

$$\phi = 6.63 \times 10^{-34} \times 3 \times 10^{8}/6800 \times 10^{-10}$$
$$= 2.92 \times 10^{-19} \text{ joule.}$$
$$= 1.82eV.$$

Example 2. *Is it possible to liberate an electron from a metal surface of work function 4.8eV with an incident radiation of (a) wavelength 2000A° and (b) wavelength 5000A° ?*

The energy of the incident radiation of wavelength λ is given by

$$E=h\nu=hc/\lambda.$$

(*a*) $$E=6.63\times 10^{-34}\times 3\times 10^{8}/2000\times 10^{-10}$$
$$=9.95\times 10^{-19}\ \text{joule}=6.22eV.$$

(*b*) $$E=6.63\times 10^{-34}\times 3\times 10^{8}/5000\times 10^{-10}$$
$$=3.98\times 10^{-19}\ \text{joule}=2.49\ eV.$$

The electron can be liberated only and only when the incident energy of the radiation is greater than the work function of the metal surface. Hence the electron will be liberated only in the first case ($E>\phi$).

Example 3. *When yellow light from a sodium lamp falls over a photocell, a negative potential of 0.30 volt is needed to stop all the electrons from reaching the collector. What potential will be needed to stop the electrons if light with =4500 A° is used ?*

We know that the energy of the incident photons is partly used against work function ϕ and rest is given to the electrons as *K.E.* To stop the electrons, a repelling field is applied. They will be stopped by a negative potential V, if $eV=K.E.$ Hence we have

$$h\nu=\phi+K=\phi+eV \quad \text{or} \quad hc/\lambda_1=\phi+eV_1.$$

$$\therefore \qquad hc(1/\lambda_1-1/\lambda_2)=e(V_1-V_2).$$

In our case $e=1.6\times 10^{-19}$C, $h=6.63\times 10^{-34}$ J.sec, $c=3\times 10^{8}m/s$, $\lambda_1=5.893\times 10^{-7}m$, $\lambda_2=4.5\times 10^{-7}m$, and $V=0.3$ volts, hence the substitution of these values in the above relation gives

$$V_2=0.953 \text{ volt.}$$

Example 4. *A pd of 100 volts is applied across the plate and cathode of a high vacuum diode with plane parallel electrodes separated by a distance of 4 cm. Find the potential at the midplane between the electrodes when the electron density at that plate is (a) zero, (b) 10^{12} electrons/m^2.*

The electric charge density due to space charge electrons and *pd* between the electrodes are given by

$$d^2V/dx^2=-\rho/\epsilon_0.$$

If the temperature is so small that the electron emission is practically zero. Thus, Poisson's equation becomes

$$d^2V/dx^2=0.$$

Integrating twice, we get $V=ax+b$.

Using boundary conditions : (*i*) when $x=0$, $V=0$ (potential of cathode) ; (*ii*) when $x=d$ (the separation of electrodes), $V=V_0$ (potential of anode), we get

$$a=V_0/d \text{ and } b=0. \quad \text{Hence } V=V_0x/d.$$

If the temperature of the filament is greater, then the electron density is not zero and has value $-\rho$.

$$\therefore \qquad d^2V/dx^2=+\rho/\epsilon_0.$$

Integrating twice, we get $V=\rho x^2/2\epsilon_0+a_1x+b_1$.

Using boundary conditions, we get

$$b_1=0 \quad \text{and} \quad a_1=V_0/d-\rho d/2\epsilon_0.$$

$$\therefore \quad V=\frac{\rho x^2}{2\epsilon_0}+\frac{V_0}{d}x-\frac{\rho d}{2\epsilon_0}x=\frac{V_0}{d}x-\frac{\rho x}{2\epsilon_0}(d-x)$$

Potential at the mid-point

(*a*) when $\rho=0$, $V=V_0\,(x/d)=V_0(\frac{1}{2})=100\times\frac{1}{2}=50$ volts.

(*b*) when $\rho=10^{12}$ electrons/meter^3,

$$V=100\times\frac{0.02}{0.04}-\frac{10^{12}\times1.6\times10^{-19}\times0.02}{2\times8.85\times10^{-12}}[0.04-0.02]$$

$=46.38$ volts.

Example 5. *A tungsten filament of diameter 1.5 mm and length 6 cm is concentric with a cylindrical plate in vacuum diode. When filament temperature is 2400°K, the plate current is 100mA with the plate voltage of 120 volts. Calculate the plate current with plate voltage of 240 and 480 volts.*

Emission current density $j==AT^2e^{-b/T}$.

For tungsten, $A=60.2\times10^4$ and $b=52400$

$j=60.2\times10^4\times(2400)^2e^{-52400/2400}=1142.7$ amp/m^2.

Hence emission current $i=j\times$ Area of cathode surface $(2\pi rl)$

$=1142.7\times2\times3.14\times0.75\times10^{-3}\times0.06$

$=322.93\ mA$.

As with plate voltage of 120 volts, the current is $100mA$, hence under space charge limited condition

$$i_b=kV_b^{3/2} \quad \text{or} \quad 100=k(120)^{3/2}$$

At $V_b=240$ volts, $\quad i_b=k(240)^{3/2}=100(2)^{3/2}=282.8\ mA$

At $V_b=480$ volts, $\quad i_b=k(480)^{3/2}=800\ mA$.

These results show that i_b at $V_b=240$ volts is less than the emission current, and at $V_b=480$ volts is greater than the emission current since the plate current is temperature limited hence even at 480 volts, the plate current is 322.93 *mA*, the emission current.

Example 6. *A vacuum diode has a cylindrical plate of length 6 cm and diameter 2.0 cm. A tungsten filament of length 6 cm and diameter 0.04 cm is at the axis of the cylindrical plate, and is at a temperature of 2400°K. Calculate (a) saturation current and (b) minimum plate voltage required to produce this current.*

Emission current density $j=AT^2e^{-b/T}$.

$$j=60.2\times10^4\times(2400)^2e^{-52400/2400}=1142.7 \text{ amp/m}^2.$$

Emission current $i=j\times2\pi rl=1142.7\times2\times3.14\times0.02\times10^{-2}\times0.06$

$=0.086$ amp=Saturation current

Under space charge condition, the plate current

$$i_b=14.6\times10^{-6}\times\frac{l}{r_a}.\ \frac{V_b^{3/2}}{\beta^2} \text{ amp.}$$

where β depends on radius of anode r_a and radius of cathode r_k as

$$\beta=\log_e(r_a/r_k)-\tfrac{2}{5}[\log_e(r_a/r_k)]^2+\ldots\ldots$$

$$=\log_e(50)-\tfrac{2}{5}(\log_e 50)^2+\ldots=1.09 \text{ (Approx).}$$

If the minimum plate potential required for saturation current is V_s, then

$$0.086=14.6\times10^{-6}\times\frac{0.06}{0.01}\times\frac{V_s^{3/2}}{(1.09)^2}$$

$\therefore\quad V_s^{3/2}=1166$ Approx. or $V_s=110.8$ volts.

Example 7. *The slope of anode and mutual characteristics of a valve are 0.02 mA/volt and 1 mA/volt respectively. Calculate the amplification factor of the valve.*

The slope of anode characteristic $=dI_b/dE_b$

$=0.02$ *mA*/volt.

The slope of mutual characteristics $=dI_b/dE_c=1$ *mA*/volt.

$$\therefore\quad \text{Amplification factor } \mu=\frac{dE_b}{dE_c}=\frac{dI_b}{dE_c}\times\frac{dE_b}{dI_b}$$

$$=\frac{(dI_b/dE_c)}{(dI_b/dE_b)}=\frac{1}{0.02}=50.$$

Example 8. *The table given below gives the plate currents for different values of plate voltage for grid voltages of zero and −2.0*

volts. Calculate the coefficients of the triode at plate voltage of 100 volts.

Plate voltage E_b in volts	80	100	120	140
Plate current I_b in mA for grid potential E_c=0 volts	8	10.9	13.8	16.8
Plate current I_b in mA for E_c=—2 volts	2 2	5.1	8	10.8

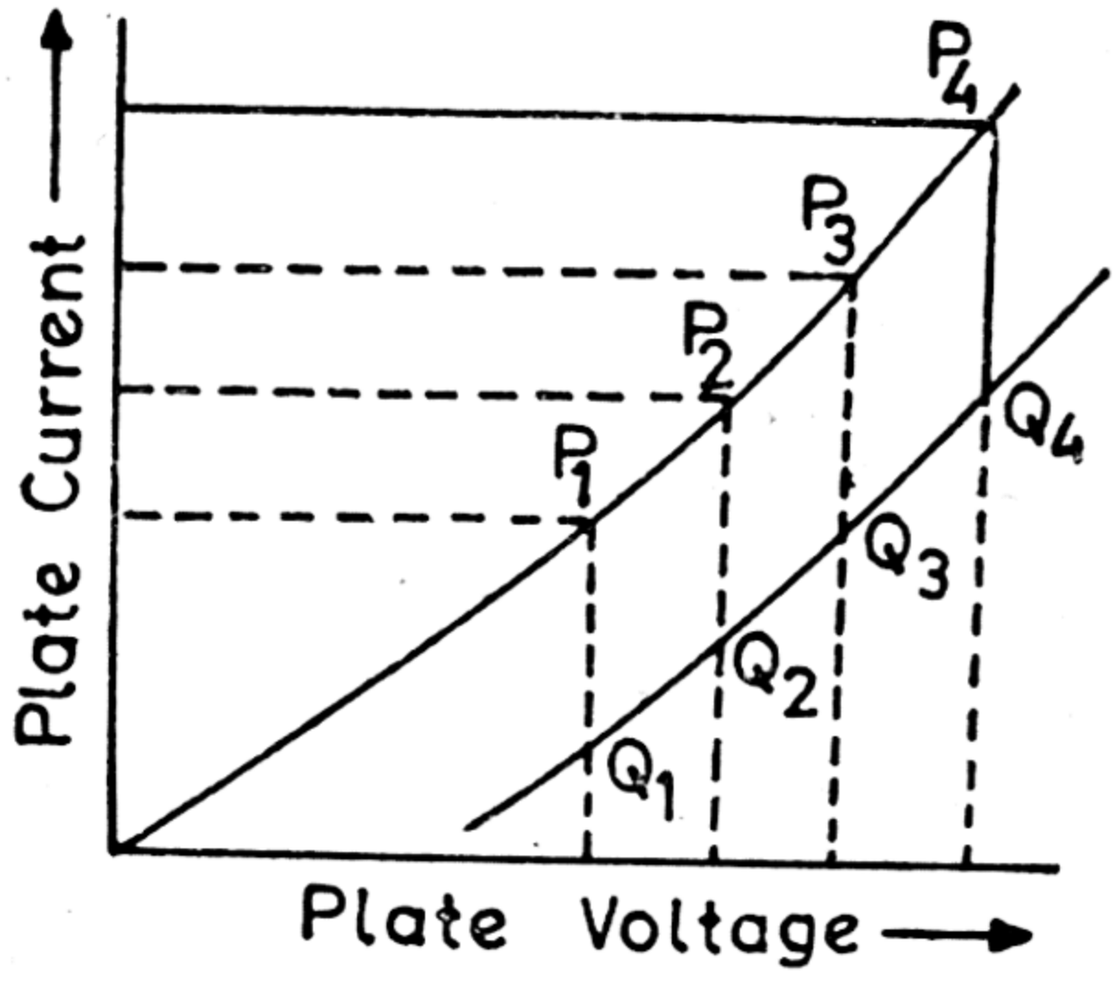


Fig. 1.31.

For E_C=0 volts, at E_b=100 volts.

$\Delta E_b = P_3 - P_1 = 120 - 80$

$= 40$ volts.

$\Delta I_b = (I_b)$ at $P_3 - (I_b)$ at P_1

$= 13.8 - 8 = 5.8\ mA.$

$$\therefore r_b = \frac{\Delta E_b}{\Delta I_b} = \frac{40}{5.8 \times 10^{-3}}$$

$= 6.9\ k\Omega.$

and $g_m = (\Delta I_b / \Delta E_c)_{E_b}$

$$= \frac{(10.9 - 5.1) \times 10^{-3}}{(0+2)}$$

$= 2.9$ milli-mho

Hence $\mu = r_p \times g_m = 6.9 \times 10^3 \times 2.9 \times 10^{-3} = 20.01.$

Also we have $\mu = \left(\frac{\Delta E_b}{\Delta E_e}\right)_{I_b} = \frac{120 - 80}{0 - (-2)} = 20,$

which is same as obtained from relation $\mu = r_p \times g_m$.

Example 9. *The plate resistance of a 6J5 triode is 8.0 kilo ohms and the transconductance is 2.8 millimhos. If only the plate voltage is increased by 60 volts, what is the increase in plate current ? What change in grid voltage will now bring the plate current to its former value ?*

As plate resistance $r_p = (\Delta E_b / \Delta I_b)_{E_c}$ = constant = 8000 ohms

$\therefore\ \Delta I_b = 60/8000 = 7.5\ mA.$

As mutual conductance $g_m = (\Delta I_b / \Delta E_c)_{E_b}$ = constant

$= 2.8 \times 10^{-3}$ mhos.

$\therefore\ \Delta E_c = 7.5 \times 10^{-3} / 2.8 \times 10^{-3} = 2.68$ volts.

This result shows that the change of 50 volts on the plate supply is equivalent to the change of 2.68 volts on the grid supply.

Example 10. *The characteristic curves of a triode valve can be represented by the relation* $I_a=0.004\ (V_a+40\ V_g)^{3/2}$ *mA, where* I_a, V_a *and* V_g *are the instantaneous values of anode current, anode voltage and grid voltage respectively. If* $V_a=250$ *volts and* $V_g=-3$ *volts, determine the valve's parameters.*

Plate (anode) current $I_a=0.004\ (V_a+40\ V_g)^{3/2}\times10^{-3}$ amp

or $(\partial I_a/\partial V_a)_{V_g}=0.004\times10^{-3}\times\frac{3}{2}\times(V_a+40\ V_g)^{1/2}$.

$\therefore$ Plate resistance $r_p=r_a=(\partial V_a/\partial I_a)$

$=\{0.006\times10^{-3}\ [250+40\ (-3)]^{1/2}\}^{-1}$

$=14.617$ kilo ohms.

As $(\partial I_b/\partial V_g)_{V_a}=0.004\times10^{-3}\times\frac{3}{2}\times40\ (V_a+40\ V_g)^{1/2}$.

$\therefore$ Transconductance $g_m=(\partial I_a/\partial V_g)_{V_a}$

$=0.24\times10^{-3}[250+40\ (-3)]^{1/2}$

$=2.736$ milli-mhos.

Hence amplification factor $\mu=r_p\times g_m=14.6\times2.736=40$. It can also be obtained by differentiating the given relation, keeping I_a as constant, as

$$\mu=-(\partial V_a/\partial V_g)_{I_a}=40.$$

Here—ve sign is introduced as V_g is —ve.

Example 11. *If the characteristic curves of a triode valve are given by the expression* $I_a=a\ (V_a+\mu V_g)^{3/2}$, *where* I_a *is in mA and* V_a *and* V_g *are in volts. Show that the value of* g_m *is proportional to* $I_a^{1/3}$. *If the measured values of* r_a *and* g_m *are 30 kilo ohm and 3 mA/V respectively at an anode current of 2mA, find the constants a and* μ *in the given expression,*

$$I_a=a(V_a+\mu V_g)^{3/2}mA=a'(V_a+\mu V_g)^{3/2}A.$$

$\therefore$ $g_m=(\partial I_a/\partial V_g)_{V_a}=a'\frac{3}{2}\ \mu(V_a+\mu V_g)^{1/2}$

$=\frac{3}{2}a'\mu(I_a/a')^{1/3}=\frac{3}{2}(a')^{2/3}\ \mu I_a^{1/3}$ ...(*i*)

or $g_m\propto I_a^{1/3}$.

As $(\partial I_a/\partial V_a)_{V_g}=a'\ .\ \frac{3}{2}(V_a+\mu V_g)^{1/2}$

or $r_p=r_a=(\partial V_a/\partial I_a)_{V_g}=\frac{2}{3}(a')^{-1}\ (V_a+\mu V_g)^{-1/2}$. ...(*ii*)

As $r_a=30$ kilo ohm, hence $(a')^{-1}\ (V_a+\mu V_g)^{-1/2}=\frac{3}{2}\times3\times10^4$

or $a'\ (V_a+\mu V_g)^{1/2}=\frac{2}{9}\times10^{-4}$. ...(*iii*)

As $I_a=2mA$, hence $a'\ (V_a+\mu V_g)^{3/2}=2\times10^{-3}$ ...(*iv*)

As $g_m=3mA/V$, hence $\frac{3}{2}\ a'\ \mu(V_a+\mu V_g)^{1/2}=3\times10^{-3}$

or $a'\ \mu(V_a+\mu V_g)^{1'2}=2\times10^{-3}$. ...(*v*)

$\therefore$ $\mu=2\times10^{-3}/(\frac{2}{9}\times10^{-4})=90$.

Solving equations we get $a'=2.3\times10^{-6}$ or $a=2.3\times10^{-3}$.

Example 12. *A triode, whose plate characteristics are given in the Fig. 1.32, is operated at a bias of—20 volts and a plate supply of 300 volts. If the load resistance is 4 kilo ohms, what are the quiescent current and voltage values ?*

We know that the potential e_b that exists between the plate

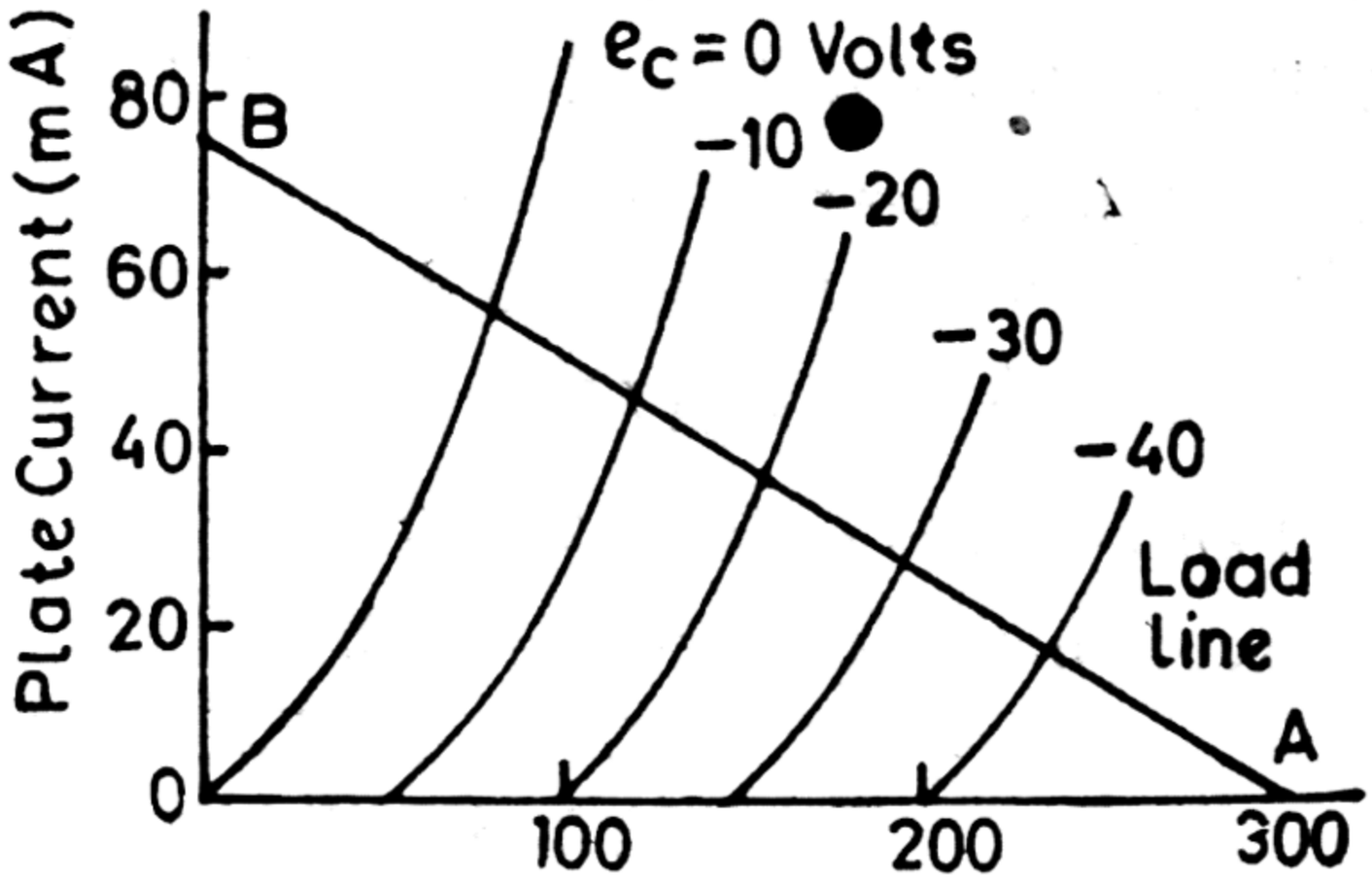


Fig. 1.32.

and cathode will depend upon both the magnitude of battery supply E_{bb} and the current i_b in the load resistor R_L, as

$$e_b = E_{bb} - i_b R_L.$$

Corresponding to $i_b = 0$, $e_b = E_{bb}$, it gives us one point on the load line. Corresponding to $e_b = 0$, $i_b = E_{bb}/R_L = 300/4000 = 75 mA$, it is the largest current possible. The figure shows that the largest current is 90 *mA*, hence the second point on the load line is found by choosing $i_b = 75$ *mA*.

The line *AB* is now drawn through the points (300,0) and (0, 75) on the Fig. 1.32. This line is found to cut the curve corresponding to $E_c = -20$ at a plate current of 38 *mA* and plate voltage of 149 volts. The plate voltage obtained from the above relation corresponding to the 38 *mA* is $e_b = 300 - 38 \times 4 = 148$ volts which is in close agreement with the value obtained from graph directly.

Oral Questions—

1. What will happen if you fill up a gas in photo-tube ?
2. What is meant by the Edison effect ?
3. What do you mean by space charge in a diode ? How can it be reduced ?
4. What are the advantages of directly heated and indirectly heated cathodes ?

5. Explain the terms: Control grid, suppressor grid, screen grid and grid bias.

6. Why do you require fixed grid bias of greater value than the maximum value of the input signal voltage ?

7. What is the difference between the static and dynamic characteristic of a vacuum tube ?

8. What factors affect the tube parameters μ, g_m and r_p ? How are they related ?

9. What is meant by interelectrode capacitance ?

10. How does the suppressor grid in pentode reduce the effect of secondary emission ?

11. What is a beam tube ?

12. What is a magnetron ?

13. What is the use of filling a gas in vacuum tubes ?

14. What is the difference between a triode and a thyraton ?

15. What are the uses of V.R. tubes ?

16. By what other names the followings are also known : tetrode, V.R. tube, cold cathode gas triodes.

17. What are they called, diode, triode. tetrodes. pentodes, V.R. tubes and beam power tube ?

Problems

1. Find whether photcelectrons will be emitted from metal surface of work function 6.6 eV when the incident radiation has a wavelength of 2300A.° (No)
($h=6.6\times10^{-27}$ ergs sec)

2. Calculate the work function of the photo cathode, if the electrons emitted by a light of 4000A° are completely stopped by a negative potential of 1.3 volts. (1.8 eV)

3. In a diode valve, the cathode and anode are parallel planes separated by a distance of 4 mm. The pd between them is 400 volts. Calculate the time of flight of electrons and acceleration acting on them when emitted from cathode with zero initial velocity. (0.674×10^{-9} sec, $1.757\times10^{16} m/sec^2$)

4. What will be the current density in amp/cm^2 of platinum filament whose work function is 4.1 eV and temperature is 2000°K (emission constant = 120.1 amp/cm^2°K^2). (224.7 amp/m^2)

5. A thermionic diode gives an emission current of 95 mA at 1050°K and 275 mA at 1150°K. Find the work function of the material in eV. (0.92 eV)

6. By how many electron volts must the work function of a surface of a metal change in order to reduce the emission from the surface at 2400°K by 10%. (0.02 eV)

7. At flat disc shaped oxide cathode with one side coated has a diameter of 3 mm. If the observed emission or saturation current is 5.7 mA. find the operating temperature of the cathode. (988°K)

8. A *6J5* triode valve has an amplification factor of 20. What grid voltage will produce cut off, if the 200 volts supply is connected with the plate ? (-10 $volts$)

9. A vacuum tube has an amplification factor 7. If the grid voltage is changed from —12 to —3 volts, what will be the effect on the plate current ? What is the change in plate voltage required to return the plate current to the original value ?

10. The plate resistance of a *6J5* triode is 10 kilo ohms and the transconductance of 2.6 millimhos If the plate voltage is increased by 50 volts, calculate (a) the increase in plate current, (b) change in grid voltage to regain the former plate current.

11. The following table gives two amplification characteristic curves of a triode. Calculate the tube constants at grid voltage of zero volt.

Grid voltage	—20V	—10V	10V	20V
Anode voltage for $I_b = 5mA$	370V	300V	220V	110V
Anode voltage for $I_b = 10mA$	415V	340V	260V	145V

12. For a certain triode change of 10 volts in the plate voltage gives rise to a change of 0.5 *mA* in the plate current. A change of 0.4 volt in the grid voltage restores the plate current to its original value. Find the approximate values of the constants of the triode,

(25.2×10^4 ohms, 12.5×10^{-4} mhos)

2

Rectifiers and Filters

2.1. RECTIFIERS

Rectification is the process in which alternating current is converted into unidirectional current by means of electrical devices. The device used for this purpose is called *rectifier*. It offers a low resistance to the current in one direction but a very high resistance to the current in the opposite direction. An ideal rectifier offers zero resistance in forward direction and infinite resistance in backward direction. It is like an synchronized switch which allows the current in forward direction. Due to this nature the rectifiers are represented by the symbol →|, where arrow points in the direction of forward current flow.

The principle of operation of rectifier may be either mechanical, thermal, chemical or electronic. Of these, the electronic principle is the one most generally used in electronic equipments. They may be classified into two groups (*i*) vacuum rectifiers, (*ii*) gas and vapour rectifiers. We will discuss semiconductor rectifiers in chapter 5.

2.2. VACUUM RECTIFIERS

If an alternating current is applied between the plate and cathode of a diode, the current will flow only during the portion of the cycle when the plate is positive. Thus the current flows during the half cycle only and the diode rectifier acts alternately as a conductor and an insulator. Such a rectifier is called *half wave rectifier*. If we use two diodes such that one is active for first half and other is active for the second half of the cycle, the arrangement is called *full wave rectifier*.

1. **Half Wave Vacuum Rectifier.** The basic circuit for half wave rectifier is shown in Fig. 2.1. Primary P of a power transformer T_r is connected to the input ac, the ac to be rectified. The secondary S of transformer consists of a diode valve V_1 and load

resistance R in series. When end A of transformer is positive and end B negative, the current will flow through the circuit or the circuit is conducting. When A is negative the current will not flow

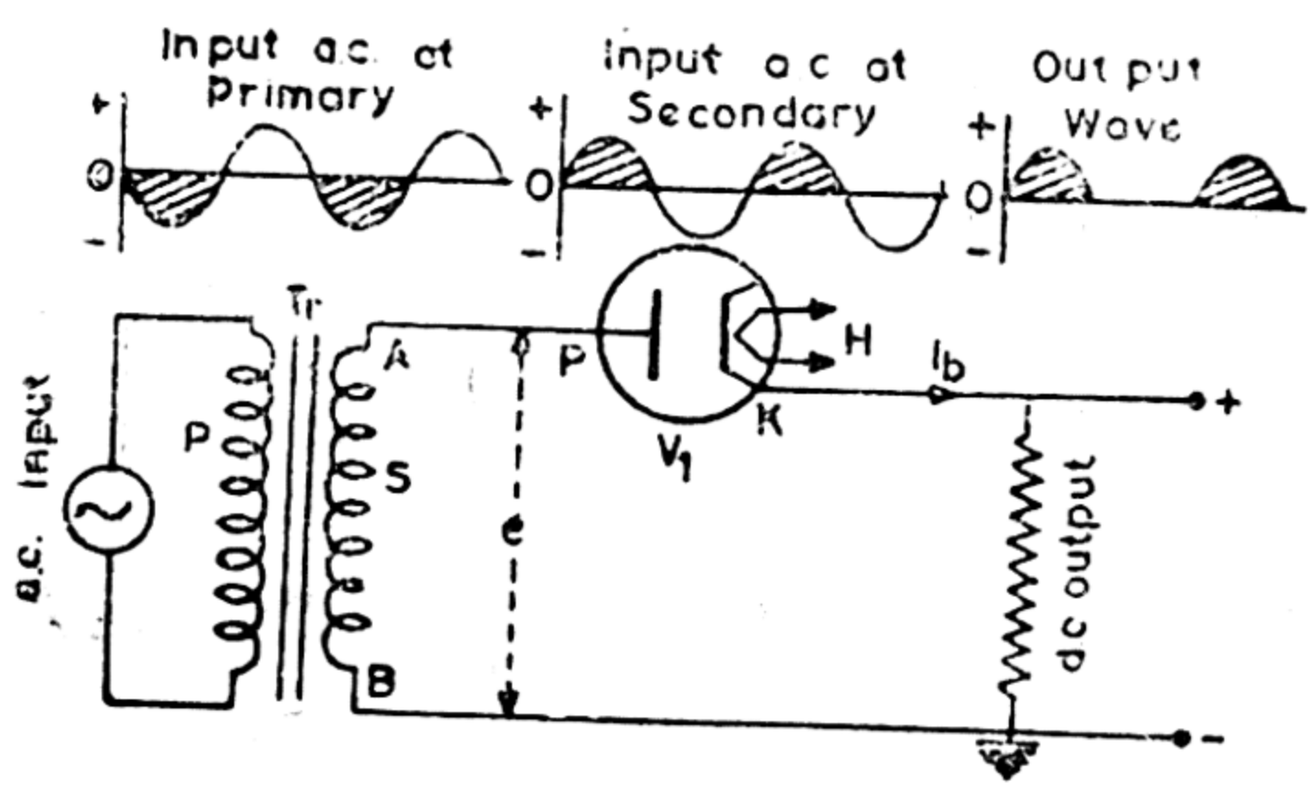


Fig. 2.1.

and the circuit is nonconducting. Thus the current in load resistance R will flow only in one direction, but with continuous change in amplitude (in pulse form) corresponding to the change in input. The input and the rectified output are as shown in Fig. 2.2). Since half of the portion of wave is obtained in the form of dc, hence it is called as *half wave rectifier*.

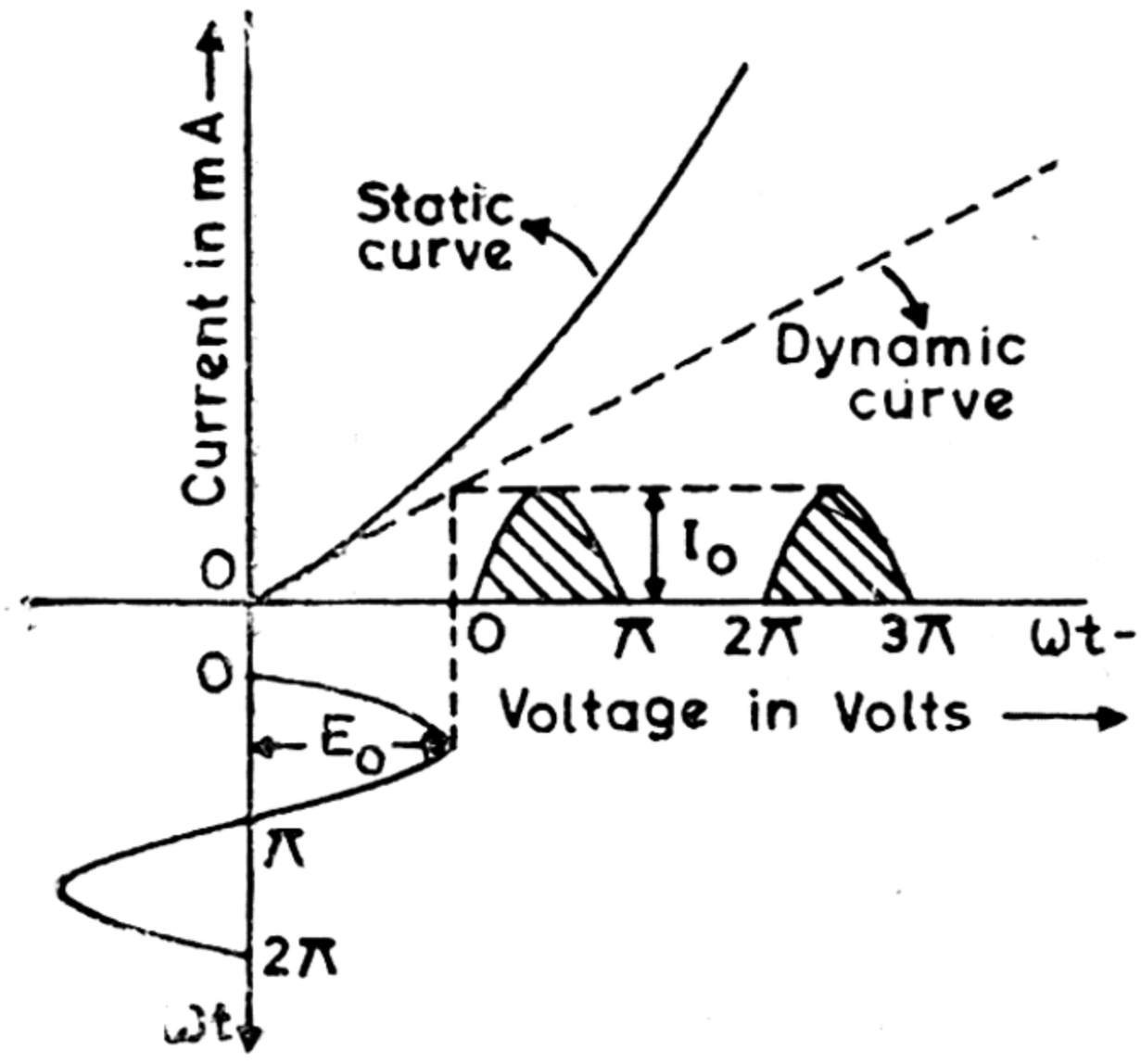


Fig. 2.2.

Regardless of the shape of the static-characteristic or the wave form of the input voltage, the output wave can be obtained graphically from the dynamic characteristic. Consider a point on the input voltage curve, draw a line from this point parallel to the current axis. If the point was on the positive half side, this line will cut the dynamic curve at a point. Now draw from this point a line ∥ to the voltage axis. In this way we can obtain the points corresponding to the points on the input wave and hence the output current wave.

Let the applied input voltage be $e = E_0 \sin \omega t$. If r_p is the plate resistance and R the load resistance, then the instantaneous current i_b through resistance R, is given by

$$i_b = \frac{\text{voltage}}{\text{total resistance}} = \frac{e}{(r_p + R)} = \frac{E_0}{(r_p + R)} \sin \omega t$$

$$= I_0 \sin \omega t \qquad [\text{where } I_0 = E_0/(r_p + R)] \qquad \ldots(1)$$

Since this current (i_b) flows only for half cycle, hence

$$i_b = I_0 \sin \omega t \quad \text{during } 0 \leqslant \omega t \leqslant \pi,$$

and $$i_b = 0 \quad \text{during } \pi \leqslant \omega t \leqslant 2\pi. \qquad \ldots(2)$$

Average of dc values. The average value of current (I_{av}) through R can be given by

$$I_{av} = I_{dc} = \frac{\int_0^T i_b \, dt}{\int_0^T dt} = \frac{\int_0^{T/2} I_0 \sin \omega t \, dt + \int_{T/2}^{T} 0.dt}{\int_0^{T=2\pi/\omega} dt}$$

$$= \frac{-\frac{I_0}{\omega}\Big[\cos \omega t\Big]_0^{T/2=\pi/\omega} + 0}{2\pi/\omega} = \frac{I_0}{\pi} = \frac{E_0}{\pi(r_p + R)}. \qquad \ldots(3)$$

Hence the dc or average voltage at the load R will be

$$E_{dc} = I_{dc} R = -\frac{I_0 R}{\pi} = \frac{E_0 R}{\pi(r_p + R)} = \frac{E_0}{\pi(1 + r_p/R)}. \qquad \ldots(4)$$

These two equations give the dc or average power across the load R.

$$\therefore \quad P_{dc} = E_{dc} I_{dc} = I^2_{dc} \, R = E_0^2 R / \pi^2 (r_p + R)^2. \qquad \ldots(5)$$

rms value of current. The root mean square or effective value of current in the circuit can be given as

$$I^2_{rms} = \frac{\int_0^T i_b^2 \, dt}{\int_0^T dt} = \frac{\int_0^{T/2} I_0^2 \sin^2 \omega t \, dt + \int_{T/2}^{T} 0 \, dt}{\int_0^T dt} = \frac{I_0^2}{4},$$

or $$I_{rms} = I_0/2 = E_0/2\,(r_p + R). \qquad \ldots(6)$$

Thus the I_{rms} is greater than the dc component I_{dc}.

Frequency components of rectified output. The output current of the rectifier can be assumed as consisting of several harmonics of frequency and can be written in terms of Fourier series as

$$i_b = A_0 + \Sigma A_n \cos n\omega t + \Sigma B_n \sin n\omega t, \qquad \ldots(7)$$

where the coefficients A_0, A_n and B_n are given by

$$A_0 = \frac{1}{T}\int_0^T i_b \, dt \; ; \; A_n = \frac{2}{T}\int_0^T i_b \cos n\omega t \, dt$$

and $$B_n = \frac{2}{T}\int_0^T i_b \sin n\omega t \, dt.$$

Substituting the values of ib from Eq. (2) in these integrals and solving, we get Eq. (7) as

$$ib = I_0\left[\frac{1}{\pi} + \frac{1}{2}\sin\ \omega t - \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t \ldots\right] \quad .. (8)$$

The first term of the series gives the direct or average component and is equal to the Eq. (3). The second term has its frequency same as that of the supply with peak value $I_0/2$. The rms value of this fundamental frequency component will be

$$= \frac{\text{Peak value}}{\sqrt{2}} = \frac{I_0/2}{\sqrt{2}} = \frac{I_0}{2\sqrt{2}} .$$

The third term gives the second harmonic term. The other frequency terms are also there in the equation. So *through the load resistance R, dc as well as ac components will pass.*

Rectifier efficiency. The efficiency of the rectifier is defined as

$$\eta = \frac{\textit{Direct output power}}{\textit{Alternating input power}} = \frac{P_{de}}{P_{ac}} . \quad \ldots(9)$$

Some of the power from the ac source is dissipated at the plate of the rectifier and the remaining part at the load resistance R. So the total input ac supplied by the transformer can be given as

$$P_{ac} = I^2_{rms}\, r_p + I^2_{rms}\, R = \tfrac{1}{4}\, I_0^2\, (r_p + R).$$

$$\therefore \qquad \eta = \frac{I_0^2 R}{\pi^2} \times \frac{4}{I_0^2(r_p + R)} .$$

$$\text{or} \qquad \%\ \text{of}\ \eta = \frac{4}{\pi^2(1 + r_p/R)} \times 100 = \frac{40.6}{(1 + r_p/R)} . \quad \ldots(10)$$

This gives the theoretical efficiency of a half wave rectifier which increases as $(r_p/R) \to 0$. Thus the maximum efficiency will be 40.6%. But the rectifier delivers maximum output power when the load resistance is equal to the plate resistance of the diode $(R = r_p)$, the efficiency is now reduced to 20.3 percent only.

Ripple Factor. This represents the fluctuating, pulsating or alternating part in the output of a rectifier. As obtained from Fourier analysis the output wave consists of alternating components, called as ripples. So *ripple factor is a measure of the fluctuating or pulsating component, i.e., purity of output.* It is defined as the ratio of the effective value of the *ac* components of the current or voltage to the direct or average value of the current or voltage, *i.e.*,

$$\textit{ripple factor}\ (r) = \frac{\textit{effective value of ac components}}{\textit{dc value of the components}} \quad \ldots(11)$$

The output current of a rectifier (I_{rms}) consists of the direct component (I_{dc}) and effective value of all *ac* components (I'_{rms}).

$$I^2_{rms}=I^2_{dc}+I'^2_{rms}$$

or
$$I'_{rms}=(I^2_{rms}-I^2_{dc})^{1/2}$$

$$\therefore \qquad r=\frac{I'_{rms}}{I_{dc}}=\left(\frac{I^2_{rms}}{I^2_{dc}}-1\right)^{1/2}. \qquad \ldots(12)$$

So for a half wave rectifier

$$r=\left[\left(\frac{I_0/2}{I_0/\pi}\right)^2-1\right]^{1/2}=\left[\frac{\pi^2}{4}-1\right]^{1/2}=1.21.$$

This shows that the effective (or *rms*) value of the *ac* component of the output exceeds the *dc* value of the output. Thus we see that the half-wave rectifier is not a suitable device for obtaining *dc*.

Peak Inverse Voltage. Peak Inverse voltage (*PIV*) is defined as the maximum voltage present across the diode ends during the period of non-conduction. For half-wave rectifier the non-conducting period is during the negative part of applied alternating wave and its peak value is equal to E_0. Actually this gives the voltage at which the insulation of tube (diode) must withstand.

Voltage Regulation. It is a measure of the ability of a rectifier to maintain a specified output voltage with the variation of load resistance and is defined as

$$Voltage \quad regulation=\frac{output\ at\ no\ load-output\ at\ full\ load}{output\ at\ full\ load}$$

In a perfect voltage regulator the output voltage does not change with the load resistance. For a half-wave rectifier we have

$$E_{dc}=\frac{E_0}{\pi(1+r_p/R)}=\frac{E_0}{\pi}\left(1+\frac{r_p}{R}\right)^{-1}=\frac{E_0}{\pi}\left(1-\frac{r_p}{R}\right).$$

$$=\frac{E_0}{\pi}-\frac{E_0 r_p}{\pi R}=\frac{E_0}{\pi}-I_{dc}.\,r_p.$$

Assuming $r_p<<R$, the higher powers of (r_p/R) are here neglected. For no load (*open circuit*, $R=\infty$), $I_{dc}=0$ and $E_{dc}=E_0/\pi$.

Let I is current at full load condition so drop in voltage at plate resistance $r_p=I.r_p$. Hence

$$voltage \quad regulation=\frac{(E_0/\pi)-[(E_0/\pi)-I.r_p]}{(E_0/\pi)-I.r_p}=\frac{I.r_p}{E_0/\pi-I.r_p} \qquad \ldots(13)$$

Thus the voltage regulation is very poor in half-wave rectifiers.

2. **Full-Wave Rectifier.** When full input *ac* is converted into rectified *dc*, the diode is called *full-wave rectifier*. Two diode valves

are employed for this purpose as shown in Fig. 2.3. In this case the power transformer has central tap *C* at secondary winding. The cathodes of the two identical diodes V_1 and V_2 are connected together and this common junction to one end of load resistance *R*. The other end of *R* is connected to the central tapping *C*. Thus only half of the secondary voltage appears between the plate and cathode of each diode. *C* can be considered as a reference zero potential for the alternating voltage in the secondary *S* of the transformer.

The voltage on each diode will be equal in magnitude but opposite in direction. At any instant, if *A* is at some positive potential relative to *C*, *B* will be at equal negative potential relative to *C*. This gives that the plate of diode V_1 is positive relative to the cathode junction *C* so electron flow in circuit is due to the diode V_1 and not due to V_2. A current thus flows across the load *R*. During the next half cycle of alternating voltage the plate of diode V_2 has

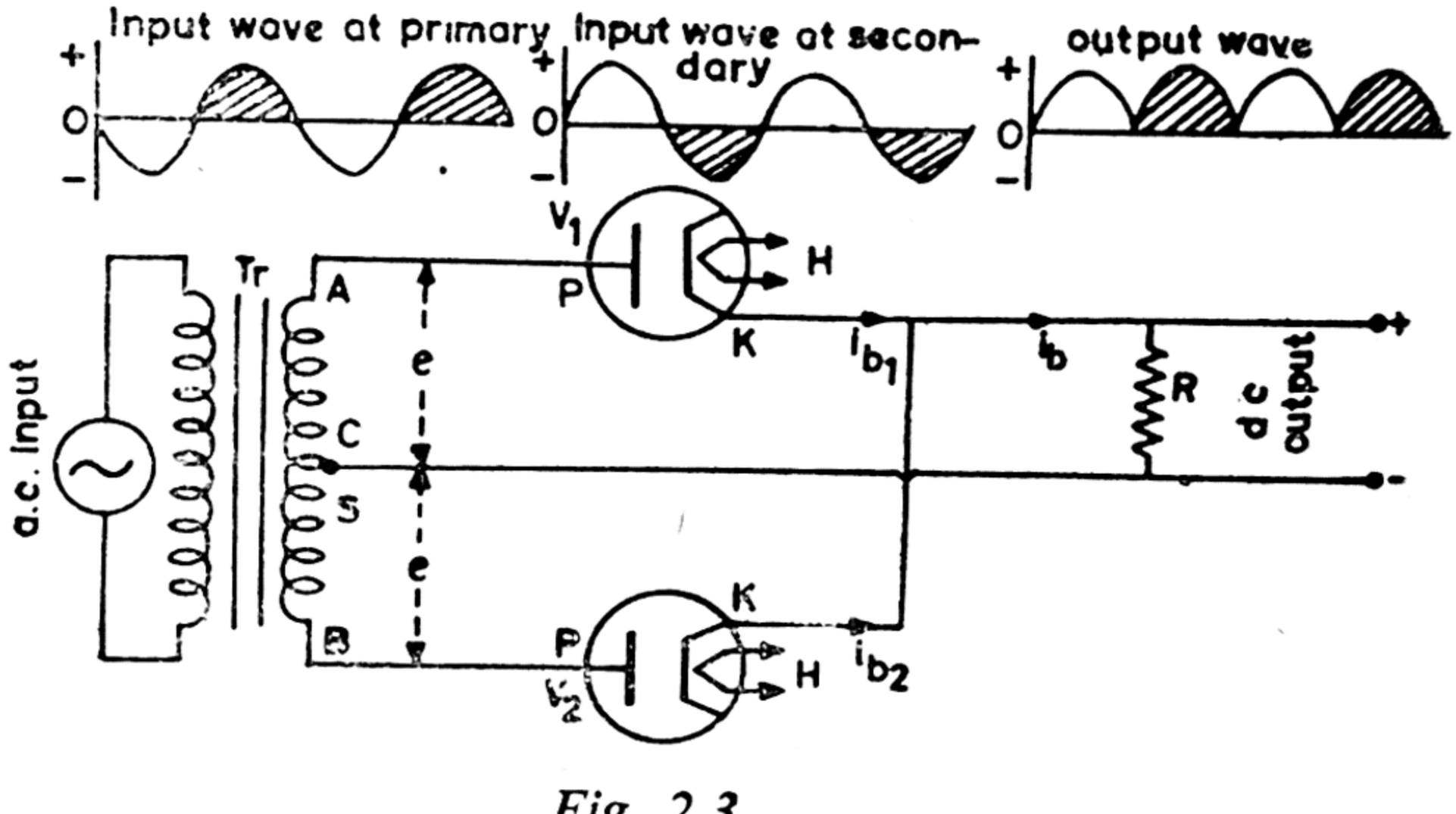


Fig. 2.3.

positive potential relative to cathode junction *C*. Again current flows across the load *R* in the same direction. Thus the voltage across *R* is always in the same direction, but fluctuating. The output of each diode can be understood separately as in Fig. 2.2 for each half cycle.

Now-a-days for fullwave rectification a double diode is used in place of two separate diode valves (Fig. 2.4). It consists of two separate plates and a single common cathode and filament (*heater*). For half of the *ac* cycle plate 1 is at positive potential, it attracts the electrons and constitutes current in circuit (during this interval plate 2 is at negative

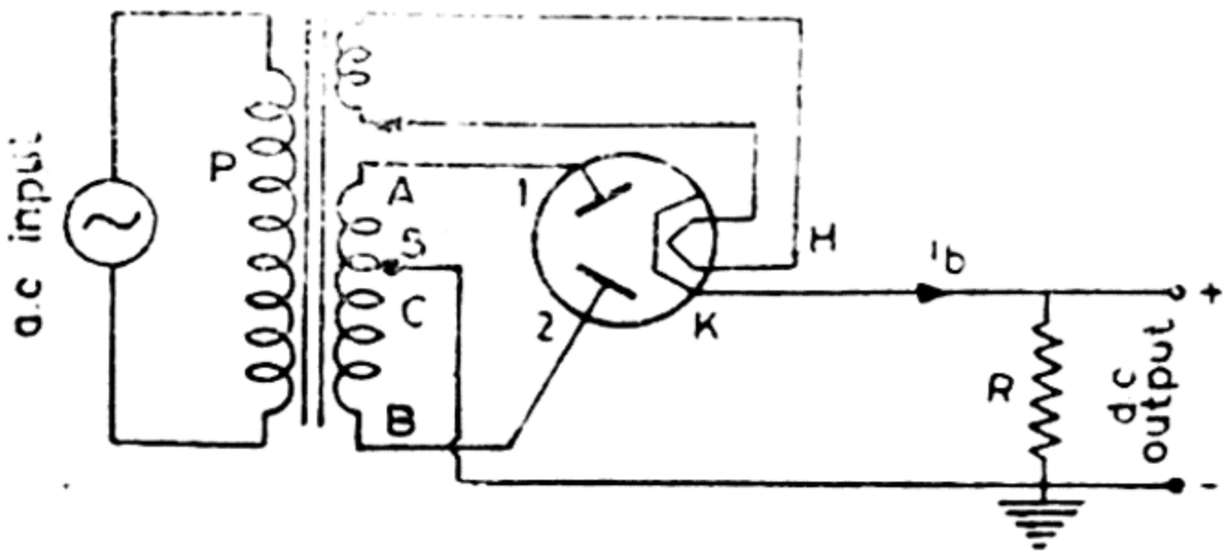


Fig. 2.4.

potential). For rest half of the *ac* cycle plate 2 is at positive potential and constitutes current in the same direction (during this interval plate 1 is at negative potential). Thus the current flows in the circuit during alternate half cycles due to two diodes. This circuit gives output for entire (full) cycle, hence called *full-wave rectifier*. The waveform of the output is also similar to that shown in Fig. 2.3. The current through R is fluctuating or pulsating, which can be minimised or removed using smoothing circuits, called *filters*, will be given in the article 2.4.

As both the tubes ars considered to be identical, also considering that the dynamic characteristic is linear from each diode, the mathematical analysis of full wave rectifier can be given as follows :

Let $e=E_0 \sin \omega t$ be the applied alternating potential at each diode, the plate resistance be r_p for each diode and the load resistance be R. The instantaneous value of current through the load will be equal to the sum of the currents due to diodes V_1 and V_2 at that instant. These currents i_{b_1} and i_{b_2} are given by the relations

$$\left.\begin{aligned} i_{b_1} &= \frac{E_0}{(r_p+R)} \sin \omega t = I_0 \sin \omega t \\ i_{b_2} &= \text{zero} \end{aligned}\right\} \text{during } 0 \leqslant \omega t \leqslant \pi \qquad \ldots(14)$$

and also $i_{b_1}=\text{zero}$

$$\left.\begin{aligned} i_{b_2} &= \frac{E_0}{(r_p+R)} \sin (\omega t+\pi) = -I_0 \sin \omega t \ldots \\ & \end{aligned}\right\} \text{during } \pi \leqslant \omega t \leqslant 2\pi \qquad \ldots(15)$$

Average or dc value. The average values of current (I_{dc}) through R can be given by

$$\begin{aligned} Idc &= \frac{1}{T}\int_0^{T/2} ib_1\, dt + \frac{1}{T}\int_{T/2}^{T} ib_2 dt \\ &= \frac{1}{T}\left[\int_0^{T/2} I_0 \sin \omega t\, dt + \int_{T/2}^{T} (-I_0 \sin \omega t) dt\right] \\ &= 2I_0/\pi = 2E_0/\pi\ (r_p+R). \end{aligned} \qquad \ldots(16)$$

So the *dc* or average voltage at the load R will be

$$Edc = IdcR = \frac{2I_0R}{\pi} = \frac{2E_0R}{\pi(r_p+R)} = \frac{2E_0}{\pi(1+r_p/R)} \qquad \ldots(17)$$

and $P_{dc}=E_{dc}.I_{dc}=I_{dc}^2R=4E_0^2R/\pi^2(r_p+R)^2$. $\ldots(18)$

rms value of current. The *rms* (or effective) value of current in the circuit can be given as

$$I_{rms}^2 = \frac{1}{T}\left[\int_0^{T/2} (ib_1)^2\, dt + \int_{T/2}^{T} (ib_2)^2\, dt\right]$$

$$I_{rms}=\frac{2}{T}\int_0^{T/2} I^2 \sin^2 \omega t\, dt=\frac{I_0^2}{2}$$

or
$$I_{rms}=\frac{I_0}{\sqrt{2}}=\frac{E_0}{\sqrt{2(r_p+R)}}. \qquad ...(19)$$

Frequency components of rectified output. Fourier analysis of the output waveform of the full-wave rectifier will be having frequency components depending upon the resultant due to the waveform of each diode output

$$ib_1=I_0\left[\frac{1}{\pi}+\frac{1}{2}\sin\ \omega t-\frac{2}{3\pi}\cos 2\ \omega t-\frac{2}{15\pi}\cos 4\ \omega t...\right]. \qquad ...(20)$$

Since ib_2 is 180° out of phase with ib_1 and hence

$$ib_2=I_0\left[\frac{1}{\pi}-\frac{1}{2}\sin \omega t-\frac{2}{3\pi}\cos 2\omega t-\frac{2}{15\pi}\cos 4\omega t...\right]. \qquad ...(21)$$

Therefore the total current

$$ib=I_0\left[\frac{2}{\pi}-\frac{4}{3\pi}\cos 2\omega t-\frac{4}{15\pi}\cos 4\omega t...\right]. \qquad ...(22)$$

This equation shows that lowest frequency term is of 2ω and the odd harmonics are not present.

Rectifier efficiency. The percentage of efficiency is defined as

$$\%\ \text{of}\ \eta=\frac{P_{dc}}{P_{ac}}\times 100=\frac{I_{dc}^2 R}{I_{rms}^2(r_p+R)}\times 100$$

$$=\frac{(2I_0/\pi)^2 R\times 100}{(I_0/\sqrt{2})^2(r_p+R)}=\frac{81.2R}{r_p+R}=\frac{81.2}{[1+(r_p/R)]}. \qquad ...(23)$$

Ripple factor. The ripple factor is given by

$$r=\left[\left(\frac{I_{rms}}{I_{dc}}\right)^2-1\right]^{1/2}=\left[\left(\frac{I_0/\sqrt{2}}{2I_0/\pi}\right)^2-1\right]$$

$$=[(1.117)^2-1]^{1/2}=0.48. \qquad ...(24)$$

Peak Inverse Voltage. In the case of full-wave rectifier when one diode (say V_1) is conducting, the other diode (V_2) will be non-conducting. The Kirchhoff's laws can be applied to Fig. 2.3, the equation obtained will be

$$e_{b_1}-e_{b_2}+2E_0 \sin \omega t=0, \qquad ...(25)$$

where eb_1 and eb_2 are the potential drops across V_1 and V_2 respectively. If the diode V_1 is conducting, we can consider the potential drop across it as zero, hence the potential drop across the non-conducting diode V_2 will be

$$eb_2=2E_0 \sin \omega t \qquad \text{or} \qquad (eb_2)_{max}=2E_0.$$

Similarly for the diode V_2 as conducting and V_1 non-conducting

$$e_{b_1} = -2E_0 \sin \omega t \qquad \text{or} \qquad (e_{b_1})_{max} = 2E_0.$$

So the peak inverse voltage (*PIV*) for the diode used for full wave rectifier circuit is $2E_0$, *i e.*, each diode must have sufficient insulation to withstand this voltage.

Voltage Regulation. The voltage regulation is given by

$$\textit{voltage regulation} = \frac{(E_{dc})_{no\ load} - (E_{dc})_{full\ load}}{(E_{dc})_{full\ load}}.$$

Here $$E_{dc} = \frac{2E_0 R}{\pi(r_p + R)} = \frac{2E_0}{\pi}\left(1 - \frac{r_p}{R}\right) \qquad (\text{when } r_p \ll R)$$

$$= \frac{2E_0}{\pi} - \frac{2E_0}{\pi} \cdot \frac{r_p}{R} = \frac{2E_0}{\pi} - I_{dc}.r_p.$$

At no load (*open circuit* $R = \infty$), $I_{dc} = 0$, and $E_{dc} = 2E_0/\pi$.

At full load we have,

$$(E_{dc})_{full\ load} = (E_{dc})_{no\ load} - (I_{dc})_{full\ load}.r_p$$

$$\therefore \quad \textit{Voltage regulation} = \frac{(I_{dc})_{full\ load}.r_p}{[(2E_0/\pi) - (I_{dc})_{full\ load}.r_p]} \qquad \ldots (26)$$

Comparison of Half-Wave and Full-Wave Rectifiers

(*i*) In half-wave rectifier, the current flows through the secondary of the transformer always in the same direction, the *dc* saturation of the transformer core takes place. This produces hysteresis loss and harmonics in the secondary output and reduces the efficiency of transformer. Whereas in a full-wave rectifier equal currents flow through the two halves of the centre tapped secondary of the transformer in opposite directions, hence no core saturation exists.

(*ii*) The average or *dc* values of currents and voltages are double in case of full-wave in comparison of half-wave rectifier.

(*iii*) Effective or *rms* value of output current in full-wave rectifier is $\sqrt{2}$ times the value for half-wave rectifier.

(*iv*) The *dc* power delivered to the load by a full-wave rectifier is 4 times the power delivered by a half-wave rectifier.

(*v*) The fundamental angular frequency term is eliminated in full-wave rectifier. The second harmonic component (2ω) in the output of full-wave rectifier is easier to filter out.

(*vi*) Efficiency of full-wave is twice that of half-wave rectifier.

(*vii*) The ripple factor of full-wave rectifier is quite low (0.48) than of a half-wave rectifier (1.21). The full-wave rectifier converts

higher percentage of power into desired *dc* power with smaller portions as undesired *ac*. So for a half-wave rectifier a comparatively expensive smoothing filter is required.

(*viii*) The full-wave rectifier diode must be chosen such that its insulation strength is sufficient to withstand the peak inverse voltage of $2E_0$. While *PIV* is E_0 only for a diode of half-wave rectifier.

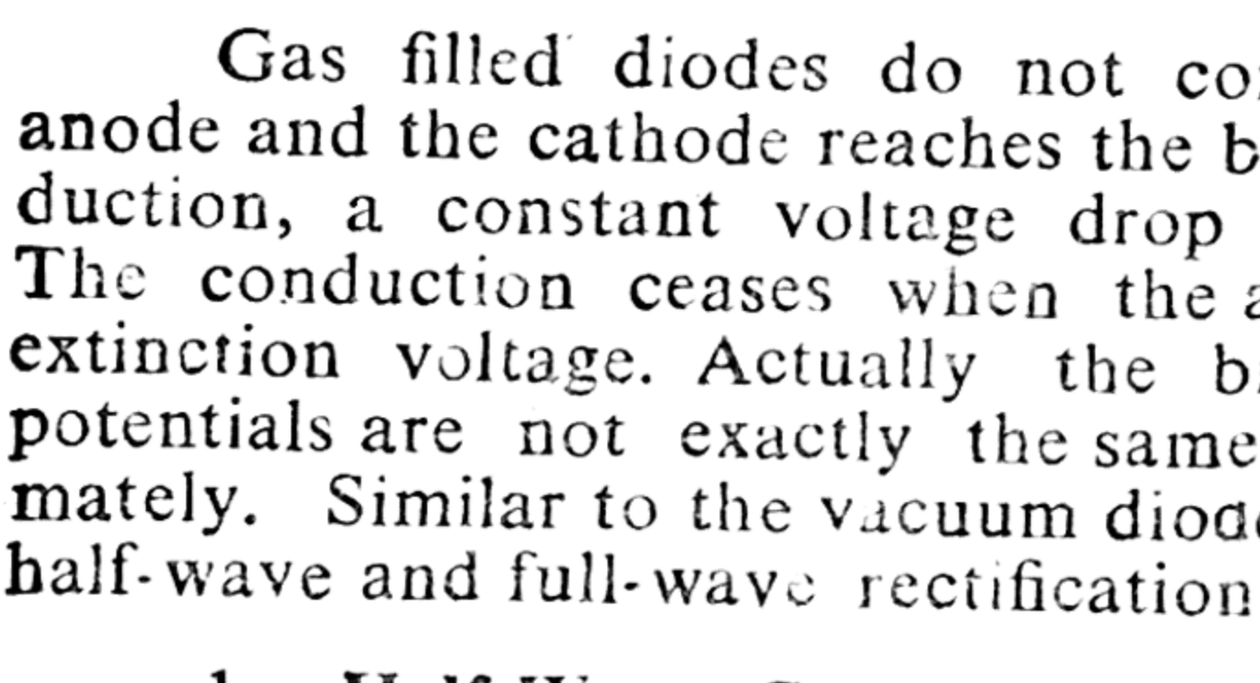
(*ix*) The half-wave has poor voltage regulation in comparison to a full-wave, *i.e.*, the change in the output of a full-wave is small than a half wave rectifier.

The half wave rectifier has the chief advantage of simplicity and low design cost and low *PIV* value compared with a full-wave rectifier. But the above points clearly indicate that a full-wave rectifier has *higher efficiency, higher dc output values less core saturation, less ripple factor and high voltage regulation*. So for a better rectification full-wave rectifier is always preferred than a half-wave rectifier.

2.3. GAS DIODE RECTIFIERS

Gas filled diodes do not conduct until the pd between the anode and the cathode reaches the breakdown value. During conduction, a constant voltage drop exists between the electrodes. The conduction ceases when the applied pd becomes less than extinction voltage. Actually the breakdown and the extinction potentials are not exactly the same, but one can assume it approximately. Similar to the vacuum diodes, gas diodes are also used in half-wave and full-wave rectification.

1. Half Wave Gas Rectifier : The circuit diagram of the half-wave rectifier with a load is shown in Fig. 2.5. A battery of voltage E_0 is connected in series with an ideal rectifier, so the current-voltage characteristic of the combination is the same as that of the idealized gas type rectifier (Fig. 2.5 *b*).



(*a*) (*b*)

Fig. 2.5.

If the supply voltage is given by $es = E_{sm} \sin \omega t$, then the load current

$$ib = [E_{sm} \sin \omega t - E_0]/R. \qquad ...(27)$$

Because of this voltage drop E_0 across the tube during conduction, the tube conducts for less than half cycle, say between angles α_1 and α_2. These angles depend on the ratio E_0/E_{sm}. Here α_1 is the angle at which the tube fires (*or starts conducting*) and α_2 at which conduction ceases.

Direct plate current can be obtained by taking the average of *ib* as

$$I_{dc}=\frac{1}{2\pi}\int_{\alpha_1}^{\alpha_2}\frac{E_{sm}\sin\omega t-E_0}{R}d\ (\omega t).$$

If $E_{sm}\gg E_0$, little error will be made by assuming as $\alpha_1=0$ and $\alpha_2=\pi$. We thus have

$$I_{dc}=\frac{1}{2\pi}\int_0^{\pi}\frac{E_{sm}\sin\omega t-E_0}{R}d(\omega t)=\frac{E_{sm}}{\pi R}-\frac{E_0}{2R}. \qquad \ldots(28)$$

$$\text{The output } dc \text{ power } P_{dc}=I_{dc}^2\,R=\frac{E_{sm}^2}{\pi^2 R}\left[1-\frac{\pi E_0}{2E_{sm}}\right]^2 \qquad \ldots(29)$$

$$\text{The input power } P_i=P_{ac}=\frac{1}{2\pi}\int_0^{\pi}e_s\,ib\,d\,(\omega t)$$

$$=\frac{1}{2\pi}\int_0^{\pi}E_{sm}\sin\omega t\ \frac{(E_{sm}\sin\omega t-E_0)}{R}d(\omega t)$$

$$=\frac{E_{sm}^2}{4R}\left[1-\frac{4E_0}{\pi E_{sm}}\right]. \qquad \ldots(30)$$

$$\text{Hence the rectifier efficiency } \eta=\frac{P_{dc}}{P_{ac}}=\frac{4(1-\pi E_0/2E_{sm})^2}{\pi^2(1-4E_0/\pi E_{sm})}$$

$$=40.6\ (1-1.87\ E_0/E_{sm})\% \qquad \ldots(31)$$

It is independent of load resistance and load current.

2. **Full-Wave Gas Rectifier.** The circuit diagram of full-wave gas rectifier is same as that of full-wave vacuum tube rectifier. In this case the one plate will conduct for the first half of the cycle and the second plate for the second half of the cycle. Thus we have output voltage in each half. On account of the voltage drop E_0, each plate conducts for less than half cycle and we thus have load current as shown in Fig. 2.6. The average load current

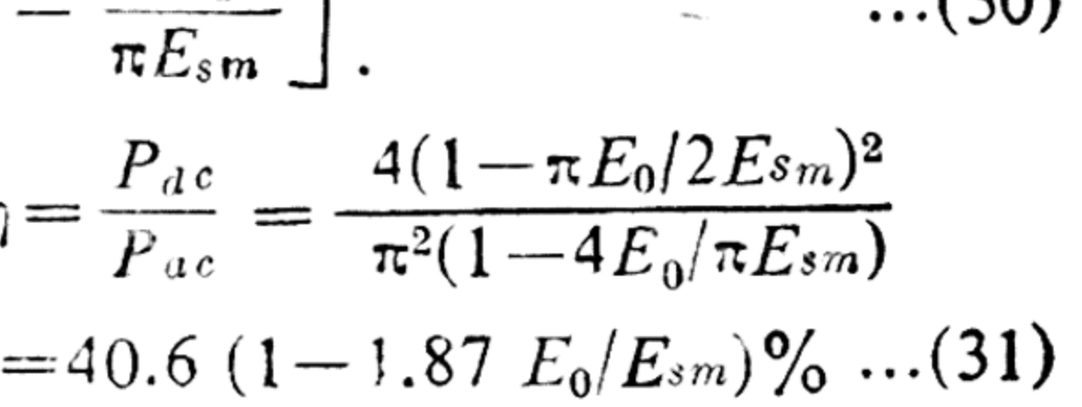


Fig. 2.6.

$$I_{dc}=\frac{2}{2\pi}\int_0^{\pi}\frac{E_{sm}\sin\omega t-E_0}{R}d\ (\omega t)=\frac{2E_{sm}}{\pi R}\left[1-\frac{\pi E_0}{2E_{sm}}\right]^2 \qquad \ldots(32)$$

$$\therefore \qquad P_{dc}=RI_{dc}^2=\frac{4E_{sm}^2}{\pi^2 R}\left[1-\frac{\pi E_0}{2E_{sm}}\right]^2. \qquad \ldots(33)$$

$$P_{ac}=2\frac{1}{2\pi}\int_0^{\pi}e_s\,ib\,d\,(\omega t)$$

$$=\frac{1}{\pi}\int_0^{\pi}E_{sm}\sin\omega t\ \frac{(E_{sm}\sin\omega t-E_0)}{R}d(\omega t)$$

$$P_{ac}=\frac{E_{sm}^2}{2R}\left[1-\frac{4E_0}{\pi E_{sm}}\right]. \qquad ...(34)$$

$$\therefore \quad \text{Rectifier efficiency } \eta=\frac{P_{dc}}{P_{ac}}=\frac{8(1-\pi E_0/2E_{sm})^2}{\pi^2(1-4E_0/\pi E_{sm})}$$

$$=81.2(1-1.87E_0/E_{sm})\%. \qquad ...(35)$$

Comparison with Vacuum Rectifiers. The gas rectifiers are used for high power applications, while the vacuum rectifiers are used for high voltage and low power arrangements. Gas type rectifiers are most useful in applications where E_0 is small compared with E_{sm}. Generally hot cathode mercury vapour rectifiers are used as they require lower filament power, has small value of E_0 and is cheaper in cost.

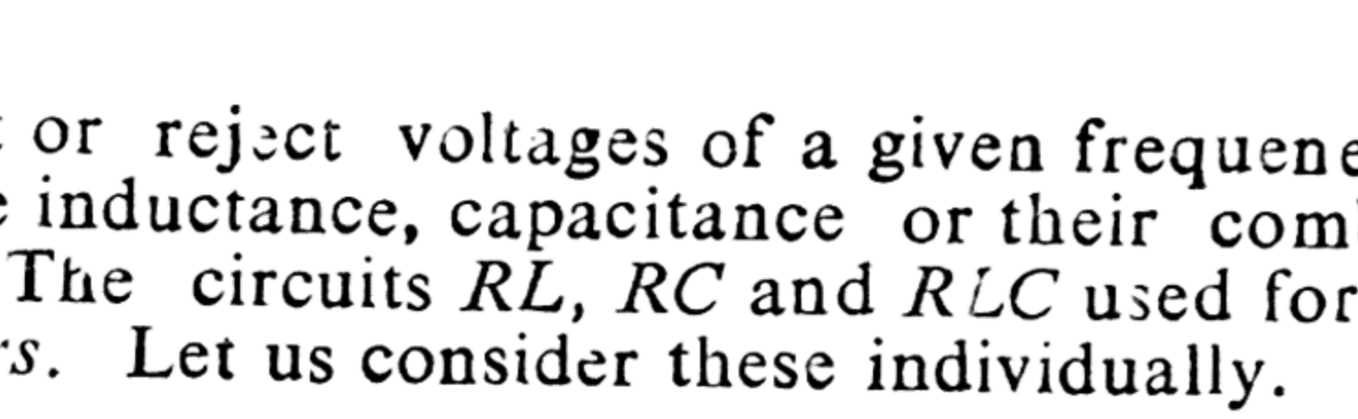
The limitation to the use of mercury vapour type of gas rectifiers is that the electrons in the plasma of the discharge perform oscillations of very high frequencies. In some cases these oscillations are fully suppressed by connecting *rf*-chokes in series with each anode lead.

2.4. FILTERS

To select or reject voltages of a given frequeney or range of frequencies the inductance, capacitance or their combinations are widely used. The circuits *RL*, *RC* and *RLC* used for this purpose are called *filters*. Let us consider these individually.

R-L circuit may be used in two ways shown in Fig. 2.7. The impedance of the circuit is $Z=(R^2+X_L^2)^{1/2}$.

The current through L and R is $i=e_{in}/Z$.



(*a*) (*b*)

Fig. 2.7.

In the circuit Fig. 2.7 (*a*), the output voltage is

$$e_{out}=iR=e_{in}\ R/(R^2+\omega^2L^2)^{1/2}.$$

$$\therefore \quad \frac{e_{out}}{e_{in}}=\frac{R}{(R^2+\omega^2L^2)^{1/2}}. \qquad ...(36)$$

This ratio is zero at $\omega\to\infty$ and unity at $\omega=0$, hence the combination (*a*) is called *low pass RL-filter*. The cut-off frequency for this is defined as that frequency for which $R=\omega L$ or

$$f_c=R/2\pi L. \qquad ...(37)$$

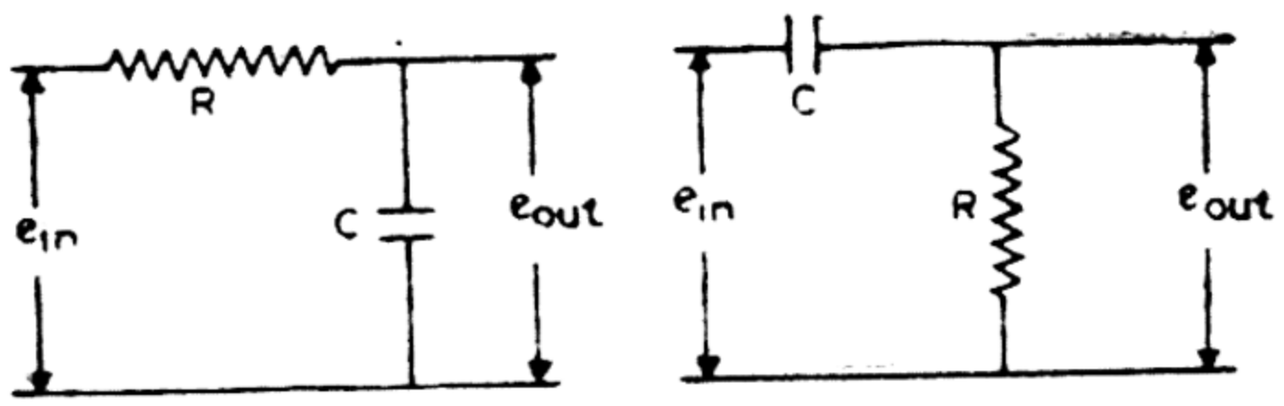
For the *RL*-circuit shown in Fig. 2.7 (*b*),

$$\frac{e_{out}}{e_{in}} = \frac{\omega L}{(R^2+\omega^2 L^2]^{1/2}} . \qquad \ldots(38)$$

This ratio is zero at $\omega=0$ and unity at $\omega\to\infty$, hence the combination (*b*) is called *high-pass RL filter*, the cut-off frequency is again defined as $f_c=R/2\pi L$.

RC-circuit may be used in two ways as shown in Fig. 2.8. The impedance of circuit

$Z=(R^2+X_c^2)^{1/2}$. The current in the circuit $i=e_{in}/Z$.



(*a*) (*b*)

Fig. 2.8.

The output voltage for the filter shown in Fig. 2.8 (*a*) is

$$e_{out}=iX_C=e_{in}/\omega C(R^2+1/\omega^2C^2)^{1/2}.$$

$$\therefore \quad \frac{e_{out}}{e_{in}}=\frac{1}{\omega C(R^2+1/\omega^2C^2)^{1/2}} . \qquad \ldots(39)$$

This ratio is unity at $\omega=0$ and zero at infinite frequency, hence the combination 2.8 (*a*) is called *low pass RC-filter*.

The cut-off frequency is defined as

$$R=X_C \quad \text{or} \quad f_c=1/2\pi RC. \qquad \ldots(40)$$

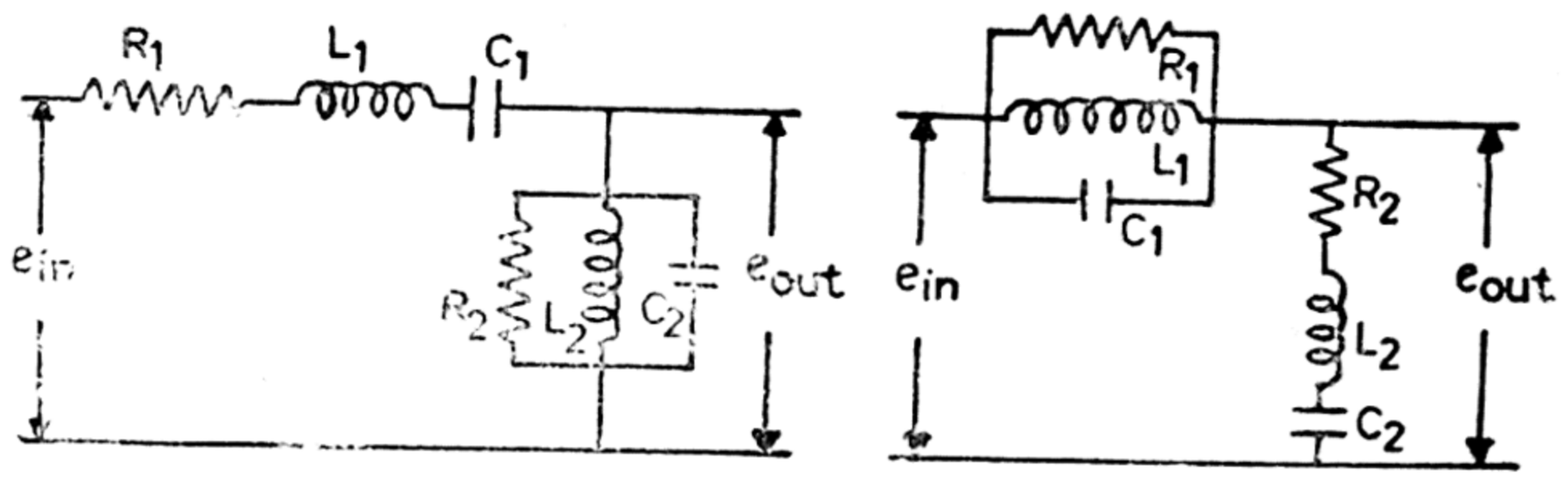
For *RC*-circuit shown in Fig. 2.8(*b*), we have

$$\frac{e_{out}}{e_{in}}=\frac{R}{(R^2+1/\omega^2C^2)^{1/2}} . \qquad \ldots(41)$$

This ratio is unity at $\omega\to\infty$ and is zero at $\omega=0$, hence the combination 2.8 (b) is called *high-pass RC-filter*.

Circuit 2 9 (a) consists of a series R_1 L_1 C_1 circuit and a parallel R_2 L_2 C_2 circuit. To have identical resonant frequencies L and C are so adjusted that $L_1C_1=L_2C_2$. Since the series circuit represents minimum Z and the parallel circuit represents maximum impedance for the resonance frequency $\omega_0\ (=1/\sqrt{LC})$ therefore e_{out}/e_{in} is maximum. This ratio decreases for frequencies greater than or less than ω_0. Thus a band of frequencies $\omega_0\pm d\omega$ is passed by the filter and other frequencies are rejected by this combination. Due to this property this combination is called *band-pass*

filter. This band pass filter becomes *band rejector circuit* if we interchange the positions of series and parallel circuits Fig. 2.9 (*b*). In this circuit e_{out}/e_{in} is minimum for frequency ω_0. Therefore a band of frequencies about ω_0 is rejected and all other frequencies being passed.



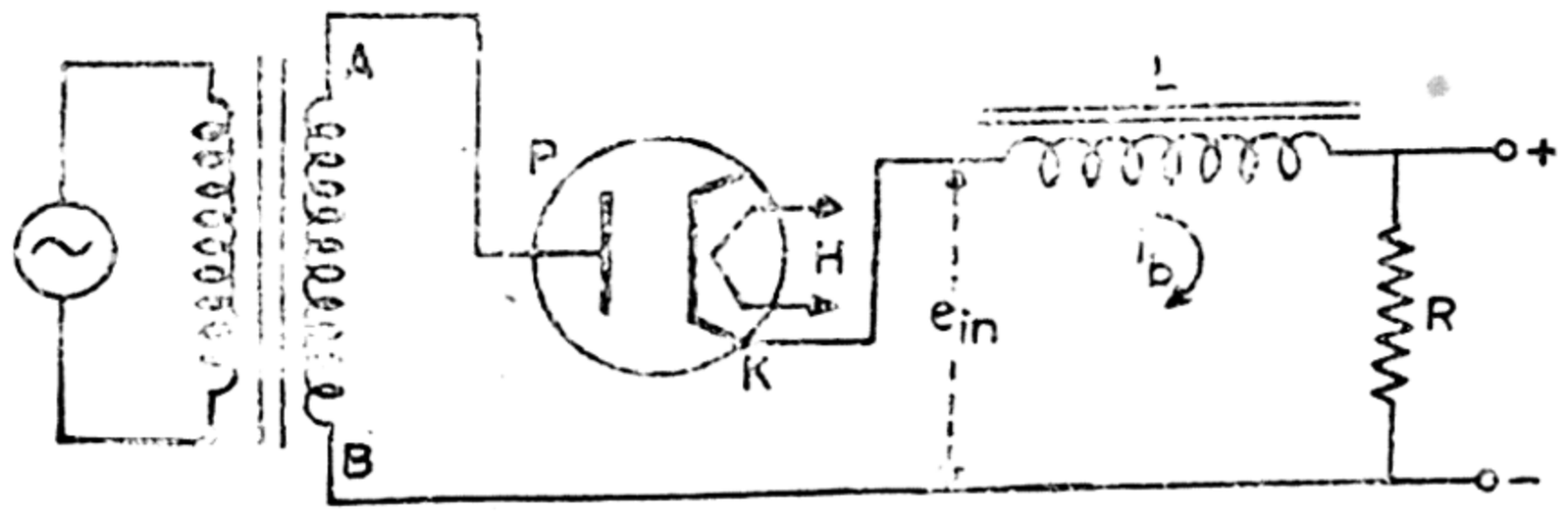
(*a*) (*b*)

Fig. 2.9.

The output of the rectifier consists of dc component and the ac or harmonic components, called ripples. While in most of the applications a pure dc free from ripples, is required. The purpose is served with the help of *low-pass filters*. These filters decrease ripple (by *by—passing* or *preventing the ac components*) and improve the efficiency of the reaction. The most common arrangements of filters used for this purpose are : (*a*) *Series inductor*, (*b*) *Shunt capacitor*, (*c*) *L-Section and* (*d*) Π-*section filter*.

(*a*) **Series Inductor Filters**. The simplest form of the filter circuit is to connect a choke (inductor) in series with the load. The inductance (L) of choke is quite high than the resistance (R_L). *In order to maintain a high value of L for a wide variation in current flow, some chokes use an iron core with a small air gap to prevent saturation.* The inductor offers a high series impedance

(*a*)



(*b*)

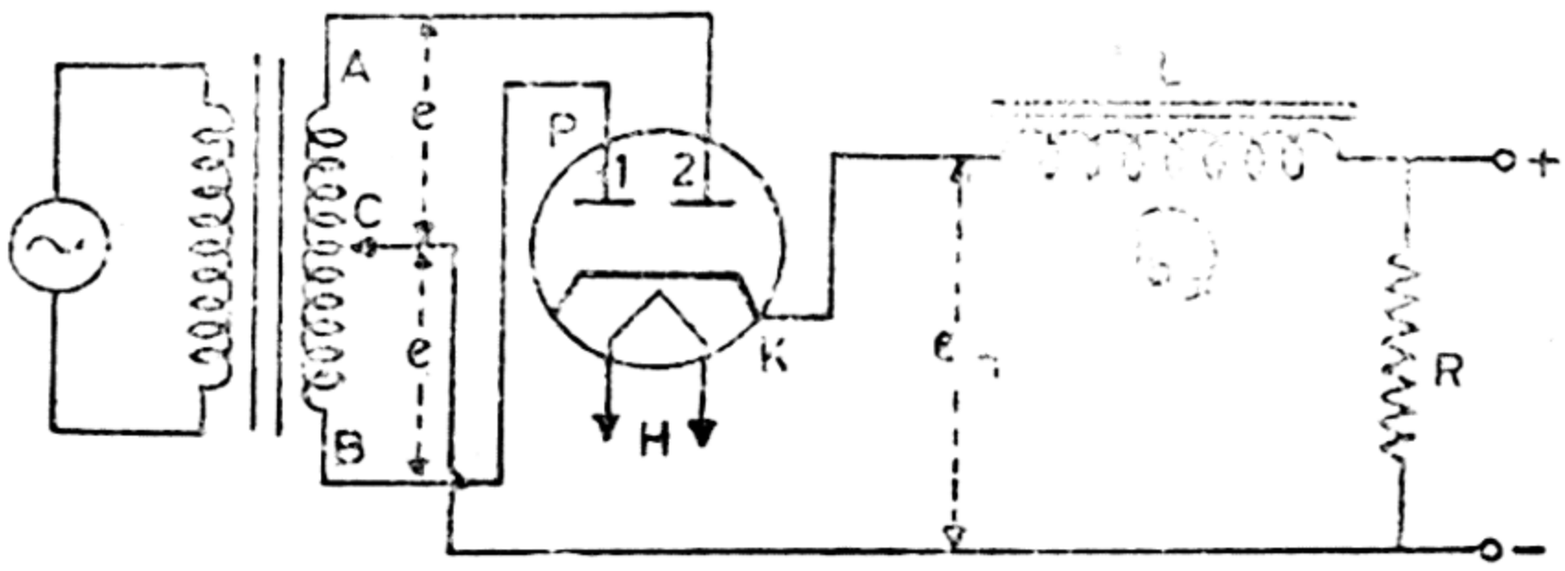


Fig. 2.10.

[$Z=\sqrt{R_L^2+\omega^2 L^2}$] in path of ac or harmonic components and opposes any change of current passing through it. Actually an inductor stores the energy in the form of magnetic field when the current raises from the average value and releases this stored energy to the load when the current falls below its average and thus reduces the ripples from the output. The Fig. 2.10 shows the circuit diagrams of a half-and full-wave rectifier with the series inductor filter.

If $e=E_0 \sin \omega t$ is the applied input at the diode, then the output without filter will be as shown by dotted curves. In the case of half wave rectifier with series inductor, the growth of current exists from 0 to $\pi/2$ part of the output which is opposed by the inductance coil and from $\pi/2$ to π there is a fall in current which is not allowed by the coil, so period of the current conduction is increased from 0 to π to 0 to $(\pi+\theta_2)$. θ_2 is known as *cut-off angle* and its value increases with the increase in L. Thus the gap of π to 2π part of the cycle in the output has been filled to some extent by introducing a series inductor in the circuit due to the induced emf $L\, di/dt$, as shown by continuous lines in Fig. 2.11 (*a*).

In case of full wave rectifier, since for each half cycle the period of conduction is beyond π radian. The period of conduction in the second rectifier will start before the current in first has ceased. Thus the current through the inductor and load resistance

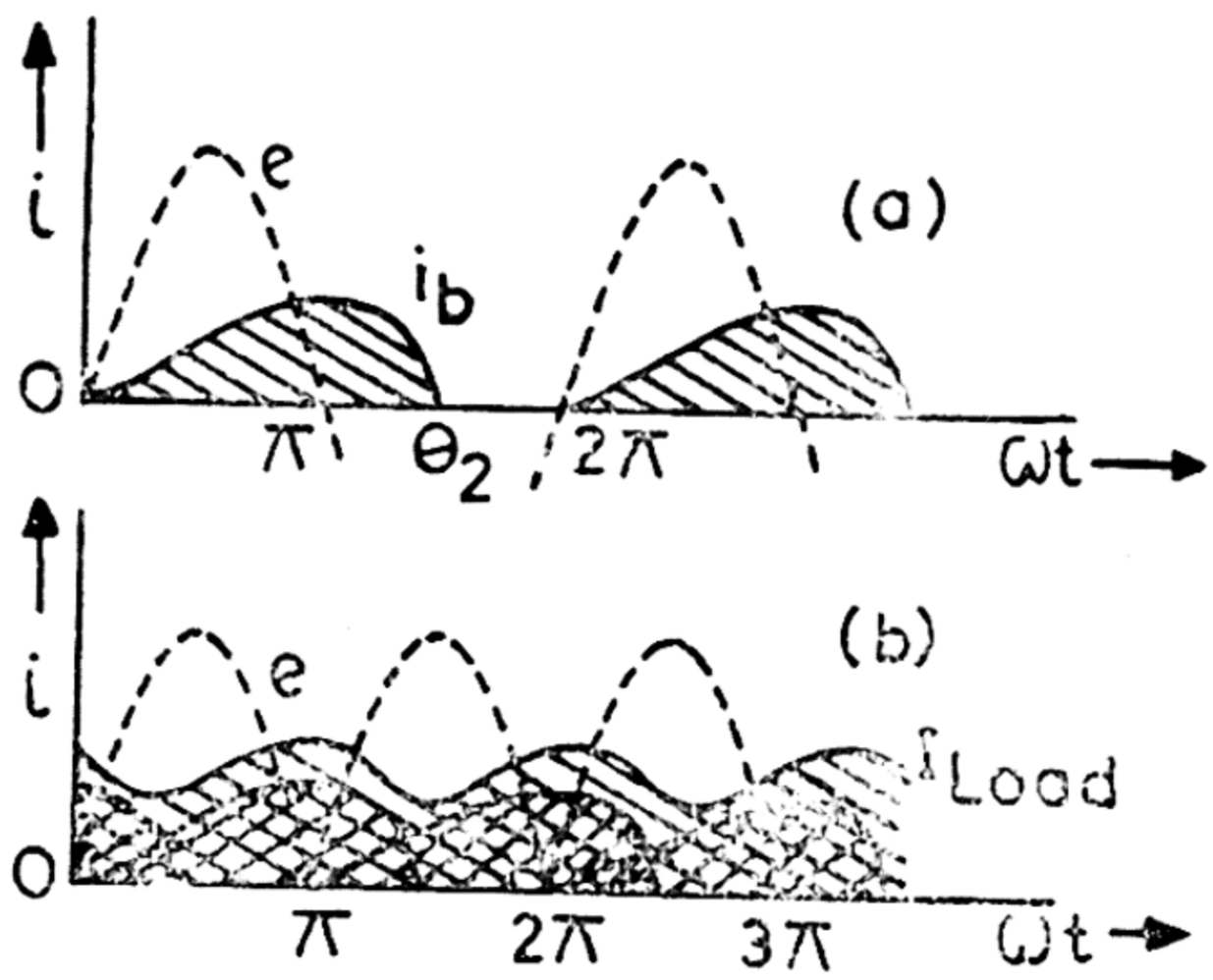


Fig. 2.11.

never drops to zero, as shown by continuous curve in Fig. 2.11 (*b*).

dc output. Fourier analysis of the output waveform for full wave rectifier is given by

$$e=\frac{2E_0}{\pi}\left[1-\frac{2}{3}\cos 2\omega t-\frac{2}{15}\cos 4\omega t.....\right]. \qquad ...(42)$$

The first term is the dc component of the input filter voltage.

If R_L be the resistance of inductor (or choke) and R is load resistance, then the (dc) current and voltage across the load can be given as

$$I_{dc} = 2E_0/\pi\,(R_L+R),$$
$$= 2E_0/\pi R. \qquad \text{(If } R_L \ll R\text{)}$$

and
$$E_{dc} = I_{dc}\ R = 2E_0R/\pi\,(R_L+R),$$
$$= 2E_0/\pi = 0.637\ E_0 \text{ (when } R_L \ll R) \qquad \ldots(43)$$

As R_L will always be quite small than the load resistance R,

$$\therefore\ E_{dc} = \frac{2E_0}{\pi}\left(1+\frac{R_L}{R}\right)^{-1} = \frac{2E_0}{\pi}\left(1-\frac{R_L}{R}\right)$$

$$= \frac{2E_0}{\pi} - \frac{2E_0R_L}{\pi R} = \frac{2E_0}{\pi} - I_{dc}\ R_L. \qquad \ldots(44)$$

Voltage Regulation. When there is no load I_{dc} is zero, and $(E_{dc})_{\text{no load}} = 2E_0/\pi$, therefore the voltage regulation

$$V.R. = \frac{(E_{dc})_{\text{no load}} - (E_{dc})_{\text{full load}}}{(E_{dc})_{\text{full load}}} = \frac{I_{dc}R_L}{2E_0/\pi - I_{dc}R_L}. \qquad \ldots(45)$$

As the inductor (choke) resistance is quite small, hence the regulation factor is also reduced, which means a better regulation.

Ripple Factor. As the amplitudes of the other higher harmonic terms are very small, hence their effects for producing ripple will also be very small, so only first ac component, *i.e.*, $(2E_0/\pi)\ (\frac{2}{3}\cos 2\omega t)$ is used for calculations. While neglecting higher harmonic terms, we have considered that the impedance due to inductance at high frequencies was also high and thus there was a better filteration for these terms. From the elementary circuit theory we can write the alternating current through the load resistance R as

$$i_{ac} = \frac{4E_0}{3\pi}\,\frac{\cos(2\omega t-\phi)}{[R_1^2+(2\omega L)^2]^{1/2}},$$

where $R_1 = (R+R_L)$ and $\phi = \tan^{-1}\ (2\omega L/R_1)$. This is the angle by which the current lags behind the alternating voltage.

Thus the effective or rms value of ac at the output can be written as

$$I_{rms} = \frac{4E_0}{3\pi\sqrt{2}[R_1^2+(2\omega L)^2]^{1/2}}.$$

Hence ripple factor $r = I_{rms}/I_{dc}$

$$= \left[\frac{4E_0}{3\pi\sqrt{2}\,[R_1^2+(2\omega L)^2]^{1/2}}\right] \Big/ \left(\frac{2E_0}{\pi R_1}\right)$$

$$= \frac{\sqrt{2}}{3}\ \frac{1}{\sqrt{[1+\{2\omega L/(R+R_L)\}^2]}}. \qquad \ldots(46)$$

From this equation, we have

(*i*) If $(R_L+R) \gg 2\omega L$, the ratio $2\omega L/(R+R_L)$ can be neglected in comparison to 1, so $r=\sqrt{2}/3=0.471$, it was 0.48 for without filter, so under such a condition the filtering is poor.

(*ii*) If $2\omega L \gg (R_L+R)$, so I can be neglected in comparison to the ratio $(2\omega L/R_1)^2$ in the denominator so,

$$r=\frac{\sqrt{2}}{3}\ \frac{R+R_L}{2\omega L}=0.236\ \frac{R+R_L}{\omega L}.$$

This shows that r can be decreased by using a choke coil with high inductance and low resistance. Being ω in the denominator, the higher frequencies will have less ripple factor. The ripple factor decreases as we decrease R or increase load current. In the above derivations we have neglected number of facts, *e.g.*, tube resistance, transformer resistance and transformer leakage reactance, etc.

(*b*) **Shunt Capacitor Filter.** A capacitor connected in parallel with the load resistance serves, effectively as a filter for ac components. The simple circuit for full wave is shown in Fig 2.12. The reactance offered by capacitor $Xc=1/\omega C$, so for dc ($\omega=0$) it offers an infinite resistance, while for any frequency (ac) its value decreases with increase in frequency. So if the resistance due to condenser is very small in comparison of load resistance (*i e.*, $1/\omega C \ll R$) then most of the ac or harmonic components will by-pass through condenser and will not appear through R. This will improve the d.c. output.

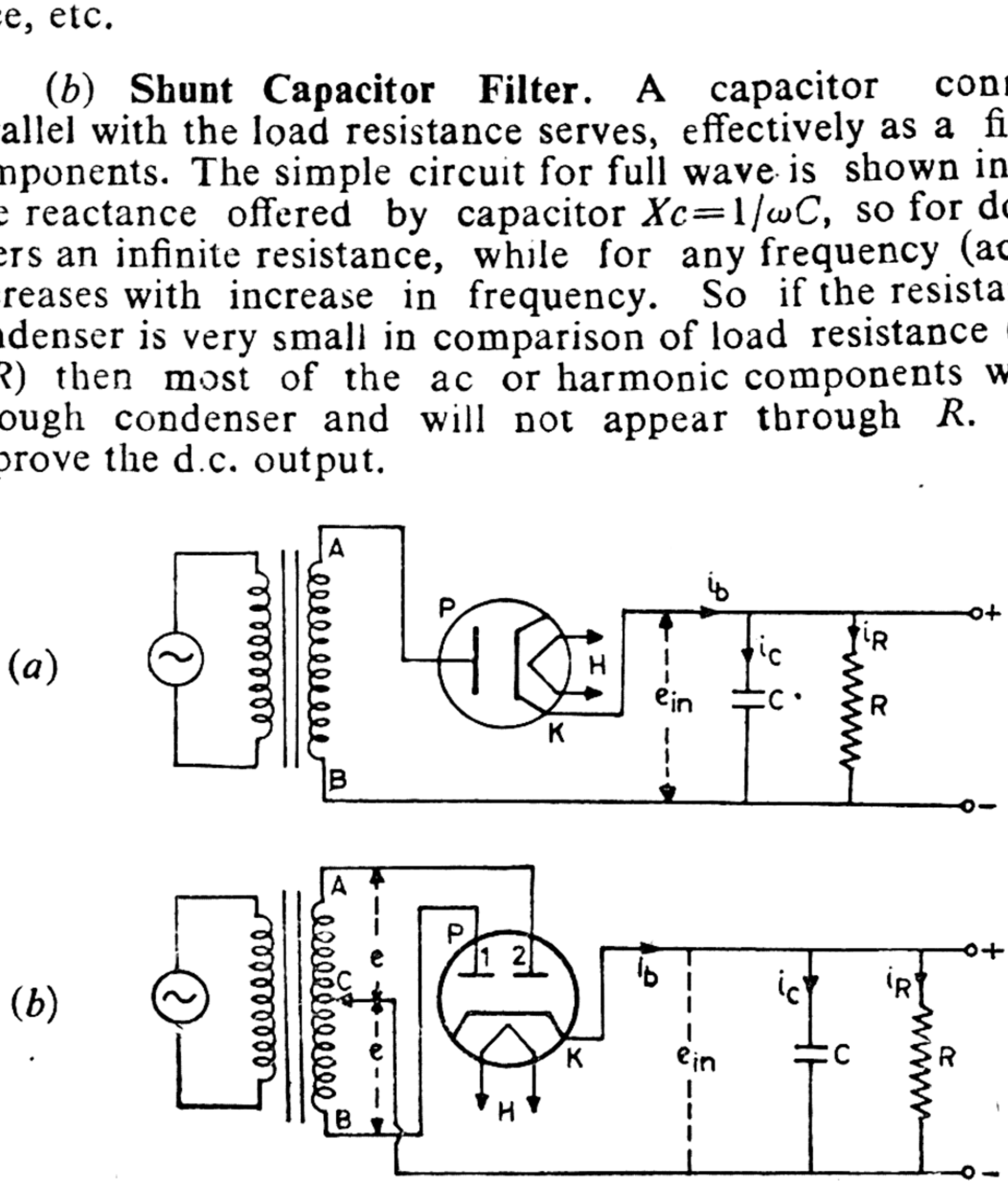


Fig. 2.12.

As the output potential from rectifier [Fig. 2.12 (*a*)] increases, the capacitor begins to charge (charging of condenser) and

thus stores electrostatic energy. The potential across the condenser plates will be equal to the potential of the output of the rectifier. This will make cathode to be sufficiently positive as it is directly connected to the plate of the condenser. During the fall in potential of rectifier output the diode is disconnected (stop conduction), as the cathode is sufficiently positive. In this period the capacitor discharges through the load resistance and delivers its stored energy and maintains the load voltage at a high value for a longer period. The rate of fall of voltage will depend upon the time constant (CR) of the circuit. If it is higher than the period of the applied ac, the fall in potential at the load will be small and thus maintaining a constant voltage over the full cycle.

Now consider the effect for next cycle when the ac input at the condenser increases, the plate current will only flow when the plate potential is more than the cathode potential (since cathode was positive with respect to plate but due to discharge of condenser the cathode potential will come down). Thus the diode conduction (current flow) will be only for a short period (θ_1 to θ_2) and that too in the form of pulses, Fig. 2.13, and from θ_2 to $(2\pi+\theta_1)$ the discharge of condenser will take place. During this discharge the source is disconnected so the tubes are acting as an automatic switchs.

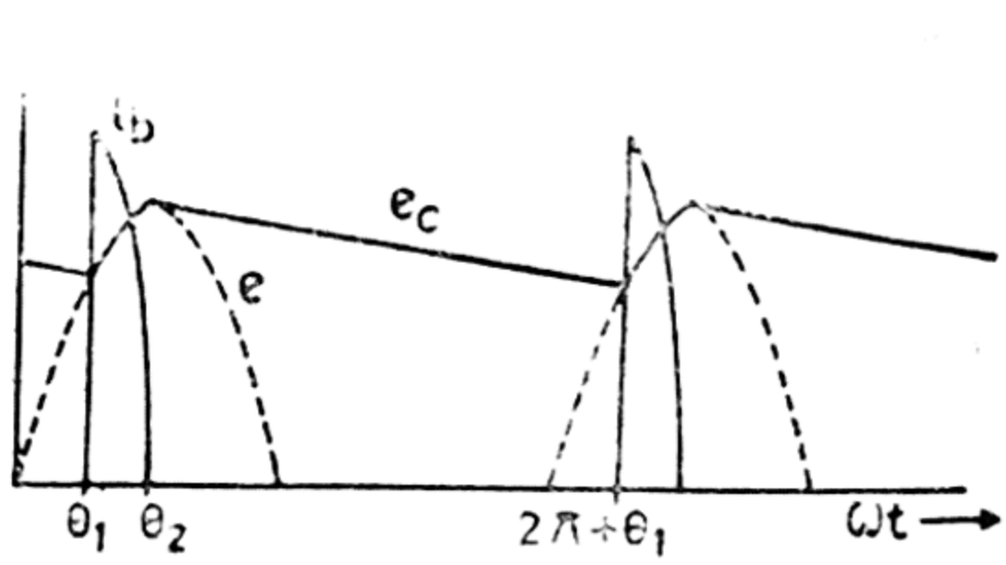


Fig. 2.13.

During the charging interval $i_b=i_C+i_R$ and since the diode switch is closed, the supply voltage $e=e_C=E_0 \sin \omega t$, hence

$$i_b=C\, de_C/dt+e_C/R. \qquad \ldots(47)$$

Thus during the conducting interval

$$i_b=\omega CE_0 \cos \omega t+(E_0/R) \sin \omega t, \quad \omega t_1 \ll \omega t \ll \omega t_2.$$

$$=\frac{E_0}{R}\sqrt{(\omega^2C^2R^2+1)} \sin (\omega t+\phi), \text{ where } \phi=\tan^{-1}\omega CR.$$

At $\omega t=\theta_2$, $i_b=0$ and we have

$$\omega CE_0 \cos \theta_2+(E_0/R) \sin \theta_2=0 \qquad \text{or } \theta_2=\tan^{-1}(-\omega RC)$$

$$\therefore \quad \phi=\tan^{-1}(\omega RC)=\pi-\tan^{-1}(-\omega RC)=\pi-\theta_2$$

and $$i_b=\frac{E_0}{R}\sqrt{(1+\omega^2R^2C^2)} \sin (\omega t_2-\omega t) \qquad \omega t_1<\omega t<\omega t_2.$$

For the period between ωt_2 to $2\pi+\omega t_1$ the rectifier is not conducting, *i.e.*, $i_b=0$ and solution of Eq. (47) becomes

$$e_C=Ae^{-t/RC}.$$

At $\omega t = \omega t_2$, $e_C = E_0 \sin \omega t_2$, hence we have

$$e_C = E_0 \sin \omega t_2 \, e^{-(\omega t - \omega t_2)/RC} \quad \omega t_2 < \omega t < (2\pi + \omega t_1).$$

At $\omega t = \omega t_1 + 2\pi$, $e_C = E_0 \sin \omega t_1$, hence we have

$$\sin \omega t_1 = \sin \omega t_2 e^{-(\omega t_1 + 2\pi - \omega t_2)/RC}. \quad \ldots(48)$$

This relation shows that ωt_2 decreases and ωt_1 increases with ωRC, *i.e., the conduction angle is short for large values of capacitance.* The dc voltage across the load can be calculated by averaging the capacitor voltage e over a cycle, as

$$E_{dc} = \frac{1}{2\pi} \int_{\omega t_1}^{\omega t_2} E_0 \sin \omega t \, d(\omega t)$$

$$+ \frac{1}{2\pi} \int_{\omega t_2}^{2\pi + \omega t_1} E_0 \sin \omega t_2 \, e^{-(\omega t - \omega t_2)/\omega RC} \, d(\omega t).$$

After integrating and substituting the values from Eqs. (47) and (48) we get

$$E_{dc} = \frac{E_0}{2\pi} - \sqrt{(1 + \omega^2 R^2 C^2)} \, [1 - \cos (\omega t_2 - \omega t_1)]. \quad \ldots(49)$$

This relation shows that $E_{dc} = E_0/2\pi$ for low values of C and reaches to the peak value of the input voltage for large values of C. High value of ωRC is needed for a small ripple.

The operation of the full-wave circuit (Fig. 2.13. *b*) is similar to that of the half-wave circuit with the discharge time of the capacitor running only to $\pi + \omega t_1$ instead of $2\pi + \omega t_1$. The cut-in angle can be recalculated by the use of π in place of 2π.

$$\sin \omega t_1 = \sin \omega t_2 e^{-(\omega t_1 + \pi - \omega t_2)/\omega RC} \quad \ldots(50)$$

The dc voltage can be calculated by averaging the capacitor voltages from ωt_1 to $\pi + \omega t_1$ and noting that there are two such pulses per cycle.

$$E_{dc} = \frac{E_0}{\pi} \sqrt{(1 + \omega^2 R^2 C^2)} \left[1 - \cos (\omega t_2 - \omega t_1) \right]. \quad \ldots(51)$$

Thus the variation in E_{dc} by the load resistance R are much smaller for the full wave rectifier than for the half-wave rectifier, *i.e., the voltage regulation is better for the full-wave circuit.*

Because of the complex waveforms, the output wave is taken to be a triangular waveform for nearest approximate calculations. These are made of portions of straight lines, as shown in Fig. 2.14. with peak value E_0 (maximum voltage at the capacitor). E_R is the

fall (discharge) in voltage across the condenser or at the load. So the average or dc value of potential for such output is given by

$$E_{dc} = E_0 - E_R/2. \qquad \ldots(52)$$

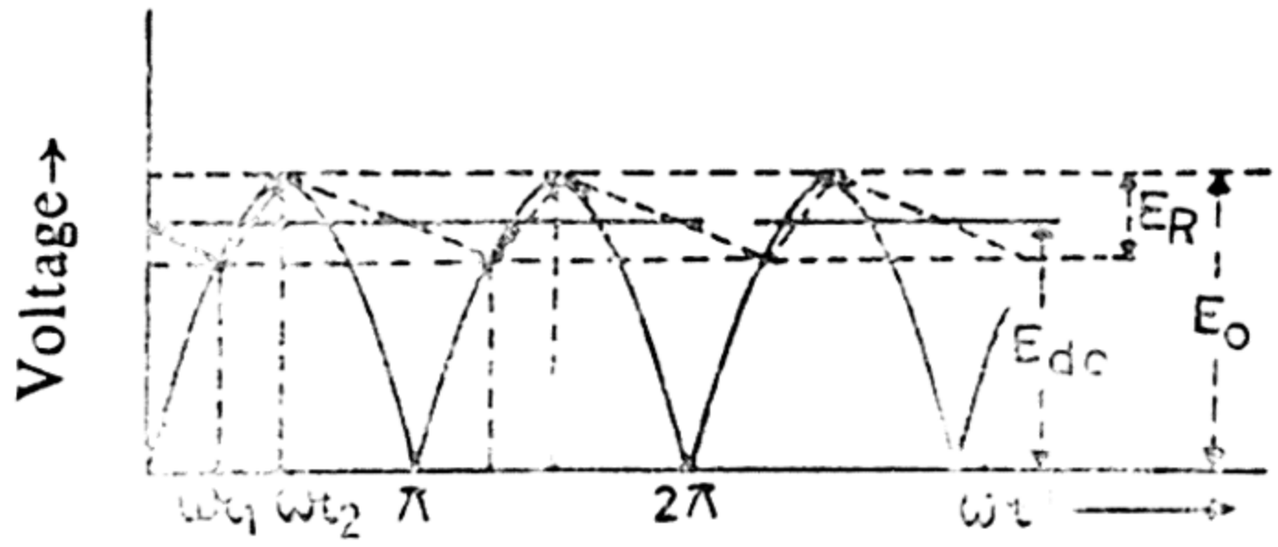


Fig. 2.14.

The rms value of triangular wave is given by

$$E_{rms} = E_R/2\sqrt{3}. \qquad \ldots(53)$$

i.e., the rms value of the ripple component of the triangular wave depends on the peak value (E_R) of wave and independent of slope or length of the straight lines.

I_{dc} = Rate of loss of charge from capacitor = loss of charge due to a drop in potential by E_R divided by $T/2$, i.e., half period.

$$\therefore \quad I_{dc} = 2CE_R/T = 2C.f.E_R \qquad (\text{as } f \text{ is the frequency})$$

So dc potential across the load R is given by

$$E_{dc} = I_{dc}\ R = 2CfE_R\ R.$$

The *ripple factor* is given as

$$r = \frac{E_{rms}}{E_{dc}} = \frac{E_R/2\sqrt{3}}{2Cf.E_R.\mathrm{R}} = \frac{1}{4\sqrt{3}CfR}. \qquad \ldots(54)$$

This shows that ripple factor raises inversely as time constant CR. Therefore the ripples will be less when the value of C or R is high (*i.e., with low load current*) and shunt capacitor filter is only suitable for low output (load) currents.

As $I_{dc} = 2C.f.E_R$ or $E_R = I_{dc}/2fC$, therefore, the output voltage

$$E_{dc} = E_0 - I_{dc}/4fC. \qquad \ldots(55)$$

Thus the output potential falls linearly with the increase in direct current. Thus a simple capacitor filter will possess a good regulation with a high value of the condenser C. It is found that the regulation of series inductor is better than the shunt capacitor filter.

(*c*) **The L-section or Inductor Input Filter.** This type of filter is the combination of series inductor and the shunt capacitor

filters. It gives very low value of ripples at all loads and hence used frequently in the electronic equipments. The circuit for full-wave

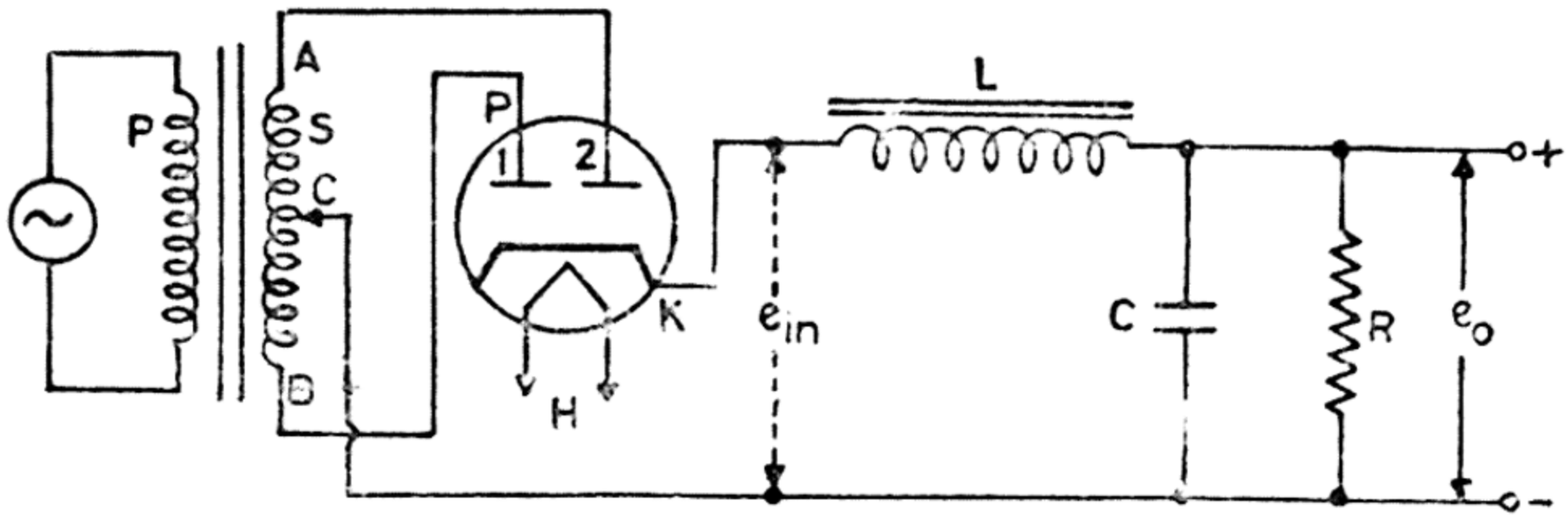


Fig. 2.15.

L-section filter is shown in Fig. 2.15. The inductor *L* offers a high impedance to the *ac* or harmonic components and thus reduces these components from the output. The shunt capacitor bypasses the *ac* components around the load. Thus the ripples in output are reduced considerably. The *dc* voltage and ripple factor may be calculated by taking the first two terms in the Fourier series representation of the output voltage of full-wave rectifier as

$$e=(2E_0/\pi)-(4E_0/3\pi)\cos 2\omega t. \qquad ...(56)$$

It is equivalent to a *dc* source in series with an *ac* source having twice the power line frequency. If the sum of the tube, transformer and choke resistances is R, then

$$E_{dc}=2E_0/\pi-I_{dc}\,R. \qquad ...(57)$$

The value of X_L and X_C at second harmonic of supply frequency (*i.e.*, for a supply frequency ω) are $2\omega L$ and $1/2\omega C$ respectively. The input impedance of the filter circuit at the second harmonic frequency 2ω is

$$Z_2=2j\omega L-\frac{jR/2\omega C}{R-j/2\omega C} \qquad \text{or} \qquad |Z_2|=(4\omega^2LC-1)/2\omega C.$$

The second harmonic emf allows the ac current in the inductor L, given as

$$(I_L)_{rms}=\frac{4E_0}{3\sqrt{2}\pi}\cdot\frac{1}{Z_2}=\frac{8\omega CE_0}{3\pi\sqrt{2(4\omega^2LC-1)}}$$

If we assume that the entire ac passes through the capacitor then $I_C\simeq I_L$, where I_C, I_L and I_R are the alternating currents through C, L and R respectively. Thus

$$I_R=\frac{(I_C)_{rms}}{2\omega CR}=\frac{(I_L)_{rms}}{2\omega CR}=\frac{4E_0}{3\pi\sqrt{2}R(4\omega^2LC-1)}$$

$$\therefore \text{ Ripple factor } r=\frac{E_{rms}}{E_{dc}}=\frac{4E_0/3\pi\sqrt{2}(4\omega^2LC-1)}{2E_0/\pi}$$

$$=0.47/(4\omega^2LC-1) \qquad ...(58)$$

Since the inductor filter has low ripple on heavy loads and the capacitor filter has low ripple on light loads, hence it is reasonable to assume that the inductor input filter has suitable ripple characteristics at all loads. For light loads ($R \simeq \infty$), the capacitor charges to the peak voltage with the *dc* output E_0. As R is decreased, the diode switches the supply voltage on to the filter and capacitor C is charged to the peak. The *dc* voltage will be lowered. The diode action is as for the shunt capacitor filter. As the load current is further increased, the discharge of the capacitor is more rapid and the effect of the indicator is appreciable. The upper part of Fig. 2.16 shows the variation in E_{dc} with I_{dc}. Here Current I_k indicates the point at which continuity of load current is obtained and the filter begins to act more like an indicator filter and less a shunt capacitor filter. Because of the good regulation at current values above I_k, the rectifier should be operated at the higher load currents. The lower part of Fig. shows the output current for L-section filter.

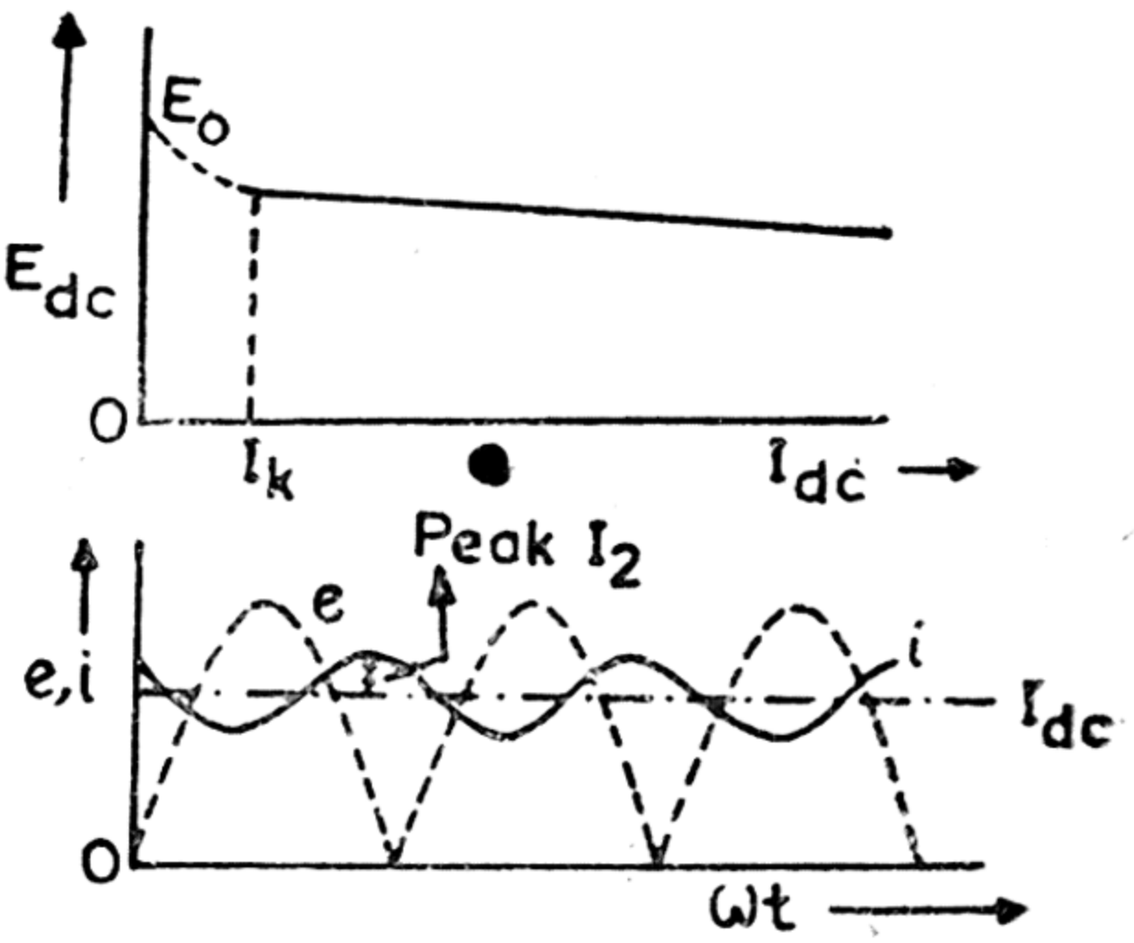


Fig. 2.16.

To determine I_k, let us equate the direct current I_{dc} and peak-second harmonic current I_2, as

$$(2E_0/\pi)/R = (4E_0/3\pi)/Z_2.$$

It gives $R = 1.5Z_2$, *i.e.* the current in the input inductor becomes continuous and the output voltage levels off at a load having resistance equal to 1.5 times the second harmonic impedance of the filter circuit. Thus the critical value of the inductor input $L_k = Z_2/2\omega = R/3\omega$. $\therefore$ $I_k = 2E_0/\pi R = 2E_0/3\pi\omega L$.

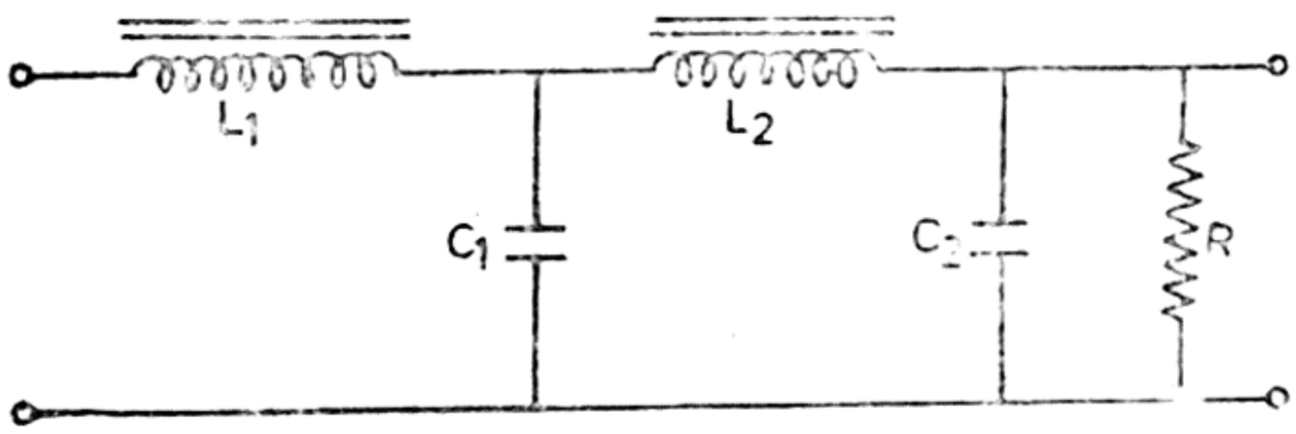


Fig. 2.17.

The value of ripples may be further reduced by adding more stages of L-sections. Each section reduces the ripple factor by $1/(4\omega^2 LC - 1)$. The two stages L-section filter circuit is shown in Fig. 2.17.

(*d*) **The Π-section or Capacitor Input Filter.** When an input C_1 is added to *L*-section filter, it forms a Π-section or *capacitor input filter.* A full wave rectifier with capacitor input or Π-section is shown in Fig. 2.18. The use of capacitor C_1 is to obtain higher output voltage for a given input, that can be obtained from *L*-section filter. It is possible for light load (high *R*) as the voltage maintained by this capacitor approaches E_0. For heavy current load (low *R*), it will fall below E_0 and thus loose regulation unless a high input capacitance is used.

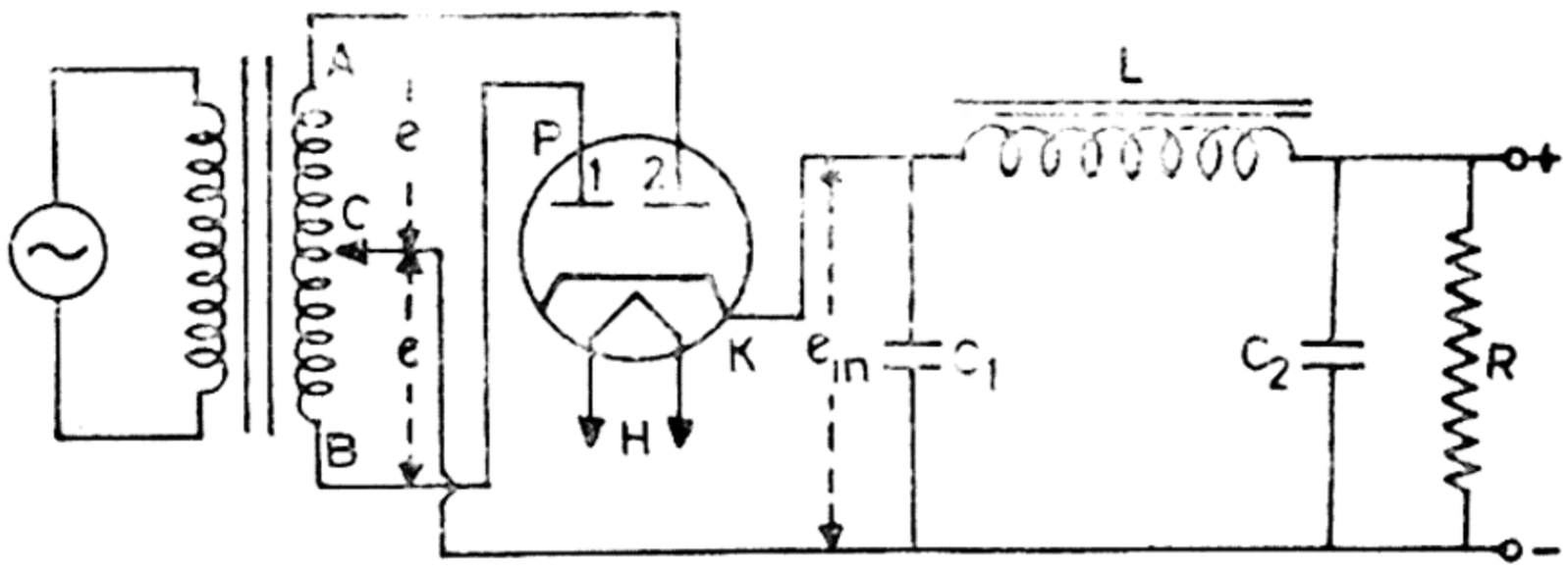


Fig. 2.18.

Capacitor C_1 draws a charging current pulse twice in a complete cycle and charged to the peak value E_0 of the input voltage. It then discharges through the inductor *L* and load *R* until the second diode connects the source on the next half cycle. During each conduction interval, capacitor C_1 charges to the peak value E_0. After this neither tube conducts nor capacitor begins to discharge. The inductor serves to maintain an almost constant discharge current. During the time in which the capacitor voltage reaches to an appropriate value, one of the tube starts conducting and thus the capacitor is again charged.

The discharge current of C_1 is difficult to evaluate. The output potential is assumed approximately that from the capacitor C_1 decreased by the $I_{dc}\ R_L$ drop in the inductor.

$$\therefore \quad E_{dc} = E_0 - E_R/2 - I_{dc}R_L \qquad \ldots(59)$$

$$= E_0 - I_{dc}(R_L + 1/4fC_1). \qquad \ldots(60)$$

This relation shows that the regulation is good for higher values of C_1.

If the ripple wave be considered triangular. It has a Fourier series given by

$$e = -\frac{E_R}{\pi}\sin 2\omega t + \frac{E_R}{2\pi}\sin 4\omega t + \ldots \qquad \ldots(61)$$

It is reasonable to neglect all harmonics except the second as their contributions are very small. This voltage is applied across $L-C_2$ filter. Generally the capacitive reactance $1/\omega C_2$ is made small compared with the load resistance *R* to prevent cyclic changes in output current from affecting the output voltage ωL is made

large compared with $1/\omega C_2$ to provide a large reduction in the ripple components of voltage. Thus

$$2\omega L \gg 1/2\omega C_2 \ll R. \qquad \text{...(62)}$$

The ripple voltage across C_2 or across R is

$$E_2 = E_{rms} = \frac{-E_R}{\sqrt{2\pi}} \times \frac{-j/2\omega C_2}{j2\omega L - j/2\omega C_2} = \frac{E_R}{\sqrt{2\pi}} \frac{1}{4\omega^2 LC_2 - 1}$$

$$= \frac{I_{dc}}{2\sqrt{2\pi f C_1}} \frac{1}{4\omega^2 LC_2 - 1} = \frac{I_{dc}}{2\sqrt{2\omega C_1}(4\omega^2 LC_2 - 1)}$$

$$\therefore \quad \text{Ripple factor} = \frac{E_{rms}}{E_{dc}} = \frac{1}{2\sqrt{2\omega RC_1}\,(4\omega^2 LC_2 - 1)}. \qquad \text{..(63)}$$

This shows that the ripple factor varies inversely as the load R or directly as the load current. It is also a function of C_1, C_2, and L.

2.5. POWER SUPPLY

The power supply unit is an essential type of radio and electronic equipment, as it supplies voltages and currents to different components of various tubes. It consists four units, namely, the transformer, rectifier, filter and voltage divider. The values of output current and voltage required will determine the rating of transformer and the type of the rectifier tube to be used. The percentage of ripple voltage allowable will determine the type of the filter circuit and its components.

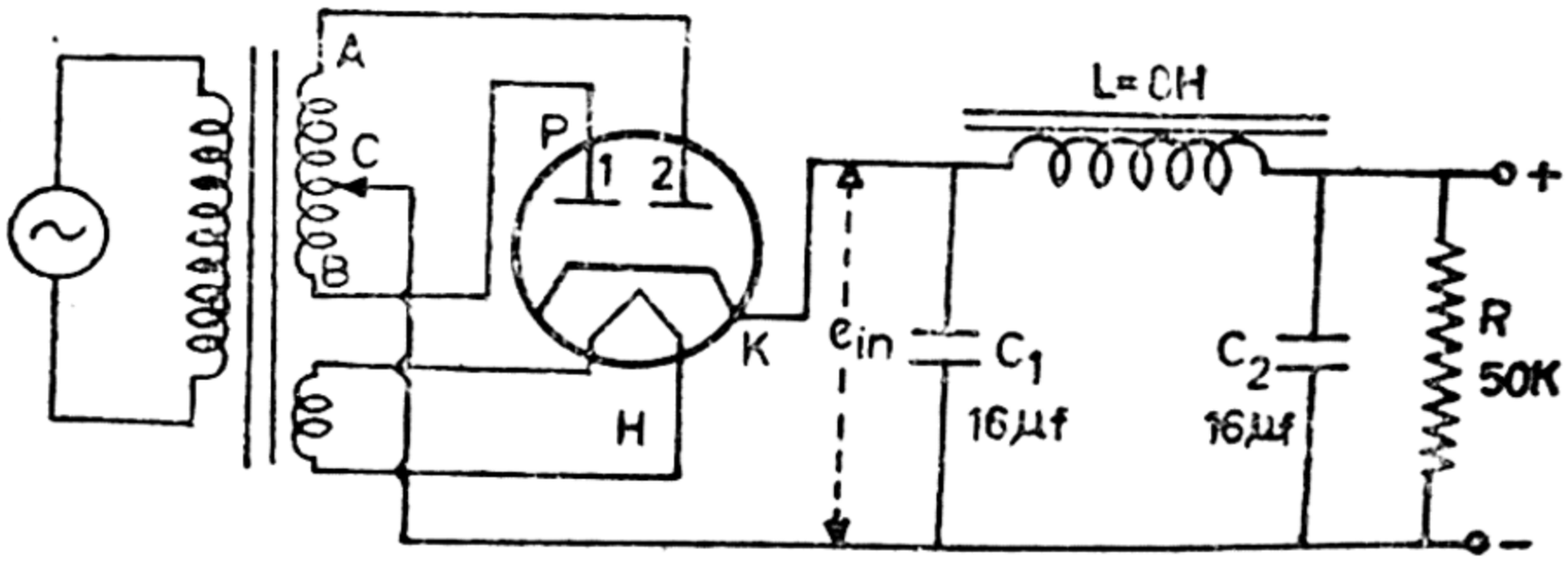


Fig. 2.19.

The circuit diagram of a power supply unit with Π-section filter is shown in Fig. 2.19. The voltage regulation can be calculated by the measuring voltage at no load and different loads. It can be shown that the voltage regulation is better if a L-section filter is also added than the Π section filter only. It can be seen that the ripple percentage varies directly as the load current and inversely as the load resistance.

The choice of various circuit elements depends upon the output voltage, voltage regulation and ripple factor. In a typical un-regulated power supply, a twin diode ($5Y3GT$) is used. The Π section filter has $L = 8H$ and $C_1 = C_2 = 16\ \mu f$. The load resistance $R_L = 50\text{k}\,\Omega$.

Electronically Regulated Power Supply. The most widely used electronically regulated power supplies provide regulations for both input and output variations. For thevoltage stabilization the *VR*-tube with a series resistance is used (Fig. 2.20). For changes in the input

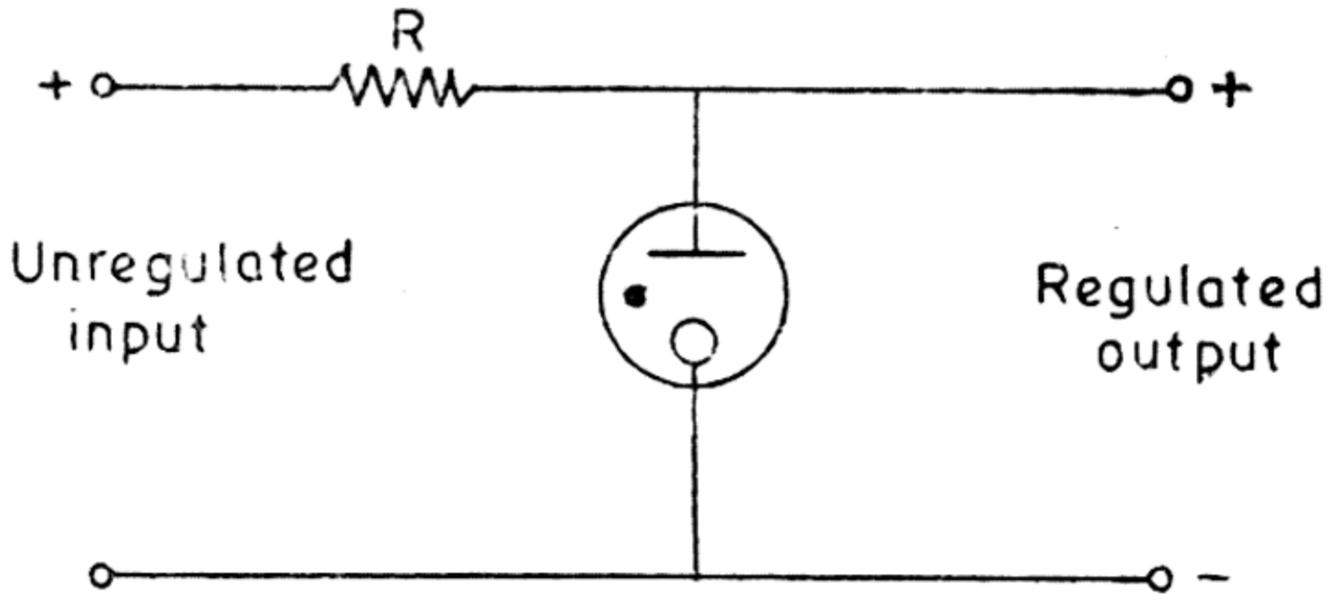


Fig. 2.20.

conditions, the voltage drop across the series resistance gets adjusted to produce a regulated output voltage. For changes in the load resistance, the current between the load and the regulating device gets adjusted to produce a constant voltage across the load. Such regulators are called shunt regulators when the devices are used in shunt or parallel to load.

In an electronically regulated power supply the unregulated potential from the rectifier and filter is applied across the terminals of the regulator. The regulation is obtained by comparing a fixed fraction of the output potential with a standard potential source (*VR*-tube). Any difference between the two is amplified by a high gain dc amplifier T_1 and applied to the control grid of the series control tube T_2 (a power triode or a pentode or beam tetrode

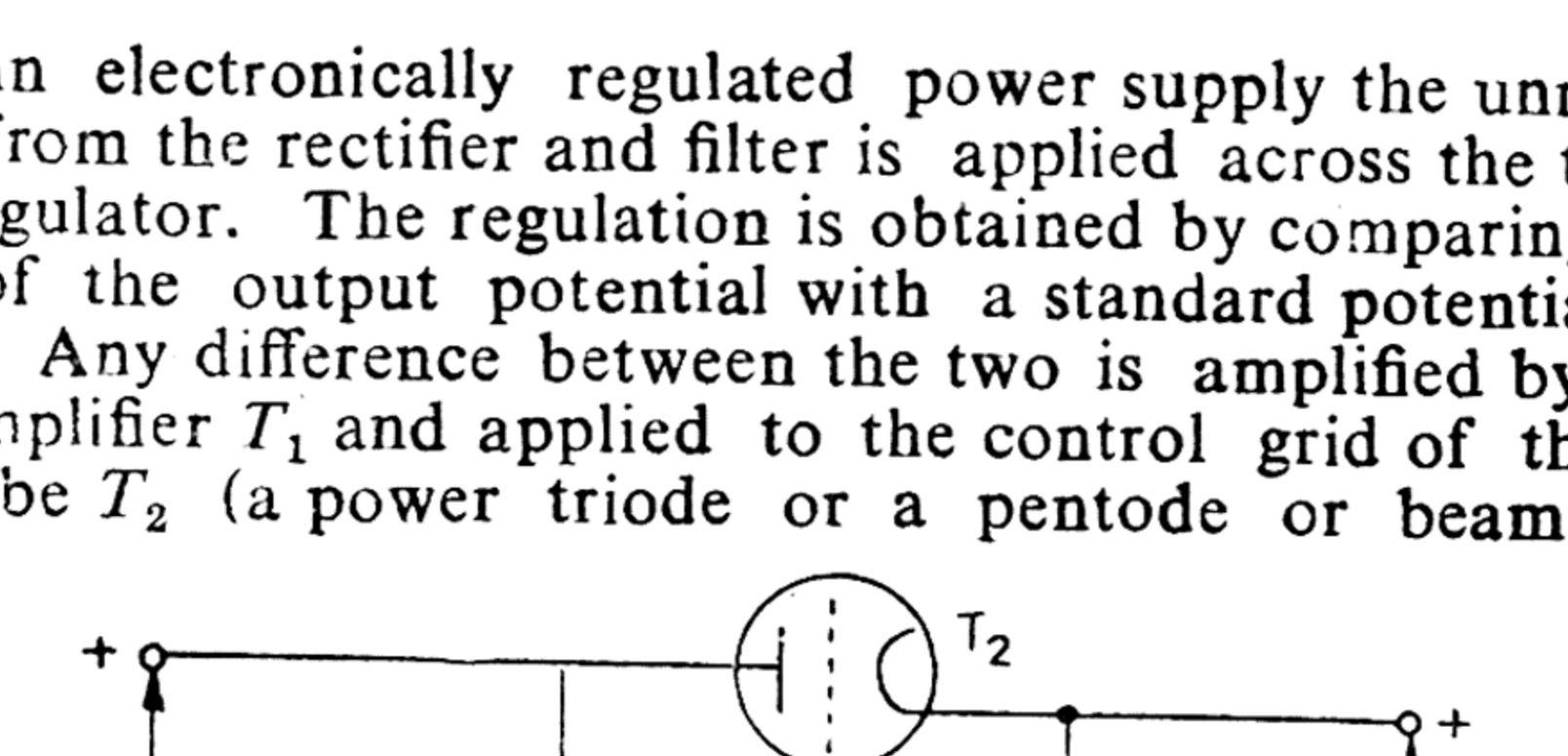


Fig. 2.21.

connected as a triode). The design of a potential regulator requires a knowledge of the characteristic curves of the control tube and also of *dc* amplifier tube. The *dc*-amplifier of sufficient gain is required to make the correction perfect. If the current require-

ments are too high for a single control tube, a number of such tubes may the connected in parallel.

Exercises

Example 1. *In a diode rectifier without filter the load resistance* $R_L = 1000\,\Omega$, *supply voltage* $e_s = 474 \sin 377\,t$ *volts, and the the source resistance* $R_s = 500\,\Omega$. *Determine* (*a*) *the dc component of load voltage,* (*b*) *the peak instantaneous diode current,* (*c*) *the peak inverse voltage across the diode and* (*d*) *the average power absorbed by the load.*

During the −ve half cycle of e_s the diode acts as an open circuit. During the +ve half cycle it acts as a short circuit and load voltage

$$e_L = \frac{R_L}{R_s + R_L}\, e_s = \frac{1000}{500+1000}\, 474 \sin\theta = 316 \sin\theta. \quad (0 \leqslant \theta < \pi)$$

The *dc* component of e_L is the average value of the voltage over a full cycle of operation.

$$E_{dc} = \frac{1}{2\pi}\int_0^{2\pi} e_L\, d\theta = \frac{1}{2\pi}\int_0^{\pi} 316 \sin\theta\, d\theta \simeq 100 \text{ volts.}$$

The peak diode current occurs when e_s is maximum +ve.

$$\therefore \quad (i_d)_{max} = \frac{(e_s)_{max}}{R_s + R_L} = \frac{474}{1000+500} = 0.316 \text{ amp.}$$

The peak inverse voltage across the diode occurs when e_s has its maximum negative value, hence

$$E_{peak\ inverse} = (e_s)_{max} = 474 \text{ volts.}$$

As no power is absorbed by the load during the—ve half cycle, the power absorbed by the load during the +ve half cycle at each instant is

$$p_L = \frac{e_L^2}{R_L} = \frac{316^2}{1000} \sin^2\theta = 100 \sin^2\theta.$$

The average power absorbed in one full cycle is

$$P_L = \frac{1}{2\pi}\int_0^{2\pi} P_L\, d\theta = \frac{100}{2\pi}\int_0^{\pi} \sin^2\theta\, d\theta = 25.$$

Example 2. *A diode* ($r_p = 500\,\Omega$) *is used as half-wave rectifier. If the rated output current is 80 mA and the rms value of voltage across the transformer is 230 volts, calculate the regulation and the efficiency of the diode. Also find the current corresponding to the output maximum power.*

$$\text{Voltage regulation} = \frac{E_{dc,\ no\ load} - E_{dc,\ full\ load}}{E_{dc,\ full\ load}} \times 100\%$$

It will be zero if there is no variation of output voltage with load. In the present case

$$E_{dc,\ no\ load} = \frac{E_0}{\pi} = \frac{\sqrt{2}\, E_{rms}}{\pi} = \frac{\sqrt{2} \times 230}{3.14} = 103.6 \text{ volts}$$

and $E_{dc,\ full\ load} = E_0/\pi - I_{dc}\, r_p = 103.6 - 0.080 \times 500$

$= 103.6 - 40 = 63.6$ volts.

$\therefore$ Regulation $= [(103.6 - 63.6)/63.6] \times 100 = 62.9\%$.

Rectification efficiency $\eta = 40.6\ [1 - I_{dc}\, r_p \pi / E_0]\%$

or $\eta = 40.6\ [1 - 40 \times 3.14/230\sqrt{2}] = 24.9\%$.

Thus the efficiency decreases from 40.6% at no load to 24.9% at full load.

The current at which the maximum power is obtained from the rectifier can be obtained by equating η to 20.3 and solving for I_{dc}. This wlll give us a value which is greater than the rated current.

Example 3 *In a full-wave vacuum tube rectifier without filter the load resistance is of 4000 ohms. If the input resistance is 800 ohms and the supply voltage is of amplitude 240 volts and frequency equal to 50 Hz. Calculate (i) peak average and rms values of current, (ii) dc power output and total power input, (iii) rectifier efficiency, (iv) form factor and (v) ripple factor.*

Maximum value of current $I_0 = \dfrac{E_0}{(R_s + R_L)} = \dfrac{240}{4800} = 50\ mA.$

Average value of current $I_{dc} = \dfrac{I_0}{\pi/2} = \dfrac{50 \times 2}{3.14} = 31.85\ mA.$

Root mean square value of current $I_{rms} = I_0/\sqrt{2} = 35.36\ mA.$

The *dc* power output $P_{dc} = I_{dc}^2 \,.\, R_L = 4.057$ watts.

Total power input $P_{in} = I_{rms}^2 (R_S + R_L) = 6.001$ watts.

Rectifier efficiency $\eta = \dfrac{P_{dc}}{P_{in}} = \dfrac{4.057}{6.001} \times 100 = 67.60\%$

$$\text{Form factor} = \frac{I_{rms}}{I_{dc}} = \frac{35.36}{31.85} = 1.11.$$

Ripple factor $r = \sqrt{[(I_{rms}/I_{dc})^2 - 1]} = \sqrt{(1.11^2 - 1)} = 0.48.$

Example 4. *A full wave vacuum diode rectifier uses a inductor filter ($L = 20$ henrys) and load resistance $R_L = 10$ kilo ohms. An alternating voltage $e = 400 \sin 314\ t$ is applied to the input. If the amplitudes of the harmonics higher than second are negligible, find (i) the dc output current, (ii) component of load current, (iii) the ripple factor and (iv) the dc output voltage.*

Neglecting terms higher than of second harmonic, the rectified output voltage of a full-wave rectifier is given by

$$e = \frac{2E_0}{\pi} - \frac{4E_0}{3\pi}\cos 2\omega t = \frac{2 \times 400}{\pi} - \frac{4 \times 400}{3\pi}\cos 628t.$$

The *dc* component of load current $I_{dc} = E_{dc}/R_L$

$= 800/3.14 \times 10^4 = 25.48 mA$

The *ac* component of load current

$$I_{rms}=E_{ac}/Z_L=\frac{4E_0}{3\pi\sqrt{2}[R_L^2+(2\omega L)^2]^{1/2}}$$

$$=\frac{4\times 400}{3\times 3.14\times 1.41[10^8+(2\times 314\times 20)^2]^{1/2}}=7.5\ mA$$

The ripple factor $r=\frac{I_{ac}}{I_{dc}}=\frac{7.5}{25.48}=0.30.$

The *dc* output voltage $E_{dc}=I_{dc}R_L=25.48\times 10^{-3}\times 10^4$
$=255$ volts.

Example 5. *Calculate the values of the elements of a low pass filter whose cut off frequency is 1000 c/sec and the terminal impedance is 600Ω.*

We know that for a low pass filter the cut off frequency $fc=1/\pi\sqrt{(LC)}$ and the impedance $R=\sqrt{(L/C)}$, hence

$$L=\frac{R}{\pi fc}=\frac{600}{3.14\times 1000}=191\ mH$$

and

$$C=\frac{1}{\pi Rfc}=\frac{1}{3.14\times 600\times 1000}=0.53\mu f.$$

Hence the required values are $L_1=95.5\ mH$, $L_2=95.5\ mH$ and $C=0.53\mu f$ in the *T*-type filter ; $L=191\ mH$, $C_1=0.265\ \mu f$ and $C_2=0.265\ \mu f$ in Π-type filter.

Example 6. *Design a power supply using a* Π*-section filter to give dc output 250 volts at 50 mA with a ripple factor not to exceed 0.01%.*

Load resistance $R_L=\frac{E_{dc}}{I_{dc}}=\frac{250}{50\times 10^{-3}}=5\times 10^3\Omega.$

% ripple factor $r=\frac{100}{\sqrt{2}\omega RC_1(4\omega^2LC_2-1)}.$

As the required value is 0.01% and the supply is of frequency 50 *Hz*, *i.e.* $\omega=2\pi f=314$, hence

$$\frac{100}{\sqrt{2}\times 314\times(5\times 10^3)\,C_1(4\times 314^2\,LC_2-1)}=0.01$$

or $$C_1(3.94\times 10^5LC_2-1)=4.5\times 10^{-3}.$$

Let us take an inductor of inductance $L=20H$ and resistance $Rc=500\Omega$. Also assuming $C_1=C_2=C$. Hence

$$7.88\times 10^6C^2-C=4.5\times 10^{-3} \quad \text{or} \quad C=24\mu f.$$

Voltage drop in choke $=I_{dc}Rc=50\times 10^{-3}\times 500=25$ volts.

Rectifier output before filteration $=250+25=275$ volts.

Example 7. *Design a power supply using a L-section filter to give dc output of 350 volts at 100 mA with a ripple that must be less than 10 volts. Use a critical value for a choke.*

The effective load resistance $R_L = E_{dc}/I_{dc} = 350/0.10 = 3500\ \Omega$

The ripple factor $r = E_{ac}/E_{dc} = 10/350 = 0.0286$.

The critical inductance $Lc = R_L/3\omega$
$= 3500/3 \times 2 \times 3.14 \times 50 = 3.72H$.

We know that $r = \dfrac{1}{6\sqrt{2\omega^2 LC}} = 0.0286$

or $$LC = 4.18 \times 10^{-5}.$$

These equations specify the minimum values of L and LC, required for filtering.

Example 8. *A choke input filter with a full-wave rectifier is used as a power supply to provide 100 mA at 300 volts. The resistance of the input choke is 80 ohms, that of smoothing choke is 400 ohms and the internal voltage drop at the rectifier tube is 30 volts when the tube is supplying full load. Calculate the voltage required at the plate of the tube when operating as full-wave rectifier.*

The potential across the input inductance $L_1 = E_{L_1} = I_S R_{L_1}$
$= 100 \times 10^{-3} \times 80 = 8$ volts.

The potential across the smoothing choke $L_2 = E_{L_2} = I_S R_{L_2}$
$= 100 \times 10^{-3} \times 400 = 40$ volts.

The output voltage at normal load for the full wave rectifier choke input filter power supply is given as

$$E_0 = 0.9\ E_S - E_T - E_{L_1} - E_{L_2}$$

or $$E_S = (300 + 30 + 8 + 40)/0.9 = 240 \text{ volts.}$$

Oral Questions—

1. How can one use a diode to rectify alternating current ?
2. Define the terms ; rectifier efficiency, ripple factor, PIV and voltage regulation.
3. Explain the principle of a full-wave rectifier.
4. Why is the full-wave rectifier more useful than half-wave rectifier ?
5. For what type of service are gaseous diode rectifiers used ?
6. What is the function of (*a*) filter circuit ? (*b*) the filter capacitor, (*c*) the filter choke ?
7. What is meant by (*a*) a low pass filter, (*b*) a choke input filter ? (*c*) a capacitor-input, (*d*) a single section filter.
8. Why does a capacitor input filter circuit have voltage regulation ?
9. For what type of service are capacitor-input circuits generally used ?
10. What is meant by (*a*) the optimum value of inductance ? (*b*) the critical value of inductance ?

11. For what type of service are inductor input circuits generally used ?

12. What are the characteristics of a choke input circuit ?

13. What is the purpose of the power supply ? Give its general requirements.

14. Name the four parts of a power supply unit ?

15. Why is the L-section filter more useful than the Π-section filter in a power supply ?

Problems :

1. A voltage 200 sin 100 πt is applied to a half-wave vacuum tube rectifier with filter. The resistance of valve is 1000Ω and load resistance is 10^4 ohms, calculate (*i*) maximum current flowing, (*ii*) average value, (*iii*) rms value of current, (*iv*) ac power input, (*v*) dc power out put, and (*vi*) ripple factor.

18.18 mA, 5.78 mA, 9.09 mA, 0.91 watt, 0.334 watt, 1.21

2. In a vacuum tube rectifier without filter, a voltage 300 sin $100\pi t$ is applied. If the tube resistance is 1000Ω and load resistance is 5000Ω. Calculate (*i*) peak, average and rms values of current. (*ii*) dc power output, (*iii*) ac power input, (*iv*) rectifier efficiency and (*v*) form factor.

50 mA, 15.92 mA, 25 mA, 1.267 watts, 3.75 watts, 33.79%, 1.57

3. A full-wave rectifier has a indicator filter (L= 20H) and load resistance R_L=20 kilo ohms. An alternating voltage e=320 sin 314t is applied to the input. If the amplitude of the harmonics higher than second is negligible, find (*i*) the dc load current, (*ii*) dc output voltage and (*iii*) ripple factor.

10.2 mA, 204 volts, 0.392

4. A half-wave rectifier with a shunt capacitor filter has the input voltage e=250 sin 314t. If C=10 μf and R_L=5000Ω. Find the values of θ_1 and θ_2 and the value of direct current. The diode plate resistance may be neglected.

42°, 94°, 41mA

5. A high pass filter for 600Ω termination is required to cut off frequencies below 20 kilo c/s. Calculate the values of the elements for T-section.

2.39 mH, 0.013 μf

6. The impedance of the low pass RL filter is 200μ H and the resistance 200$k\Omega$. Calculate the cut off frequency for this filter. At what frequency is the output voltage equal to one quarter the input voltage.

159 Mc, 615 Mc

7. Calculate the ripple voltage at the output of a single section choke input filter circuit using a 15 henry choke and a 10 μf capacitor, if the input to the filter is obtained from a full-wave rectifier operated from a 50 cycles power source. What will happen if a two section choke input filter circuit consisting two similar chokes and two similar capacitors are used ?

0.96%, 0.015%

8. Design an LC filter for a full-wave rectifier circuit supplied at 50 hertz. It is to have a 3000 ohm load at dc output 300 volts and a ripple factor of 0.005. Use a critical value for the choke.

3

Amplifiers

3.1. AMPLIFIERS AND THEIR CLASSIFICATION

An amplifier is a device which amplifies or increases magnitude of the applied input. Commonly vacuum tube (or transistor) circuits amplify the ac signal applied at the input. In vacuum tube, the control grid is capable of varying current sufficiently in the plate circuit. If a small signal (weak alternating voltage or current) is applied to the grid circuit then the grid is capable of amplifying this signal. The signal in the grid circuit is called input and the amplified signal collected in the plate circuit is called output and the tube used is called *amplifier*. Triode, tetrode, pentode valves, etc. are frequently used as an amplifier. The amplifier may be classified in different ways as :

(1) According to the use of tube.

(*a*) *Voltage amplifier*. This is used to amplify the signal voltage. The tube used as a voltage amplifier must have high amplification factor and must be operated with high impedance load.

(*b*) *Power amplifier*. A power amplifier produces a high alternating power in the output circuit at the cost of dc power, provided by the dc plate voltage (E_{bb}) ;

(*c*) *Current amplifier*. This is used to produce a large change of plate current when a small input ac potential is applied. The tube for this purpose must have high mutual conductance (g_m).

(2) According to frequency range of operation.

(*a*) *Direct coupled amplifier*. This is used to amplify signals of extremely low frequencies, or dc signals ;

(*b*) *Audio frequency amplifier*. This is used to amplify the signals of audio frequency range (20 to 20,000 c/sec) ;

(*c*) *Radio frequency amplifier.* This is used to amplify the signals from 16 kilo-cycles/sec to several mega cycles/sec.

(*d*) *Video frequency amplifier.* Such amplifiers are used in television and may amplify the ac voltages ranging from 50 c/sec to several hundered maga cycles per seconds.

(3) According to the interstage coupling.

(*a*) *Resistance capacitance (R-C) coupled amplifier ;*

(*b*) *Inductance-Capacitance (L-C) coupled amplifier ; and*

(*c*) *Transformer coupled amplifier.*

(4) According to the range of frequency to be amplified.

(*a*) *Narrow band amplifier.* The amplifier in which the band or range of frequencies to be amplified is small in comparison with the middle frequency of the band.

(*b*) *Wide band amplifier.* The amplifier in which the band of frequencies to be amplified is large compared with the middle frequency of the band.

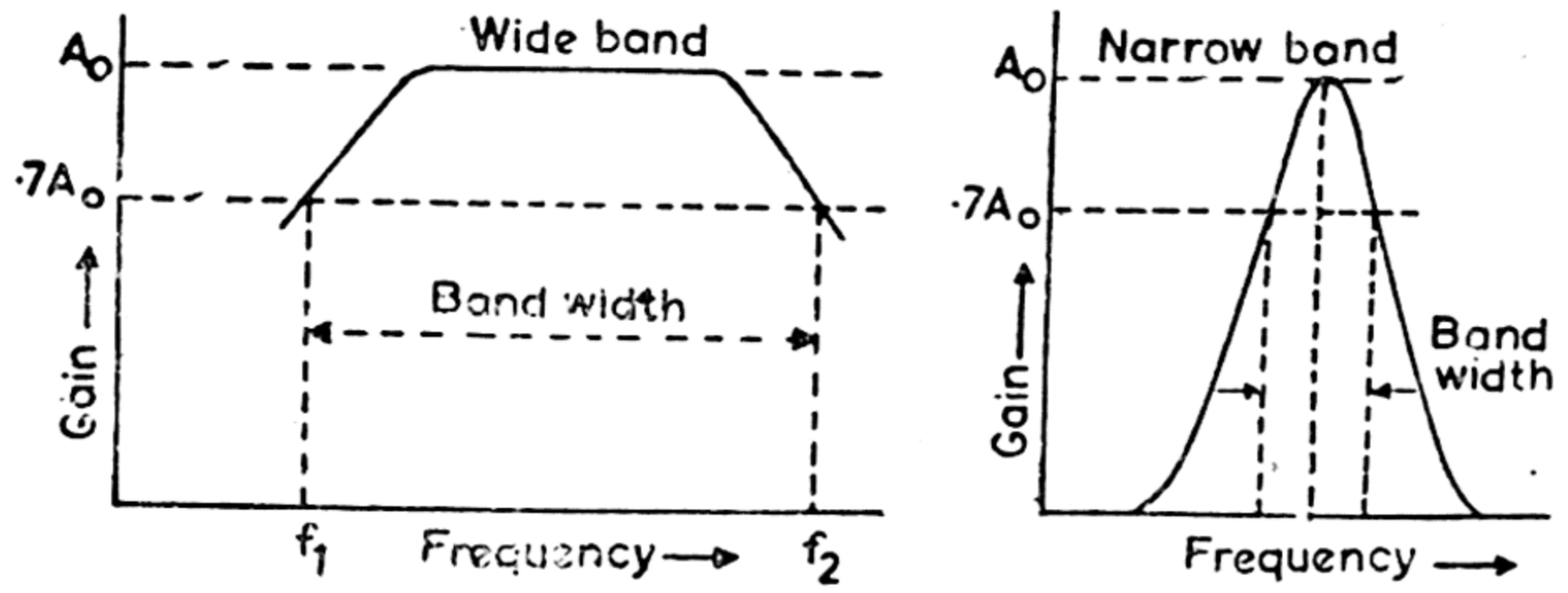


Fig. 3.1.

(5) According to the point of operation.

(*a*) *Class A-amplifier.* In such amplifiers the grid (E_{cc}) and input ac signal applied to the grid are so adjusted that the plate current should flow during the complete cycle of input.

(*b*) *Class B-amplifier.* In such amplifiers the grid basis (E_{cc}) is just equal to the cut-off potential of the tube and the plate current will flow during one half of each input cycle, during positive half only.

(*c*) *Class AB amplifier.* In such amplifiers the grid (E_{cc}) and input are so adjusted that the plate current flows for appreciably more than half and less than the entire cycle of input. These are classified as AB_1 and AB_2 amplifiers. The former does not draw grid current at any time while the latter does take grid current during some part of the input cycle.

(*d*) *Class C-amplifier.* In such amplifiers the grid bias (E_{cc}) is more negative than that the cut-off potential of the tube. The plate current flows for less than positive half of applied cycle.

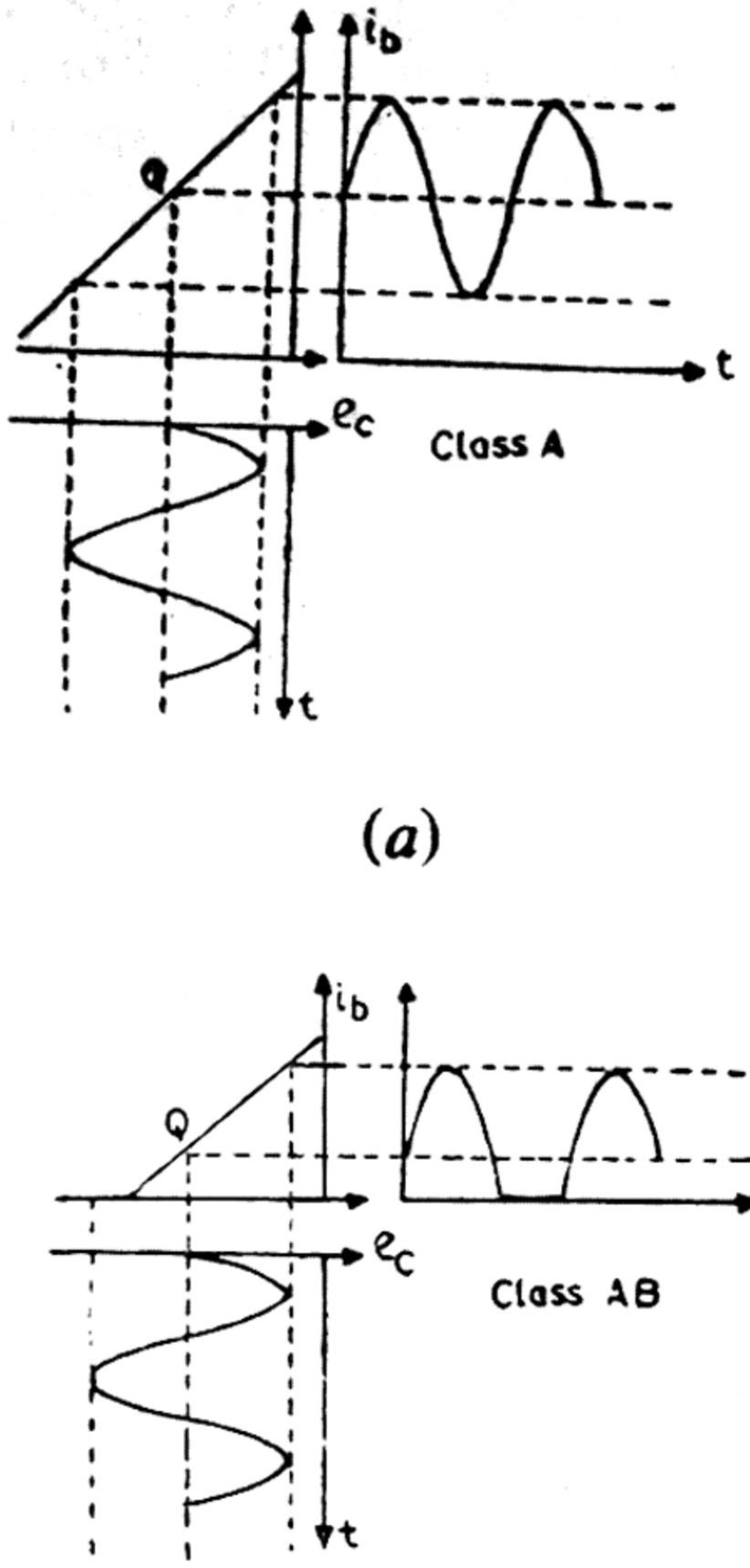


Fig. 3.2.

The variations of plate current with the grid bias and the input for these amplifiers are shown in Fig. 3.2. Here *we shall discuss only class A audio frequency amplifier in details.*

3.2. CLASS-A TRIODE VOLTAGE AMPLIFIER

The Fig. 3.3 shows the circuit diagram of triode as voltage amplifier. The ac signal (input) e_s is applied between grid-cathode. The amplified output e_0 is collected across load impedance Z_L between plate and cathode. To operate the amplifier as class *A*, the plate current i_b should flow for entire cycle of input. The necessary grid bias E_{co} is adjusted such that the total grid potential ($E_{cc}+e_s$) should always remain negative so that there is no grid current and the variation in grid-potential is entirely on the linear part of the transfer characteristic. Under these conditions due to the change in the

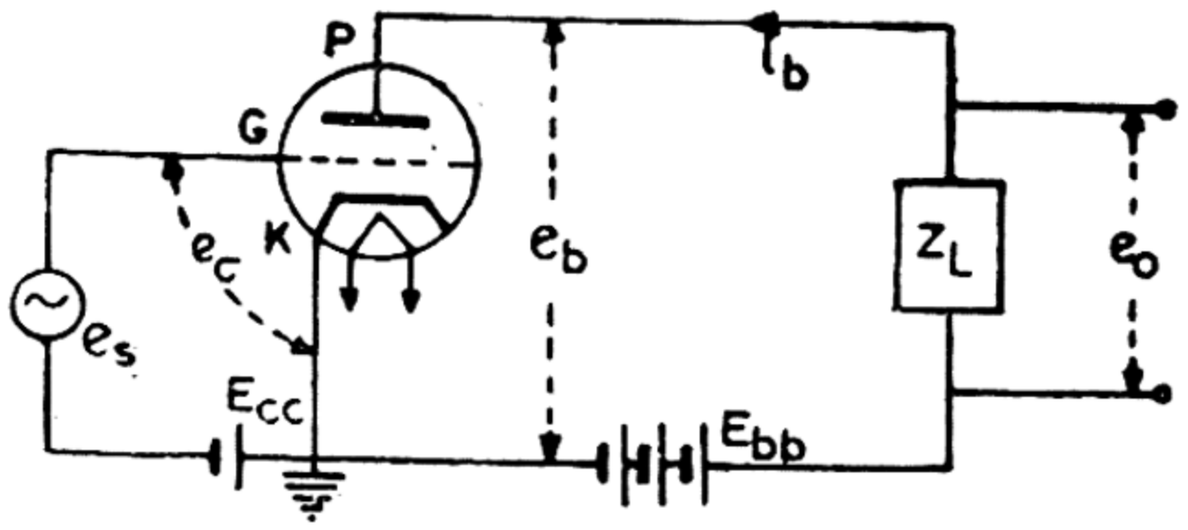


Fig. 3.3.

grid potential e_c, the plate current i_b and hence the voltage across Z_L, will also change similar to that of input signal. The output (e_0) will be free from distortion so the waveform and the frequency of the output will be exactly as that of input signal, but amplified.

The Fig. 3.4 shows the operation of the *dynamic transfer characteristic curve* (xy) of the valve at the particular load impedance Z_L. The input signal is shown as *abcde*, the operating *or quiescent point* (Q), which corresponds to $e_s=0$, is so adjusted that the entire input (e_c) should lie in the negative region of grid bias. Also the peak value of input corresponding to *b* and *d* must be within the straight part of characteristics. The output amplified wave form is shown by curve *ABCDE*.

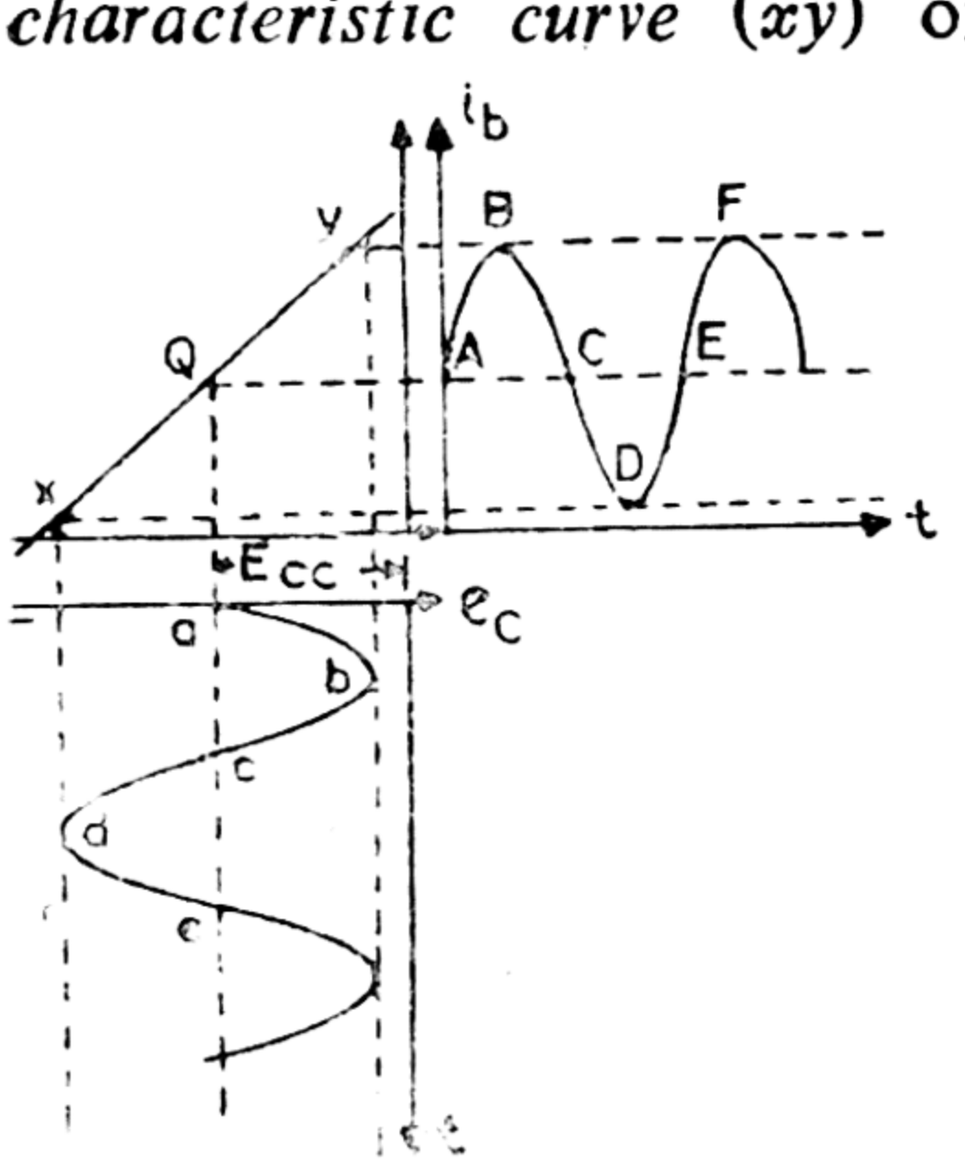


Fig. 3.4.

The Fig. 3.5 shows the class-*A* operation of amplifier on the *anode characteristic*. If i_b is the current at any instant in the plate circuit, the instantaneous potential at the plate is given by

$$e_b = E_{bb} - i_b Z_L, \qquad \ldots(1)$$

where E_{bb} is the plate supply voltage. This Eq. when plotted on the axes of anode characteristics, gives a straight line which intersects at the points, L when $i_b=0$, $e_b=E_{bb}$ and M when $e_b=0$, $i_b=E_{bb}/Z_L$. This line drawn is called *load line* and intersection with characteristics at various grid potentials gives the corresponding values of anode current and anode potential. Q is the operating point and the input at grid (e_c), is *abcde*, the corresponding plate current (i_b) is *ABCDE* and the plate potential (e_b) as $a'b'c'd'e'$.

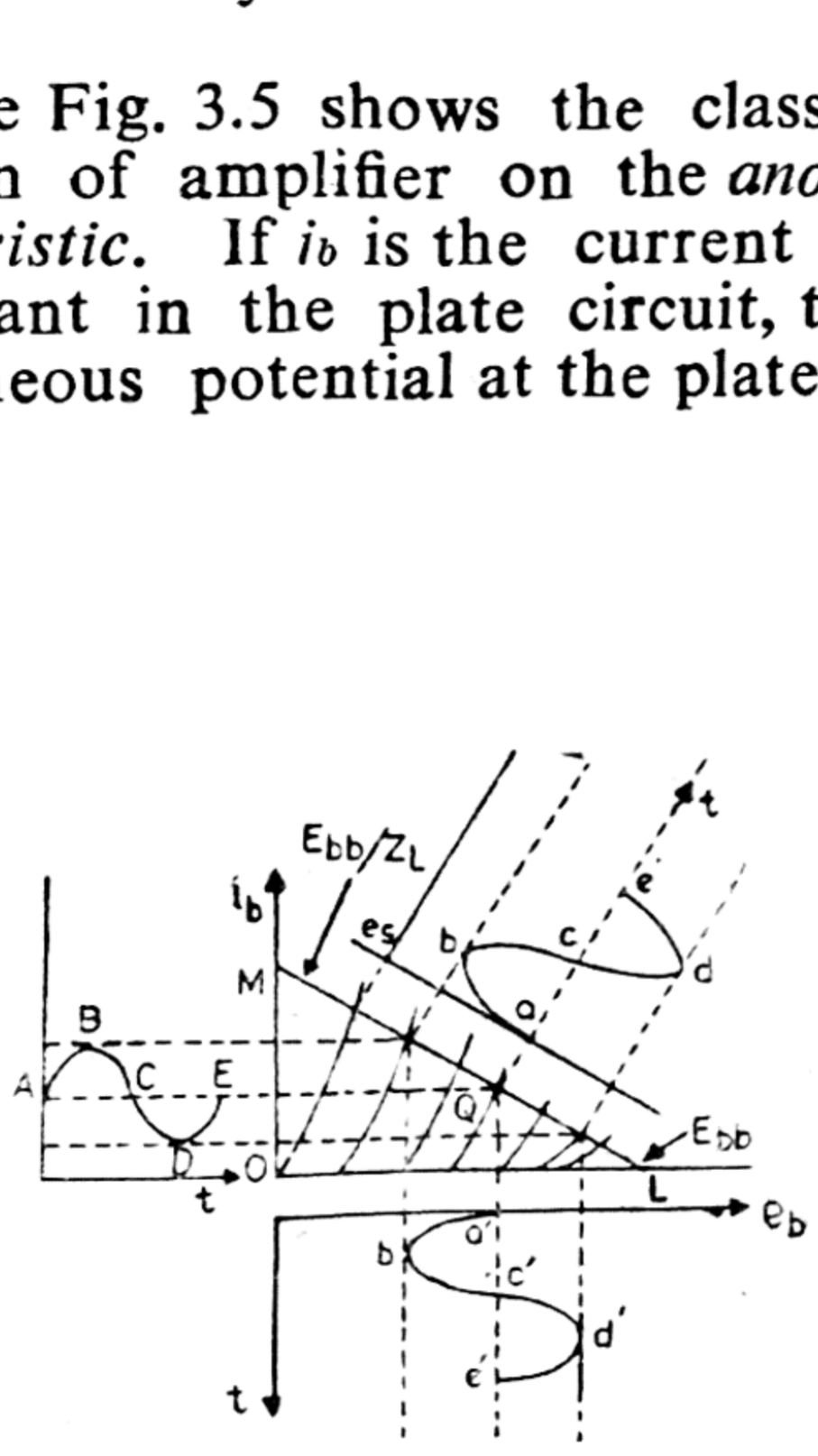


Fig. 3.5.

The phase relation analysis. The input signal voltage is generally sinusoidal ($e_s=E_0 \sin \omega t$). When the dc grid bias is E_{cc}

the potential appearing at the grid with respect to earth (cathode) will be

$$e_c = e_s - E_{cc}. \qquad \ldots(2)$$

As already required E_{cc} is always kept negative. The e_c and e_s v/s ωt are plotted in Fig. 3.6 (top).

The plate current is governed by the applied potential of plate (E_{bb}) and the grid voltage (e_c). At any instant plate current is given by

$$i_b = I_b + i_p, \qquad \ldots(3)$$

where i_p is the variation of the plate current.

Hence Eq. (1) becomes

$$e_b = E_{bb} - (I_b + i_p)\, Z_L.$$

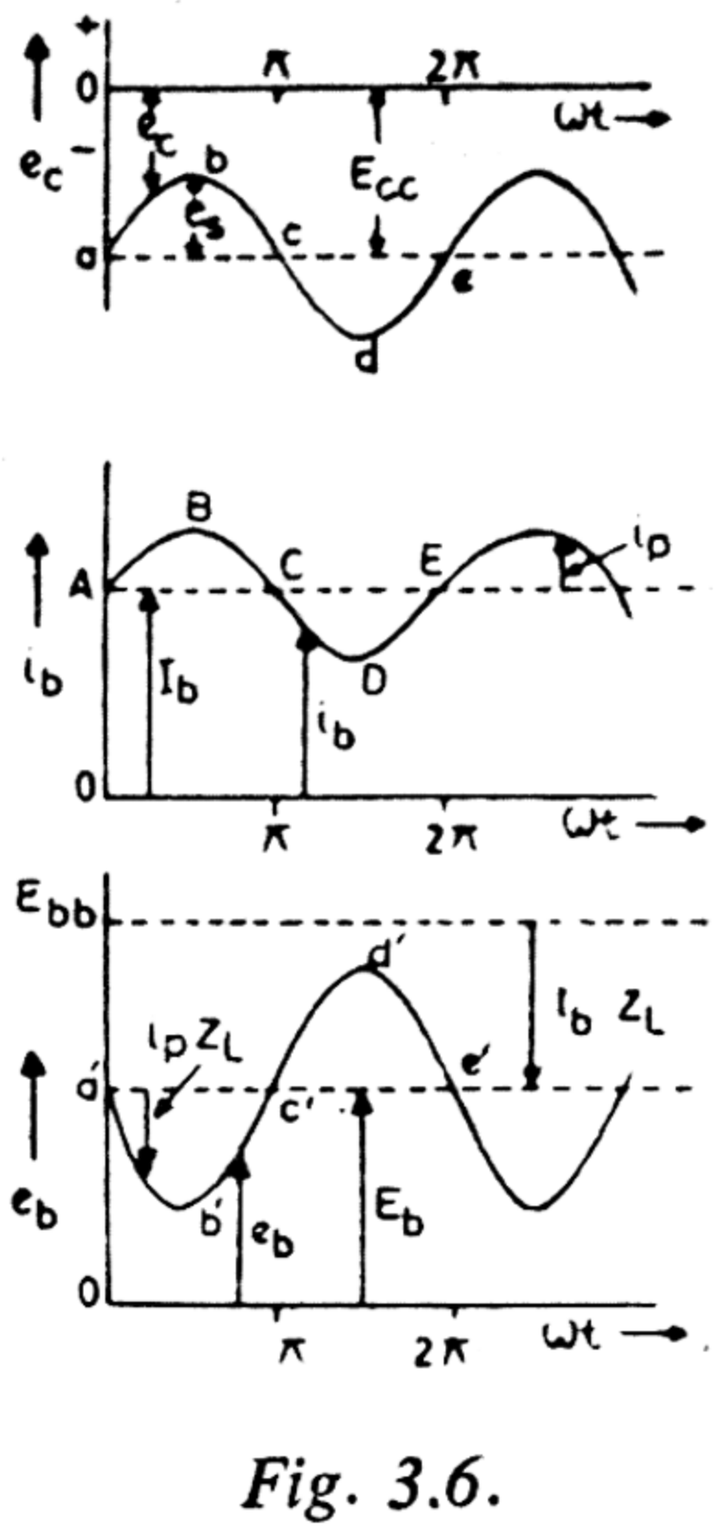


Fig. 3.6.

But corresponding to quiescent point or no signal point, we have

$$e_b = E_b = E_{bb} - I_b Z_L. \qquad \ldots(4)$$

But $e_b = E_b + e_p \quad \therefore \quad e_b = E_{bb} - I_b Z_L + e_p.$

$$\therefore \qquad e_p = -i_p\, Z_L. \qquad \ldots(5)$$

These all relations are plotted in Fig. 3.6.

The above analysis shows that the plate current rises when the grid voltage increases and vice versa as these two are in the same phase. This gives that the potential drop across load impedance will also increase. But the anode voltage (e_b) decreases with the increase of plate current (i_b) or grid potential (e_c). So the anode voltage is π radians (180°) out of phase with grid voltage.

Voltage gain. Let for a de_c change in grid potential the change in anode current be di_b and corresponding change of anode voltage is de_b, so we have from Eq. (1), that

$$de_b = -Z_L\, di_b. \qquad (\text{as } dE_{bb}=0) \qquad \ldots(6)$$

Since the plate current (i_b) is a function of plate voltage (e_b) and grid voltage e_c, hence we can write

$$i_b = (e_b,\ e_c). \qquad \ldots(7)$$

For a small change in anode current we thus have

$$di_b = \left(\frac{\partial i_b}{\partial e_c}\right)_{e_b} de_c + \left(\frac{\partial i_b}{\partial e_b}\right)_{e_c} de_b. \qquad \ldots(8)$$

But $(\partial ib/\partial ec)_{eb}=g_m$, the transconductance and $(\partial ib/\partial eb)_{ec}=1/r_p$. where r_p is the plate resistance. Hence Eq. (8) in terms of g_m and r_p will be

$$dib=g_m\, dec+(1/r_p).\, deb=g_m\, dec-Z_L\, dib/r_p$$

or
$$dib=\frac{r_p.g_m.dec}{r_p+Z_L}=\frac{\mu dec}{r_p+Z_L}. \qquad ...(9)$$

Now from Eq. (6), we have

$$deb=-\mu Z_L\, dec/(r_p+Z_L). \qquad ...(10)$$

The voltage gain or voltage amplification (A) of an amplifier is defined as the ratio of output ac voltage to the input ac voltage.

$$\therefore \quad \text{Voltage gain } (A)=\frac{deb}{dec}=-\frac{\mu Z_L}{r_p+Z_L}=\frac{-\mu}{(1+r_p/Z_L)}. \qquad ...(11)$$

The negative sign shows that the input and output voltages are 180° out of phase, *i.e., the amplifier shifts the phase of the output voltage by 180°*

In case the load is pure resistive, load $Z_L=R_L$ and we have the voltage gain as

$$A=-\frac{\mu}{(1+r_p/R_L)}. \qquad ...(12)$$

(*i*) If $R_L \ll r_p$ (or $R_L \to 0$), $deb \to 0$ and also $A \to 0$; (*ii*) If $R_L=r_p$, $A=-\mu/2$; and (*iii*) If $R_L \to \infty$, $A \to -\mu$. Thus the gain changes with load resistance. The theoretical maximum gain of an amplifier can be equal to the amplification factor of the tube.

As the changes about the quiescent values are respectively,

$dib=ib-Ib=ip$; $dec=ec-Ecc=es$ and $deb=eb-Eb=ep$.

Therefore the equation (9) may be written as

$$ip=\mu es/(r_p+R_L). \qquad ...(13)$$

Therefore the potential across the load will be

$$ep=ipR_L=-\mu esR_L/(r_p+R_L). \qquad ...(14)$$

Voltage equivalent diagram. The Eq. (13) of above analysis gives the current in plate circuit. It can be written as

$$\mu es=r_p ip+R_L ip. \qquad ...(15)$$

So, an alternating voltage e_s in the grid circuit is μ times effective, *i.e.*, equivalent in effect to an alternating voltage μes in the plate circuit. Thus, we can consider the valve as a generator of

alternating voltage μe_s with internal resistance r_p, resulting an alternating current in plate circuit as i_p. It is known as the *equivalent anode circuit theorem*. The load impedance (or resistance R_L) is connected in series. So the voltage of this equivalent generator μe_s is dropped across r_p and R_L. Hence Eq. (15) can be represented by a *voltage equivalent diagram* of the valve. In this diagram (Fig. 3.7a), there is a voltage generator with emf (μe_s), G the grid, K the cathode and P the plate position. The load resistance R_L is connected across this voltage generator through plate resistance r_p. The reference positive polarity and current direction are essential parts of the voltage equivalent representation. The transfer generator μe_s is assumed +ve at the cathode K. From this diagram the current in the circuit is

$$i_p = \mu e_s/(r_p + R_L),$$

so the potential drop across the load R_L and the voltage gain will be same as given by Eqs. (14) and (12) respectively. *The equivalent circuit technique is very convenient in solving complicated circuits.*

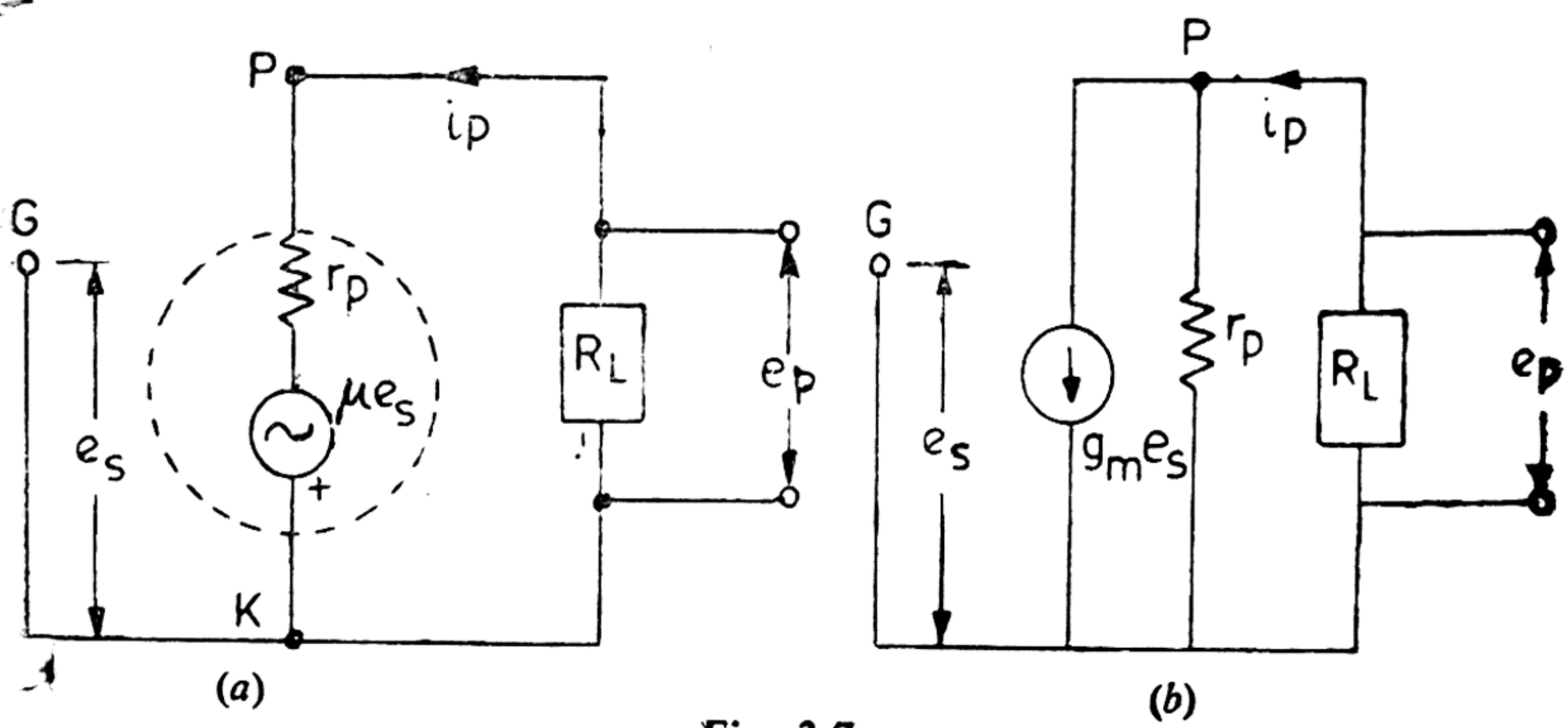


Fig. 3.7.

Current source equivalent circuit : Equation (15) may also be written as

$$\mu e_s/r_p = i_p + R_L\, i_p/r_p = i_p - e_p/r_p$$

or $g_m\, e_s = i_p - e_p/r_p$.

This corresponds to a circuit known as *current source equivalent circuit*. In this circuit, shown in Fig. 3.7 (*b*), a current source $g_m\, e_s$ supplies two branches. Thus the tube is taken to be equivalent to a constant-current source shunted by a conductance instead of the potential source with a series resistance.

Maximum output power. The ac power output can be given as

$$P_{out} = i_p^2\, R_L$$

$$= \left(\frac{\mu e_s}{r_p + R_L}\right)^2 R_L = \frac{(\mu e_s)^2 R_L}{(R_L - r_p)^2 + 4 r_p R_L}. \qquad \ldots(16)$$

Thus the P_{out} will be maximum, when the denominator is minimum, or $R_L = r_p$. Hence the maximum output power is given as

$$(P_{out})_{max} = (\mu e_s)^2/4r_p. \quad \text{.. (17)}$$

The adjustment of $R_L = r_p$ in the circuit is known as *impedance matching.*

The power gain of an amplifier is generally expressed in terms of a logarithmic unit called the *decibel*, defined as

$$\text{decibel power gain} = 10 \log_{10} (P_{out}/P_{input}) = 10 \log_{10} A_p, \quad \text{...(18)}$$

where A_p is the power gain. The bel is a unit ten times larger in size than the decibel.

3.3. INTER STAGE COUPLING IN AMPLIFIERS

High gain is necessary in most of the applications. As the gain due to an amplifier is limited, maximum value is μ when $r_p \gg R_L$ (when the pure resistive load is considered). It is possible to have higher amplifications by using more number of amplifier units in chain, one after the other unit, in such a manner that the output of one stage serves as input for the next stage and so on. This is called *cascading of amplifiers*, and the total amplifier as *cascade amplifier.* If $A_1, A_2, A_3, \ldots A_n$ be the gains of the first, second, third.... n^{th} stage of the amplifier unit respectively of the n-stage cascade amplifier, then the net gain A of this amplifier will be

$$A = A_1 A_2 A_3 \ldots A_n.$$

To have coupling between the various stages of the amplifiers, the important coupling is with the help of combination of resistance, capacitance, inductance and transformer as : (*i*) *Resistance capacitance coupling*, (*ii*) *Inductance-capacitance coupling* and (*iii*) *Transformer coupling.* Let us discuss these one by one.

3.4. RESISTANCE CAPACITANCE COUPLED AMPLIFIER

It is one of the most common and important amplifier circuit used for *wide band low frequency amplification.* In this amplifier (Fig. 3.8) the input signal e_s, to be amplified is in the grid cathode circuit. The necessary bias (operating point) is obtained with the parallel arrangement of C_k and R_k. The required plate potential is obtained from high tension source *H.T.* and resistance R_L is used as the first load resistance. The output of the amplifier is coupled to next stage with *C*, the coupling condenser and R_g, the grid leak resistance. The alternating voltage appearing across R_g serves as the input for the next stage. The

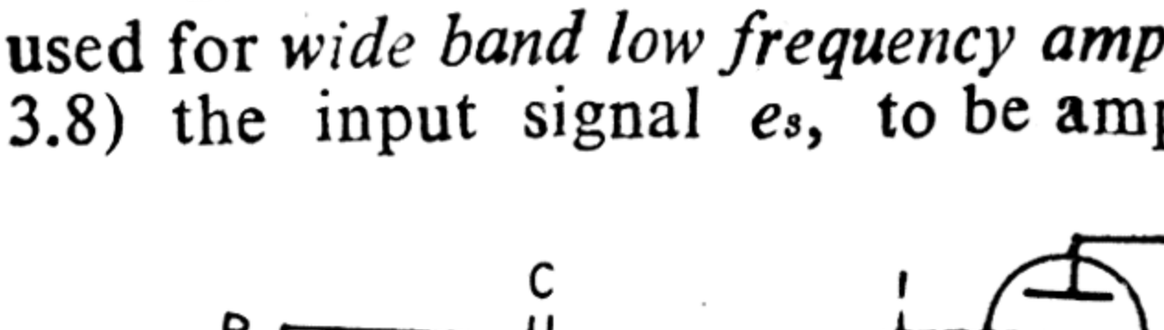


Fig. 3.8.

voltage amplification is fairly constant over a wide-range of frequencies. Let us now consider the *function of circuit components.*

(*a*) *Cathode, self or automatic biasing.* The parallel combination of a resistance R_k and a electrolytic condenser C_k provides the required negative grid bias voltage. To minimize distortion in the output, the amplifier should be operated under class *A* condition, *i.e.*, quiescent point or operating point should be adjusted to the middle portion of the dynamic transfer characteristic so that entire input signal lies only in the linear portion of this characteristic, In the absence of input signal e_s, a steady anode current i_b will flow through R_k, producing $i_b R_k$ potential across the cathode-load resistance and provides the necessary grid bias (E_{cc}) with positive polarity towards cathode. Now when e_s is also applied the anode current will vary, to maintain the voltage drop constant, a high value electrolytic condenser C_k is connected in parallel with R_k. Since C_k offers a low impedance in the path of *ac* and high for steady current, the *ac* components are by-passed through C_k and the steady or average components of current keep the voltage constant. So the parallel combination of R_k and C_k gives an automatic biasing also known as *self biasing* or *cathode biasing*.

(*b*) *Load resistance* (R_L). For successful operation of the amplifier circuit an impedance should be used in the plate circuit, across which the output can be tapped. The impedance may be a pure resistance. This also gives the necessary plate potential through the high tension battery.

(*a*) *Coupling condenser and Resistance.* The large values of *C* and R_g serve the purpose of coupling between the stages. *C* isolates the grid of the second stage from the anode (*H.T.*) circuit of the first stage for all *dc* potentials, so also known as *blocking condenser.* It offers impedance ($1/\omega C$) which is very small even for low frequency of the input, so allow the *ac* components of its input and prevents the *dc* components. The resistance R_g is much greater than impedance ($1/\omega C$) and load resistance R_L, and is known as *grid leak resistance.* The *dc* components passed through *C*, set up a signal voltage across the resistance R_g. The grid leak resistance also provides a path between grid and cathode of the next stage. Without this grid leak path, the accumulation of negative charges on grid, as connected on negative side of condenser plate, will make the grid negative even beyond cut off, which will stop the operation of the next stage. So R_g leaks or offers a path for the negative charges accumulated on the condenser plate. As R_g is in parallel with R_L, the effect of load resistance is also reduced due to its presence.

To consider the response of *RC* coupled amplifier the effect due to inter-electrodes capacitance is quite important. The *ac* equivalent circuit for single stage *RC*. coupled amplifier is shown in Fig. 3.9. C_{in} is the input capacitance of the next stage, C_{pk} is inter-electrodes capacitance, r_p is usual plate resistance, e_s is effective grid potential or input.

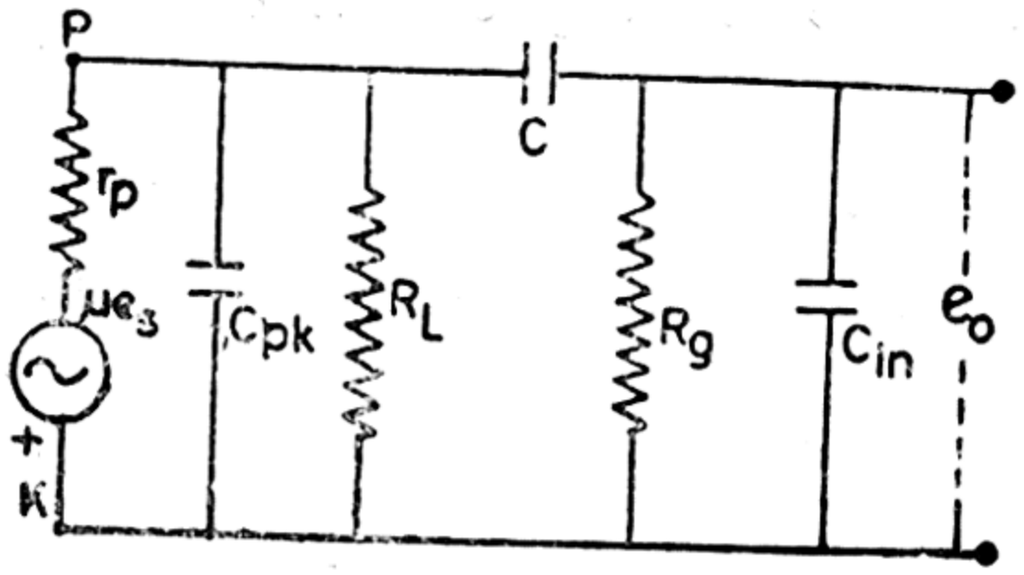


Fig. 3.9.

Voltage Amplification. We shall derive the expression for the voltage gain for the three frequency ranges of the complete audio frequency separately.

(*i*) *Middle frequency region.* As the capacitance of the coupling condenser is very large, hence at the mid—frequencies the reactance offered by it will be negligibly small in comparison to R_g. As it is in series hence can be neglected in equivalent circuit. Also the reactance due to other shunt capacitances (C_{pk} and C_{in} etc.) is very high, these are also omitted in the reduced equivalent circuit (Fig 3.10). Thus the effective load impedance of the circuit (Z_m) will be due to R_L and R_g only.

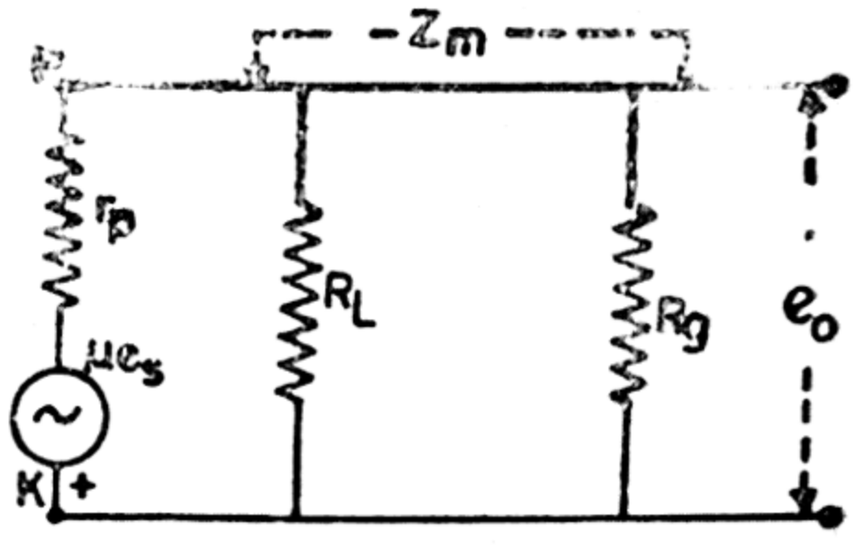


Fig. 3.10.

$$\therefore \quad \frac{1}{Z_m}=\frac{1}{R_L}+\frac{1}{R_g} \quad \text{or} \quad Z_m=\frac{R_L R_g}{R_L+R_g}. \qquad \ldots(19)$$

The voltage drop across Z_m will be the output voltage since R_L and R_g are connected in parallel.

$$e_0=-\mu e_s Z_m/(r_p+Z_m), \qquad \ldots(20)$$

Therefore the voltage gain of the amplifier in the mid-frequency region is given by

$$A_m=e_0/e_s=-\mu Z_m/(r_p+Z_m)$$

$$=-\frac{\mu R_L R_g}{r_p R_L+r_p R_g+R_L R_g}=-g_m R_{eq} \qquad \ldots(21)$$

where $$R_{eq}=\frac{r_p R_L R_g}{r_p R_L+r_p R_g+R_L R_g}. \qquad \ldots(22)$$

This expression for voltage gain is independent of frequency, hence remains constant for middle frequency variations.

(*ii*) *Low frequency region.* At low frequency the reactance due to coupling condenser ($1/\omega C$) is not small hence can not be neglected. The reactance due to shunt capacitance C_{pk} and C_{in} will be high (even more than that at mid-frequency), hence omitted in the reduced equivalent circuit as in Fig. 3.11. The effective load impedance of the circuit (Z_l) will be given by

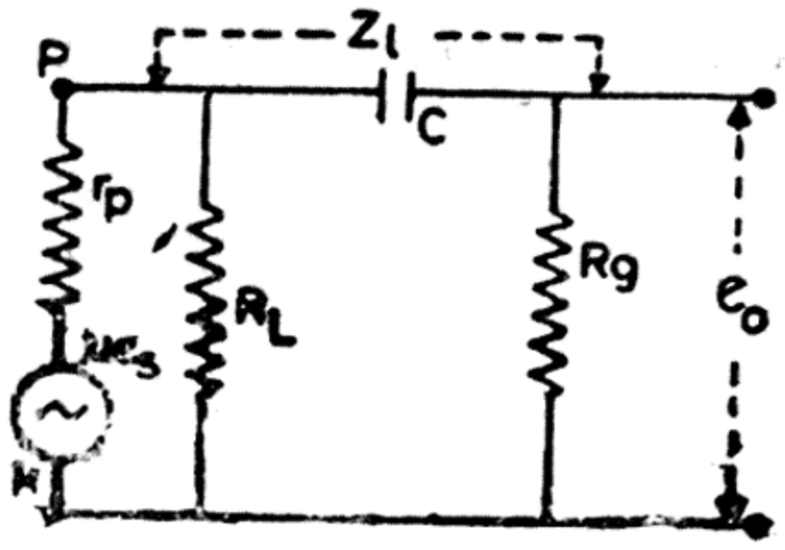


Fig. 3.11.

$$\frac{1}{Z_l}=\frac{1}{R_L}+\frac{1}{(R_g-j/\omega C)} \text{ or } Z_l=\frac{R_L(R_g-j/\omega C)}{R_L+(R_g-j/\omega C)} \qquad ...(23)$$

If i_p is the current in plate cathode circuit then the voltage drop at this impedance will be

$$e_p=i_pZ_l=-[\mu e_s/(r_p+Z_l)\ ]Z_l.$$

If e_0 is voltage output across R_g (to be coupled with the next stage), then we haue

$$\frac{e_0}{e_p}=\frac{R_g}{(R_g-j/\omega C)} \text{ or } e_0=\frac{\mu e_s Z_l R_g}{(r_p+Z_l)\ (R_p-j/\omega C)}. \qquad ...(24)$$

So the voltage gain at low frequency will be

$$A_l=\frac{e_0}{e_s}=-\frac{\mu Z_l R_g}{(r_l+Z_l)\ (R_g-j/\omega C)}$$

$$=-\frac{\mu R_L R_g}{r_pR_L+r_pR_g+R_LR_g-j(r_p+R_L)/\omega C} \qquad ...(25)$$

Thus the gain at low frequencies is less than the gain at mid-frequencies. It decreases with the decrease in frequency.

$$\therefore \qquad \frac{A_l}{A_m}=\frac{1}{1-j/\omega CR_{eg}}, \qquad ...(26)$$

where $$R_{eg}=\frac{r_pR_L+r_pR_g+R_LR_g}{r_p+R_L}=\frac{r_pR_L}{r_p+R_L}+R_g. \qquad ...(27)$$

Let there be a low frequency f_l, at which $R_{eg}=1/\omega_l C$, *i.e.*,

$$f_l=1/2\pi CR_{eg}.$$

Putting for CR_{eg} in equation (26), we have

$$\frac{A_l}{A_m}=\frac{1}{1-jf_l/f}, \quad \text{or} \quad \left|\frac{A_l}{A_m}\right|=\frac{1}{\sqrt{[1+(f_l/f)^2]}}$$

and when $f=f_l$

$$|\ A_l/A_m\ | \ =1/\sqrt{2}=0.707=70.7\%. \qquad ...(28)$$

This low frequency f_l at which the gain falls to $1/\sqrt{2}$ or 70.7% of its mid-frequency gain, the power is half, is called as the *lower cut off frequency* of the amplifier.

(*ii*) *High frequency region*. At high frequencies the reactance offered by coupling condenser is very small, *i.e.*, negligible. The shunt capacitance is low and the reactance due to these shunt capacitances is also not low, as frequency is high, hence cannot be neglected in the equivalent circuit (Fig. 3.12). The effective load impedance Z_h offered at high frequencies will be due to R_L, R_g and C_g (sum of all shunt capacitances in the circuit, *e.g.*, C_{pk}, C_{in}, etc.) and is given by

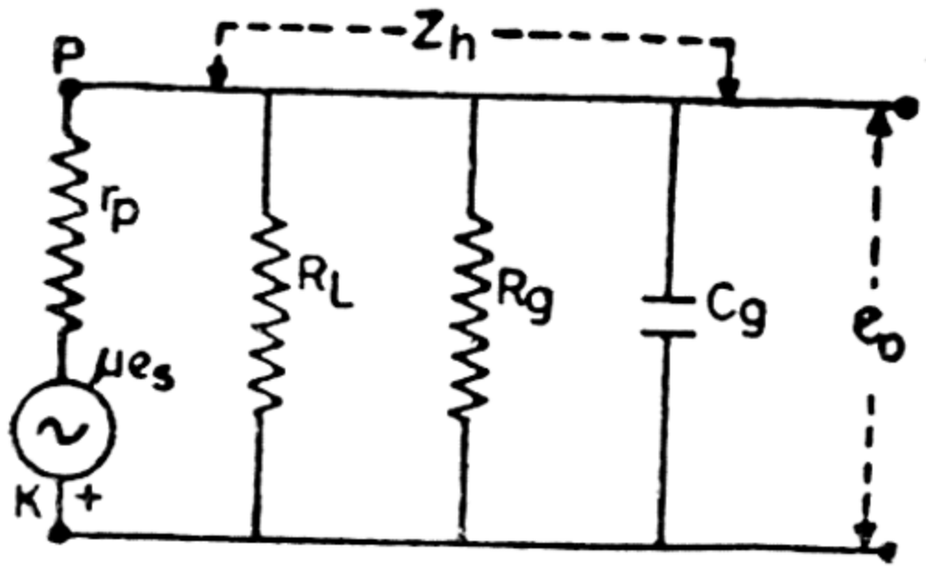


Fig. 3.12.

$$\frac{1}{Z_h}=\frac{1}{R_L}+\frac{1}{R_g}+j\omega C_g \quad \text{or} \quad Z_h=\frac{R_L R_g}{R_L+R_g+j\omega C_g R_L R_g} \qquad \ldots(29)$$

At high frequencies the voltage gain is

$$A_h=-\frac{\mu Z_h}{r_p+Z_h}=-\frac{\mu R_L R_g}{r_p R_L+R_L R_g+r_p R_g+j\omega C_g R_L R_g r_p} \qquad \ldots(30)$$

$$\therefore \frac{A_h}{A_m}=-\frac{1}{1+j\omega C_g R_L R_g r_p/(r_p R_L+R_L R_g+r_p R_g)}=\frac{1}{1+j\omega C_g R_{eq}},$$

where $R_{eq}=R_L R_g r_p/(r_p R_L+R_L R_g+r_p R_g)$.

The gain at high frequencies is less than the gain at mid-frequency. It also decreases with increase in frequency due to shunt capacitances. Let there be a high frequency f_h at which $R_{eq}=1/\omega_h C_g$, *i.e.*,

$$f_h=1/2\pi C_g R_{eg}. \qquad \ldots(31)$$

Putting for $C_g R_{eq}$ in the above expression, we have

$$\frac{A_h}{A_m}=\frac{1}{1+jf/f_h} \quad \text{or} \quad \left|\frac{A_h}{A_m}\right|=\frac{1}{\sqrt{[1+(f/f_h)^2]}}, \qquad \ldots(32)$$

and when $f=f_h$.

$$|A_h/A_m| = 1/\sqrt{2}=0.707=70.7\%. \qquad \ldots(33)$$

This high frequency f_h at which the gain falls to $1/\sqrt{2}$ or 70.7% of its mid-frequency gain, the power is half, is called as the *higher cut off frequency of the amplifier*.

Frequency response curve. The variation of gain with frequency is called the *response curve*. It is the characteristic of amplifier. The frequency response curve for *R-C* coupled amplifier is shown in Fig. 3.13. At lower frequencies the reactance due to coupling condenser ($1/\omega C$) will be finite but decreases with the increase with frequency and is negligible in the middle frequency region. So the voltage gain is reduced in lower frequency region. In the region of middle frequency the gain is independent of frequency. In the high frequency region the reactance of coupling condenser is negligible but the shunt capacitance gives

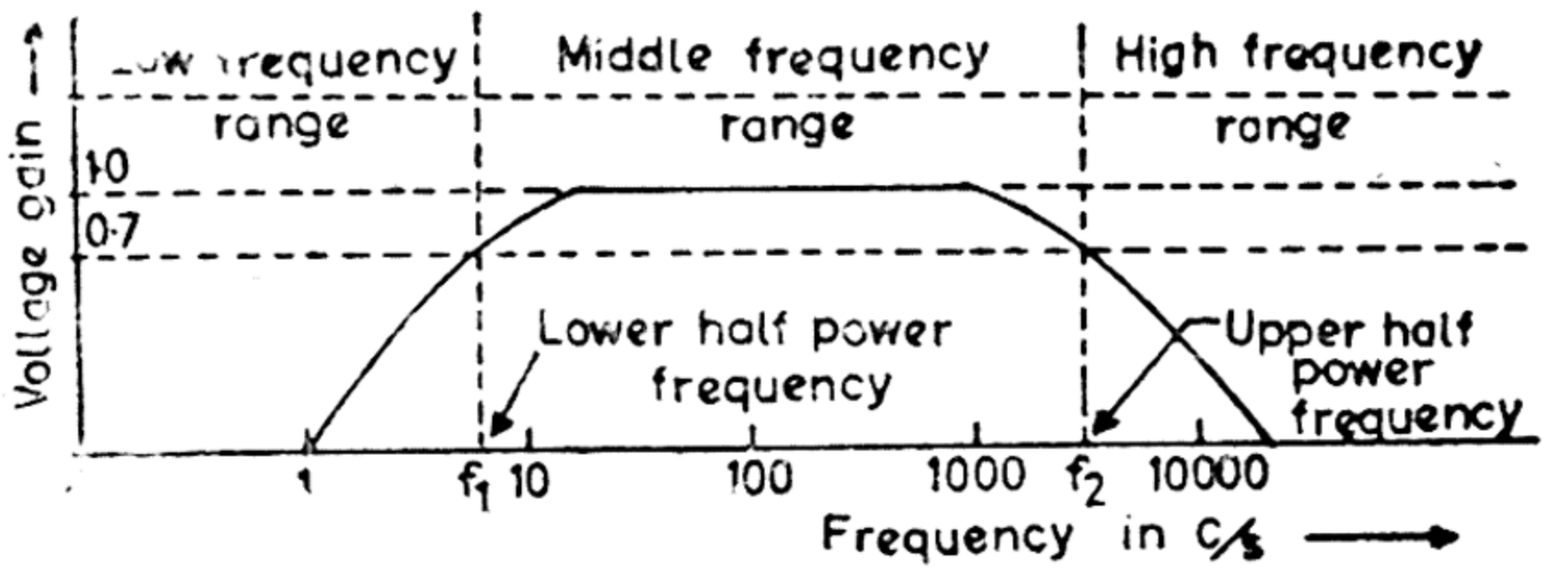


Fig. 3.13.

a small reactance which is effective. With the increase in frequency the shunt reactance ($1/\omega C_g$) decreases, which reduces the gain. To increase the middle frequency region, f_l is lowered and f_h is increased by selecting suitable circuit components. To further increase the gain in the middle frequency ranges the pentode valve is suitable since it has high amplification factor, a low plate resistance and very low interelectrode capacitance in comparison to triode valve.

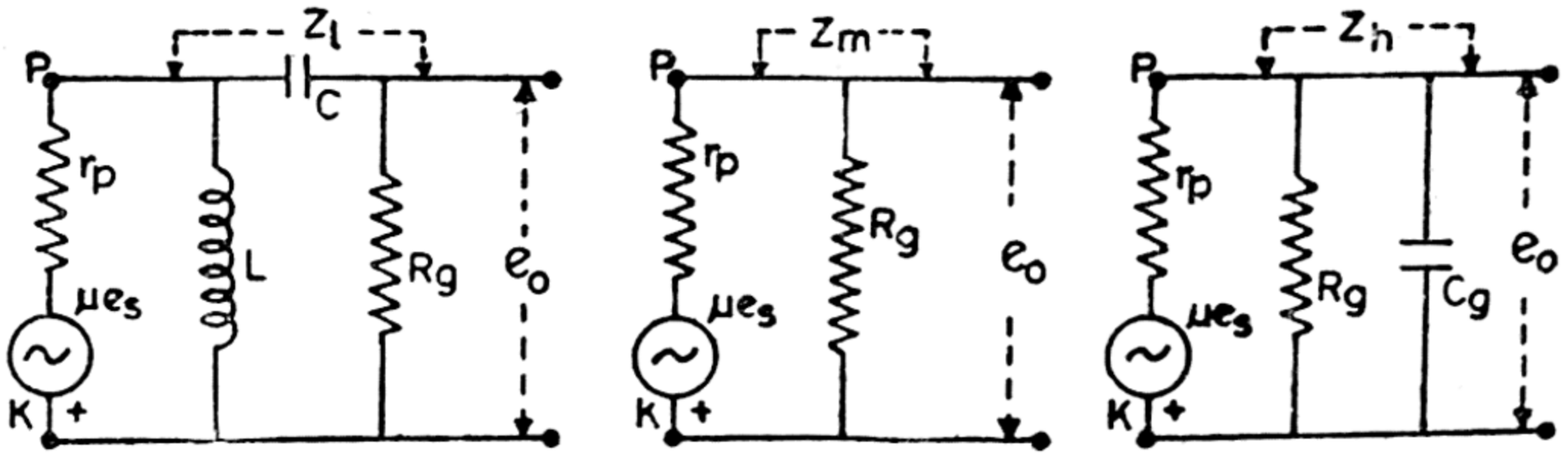
The advantages of this type of amplifier circuit are : (*i*) *The overall amplification is generally higher than can be obtained with the other coupling methods,* (*ii*) *It has uniform response over a wide range of frequencies,* (*iii*) *Its parts are low in cost and require very little space, and* (*iv*) *As there are no coils and transformers in the circuit, the amount of nonlinear distortion is minimum. It also suffers with a few drawbacks ; this amplifier is not suitable to amplify the higher frequencies and also much power is drawn from high tension battery, so it has low efficiency.*

3.5. IMPEDANCE COUPLED AMPLIFIER

To eliminate the high voltage drop between the high tension battery and the plate of the tube the coupling resistance in the *RC*-coupled amplifier is replaced with an iron core choke of high inductance and low resistance. Such an amplifier is called *impedance or inductance capacitance* (*LC*) coupled amplifier. The voltage across the coupling impedance will be small as its dc resistance is very small. The circuit diagram is same as shown in Fig. 3.8, except the inductance L in place of R_L.

The impedance offered by the coupling coil L to the signal current depends on its inductance and the frequency of the signal. To obtain high impedance at low audio—frequencies, L is made as high as possible. In *af*-amplifiers L ranges from 10 to 800 henrys.

Voltage amplification. The method of determining its voltage amplification is similar to that used for the *RC*-coupled amplifier circuit. The circuit characteristics are again observed at the low, middle and high frequencies of the audio-range. The equivalent circuit diagrams for these ranges are shown in Fig. 3.14. At the low and middle frequencies, the reactance of the shunting capacitance C_g is very high as compared with R_g and can thus be neglected. At the middle and high frequencies the reactance of the inductor L is much greater than R_g and can thus be neglected.



(*a*) (*b*) (*c*)

Fig. 3.14.

Low frequency region. Using Fig. 3.14 (*a*), we get

$$Z_l = \frac{j\omega L(R_g - j/\omega C)}{R_g + j(\omega L - 1/\omega C)}. \qquad \ldots(34)$$

The voltage drop at this impedance

$$e_p = i_p Z_l = [-\mu e_s/(r_p + Z_l)]Z_l.$$

Therefore the voltage output across R_g

$$e_o = \frac{R_g}{R_g - j/\omega C} e_p = -\frac{\mu e_s Z_l R_g}{(r_p + Z_l)(R_g - j/\omega C)} \qquad \ldots(35)$$

The voltage gain at low frequency is thus given as

$$A_l = \frac{e_o}{e_s} = -\frac{\mu R_g . j\omega L}{r_p R_g + L/C + j[\omega L(r_p + R_g) - r_p/\omega C]}. \qquad \ldots(36)$$

The voltage amplification decreases because of the reduction in the value of X_L ($=\omega L$) shunting the load and the increase in the value of X_C ($=1/\omega C$) connected in series with R_g.

Middle frequency region. Using Fig. 3.14 (*b*), we get $Z_m = R_g$ and the out put voltage

$$e_o = -\mu e_s R_g/(r_p + R_g).$$

$$\therefore \text{ Voltage gain } A_m = e_o/e_s = -\mu R_g/(r_p + R_g). \quad \ldots(37)$$

Thus the voltage amplification of the circuit at the middle frequencies is dependent largely upon the resistances in the circuit because the effects of all the reactances are negligible at these frequencies.

High frequency region. Using Fig. 3.14 (*c*), we get,

$$\frac{1}{Z_h} = \frac{1}{R_g} + j\omega C_g \text{ or } Z_h = \frac{R_g}{1 + j\omega C_g R_g}$$

$$\therefore \text{ Voltage gain} = \frac{-\mu Z_h}{r_p + Z_h} = \frac{-\mu R_g}{r_p + R_g + j\omega C_g r_p R_g}. \quad \ldots(38)$$

Thus the voltage amplification at high frequencies decreases because X_C $[= 1/\omega C_g]$ becomes relatively low at these frequencies.

Frequency Response Curve. The analysis shows that the frequency response is generally similar to that for the *RC*-coupled amplifier except that the flat region is narrower and at greater height. The decrease in gain is greater at both low and high frequencies than for *RC*-coupled amplifiers. This decrease at low frequencies is due to the relatively low value of impedance of the coupling coil. At high frequencies, it decreases due to the relatively high value of C_g.

3.6. TRANSFORMER COUPLED AMPLIFIER

The coupling between two successive stages of amplifiers, through a suitable transformer, increases the gain of the amplifier. An untuned, iron core, step-up transformer is used for coupling the stages for audio frequency ranges. Such a transformer used for coupling of the output of one stage of a cascade amplifier to the input of the next stage is called *inter stage transformer.*

The circuit arrangement of a transformer coupled amplifier is shown in Fig. 3.15. This is usually operated under class-*A* condition. It is obvious from the circuit diagram that the successive stage (or grid) of the amplifier is isolated from the H.T. of the previous stage due to this coupling. The primary winding *pp* of the step-up transformer (*T*) connected in the plate circuit of the first stage and the secondary winding *s* to the grid circuit of the next stage. Since the primary has high inductive reactance and low dc resistance, so practically very small drop in potential will exist over the resistance of the primary. This low dc potential will be sufficient to operate

the amplifier. Also by operating the tube at high plate potential, higher amplification will be obtained. Since the transformer is also step up (say ratio 1 : n) so the output voltage of the first stage will be stepped up by the transformer ratio (n) times (practically $n_{max}=3$). The gain of amplifier will be the product of amplification factor of the tube and transformer ratio, *i.e.*, $A \sim \mu n$.

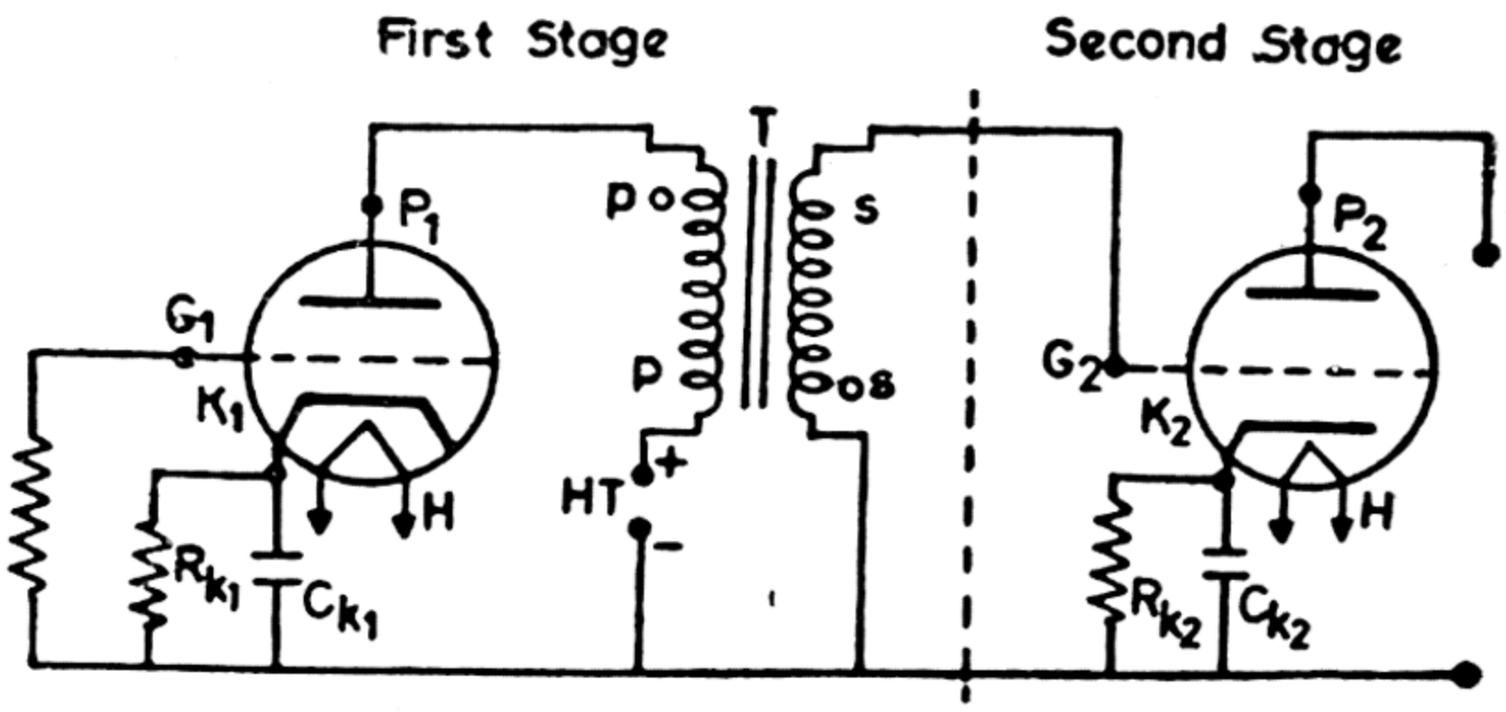


Fig. 3.15.

Voltage Amplification. The equivalent circuit of Fig. 3.15 is complicated. To make it simple to solve we assume that : (*i*) Resistance due to eddy current loss is negligibly small in comparison to the resistance of the primary of the transformer. (*ii*) Input resistance of the next stage is very high. Its effect is negligible as it is in parallel with the secondary. (*iii*) Distributed capacitance of the primary winding of the transformer is negligibly small in comparison to the capacitance reflected from the secondary to the primary.

The equivalent simplified circuit is thus obtained as shown in Fig. 3.16.

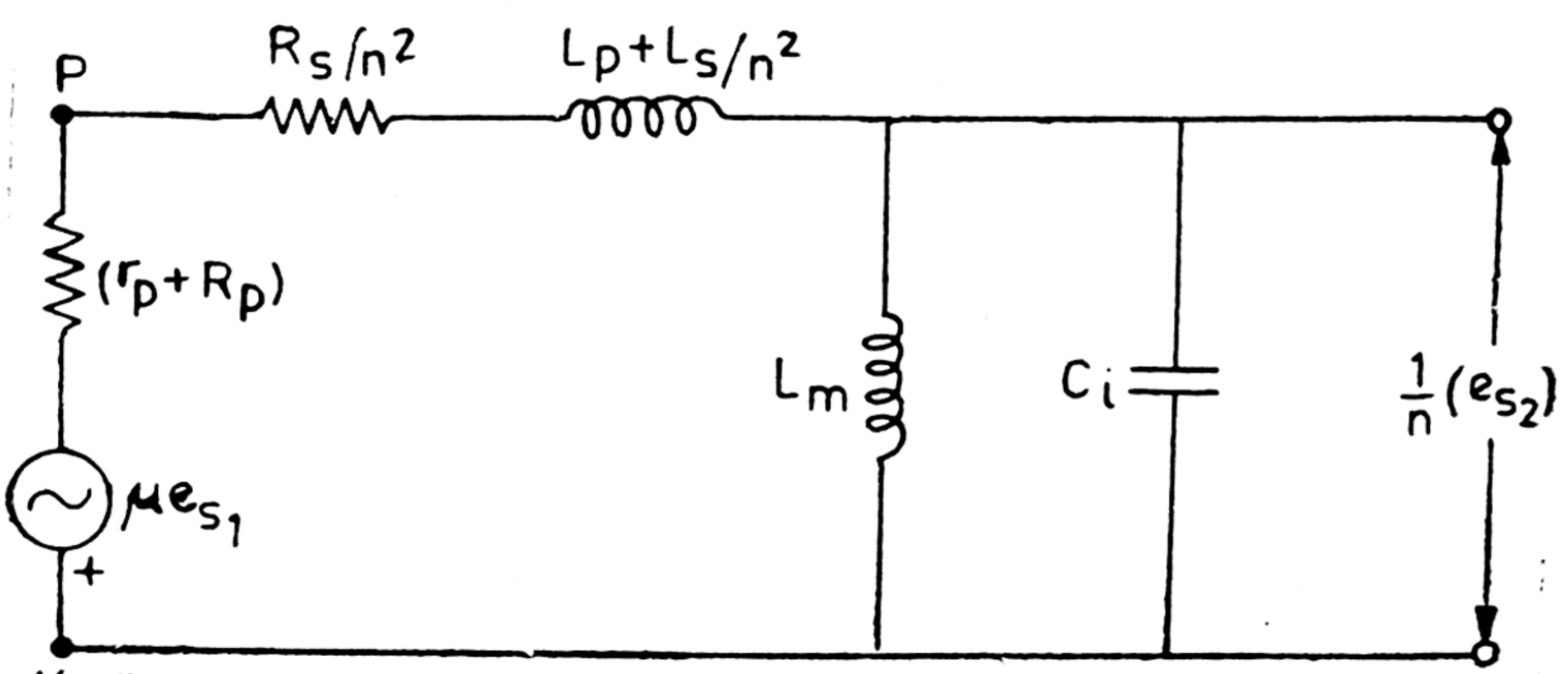


Fig. 3.16.

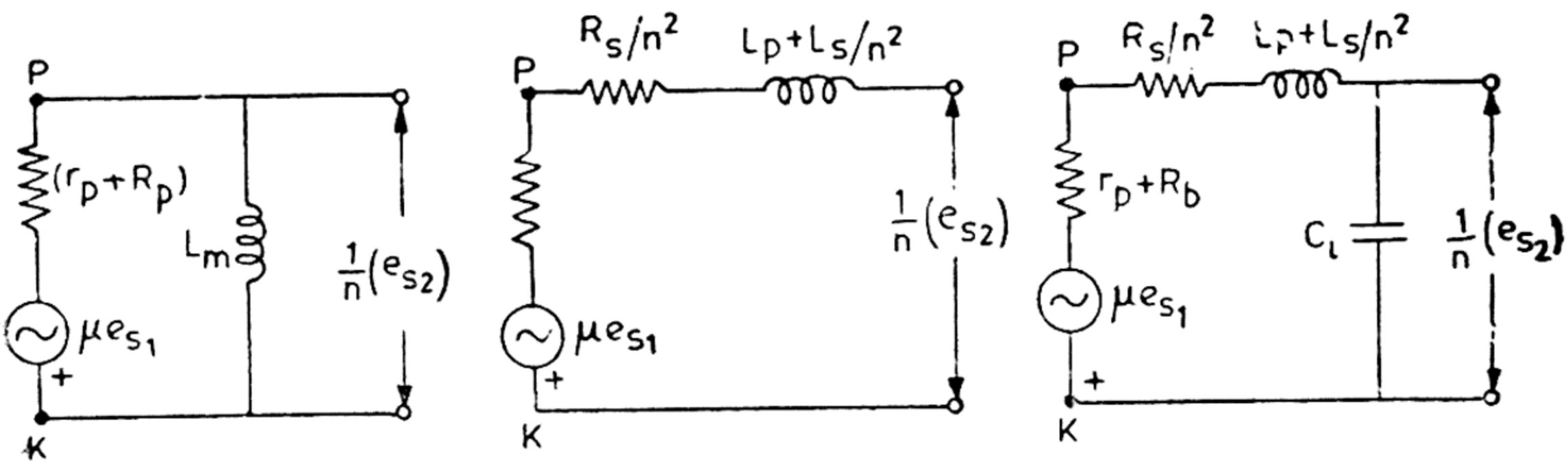
In this circuit R_p and R_s are the effective resistances of primary and secondary windings respectively and L_p and L_s the respective leakage inductances. L_m is the magnetising inductence of primary and C_i is the input capacitance of the next stage.

Let us now consider the performance of the amplifier for low middle and high frequency ranges.

(*i*) *Low frequency region.* In this region of frequency the reactance of input capacitance of the second stage is so high that it constitutes open circuit and hence can be omitted. The current flowing through R_s/n^2 and L_p+L_s/n^2 is negligible, so the voltage drop across them may be neglected. The equivalent circuit is reduced to that shown in Fig. 3.17 (*a*) The plate current i_p is given by

$$i_p=-\mu e_{s_1}/[(r_p+R_p)+j\omega L_m].$$

$\therefore$ Voltage across $L_m=j\omega L_m i_p=-\mu e_{s_1} j\omega L_m/[(r_p+R_p)+j\omega L_m]$.



(*a*) (*b*) (*c*)

Fig. 3.17.

This voltage must be equal to the output voltage e_{s_2}/n, hence

$$\frac{-\mu e_{s_1} j\omega L_m}{(r_p+R_p)+j\omega L_m}=\frac{e_{s_2}}{n}.$$

$\therefore$ Voltage gain $A_i=\dfrac{e_{s_2}}{e_{s_1}}=\dfrac{-\mu n j\omega L_m}{(r_p+R_p)+j\omega L_m}=\dfrac{-\mu n}{1-j(r_p+R_p)/\omega L_m}$ ...(39)

Its magnitude $|A_i|=\mu n\left[1+\left(\dfrac{r_p+R_p}{\omega L_m}\right)^2\right]^{-1/2}$

and phase angle $\phi=\tan^{-1}(r_p+R_p)/\omega L_m$.

If we define f_i as $2\pi f_i L_m=r_p+R_p$, then

$$A_i=\mu n/[1+(f_i/f)^2]^{1/2}. \qquad ...(40)$$

Hence f_i is known as lower half frequency.

(*ii*) *Middle frequency range.* This frequency range starts from frequency f_i upwards. In the middle frequency range the frequency is large enough to make reactance ωL_m quite large so that it may be omitted and the reactance offered by C_i is so high that it can be regarded equivalent to an open circuit and may also be omitted. The equivalent circuit thus reduces to as shown in Fig. 3.17(*b*).

In this range practically no voltage is lost, hence

$$-\mu e_{s_1} = -e_{s_2}/n.$$

$\therefore$ Voltage gain $A_m = \dfrac{e_{s_2}}{e_{s_1}} = -\mu n.$ ...(41)

(*iii*) *High frequency range.* In this region of frequency the effects of the shunting capacitances and the leakage inductances cannot be neglected. Only the L_m can be omitted on account of its high reactance. The equivalent circuit thus reduces to Fig. 3.17(*c*).

The plate current is given by

$$i_p = \frac{-\mu e_{s_1}}{(r_p + R_p + R_s/n^2) + j\,[\omega(L_p + L_s/n^2) - 1/\omega C_i]}$$

$\therefore$ Voltage across $C_i = i_p/j\omega C_i = e_{s_2}/n$ (output)

or $$\frac{e_{s_2}}{n} = \frac{-\mu e_{s_1}/j\omega C_i}{(r_p + R_p + R_s/n^2) + j[\omega(L_p + L_s/n^2) - 1/\omega C_i]}$$

$\therefore$ Voltage gain $A_h = \dfrac{e_{s_2}}{e_{s_1}} = \dfrac{-\mu n/j\omega C_i}{(r_p + R_p + R_s/n^2) + j[\omega(L_p + L_s/n^2) - 1/\omega C_i]}$ ...(42)

On account of the reduction of reactance $1/\omega C_i$ and increase of reactance $\omega(L_p + L_s/n^2)$ at high frequencies, the voltage gain decreases. The gain is maximum, when $\omega(L_p + L_s/n^2) = 1/\omega C_i$, or the

frequency $f_h = 1/2\pi[(L_p + L_s/n^2)C_i]^{1/2}$.

At this frequency the voltage gain

$$A_h = -\mu n/j\omega C_i(r_p + R_p + R_s/n^2). \quad ...(43)$$

Frequency response curve. The variation of voltage gain with frequency, a characteristic of the transformer coupled amplifier, is shown in Fig. 3.18. When the frequency is low, the effective impedance (proportional to frequency) of the primary is low, hence gain falls with the decrease in frequency. The gain is constant for a small range of middle frequency. At higher frequences the distributed shunting capacitances of the transformer windings, the inter-electrode capacitances of the tube and also the leakage inductances of the primary are the factors due to which the gain falls rapidly. These shunt capacitances and leakage induc-

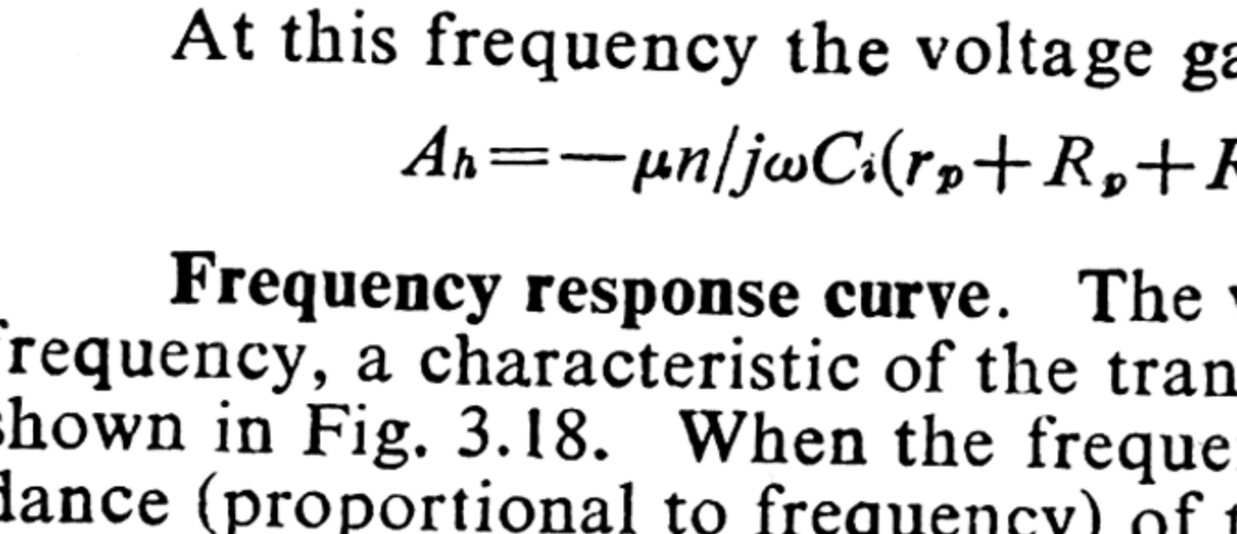


Fig. 3.18.

tances act as a series resonant circuit at high frequencies, so there exists a particular resonant frequency (f_0) for the circuit. At this resonant frequency a peak is observed in the response curve. To obtain fairly uniform curve, *i.e.*, to reduce the height of the peak the secondary resistance is increased.

Beside the above characteristics of the transformer coupled amplifier, this coupling is generally used for obtaining the impedance matching between the stages and also by using a centre tapped secondary. This is used as phase inventor in pushpull-amplifier. But the stray magnetic leakage effects, weight and high cost for the suitable transformer are the few disadvantages of this coupling.

3.7. DISTORTION IN AMPLIFIERS

The output of an ideal class-A amplifier should be of the same waveform as the input. It is expected that (*i*) the output should have all the frequencies which are contained in the input, (*ii*) the relative change in amplitude of various output components must be equal and (*iii*) the relative phase-change in all output components must be equal. Any difference between the input and output waveforms of frequency, amplitude and phase, is termed as *distortion* in amplifier. It exists in output due to the amplifier tube itself or due to various circuit elements in the circuit. These are classified as :

(*i*) **Amplitude distortion**. From the dynamic characteristic of the valve we know that for the grid signal falling within the straight portion of the characteristic, the output will have linear amplification. On the other hand the input signal falling out of the straight part will have different amplification (or flattened), Fig. 3.19. This causes amplitude distortion and also gives new frequencies in output which are not present in input. Similar non-linearity will be produced if the tube is operated under improper grid bias. If due to the grid bias, the grid becomes positive for any part of the input, grid will derive some grid current which will distort the output. If the input signal is too large, either half or both halves of the output signal may be distorted.

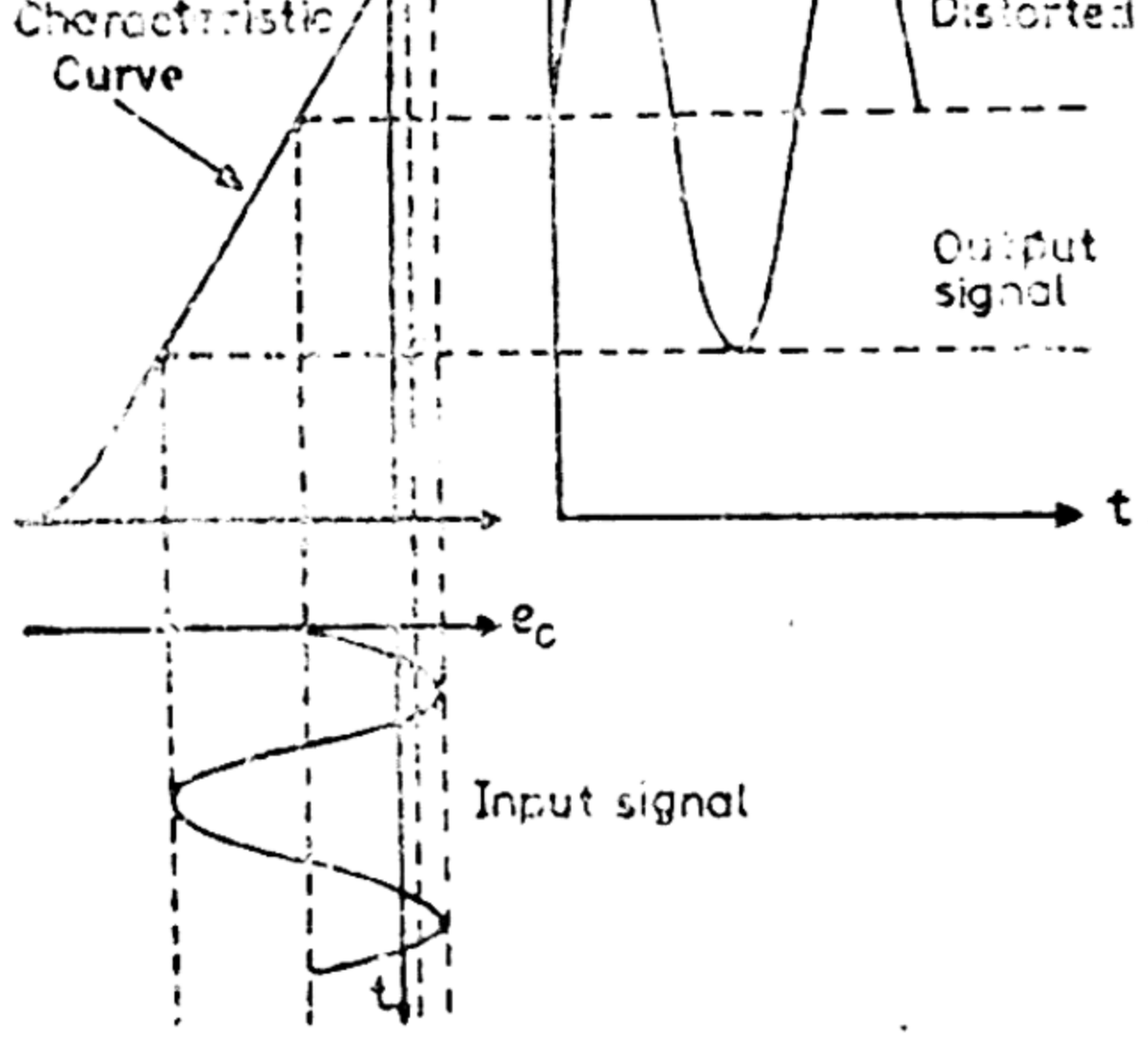


Fig. 3.19.

On account of the non-linearity of the dynamic characteristic curve of the triode, the output waveshape is no longer a sine wave. It is a complex wave consists of the original frequency and harmonics of the original frequency. The current may thus be represented as

$$i_p = a + be_s + C\,e_s^2, \qquad \ldots(44)$$

where $a = i_0$ = anode current in the absence of the signal and b, c, etc., are constants. If the input wave is a sine wave of the form

$$e_s = E_{0s} \sin \omega t \qquad \ldots(45)$$

$$\therefore \quad i_p = i_0 + bE_{0s} \sin \omega t + cE_{0s}^2 \sin^2 \omega t$$

$$= i_0 + bE_{0s} \sin \omega t + \frac{1}{2} c\,E_{0s}^2 (1 - \cos 2\omega t)$$

$$= (i_0 + \tfrac{1}{2} cE_{0s}^2) + bE_{0s} \sin \omega t + \tfrac{1}{2} cE_{0s}^2 \sin (2\omega t - \pi/2) \qquad \ldots(46)$$

Thus we see that

(*i*) The steady component of the anode current increases by an amount $\frac{1}{2} cE_{0s}^2$

(*ii*) In addition to the wanted (fundamental) signal there is a term $\frac{1}{2} cE_{0s}^2 \sin (2\omega t - \pi/2)$ which is the second harmonic component of the plate current displaced by an angle of 45° from the fundamental.

The degree of distortion produced by an amplifier is defined as the ratio of the second harmonic amplitude to the fundamental and is also known as coefficient of second harmonic distortion D_2.

$$\therefore \quad D_2\% = \frac{\frac{1}{2} cE_{0s}^2}{b\,E_{0s}} \times 100\% = \frac{50\,cE_{0s}}{b}\,\%. \qquad \ldots(47)$$

Thus the percentage distortion can be calculated if the constants b and c are obtained from the dynamic characteristic, as follows :

when $\omega t = 0$, $i_p = (i_p)_{mean} = i_0 = a$ ...(48)

$\omega t = \pi/2$, $i_p = (i_p)_{max} = i_0 + bE_{0s} + cE_{0s}^2$ ...(49)

$\omega t = 3\pi/2$, $i_p = (i_p)_{min} = i_0 - b\,E_{0s} + c\,E_{0s}^2$. ...(50)

From these equations, we get

$$(i_p)_{max} - (i_p)_{min} = 2bE_{0s} \qquad \ldots (51)$$

and $(i_p)_{max}+(i_p)_{min}-2i_0=2c\ E_0s^2$. ...(52)

$$\therefore D_2\%=\frac{(i_p)_{max}+(i_p)_{min}-2i_0}{2\,[(i_p)_{max}-(i_p)_{min}]}\times 100\%. \quad ...(53)$$

This equation holds only when the i_p is related to the applied grid voltage by equation (44).

The non-linear distortion can be reduced by proper selection of grid bias, *i.e.*, both the selections of linear portion of characteristics and —ve grid bias only. In many cases both situations may not be possible to observe then other proper circuits, *viz* , *feedback circuits*, etc., remove the possibility of such distortions.

(*ii*) **Frequency distortion**. When the amplification of different frequencies of input are not equal, there exists frequency distortion in the output. This type of distortion is mainly caused due to various reactive elements in the circuit, *viz.*, inductance, capacitance, etc. The response curve of an amplifier gives the variation of gain with frequencies, the gain decreases at low and high frequencies. If the curve is horizontal only, then there will be no frequency distortion. To decrease this distortion proper selection of the circult elements is essential or the limited range of frequency should be reproduced by the amplifier.

(*iii*) **Phase distortion**. When the phase difference of different frequencies of the input are not maintained at the output, there exists phase distortion. The various coupling reactive elements, *viz.*, inductance and capacitance, etc., cause phase change in the output, as the phase change between current and voltage depends upon the frequencies and is not constant for all frequencies. In the audio-frequency region this effect is not important, since this effect is not perceptible to human ears. This effect is very important at video frequencies, *viz.*, in television, etc., where the operation is a function of wave-shape.

3.8. FEEDBACK AMPLIFIERS

When a portion of the output energy (voltage, current) is derived back to the input circuit, this process is called as *feedback*. This process makes the input dependent on the output of the amplifier and also controls the output. When the feedback energy is in phase with input signal and aids it, it is known as direct, *regenerative* or *positive feedback*. In this case the gain of the amplifier is increased but the range of the uniform amplification is reduced, Due to this process the distortion and noise in output is increased and also the stability is reduced for the variation of input voltage. If this feedback energy is enough the process is used as an oscillator. When the feedback energy is 180° out of phase with the input signal and thus opposes it, it is known as *degenerative or negative* feedback. In this case, the gain of the amplifier is reduced but the range of the uniform amplification is increased. This also increases

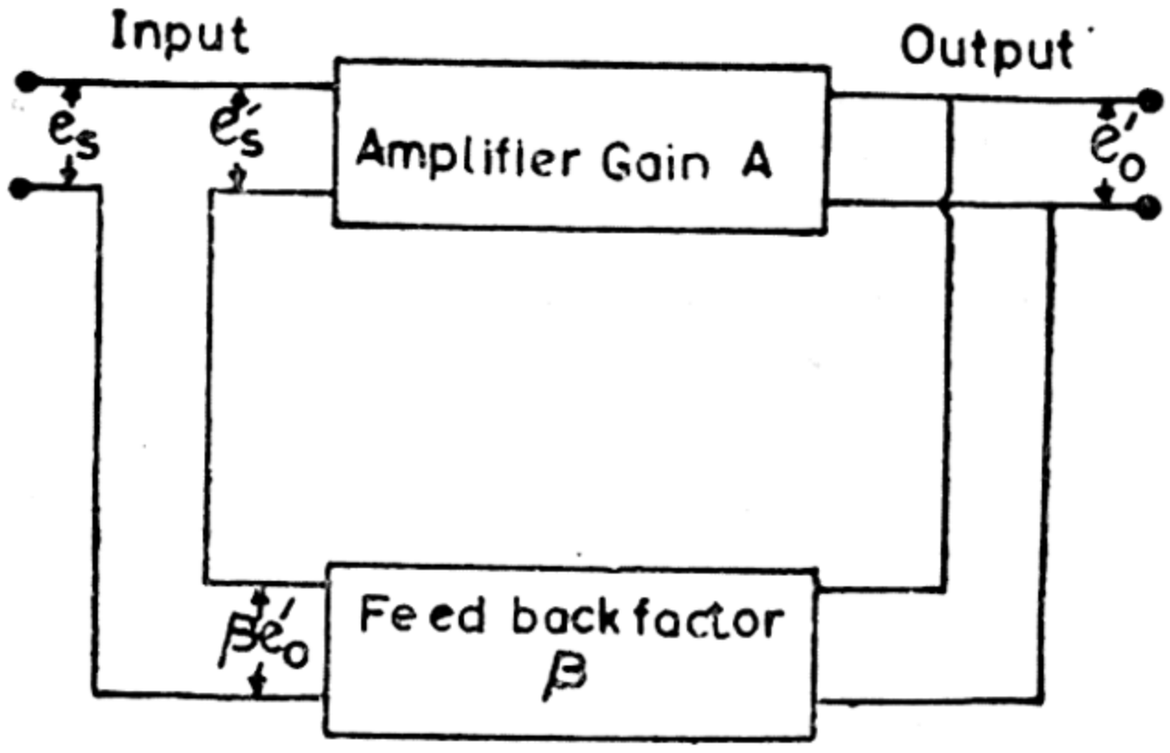


Fig. 3.20.

the stability of the amplifier. This process reduces all amplitude, frequency and phase distortions and also noise in the output.

The basic principle of the feedback amplifier can be given on the basis of the Fig. 3.20. Let gain of the amplifier in the absence of any feedback is *A*. *A* fraction βe_0 of the output (e_0) is derived back to the input circuit. The factor β is defined as

$$\beta = \frac{\text{Voltage feedback}}{\text{Output voltage}} \quad \ldots(54)$$

If e_0' be the output voltage of the amplifier with the circuit. Thus the actual input at the amplifier is equal to the sum of original input and feedback voltage.

i e. $$e_s' = e_s + \beta e_0'. \quad \ldots(55)$$

This will be amplified *A* times by the amplifier, hence

$$e_0' = A(e_s + \beta e_0') \text{ or } e_0' = Ae_s/(1 - A\beta).$$

$$\therefore \quad e_0'/e_s = A_f = A/(1 - A\beta). \quad \ldots(56)$$

A_f is the gain of the amplifier with feedback. Thus due to feedback the gain *A* of amplifier is changed by a factor $1/(1 - A\beta)$. The term $A\beta$ is known as *return ratio, loop gain feedback factor* or *loop transmission*. In most of the circuits *A* and β both are of complex nature and are functions of frequency. Following cases are possible on the basis of Eq. (56) :

(*i*) If $|1 - A\beta| < 1$, A_f will be greater than *A*. This is the condition for positive feedback.

(*ii*) If $|1 - A\beta| = 0$, A_f will become infinite, which means that output will be obtained without any external input. This is the condition of oscillator.

(*iii*) If $|1 - A\beta| > 1$, A_f will be less than *A* and $\beta e_0'$ will decrease the input voltage e_s. This is the condition of —ve feedback.

(*iv*) If $A\beta > 1$ or $A_f = -A/A\beta = -1/\beta$.

This condition of large —ve feedback gives that the amplification depends on feedback circuit only. If β is independent of

frequency, then gain A_f will also be independent of frequency and also of the tube parameters, supply voltages and other variations (which result in high stability of amplifier). So the gain is entirely controlled by the nature of feedback factor (β). Since the feedback network is generally resistive, the feedback and gain both are independent of frequency.

Advantage of Feedback Amplifiers. (*i*) The non-linear and phase distortion will be reduced by —ve feedback in the same manner as noise and hum. (*ii*) As β is generally independent of the frequency the variation in the gain due to changes in frequency, *i.e.*, the frequency distortion, will be reduced when —ve feedback is used. (*ii*) The stability of an amplifier can be improved by using —ve feedback.

Limitations of Feedback Amplifiers. (*i*) The increase in the feedback factor causes an increase in the proportion of the output voltage that is returned to the input. It thus decreases the net voltage amplification of the circuit. (*ii*) The feedback voltages are not exactly 180° out of phase with the input signal in case of —ve feedback amplifiers. Such a phase can be obtained in three stage amplifiers.

The feedback circuits are mainly of three types : (1) Current feedback circuit in which the voltage feedback from the output of the tube is proportional to the current through the load, (2) voltage feedback circuit in which the feedback voltage is proportional to the voltage across the load, and (3) compound feedback circuit.

Negative Voltage Feedback Circuit. The Fig. 3.21 shows the negative voltage feedback circuit in which a fraction of output voltage is fedback to the input. R_1-R_2 is potential divider and the voltage across R_2 is fed back to the grid circuit. (R_1+R_2) resistance is high so the potential across these will be equal to the output (e_0) of amplifier. The voltage between the grid and cathode after the feedback will be

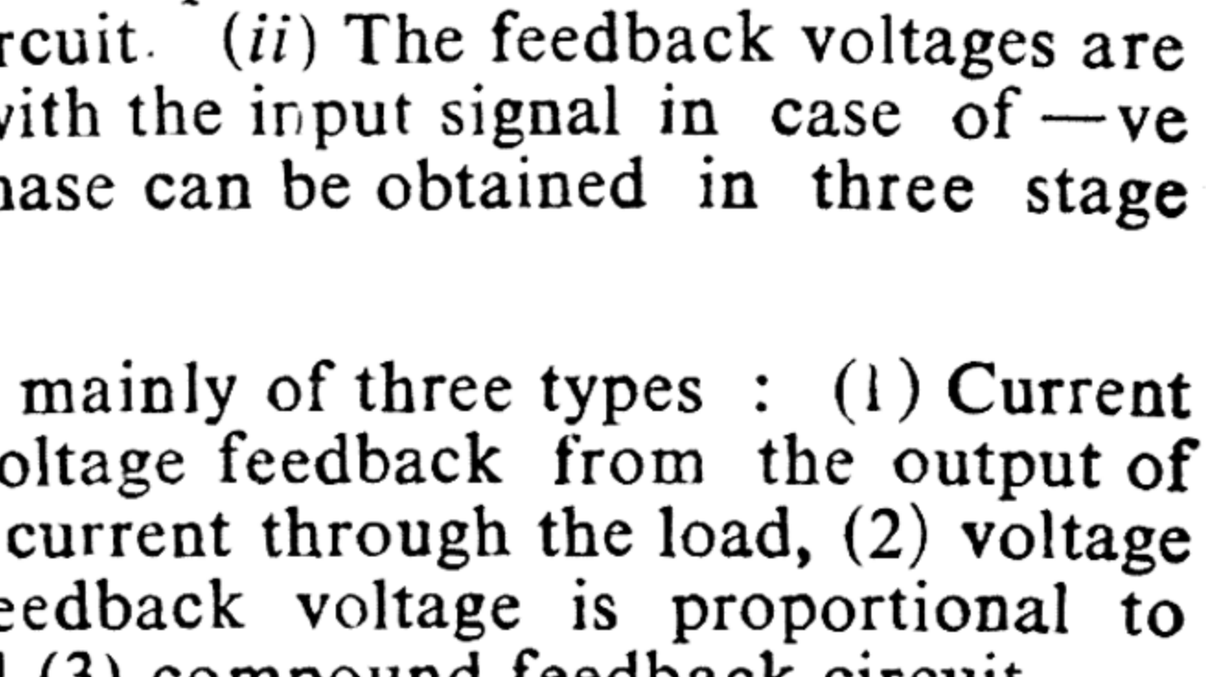


Fig. 3.21.

$$ec = e_s + R_2 e_0/(R_1+R_2). \quad \text{.. (57)}$$

This gives the value of feedback fraction (β) as

$$\beta = R_2/(R_1+R_2). \quad \text{... (58)}$$

The gain of the amplifier without the feedback is given by

$$A = -\mu R/(r_p+R). \quad \text{.. (59)}$$

The reactance of the condenser C is negligible at the frequency of operation.

The gain of the amplifier with the feedback is given by

$$A_f = \frac{A}{1-A\beta} = \frac{-\mu R/(r_p+R)}{1-[-\mu R/(r_p+R)]\beta}$$

$$= \frac{-\mu R}{r_p+(1+\mu\beta)R} = \frac{[\mu/(1+\mu\beta)]R}{r_p/(1+\mu\beta)+R}$$

$$= \frac{-\mu' R}{r_p'+R} \qquad \text{...(60)}$$

This equation shows that with the feedback the amplification factor μ is reduced to $\mu' = \mu/(1+\beta\mu)$ and the dynamic resistance r_p to $r_p' = r_p/(1+\mu\beta)$. This maintains a constant voltage gain for greater extent with the change in the load resistance.

In an amplifier, the amount of feedback, for the constant input is expressed in decibels (*db*) *as*

$$db \text{ feedback} = 20 \log_{10} (e_0'/e_0). \qquad \text{.. (61)}$$

3.9. CATHODE COUPLED AMPLIFIER (CATHODE FOLLOWER)

In this type of amplifiers the signal is applied between the grid and grounded cathode and the output is taken from across the unbypassed resistance R_K in the cathode circuit. The plate is connected either directly to H.T. battery or through resistance R_1. A bypassed condenser C_2 of large capacitance is used to ground the plate to ac signals. When the input signal is +ve the plate current will increase and thus the output voltage across the cathode resistor R_K will increase. Due to this reason the *cathode coupled amplifier* is named as *cathode follower*. The +ve signal to the input increases the voltage across R_K and makes the cathode more +ve with respect to the ground, or grid more —ve with respect to the cathode. It is opposite to the effect produced by the original +ve signal, hence this circuit is known as a *degenerative feedback, inverse feedback* or *—ve feedback amplifier.*

In order that the signal voltage may produce change in plate

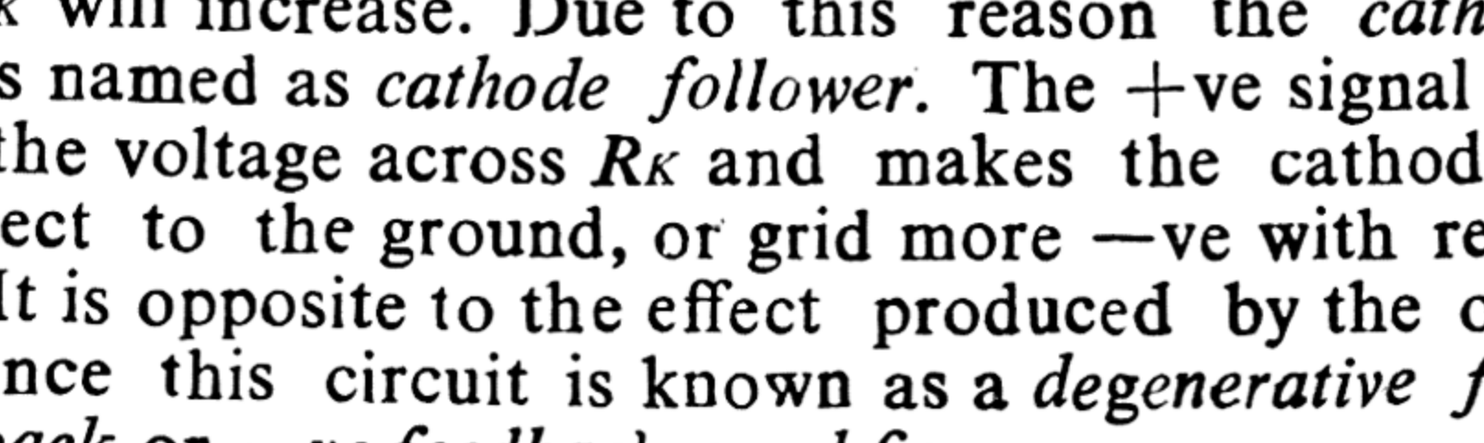


Fig. 3.22.

current, the output voltage can never equal or exceed the signal voltage, or the voltage amplification is always less than unity. The equivalent circuit 3.22 (right) shows that

$$i_p(r_p+R_K)=\mu e_s'=\mu(e_s-V_0)=\mu(e_s-i_pR_K)$$

or
$$i_p=\frac{\mu e_s}{r_p+R_K+\mu R_K}=\frac{\mu e_s}{r_p+R_K(1+\mu)} \quad \ldots(62)$$

$$\therefore \quad \text{Voltage gain } A=\frac{V_0}{e_s}=\frac{i_p \,.\, R_K}{e_s}=\frac{\mu R_K}{r_p+R_K[1+\mu]} \quad \ldots(63$$

As the denominator will always be greater than the numerator, hence A is always less than 1. The higher values are for μ and R_K, the closer the voltage amplification will approach unity.

At this stage the cathode and grid rise and fall together in potential by equal amount justifying the name cathode follower.

The expression for current may be written as

$$i_p=\frac{[\mu/(1+\mu)]e_s}{[r_p/(1+\mu)]+R_K} \,.$$

This shows a simple amplifier circuit with the equivalent potential

$$e_{eq}=[\mu/(1+\mu)]e_s \quad \ldots(64)$$

and the effective internal impedance

$$Z_{ef}=r_p/(1+\mu). \quad \ldots(65)$$

$\therefore$ Output terminal impedance

$$Z=\frac{[r_p(1+\mu)]R_K}{r_p/(1+\mu)+R_K}=\frac{A}{g_m} \,. \quad \ldots(66)$$

Circuit Characteristics. (1) The voltage amplification is less than 1, (2) It has very low input capacitance, very high input impedance and low output impedance, (3) It has low values of amplitude and frequency distortions, (4) It has uniform frequency response over a very wide frequency band, (5) It has no phase inversion, (6) It can supply relatively large signal currents to low impedance loads and can handle high input voltages without overloading, (7) It has good circuit stability.

Applications. (1) The cathode follower circuit can be used as an impedance changing or impedance matching device. (2) As it provides a more uniform response over a wider frequency range than a transformer coupled amplifier, hence is used in place of a transformer. (3) It is used as a power amplifier with less distortions.

3.10. POWER AMPLIFIER (CLASS A PUSH PULL AMPLIFIER)

Power amplifiers are used in both *a-f* and *r-f* and are classified according to the operational frequency as : *af-power amplifiers, and r-f power amplifiers.* These amplifiers can be subdivided according to the manner in which the tube is operated (such as class A, class *B*, etc.). The tubes used as power amplifiers are operated singly, in parallel or in push pull depending upon the amount of output power required. Power amplifiers are similar to voltage amplifiers in construction and operational principle. Dissimilar to voltage amplifiers the power amplifier tubes are operated with a lower value of plate load impedance as : (*i*) r_p is lower, (*ii*) larger output power is required, and (*iii*) power output is the paramount factor.

In push pull arrangement two identical tubes are connected as shown in Fig. 3.23. The grids and plates of the two tubes are connected respectively to opposite ends of the secondary of the input transformer and the primary of the output transformer. The

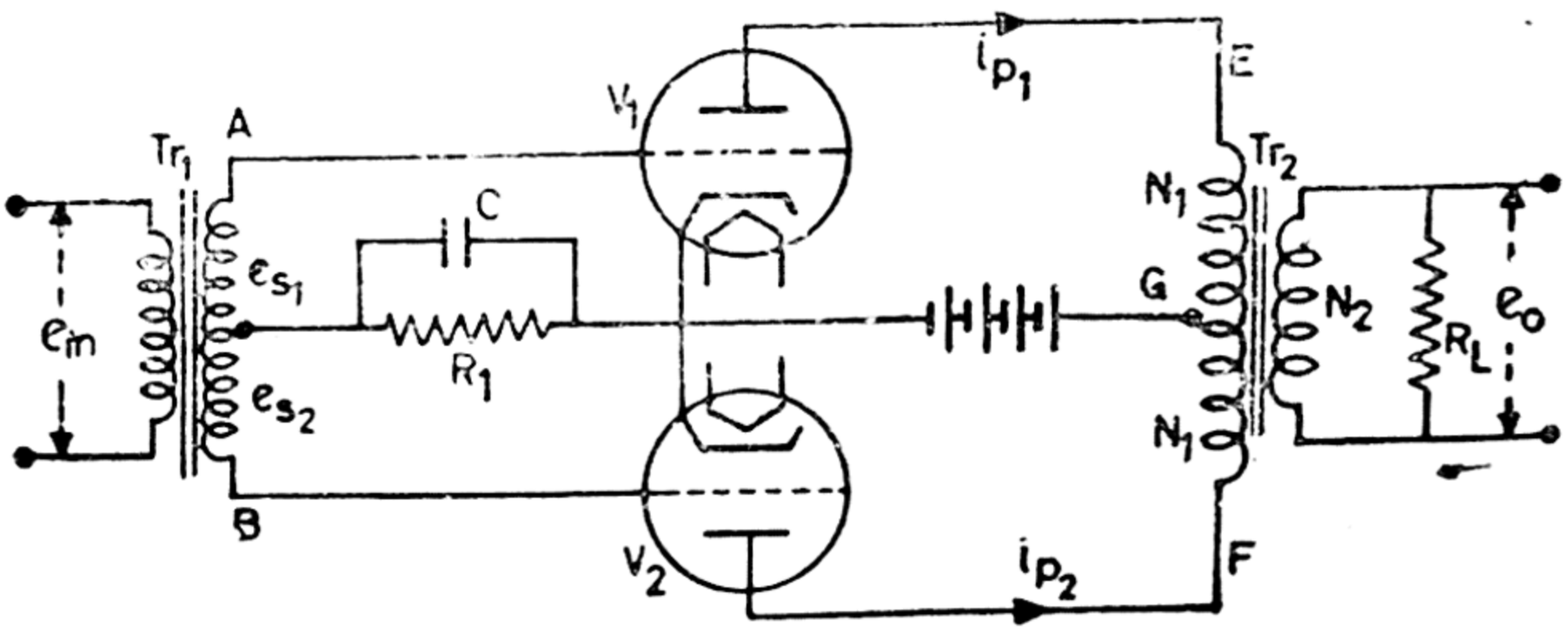


Fig. 3.23.

secondary of the input and the primary of the output transformers are connected at centres through the point connecting the cathodes of the tubes. This connection is known as balanced circuit and thus amplifier is also referred to as a *balanced amplifier.* At any instant of time the ends *A* and *B* of an input transformer are of opposite polarity. Hence e_{s_1} and e_{s_2} will always be equal and 180° out of phase with each other. If the end *A* is +ve, the end *B* will be negative. The grid of V_1 will then become +ve and thus causing an increase in plate current through section *EG* of the primary of the output transformer. The grid of B_2 becomes more −ve causing the plate current in section *EG* to decrease. As the tubes are identical and the changes in their grid voltages are equal, the variation in the plate current will thus be equal but 180° out of phase with each other. Thus we see that one tube pushes current through one-half of the primary of the output transformer while the second tube pulls an equal amount of current through the rest half, hence the name *push pull.*

If the signal input is zero, the plate currents i_{p_1} and i_{p_2} are equal and flow in opposite directions in EG and GF circuits. If ac signal is applied to the input hence to the grids, the plate current in one tube increases while decreases in the other tube. As both tubes operate over more than the linear portion of their dynamic curves, the plate currents are distorted, mainly due to the second harmonic. The phase relation of the second harmonic produced by each tube is such that they cancel each other in the output transformer.

The output transformer along with load resistance R_L is equivalent to a resistance $R_p=(2N_1/N_2)^2\ R_L$, where N_1 is the number of turns in the each half of the primary of the output transformer and N_2 that in its secondary. The equivalent circuit of the amplifier thus have generator voltages and internal impedances as the sum of the individual tubes. Therefore the plate current

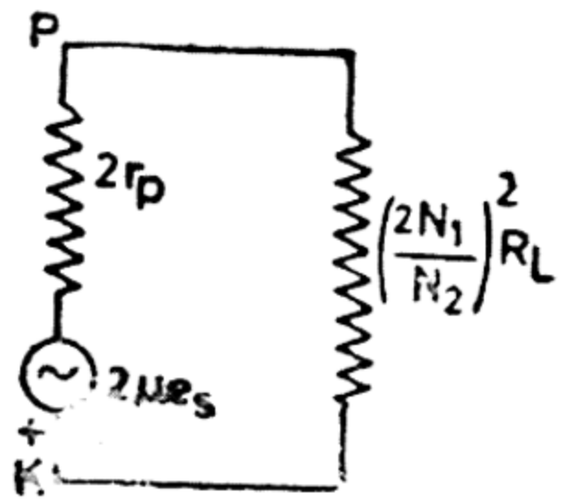


Fig. 3.24.

$$i_p=\frac{2\mu e_s}{2r_p+(2N_1/N_2)^2R_L}=\frac{\mu e_s}{r_p+2(N_1/N_2)^2R_L}.$$

Hence the total power delivered to the load

$$P_L=i_p^2\ (2N_1/N_2)^2R_L$$

$$=\left(\frac{\mu e_s}{r_p+2(N_1/N_2)^2R_L}\right)^2\left(\frac{2N_1}{N_2}\right)^2 R_L. \qquad ...(67)$$

It will be maximum when $r_p=2(N_1/N_2)^2R_L$, *i.e.*, the plate to plate load impedance is equal to twice the plate resistance of one tube.

$$\therefore\ P_{L,max}=\left(\frac{\mu e_s}{2r_p}\right)^2\left(\frac{2N_1}{N_2}\right)^2\ R_L=\frac{(\mu e_s)^2}{2r_p}, \qquad ...(68)$$

and $$i_{p,max}=\mu e_s/2r_p. \qquad ...(69)$$

Advantages. (1) It is possible to obtain more than twice the power output of a single tube amplifier by using two similar tubes in push pull. (2) The magnetizing effect of the dc on the iron core of the output transformer cancels out hence the average plate current may have any value. (3) The ac sum of this amplifier is very small. (4) It is more useful to use two small tubes in push pull rather than one large tube. (5) Pentodes and beam power tubes in push pul have high third harmonic distortion than the triodes for comparable conditions and low second harmonic distortion.

3.11. NOISE IN AMPLIFIERS

In addition to the desired signal, the output of an amplifier generally has random variations which are called *noise*. In radio reception, noise is noticed as a *hiss* and in television as *snow*. Noise places a basic lower limit on the signal levels at which a device or an amplifier may operate. Noise has no specific frequency and is distributed over broad frequency bands. It is said to be a white noise if distributed uniformly over all frequencies. For use in comparison of the noise, a term noise figure ($N\,F.$) is used, which is defined as the ratio of the signal-to-noise power at the signal input to the signal-to-noise power at the output of the device, *i.e.*,

$$N.F.=\frac{P_{is}/P_{in}}{P_{os}/P_{on}}. \qquad ...(70)$$

Noise in electronic circuits and devices is introduced by several phenomena :

(1) *Thermal noise*. Thermal noise is developed by the random thermal motions of the charge. It depends upon the temperature. The thermal noise voltage due to a resistance R is given as

$$V_{T.N.}=\sqrt{4kTR\triangle f}, \qquad ...(71)$$

where k is Boltzmann's constant$=1.38\times10^{-23}$ joules/°K, T is temperature in °K and $\triangle f$ is the frequency band in hertz over which the power is measured.

To reduce thermal noise the operating temperature should be lowered and the band width should be made as small as possible.

(2) *Shot noise*. Shot noise is due to the random nature of charge of the emitted electrons from a cathode. This noise may be reduced if the tube operates under space charge limited conditions. Shot noise is uniformly distributed throughout the useful frequency spectrum. This tube noise may be expressed in terms of resistor. The approximate expression for this resistance for triodes is given by

$$R_{eq}\approx 2.5/g_m. \qquad ...(72)$$

(3) *Gas noise*. Gas noise is produced by the random current disturbances, when the positive ions produced due to the ionization of the remaining gas in the tube penetrate the space charge. It is not appreciable above 10Mc/sec.

(4) *Partition noise*. Partition noise is caused due to the irregular division of current between the various electrodes. Thus the noise is greater in pentode.

(5) *Contact noise.* Contact noise is developed by poor or intermittent electric contact, such as from dirty or damaged switch contacts, terminals, leaky condensers, faulty resistances, etc.

(6) *Microphone noise.* This noise is produced due to the mechanical vibration of parts of circuit. When the valve is struck, its electrodes vibrate, their relative positions change and the anode current varies which produces noise.

(7) *Hum.* Hum is the extraneous output voltage produced by the associated or adjacent power circuits. It is produced (*i*) due to *a c.* present in the *d.c.* plate supply (*H.T*), (*ii*) through the inter-electrode capacitance between cathode and grid when directly heated filament operates with *a.c.*, and (*iii*) by the pickup due to stray electrostatic and magnetic fields produced by adjacent power leads.

3.12. VACUUM TUBE VOLTMETER

As the voltmeter resistance is not infinite, it must draw some power from voltage source to move the needle and thus will produce error in observation. To avoid this error a vacuum tube amplifier is used, the resulting device is called a *vacume tube voltmeter VTVM*). The circuit of a simple vacuum tube voltmeter and the variation of plate current with the grid voltage for a typical vacuum triode are shown in Fig. 3.25. This characteristic curve shows that the voltage applied between the grid and the cathode controls the

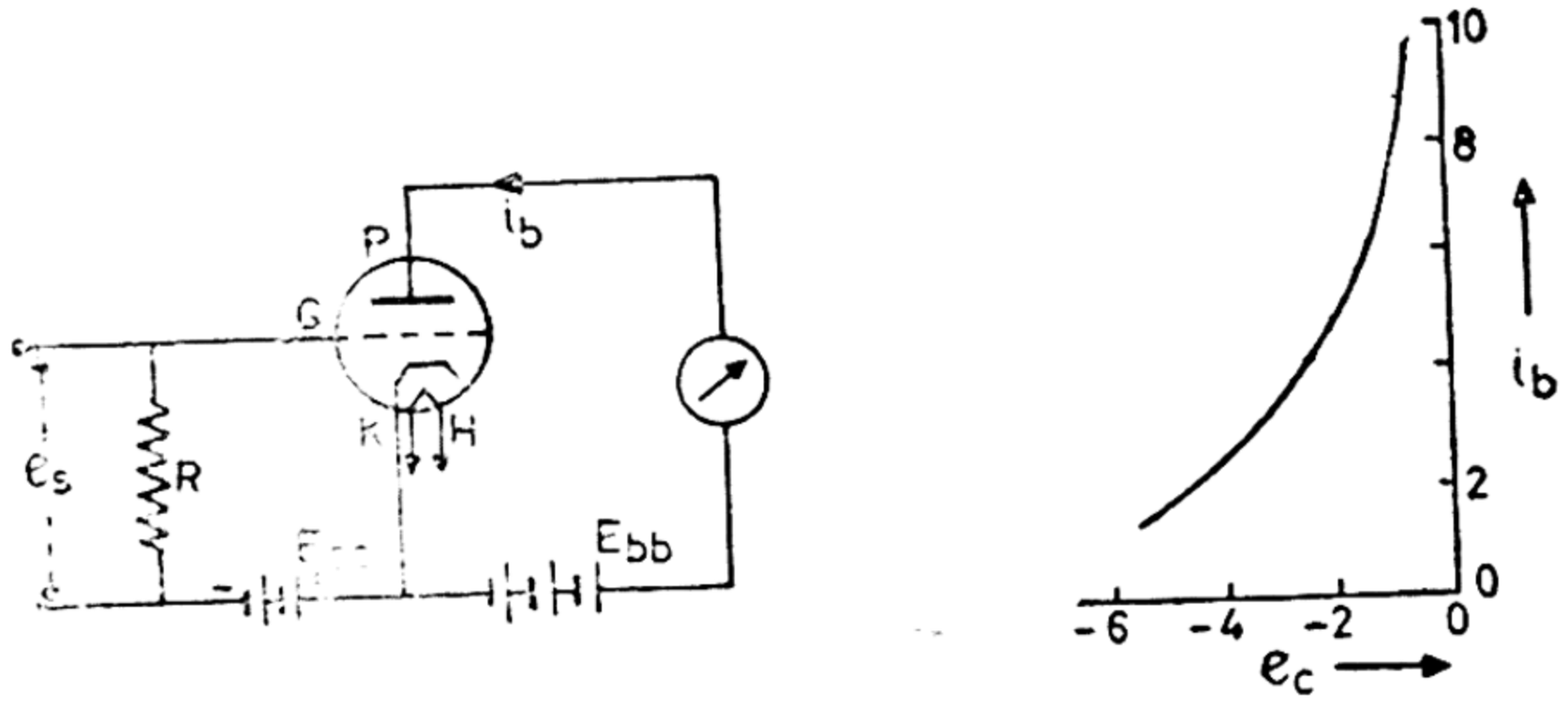


Fig. 3.25.

current through a meter and is negative always with respect to the cathode. If the meter is to indicate zero volt at the left, it is necessary to add a bias voltage in series with the unknown voltage. The bias voltage is negative and is adjusted to get no flow of current in the plate circuit when unknown voltage is zero. This circuit has limited range of input voltages, is nonlinear at the ends of the characteristic and once calibrated requires that the tube should maintain constant characteristics. These limitations can be overcome by more complex circuits.

As stream of electrons pass through the grid, some electrons are collected by the grid. If these electrons are allowed to accumulate on the grid, their negative charge would change the grid potential. To return the electrons to the cathode a resistor R is placed in the circuit between the grid and the bias voltage. The resistance is quite high and in most *VTVM*'s is 10 megohms for all voltage ranges.

VTVM is the most widely used instrument for making measurements at radio frequencies. In one type of *VTVM* the alternating voltage to be measured is first rectified by a peak type diode detector and then amplified by a dc amplifier. Such an arrangement measures the peak amplitude of the positive half cycles

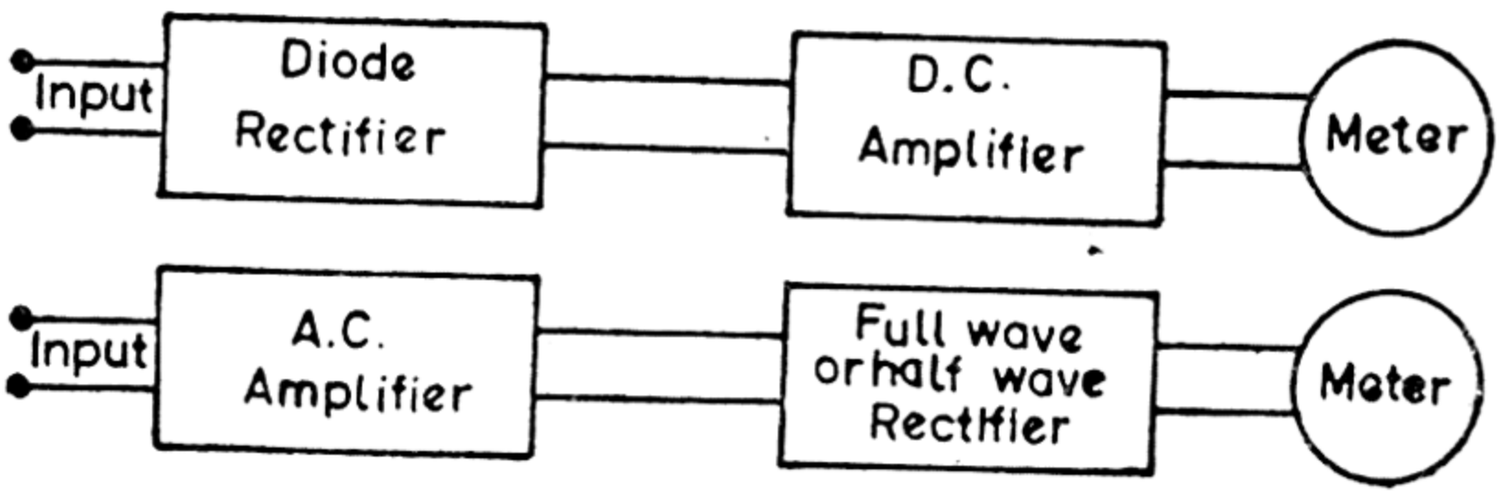


Fig. 3.26.

of the applied voltage and is useful for frequencies up to the order of 100 to 500 *Mc*/sec. In another type, the alternating voltage to be measured is first amplified by a wide band amplifier stabilized by means of —ve feedback and then is rectified by a half wave or full wave rectifier. This arrangement measures the average amplitude of the wave. The sensitivity is greater than for the previous type. It can measure *ac* voltages as small as a few microvolts. The frequency range is, however, limited by the amplifier to a maximum of about 10 *Mc*/sec.

Many *VTVM*'s also measure resistance using the simple circuit for an ohm-meter.

Equipment check. Put the *VTVM* in operating position. Before turning on, check whether meter reads zero or not. If not, turn carefully the set *screw* on the meter face. This adjustment is required very infrequently. Now the *VTVM* is on. Set the function selector switch on the dc position and wait to get the meter needle in steady position. Adjust the *meter zero* with zero adjust control. This adjustment is required frequently.

Exercises

Example 1. *The amplification factor of a triode is 100 and its anode resistance is 20 kilohms. If the load resistance is 10 kilohms and the input signal voltage is 0.4 volt, calculate the output signal voltage and the voltage amplification factor.*

The voltage amplification factor A of a triode is given by

$$A=-\mu R_L/(R_L+r_p).$$

In this problem, the amplification factor of the triode $\mu=100$, the anode resistance $r_p=20\times10^3\ \Omega$ and the load resistance $R_L=10\times10^3\Omega$. Therefore

$$A=-\frac{100\times10\times10^3}{10\times10^3+20\times10^3}=\frac{-100}{3}=-33.3.$$

It is equal to the ratio of the output and input voltages. Hence

$$V_o/V_i=A=33.3.$$

or the output signal voltage $V_o=0.4\times33.3=13.3$ volts.

Here the negative sign has been ignored as we are concerned with the magnitude only. This negative sign represents a phase change of π on amplification of the input signal.

Example 2. *A basic amplifier makes use of a triode of $\mu=10$, $r_p=10k\Omega$. This triode is replaced by anode triode of $r_p=25k\Omega$. The voltage gain of the amplifier remains unchanged. Find μ of the second triode. The load resistance in each case is $20k\Omega$.*

The voltage gain of the amplifier $A=-\mu R_L/(r_p+R_L)$.

For the first triode $A_1=-10\times20\times10^3/(10\times10^3+20\times10^3)$
$=-20/3.$

For the second triode $A_2=-\mu_2\times20\times10^3/(25\times10^3+20\times10^3)$
$=-20\mu_2/45.$

In this problem $A_1=A_2$.

$\therefore\quad -20/3=-20\mu_2/45\quad$ or $\quad\mu_2=15.$

Example 3. *Calculate the triode constants if the voltage gain to the load resistances of 10 and 10 $k\Omega$ be 5/3 and 20/7 respectively.*

The numerical value of the voltage gain of the triode $A=\mu R_L/(r_p+R_L)$.

There are two cases :

(1) $A=5/3$, when $R_L=10\ K\Omega$.

$$\therefore\quad \frac{5}{3}=\frac{\mu\times10\times10^3}{r_p+10\times10^3}. \qquad \ldots (i)$$

(2) $A=20/7$, when $R_L=20\ k\Omega$.

$$\therefore\quad \frac{20}{7}=\frac{\mu\times20\times10^3}{r_p+20\times10^3}. \qquad \ldots (ii)$$

From these equations, we get

$$r_p = 50 \times 10^3 = 50\ k\Omega$$

and $$\mu = 10.$$

As $g_m = \mu / r_p$, hence $g_m = 10/50 \times 10^3 = 2 \times 10^{-4}$ mho.

Example 4. *A resistance of 5 kΩ is connected in the anode of a triode whose plate resistance is 20 kΩ. For a given alternating voltage applied to the grid, the alternating component of the anode-cathode voltage is unaltered when the valve is replaced by another having an amplification factor twice that of original value. Find the slope resistive of this second valve. Find also the value of ac voltage to be applied between grid and cathode to get ac power output of 2 watts across the load in each case, if μ of first valve is 10.*

Amplification gain $A = -\mu R_L/(r_p + R_L)$.

As the amplification gain is the same in both the cases, hence

$$-\frac{\mu \times 5000}{20000+5000} = -\frac{2\mu \times 5000}{r_p + 5000}.$$

$\therefore$ Slope resistive of the second valve $r_p = 45000$ ohms.

ac power across load $= P_L = I_p^2 R_L = [\mu e_s/(r_p + R_L)]^2 R_L$.

Hence for the first valve, we have

$$2 = \left[\frac{10 e_s}{20{,}000+5000}\right]^2 \times 5000 = \frac{e_s^2}{1250}$$

or $$e_s = 50 \text{ volts}.$$

For the second valve, we also get $e_s = 50$ volts.

Example 5. *Calculate the voltage gain of the grounded grid amplifier. Find the output and input impedances.*

A grounded grid or cathode input amplifier and its equivalent circuit are shown in Fig. 3.27. In this circuit the grounding of the grid reduces the input capacity to C_{gk} and the output capacity to C_{gp}. We are not taking these as both are very small. Since plate current flows through the input source e_s, its internal resistance R is important and is included in the equivalent circuit. Applying Kirchhoff's second law, we get

$$\mu e_g + e_s - i_b\ (r_p + R + Z_L) = 0$$

and $$e_g - e_s + i_b R = 0.$$

Solving these two relations, we get

$$i_b = (\mu + 1)\ e_s/[r_p + (\mu + 1)R + Z_L].$$

Since the load voltage rise $e_o = -ibZ_L$ and since e_s also represents a negative rise of voltage above the common terminal, hence the voltage gain.

$$A = \frac{e_0}{e_s} = \frac{(\mu+1)Z_L}{r_p+(\mu+1)\,R+Z_L}.$$

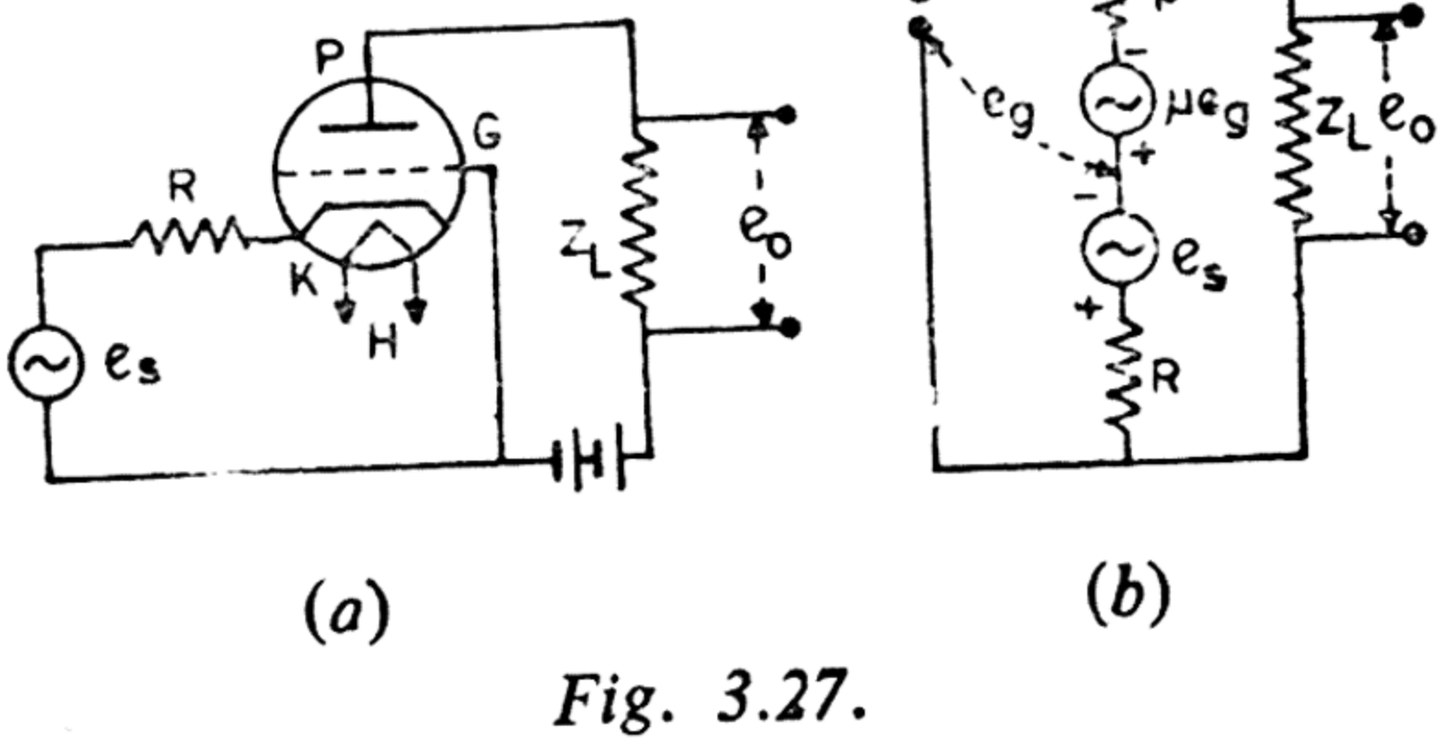


Fig. 3.27.

This result shows that (*i*) there is no phase reversal in the circuit, (*ii*) tube appears to have an amplification factor $(\mu+1)$, (*iii*) and an internal resistance or effective plate resistance

$$r_p' = r_p+(\mu+1)\,R.$$

This is the output impedance of the amplifier. As

$$\frac{e_s}{ib} = \frac{r_p+(\mu+1)R+Z_L}{\mu+1} = R+\frac{r_p+Z_L}{\mu+1}.$$

Hence the input impedance of the tube alone is

$$Z_{in} = (r_p+Z_L)/(\mu+1).$$

Thus the circuit steps down the total impedance of the plate circuit by a factor $(\mu+1)$. Hence the main feature of the circuit is a step-up impedance ratio, from low input to high output impedance *similar to a pump which lifts water from a low pressure system to a high pressure system.*

Example 6. *A triode valve requires a bias of −12 V. At the usual mean operating potential the anode current is 6.6 mA. What value resistor is necessary between H.T. and L.T. to achieve this bias and what is a suitable bypass capacitor value if the lowest frequency concerned is 100 c/s ?*

Bias resistor value can be readily obtained by the Ohm's law as

$$R_k \times 6.6 \times 10^{-3} = 12 \quad \text{or} \quad R_k = 1817 = 1800\,\Omega \text{ (Approx.)}$$

To calculate C_k, the capacitance required, a simple working rule is to consider that the reactance in ohms should be not more

than 0.1 R_k at the lowest frequency. Thus the minimum value of C_k is given by

$$1/\omega C_k = 0.1 R_k \quad \text{or} \quad C_k = 10/\omega R_k.$$

$$\therefore \quad C_k = \frac{10}{2\pi f R_k} = \frac{10}{2 \times 3.14 \times 100 \times 1800} = 9\mu f.$$

Example 7. *We want to design an R—C coupled amplifier by using a triode whose plate resistance is 10kΩ and amplification factor 20. The load resistance used in the plate circuit is also 10 kΩ. If the voltage gain desired at middle frequency is 9.9, calculate the value of grid leak resistance for the next stage, What should be the value of coupling capacitor so as to have the lower half power frequency of 10 c/s.*

The voltage gain of *R-C* coupled amplifier at mid-frequency

$$A_m = -\mu . R_L R_g / (r_p R_L + r_p R_g + R_L R_g).$$

Substituting numerical values, we get

$$9.9 = \frac{20 \times 10{,}000 \times R_g}{10{,}000 \times 10{,}000 + 10{,}000 R_g + 10{,}000 R_g}$$

or Grid leak resistance $R_g = 495{,}000$ ohms.

The value of lower half power frequency

$$f_1 = \frac{1}{2\pi C}\left(\frac{r_p R_L}{r_p + R_L} + R_g\right)^{-1} \text{ or } C = \frac{1}{2\pi f_1}\left[\frac{r_p R_L}{r_p + R_L} + R_g\right]^{-1}$$

$$\therefore \quad C = \frac{1}{2 \times 3.14 \times 10}\left[\frac{10^4 \times 10^4}{10^4 + 10^4} + 49.5 \times 10^4\right]^{-1}$$

$$= 0.0318 \mu f.$$

Example 8. *An R-C coupled amplifier uses triode having amplification factor of 50 and dynamic plate resistance of 14 kilo ohms. The load resistance is 100 kilo ohms and the grid leak resistance of the next stage is 800 kilo ohms. The total shunt capacitance is 200 μμf. Calculate for one stage of this amplifier (i) mid band gain, (ii) value of coupling capacitor to get lower half power frequency of 10 Hz, (iii) upper half power frequency and (iv) for coupling capacitor of 0.05 μf, voltage gain at frequencies of 5, 500 and 50 kHz.*

We know that

$$\frac{1}{R_{pLg}} = \frac{1}{r_p} + \frac{1}{R_L} + \frac{1}{R_g} = \frac{1}{14 \times 10^3} + \frac{1}{100 \times 10^3} + \frac{1}{800 \times 10^3}$$

or $R_{pLg} = 1.21 \times 10^4$ ohms.

Mid band gain $A_m = -g_m R_{pLg} = -\frac{-50}{14000} \times 1.21 \times 10^4 = -43.19$.

Coupling capacitance $C=1/2\pi f_1 R_{eq}$,

where $$R_{eq}=R+\frac{r_p R_L}{r_p+R_L}=800\times10^3+\frac{14\times10^3\times100\times10^3}{14\times10^3+100\times10^3}$$

$=812.3$ kilo ohms.

$$\therefore \quad C=\frac{1}{2\times3.14\times10\times812.3\times10^3}=0.0981\,\mu f.$$

Upper half power frequency $f_2=1/2\pi C_s R_{pLg}$

$=1/2\times3.14\times200\times10^{-12}\times1.21\times10^4=65.8$ kilo Hz.

For coupling capacitor $C=0.05\ \mu f$, lower half power frequency

$$f_1=\frac{1}{2\pi CR_{eq}}=\frac{1}{2\times3.14\times0.05\times10^{-6}\times812.3\times10^3}=3.92 \text{ Hz.}$$

Frequency of 5 Hz is close to the lower half power frequency, hence voltage gain

$$A=\frac{A_m}{[1+(f_1/f)^2]^{1/2}}=\frac{43.19}{[1+(3.92/5)^2]^{1/2}}=33.98.$$

Frequency of 500 Hz lies in true mid band as it is greater than 1.92 Hz and smaller than 65.8×10^3 Hz. Hence gain at this frequency may be taken as 43.19.

Frequency of 50 kilo Hz is close to the upper half power frequency, hence voltage gain

$$A=\frac{A_m}{[1+(f/f_2)^2]^{1/2}}=\frac{43.19}{[1+(50/65.8)^2]^{1/2}}=34.4.$$

Example 9. *A three-stage R-C coupled amplifier has plate resistance 10 kilo ohms, valve amplification 15, external resistance 80 klio ohms and loss in coupling capacitor 5%. Find the voltage across the last output resistance for an input of 3 milli volts.*

$$\text{Voltage across } R_L=\frac{\mu e_s R_L}{r_p+R_L}=\frac{15\times3\times10^{-3}\times80\times10^3}{10\times10^3+80\times10^3}$$

$=40\times10^{-3}$ volt.

A loss across coupling capacitor is 5%, hence

loss in capacitor $=40\times10^{-3}\times(5/100)=2\times10^{-3}$.

$\therefore$ Input for second stage $=40-2=38$ milli volts.

$$\text{Voltage across second stage } R_L=\frac{\mu e_s R_L}{r_p+R_L}=\frac{15\times38\times10^{-3}\times80\times10^3}{10\times10^3+80\times10^3}$$

$=507\times10^{-3}$ volt.

Loss in the second stage $=507\times10^{-3}\times(5/100)=25.35$ milli volts

$\therefore$ Input for the third stage $=507-25.35=481.65$ milli volts.

Voltage across third stage $R_L=\dfrac{\mu e_s R_L}{r_p+R_L}=\dfrac{15\times481.65\times10^{-3}\times80\times10^3}{10\times10^3+80\times10^3}$

$=6422\times10^{-3}$ volts.

Loss in the third stage $=6422\times5/100=321.1$ milli volts.

$\therefore$ Input for the next stage $=6422-321.1=6100.9$ milli volts.

Example 10. *A transformer coupled amplifier uses a triode amplification factor of 15 and dynamic plate resistance of 10,000 ohms. Effective primary resistance $R_p=1000\,\Omega$, effective secondary resistance $R_s=9000\,\Omega$, magnetising inductance of primary $=10$ H, leakage inductances referred to the primary $=0.1$ H, effective load capacitance across the primary $=400$ pf, and turns ratio $=2$. Calculate (i) Mid frequency range gain, (ii) lower half power frequency, (iii) resonant frequency in high frequency range, (iv) true mid band, and (v) voltage gain at frequency of 20 Hz.*

Mid frequency range voltage gain $A_m=n\mu=2\times15=30$.

The total anode resistance $r_p'=r_p+R_p=10{,}000+1000$.

$\therefore$ Lower half power frequency $f_1=\dfrac{r_p'}{2\pi L_p}=\dfrac{11000}{2\times3.14\times10}$

$=175$ *Hz.*

Resonant frequency in high frequency range is

$$f_2=\frac{1}{2\pi[L'C']^{1/2}}=\frac{1}{2\pi[0.1\times400\times10^{-12}]^{1/2}}=26.5\times10^3\ Hz.$$

$\therefore$ True mid band $=0.1f_2-10f_1=2.65\times10^3-1.75\times10^3$

$=900$ *Hz.*

$$\text{Gain at frequency of 20 Hz}=\frac{A_m}{[1+(f_1/f)^2]^{1/2}}=\frac{30}{[1+(175/20)^2]^{1/2}}$$

$=3.41$.

Example 11. *An amplifier has a voltage gain of 10,000 with normal plate supply. If the plate supply is reduced, the voltage gain reduces to 8000. When a —ve feedback is applied at the normal plate supply, the voltage gain reduces to 1/100 of its value without feedback. Calculate (i) voltage gain with feedback for the two values of the plate supply, (ii) percentage reduction in voltage gain with the reduction in the plate supply, for both conditions, with and without feedback.*

With −ve feedback the voltage gain $A_f = A/100 = A/(1-\beta A)$.

$\therefore\ 1-\beta(-10{,}000)=100$ or $\beta=0.0099$

and voltage gain with −ve feedback $A_f = -10000/100 = -100$.

When the plate supply is reduced, the voltage gain with feedback

$$A_f' = -8000/[1-0.0099\ (-8000)] = -99.75.$$

$\therefore$ % reduction in voltage gain with feedback

$$= \frac{-100-(-99.75)}{-100} \times 100 = 0.25\%$$

and % reduction in voltage gain without feedback

$$= \frac{-10{,}000-(-8000)}{-10000} \times 100 = 20\%.$$

Example 12. *In a cathode follower the different values are : amplification factor $\mu=50$, plate resistance 10,000 ohms and cathode resistance $R_k=1000$ ohms. Calculate : (1) voltage gain, (2) internal impedance, (3) output impedance.*

We know for a cathode follower that

voltage gain $A = \mu R_k/[R_k(1+\mu)+r_p]$

$$= 50 \times 1000/[1000(1+50)+10{,}000] = 0.82$$

The plate current in cathode follower $i_p = \dfrac{\mu e_s}{r_p + R_k(1+\mu)}$

or $$i_p = \frac{[\mu/(1+\mu)]\ e_s}{[r_p/(1+\mu)+R_k}\ .$$

$\therefore$ Internal impedance $= \dfrac{r_p}{1+\mu} = \dfrac{10{,}000}{1+50} = 196\,\Omega.$

Output impedance $= \dfrac{r_p R_k}{r_p + R_k(1+\mu)} = \dfrac{10{,}000 \times 1{,}000}{10{,}000+1000(1+50)}$

$= 163.93$ ohms.

Example 13. *Prove that the maximum amount of power output is obtained at the load of an amplifier (pushpull) circuit when the load resistance is equal to twice the plate resistance of one tube. If the each triode, having $\mu=12$ and $r_p=10k\,\Omega$ is fed from an anode supply of 400 volts. If the total direct current from the supply is 10mA, input signal applied to each triode is 10 volts, load resistance $R_L=5\,\Omega$ and the ratio of primary to secondary winding of output transformer is 30. Calculate (i) output voltage and (ii) the plate current efficiency.*

Total output power $P_L=\left(\frac{\mu e_s}{r_p+\frac{1}{2}R_p}\right)^2 R_p$

For the total power delivered to be maximum

$$\frac{dP_L}{dR_p}=\left(\frac{\mu e_s}{r_p+\frac{1}{2}R_p}\right)^2-\frac{(\mu e_s)^2 R_p}{(r_p+\frac{1}{2}R_p)^3}=0$$

or $R_p=2r_p=$ Twice the plate resistance.

As $R_p=(2N_1/N_2)^2 R_L=30^2\times 5=4500\,\Omega$.

Plate current $i_p=2\mu e_s/[2r_p+(2N_1/N_2)^2 R_L]$

$=2\times 12\times 20/[2\times 10^4+4500]=19.6$ mA.

As $(2N_1/N_2)\,e_0=i_p R_p$, hence the output voltage

$$e_0=19.6\times 10^{-3}\times 4500/30=2.94.$$

The output *ac* power $P_{ac}=i_p^2 R_p=(19.6\times 10^{-3})^2\times 4500=1.728$

$\therefore$ Plate circuit efficiency $\eta=[P_{ac}/Pbb]\times 100$

$$=\frac{i_p^2 R_p}{iE_{bb}}\times 100=\frac{1.728\times 100}{10\times 10^{-3}\times 400}=43.2\%$$

Oral Questions

1. What is meant by amplification ?
2. How are the amplifiers classified ?
3. What is the fundamental difference between class A, class B, class AB and class C amplifiers ?
4. Which class of amplifier operation is commonly used for *r-f* and *a-f* amplifiers in radio receivers ?
5. What is meant by load line ?
6. What is the phase difference between the input signal voltage and the anode voltage in a class A amplifier ?
7. What is the advantage of equivalent diagram of any electronic circuit ?
8. What is meant by impedance matching in an amplifier ?
9. What are the functions of R_k, C_k and R_g in *R-C* coupled amplifiers ?
10. What is the advantage of *RC*--coupled amplifier ?
11. What is the advantage of the impedance coupled amplifier over the *RC*-coupled amplifier ?
12. What are the advantages of transformer coupled amplifier ?
13. What is meant by distortion in amplifiers ? What are the causes ?

14. What is meant by feedback ? Under what conditions does it occur ? What are its advantages ?

15. What is meant by cathode follower ? By what other names is this circuit also known ?

16. What are the applications of cathode follower circuit ?

17. What is the purpose of power amplifiers ? How are they classified in terms of (*i*) operating frequency, (*ii*) tube operating characteristics ?

18. What is meant by VTVM ? Why is it better than a voltmeter ?

Problems

1. A triode used as an amplifier provides a voltage gain of 15 with a load resistance of 10,000 ohm. The gain changes to 20 when the load is increased by 10,000 ohm. Find the plate resistance and other constants of the triode.

(*10 kΩ, 30, 3×10^{-3} mho*)

2. In a triode valve the plate resistance is 10_k Ω and the anode load resistance is 30 *k*Ω, the amplification factor is 36. Calculate the voltage gain.

(*—27*)

3. A triode valve used as amplifier gives a voltage amplification of 30 with a load resistance of 50 *k*Ω and 34 with a load resistance of 85 *k*Ω. Calculate the constants of the valve.

(r_p=*20 k*Ω, μ=*42*, g_m=2.1×10^{-3} *mho*)

4. A triode has a mutual conductance of 2.5 *m*A/volt and an anode ac resistance of 20 *k*Ω. Find the load resistance which must be inserted in the anode circuit to obtain a stage gain 30.

(*30 k*Ω)

5. A triode valve has $\mu = 30$ and $r_p = 3$ *k*Ω. It is used to amplify an ac signal of 0.05 volts. Calculate the output voltage if the load resistance is 50 *k*Ω.

(*1.4 volts*)

6. The slopes of the mutual and anode characteristics of a triode valve are 2.4 *m*A/V and 0.05 *m*A/V respectively. What value of anode load resistance will be required in a simple voltage amplifying stage using this valve to give a stage gain 30.

(*33.33 k*Ω)

7. Alternative triode valves are available for simple amplifier. Valve **A** has an amplification factor of 33 and a slope resistance of 60 kilo ohm, while valve *B* has an amplification factor of 20 and a slope resistance of 15 kilo ohms. If the amplifier is to operate with an anode load of 30 kilo ohm, find which valve will provide the higher gain. At what value of anode load the two valves give the same gain.

(*B valve, 51.23 kilo ohm*)

8. Find the maximum power which can be obtained for a signal of 1 volt by triode having r_p=2000 ohms and amplification factor 10.

(*0.0125 watt*)

9. An *RC*-coupled amplifier has the following value of circuit components : μ=20, r_p=10 *k*Ω, R_g=100 *k*Ω, R_g=50 *k*Ω, C=0.001 μ*f* and C_{eq}=400*pf*. Find the gain of the amplifier at 12 *kc/s*.

(*—14.9*)

10. If the circuit components of a *R-C* coupled amplifier are : μ=200, r_p=100 *k*Ω R_L=250 *k*Ω, C=0.01 μ*f* and R_g=0.5 *M*Ω, find the gain of this amplifier at frequencies 50 and 1000 *kc/s* in decibels.

(*42, 41*)

11. An *RC*-coupled amplifier uses pentode having amplification factor μ=100, dynamic plate resistance r_p=500 *k*Ω, load resistance R_L=100Ω, grid

leak resistance of each stage R_g=500 $k\Omega$, coupling capacitance C=0.1 μf, plate to cathode capacitance C_{pk}=3$\mu\mu f$, input capacitance C_{in}=5$\mu\mu f$ and writing capacitance C_w=2$\mu\mu f$. Calculate lower and upper half power frequencies, mid band gain and gain band width product.

(2.73 Hz, 222.8 k Hz, −14.29, 3.18 × 10⁶ Hz)

12. A valve having an amplification factor of 7 and an anode slope resistance of 7000Ω is to be used as a low frequency amplifier. Anode circuit transformer has turns ratio of 3 and its low resistance winding has an inductance of 10 *H*. Calculate the gain for voltages of frequency 100 *c/s*. *(14)*

13. A change of anode supply voltage from 300 V to 270 V reduces the gain of a amplifier from 700 to 650. If one fourteenth of the output voltage is introduced at the input as negative feedback, find the resulting gain at 300 V and 270 V anode supply, *(13.72, 13.71)*

14. An amplifier has a voltage gain of 200, distortion of 10% and a normal input of 0.5 volts. If a negative feedback with α=0.05 is applied, then find the new gain and distortion. What will be the input voltage for the same output voltage in the presence of feedback. *(18.18, 0.909%, 5.5 volts)*

15. A vacuum tube has a transconductance of 2500μ mhos and a plate resistance of 80,000 ohms. What is ths ac power output if the peak value of the input signal is 16.5 volts and R_L is 7000 ohms ? What value of plate load resistance would be required for the tube to produce maximum power output ? Calculate the maximum power output corresponding to an input signal with a peak value of 16.5 volts. *(5 watts ; 80,000 ; 17 watts)*

4

Oscillators, Modulators, Detectors and Receivers

4.1. OSCILLATORS

The electronic device employed for generating electrical oscillations is called as *electric oscillator*. In this device *dc* power is converted into an *ac* power. This function is reverse of the function of a rectifier which converts *ac* power into *dc* power, due to this reason the oscillator is also known as *invertor*. Oscillators are used for variety of purposes, *e.g.*, in (*i*) *Radio transmitters and receivers* (*ii*) *Television transmitters and receivers*, (*iii*) *Radar*, *etc.*

We know that the combination of a capacitor C fully charged initially and an inductor L gives electrical oscillations. In these oscillations the electrical energy associated with the capacitor is completely transferred to the magnetic field linked with the inductor. The latter then transferred to the former. This process will continue and is known as *electrical oscillation*. In practice there is some resistance R of the coil, hence the frequency of oscillations is thus given by

$$f=\frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)}. \qquad \ldots(1)$$

This damping is due to resistance R because of losses due to (*i*) heating effect in the resistance R and (*ii*) the generation of electromagnetic waves when frequency is high. To maintain continuous oscillations the power must be fed to the circuit in some way so that the losses due to R are overcome. One way for producing continuous oscillations is the use of the negative resistance, equal to the positive resistance R of L-C circuit. The oscillator in which a —ve resistance is produced by the amplifier tube to neutralize the positive resistance of the oscillating circuit is known as a *negative resistance oscillator*. The other method for the compensation of energy loss of the L-C circuit is the use of positive feedback circuit. This type of oscillator is known as *positive feedback type oscillator*.

Requirements of an oscillator. (*a*) *Feedback device.* To maintain continuous oscillations +ve feedback is to be introduced from output to the input circuit of the feedback amplifier, shown in Fig. 4.1. If A is the voltage gain of the amplifier and β is the feedback ratio, the feedback voltage supplied to the input of the amplifier is given by

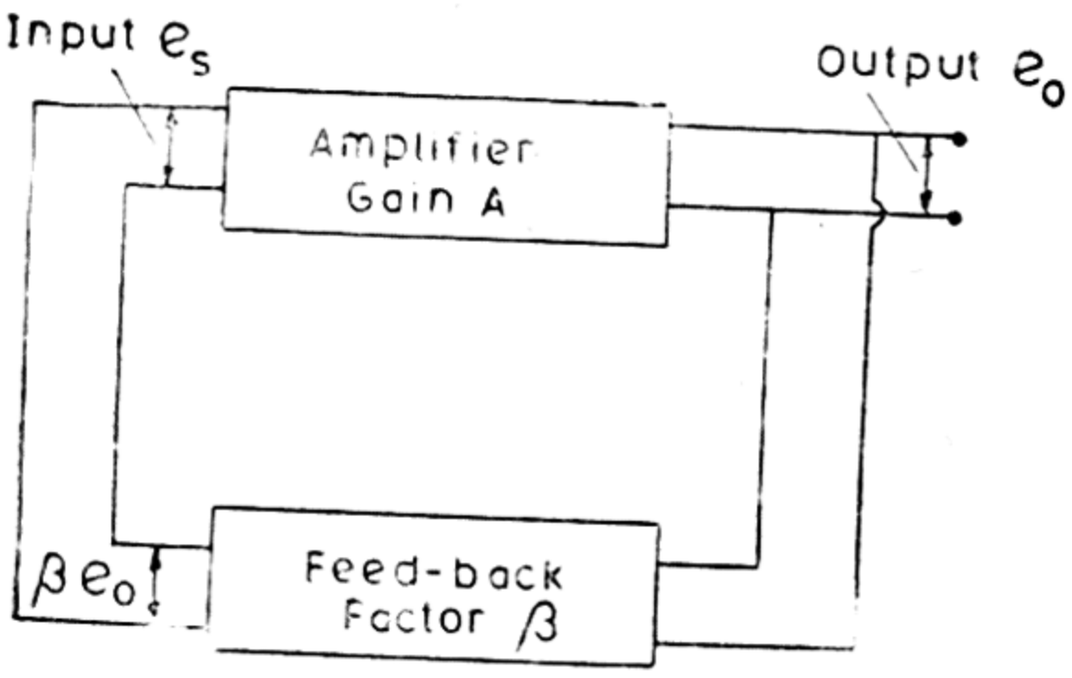


Fig. 4.1.

$$e_f = e_s = \beta e_0 = \beta(Ae_s)$$

or $$e_s\,(1-\beta A)=0.$$

For an output $e_s \neq 0$, we have

$$1-\beta A=0 \text{ or } \beta A=1 \quad \ldots(2)$$

This is known as *Barkhausen criterion for sustained oscillations.*

Thus to maintain sustained (*continuous*) oscillations the gain frequency, and phase shifts should be such that $A\beta = 1$ and the phase shift of amplifier should be equal to negative phase shift of feedback circuit, hence the overall phase shift around the complete cycle must be zero or a multiple of 2π. For a good frequency stability $d\theta/d\omega$ should be large, so that a large change in phase angle is obtained for a small shift in frequency.

To examine Barkhausen criterion analytically, consider that e_s is the voltage between the grid and the cathode. The output current is $g_m e_s$ if we assume the linear operation of the circuit. An input current i_p in the feedback circuit will produce certain voltage across the output terminals of the circuit. The transfer impedance of the circuit

$$Z_T = -e_s/i_p,$$

where — ve sign is due to phase shift of 180° in feedback circuit. Thus for continuous oscillations

$$(g_m e_s + e_0/r_p)\, Z_T = -e_s. \quad \ldots(3)$$

Since $e_0 = Ae_s$, $r_p g_m = \mu$, hence Eq. (3) becomes

$$g_m Z_T(1 + A/\mu) = -1. \quad \ldots(4)$$

This is the *first form of the Barkhausen criterion* for the continuous oscillations.

Another expression can be obtained as $e_0 = -i_p Z$.

$$\therefore \quad (g_m e_s + Ae_s\, g_m/\mu)Z = -Ae_s$$

or $$g_m Z\left(\frac{1}{A}+\frac{1}{\mu}\right) = -1 \quad \text{or} \quad \left[\frac{1}{g_m Z}+\frac{1}{\mu}\right) = -\frac{1}{A}. \quad (5)$$

This is *another form of the Barkhausen criterion*. If the ratio $e_f/e_0=\beta$, the fraction of the output voltage that is feedback into the input circuit through the feedback circuit, then

$$\left(\frac{1}{g_m Z}+\frac{1}{\mu}\right)=-\beta \quad \text{or} \quad Z+\frac{r_p}{1+\beta\mu}=0. \qquad \ldots(6)$$

Eqs. (5) and (6) give

$$\beta A=1.$$

(*b*) *Amplitude Limiting Device.* As soon as the oscillator is switched on, oscillations start gradually and the amplitude increases. When the amplitude of oscillation has built up, the output power gets reduced. At large input amplitude, fed from output, the plate current is cut off at each —ve peak of grid voltage and at +ve peak, current flows which reduces the anode current and hence the power delivered to the tank circuit (*L*·*C* circuit). Thus the amplitude of oscillation soon becomes steady, so that the energy delivered to the tank circuit by one pulse of anode current is equal to energy lost by the tank circuit in one oscillation.

(*c*) *Grid bias.* The combination of the grid capacitance (C_g) and grid resistance (R_g) is used for self biasing of the oscillator, which results in self starting oscillations. As soon as the circuit is switched on, the grid bias is zero. The oscillations start growing. During the first +ve half of the generated voltage, capacitor C_g is charged. The capacitor will therefore have a steady *pd* developed across it. The plate of the capacitor joined to the grid will be —vely charged as it collects electrons. During the —ve half cycle of the generated voltage, grid current does not flow and the capacitor discharges through the grid resistance R_g. The time constant R_gC_g is so chosen that a steady bias is maintained. With the each half cycle of oscillation, the amplitude of oscillation increases continuously. The grid bias also increases gradually until an equilibrium position is reached when the charge received by the capacitor in the +ve half cycle is exactly same as the charge removed to R_g during the —ve half cycle. At this time the bias and amplitude of oscillation become steady. The bias obtained in this manner is at or beyond cut-off.

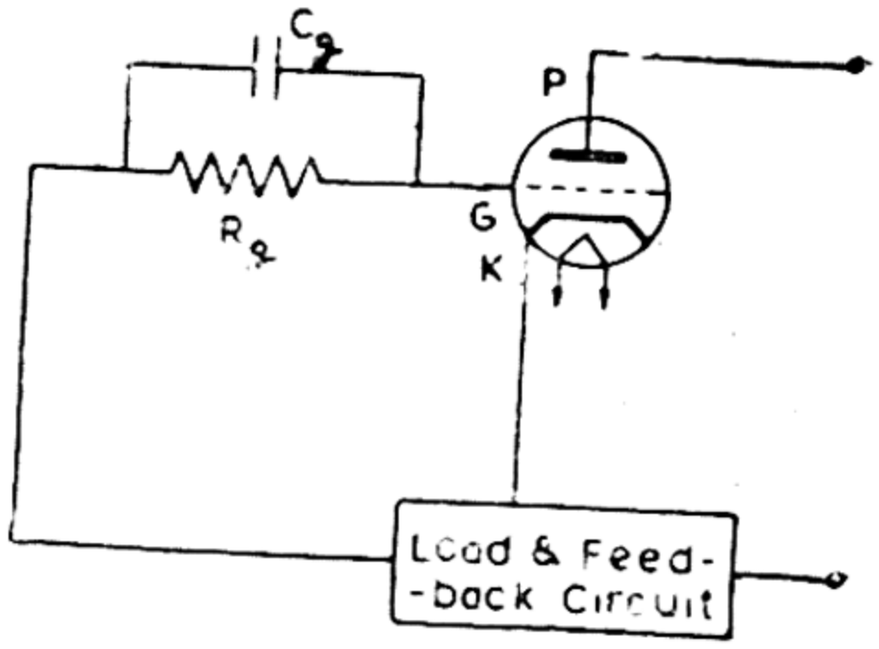


Fig. 4.2.

If the time constant R_gC_g is too large, the bias becomes very high after a number of cycles of oscillation. At this higher bias, the oscillations die out and the capacitor slowly but continuously discharges through R_g and bias reduces. When capacitor has discharged sufficiently, the oscillations again start. In this way we get

discontinuous oscillations. Hence to eliminate this discontinuity, time constant R_gC_g should be reduced suitably.

4.2. TYPES OF OSCILLATOR CIRCUITS

Vacuum tube oscillator circuits are broadly divided into two groups :

1. Producing sinusoidal wave. These may be classified as :

(*i*) *Feedback oscillator.* It is the combination of tuned oscillator with a +ve feedback. The feedback oscillators commonly used are : (*a*) tuned grid, (*b*) tuned plate, (*c*) Hartley, (*d*) Colpitts, (*e*) complex types using more than one tuned circuits, *e.g.*, the tuned grid tuned plate oscillator.

(*ii*) *Negative resistance oscillator.* The circuit element is said to possess —ve resistance when the increase in voltage decreases the current through it. The commonly used types are : (*a*) dynatron, (*b*) transitron, (*c*) push pull circuit and (*d*) resistance capacitance circuit.

(*iii*) *Crystal oscillator.* To maintain the frequency of oscillations to a fixed value, crystal oscillators are required. Actually crystal does not produce oscillations but controls the output frequency.

(*iv*) *Magnetostriction oscillator.* This circuit is based on the principle that a change in magnetization will cause a magnetic material to expand or contract. On the other side a contraction or expansion of a magnetic material will cause a change in magnetization.

(*v*) *Ultra high frequency oscillator.* Above mentioned oscillators cannot be used to produce ultra high frequency oscillations. These are obtained by using (*a*) +ve grid, (*b*) magnetron, (*c*) velocity modulation, (*d*) resonant cavities and (*e*) resonant lines.

2. Producing non-sinusoidal wave. These are also named as *relaxation oscillator* circuits. The types most commonly used are : (*i*) Vander Pol, (*ii*) multivibrator, (*iii*) glow tube discharge, (*iv*) arc tube discharge, (*v*) saw-tooth wave generator, (*vi*) rectangular or square wave generator. These are generally used as electronic timing and control circuits in television, radar, oscilloscope and industrial control equipment.

Let us discuss few oscillator commonly used.

(A) **The Tuned Grid Oscillator.** Fig. 4.3 shows the circuit diagram of a tuned grid oscillator. To maintain sustained oscillations, the amplifier output is fed back to its input and the feedback energy supplied by the plate circuit to the grid circuit should

be in proper phase and of proper magnitude. In this oscillator a L-C tank circuit (L_1-C_1 *in parallel*) is placed in the grid circuit, an inductance coil L placed in the plate circuit is coupled to the coil L_1 of the tuned circuit by mutual inductance M, which provides the necessary feedback from the plate circuit to the grid circuit. The C_gR_g combination in the grid circuit is placed for self-biasing of the oscillator.

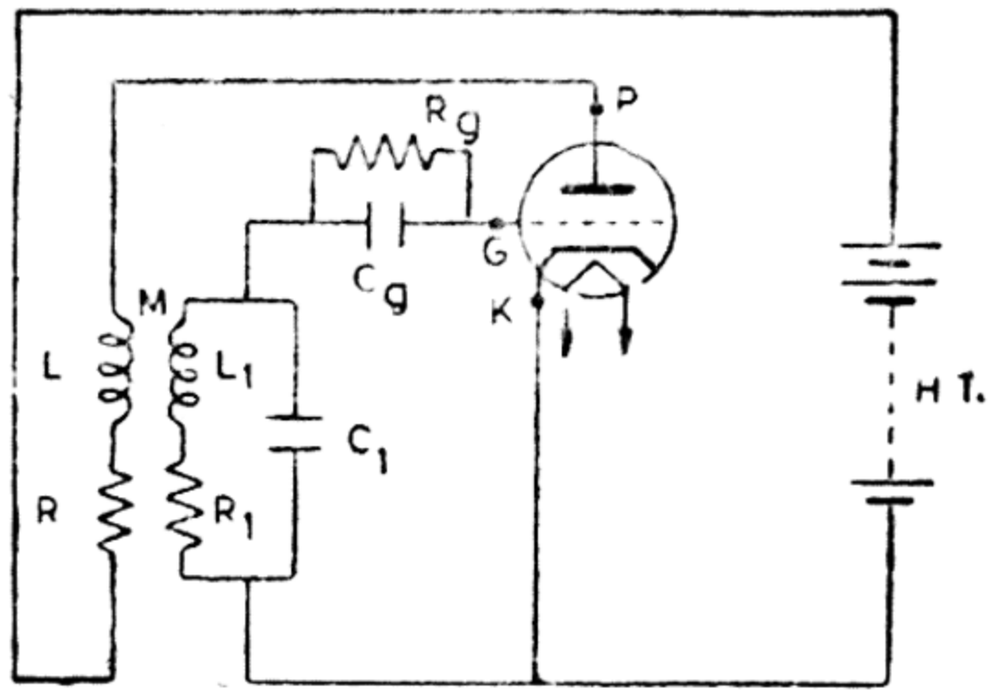


Fig. 4.3.

On switching the H.T. supply, a transient current occurs in the coil L_1, the $L_1C_1R_1$ circuit will then beign to execute damped oscillations. The resulting variations in grid potential will cause corresponding changes in anode current. These variations pass through L and due to mutual coupling will induce emfs in L_1 of the same frequency. If the feedback energy, depending on M and g_m of the valve, is of sufficient quantity to overcome the losses due to R, and the feedback is in correct phase, the sustained oscillations will be maintained. This is the application of coil L, known as *reaction coil.*

Let e_s be the voltage between grid and cathode of the tube. Due to this voltage at the input, current i_p flows in the plate circuit. As this current passes through the reaction coil L, voltage $j\omega M i_p$ is induced between the grid and cathode through mutual inductance M. If this emf is greater than or equal to the input voltage e_s, and is in the same phase as e_s, the sustained oscillations are maintained. The tube and coupling (*like transformer*) each makes a phase change of π, thus the feedback voltage is in phase with original voltage.

Applying second Kirchhoff's law to meshes (1) and (2) of the equivalent circuit 4.4, we get

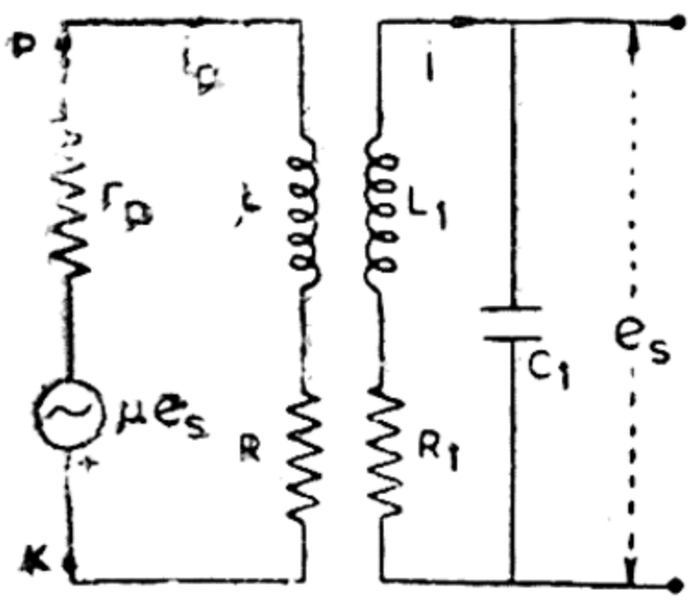


Fig. 4.4.

$$\mu e_s = i_p[R + r_p + j\omega L] - j\omega M\, i. \quad \ldots (7)$$

$$0 = i\,[R_1 + j\omega L_1 + 1/j\omega C_1] - j\omega M i_p \quad \ldots (8)$$

As the output voltage e_0 = pd across capacitance $C_1 = i/j\omega C_1$ = input voltage e_s across grid and cathode by feedback process. Hence from Eq. (7) we have

$$\mu i/j\omega C_1 = i_p\,[R + r_p + j\omega L] - j\omega M i$$

or $$i[\mu/j\omega C_1 + j\omega M] = i_p[R + r_p + j\omega L] \ldots (9)$$

Eliminating i and i_p from relations (8) and (9), we get

$$j\omega M(\mu/j\omega C_1 + j\omega M) = (R + r_p + j\omega L)\,(R_1 + j\omega L_1 + 1/j\omega C_1).$$

If $R \ll r_p$, then we have

$$\mu M/C_1 - \omega^2 M^2 = (r_p + j\omega L)\ (R_1 + j\omega L_1 + 1/j\omega C_1) \qquad \text{...(10)}$$

Equating the real parts, we have

$$\mu M/C_1 - \omega^2 M^2 = r_p R_1 - \omega^2 L L_1 + L/C_1$$

or $$\mu M - \omega^2 M^2 C_1 = r_p R_1 C_1 - \omega^2 L L_1 C_1 + L.$$

At resonance $\omega L_1 = 1/\omega C_1$ or $\omega^2 L_1 C_1 = 1$, hence

$$\mu M - M^2/L_1 = r_p R_1 C_1.$$

As M^2/L_1 is small compared with μM, hence

$$\mu M = r_p R_1 C_1 \quad \text{or} \quad M = r_p R_1 C_1/\mu = C_1 R_1/g_m. \qquad \text{...(11)}$$

This relation gives the maximum value of M, required for the sustained oscillations. The transconductance of the valve

$$g_m \gg C_1 R_1/M. \qquad \text{...(12)}$$

Equating imaginary parts of Eq. (10), we get

$$0 = \omega L R_1 + \omega L_1 r_p - r_p/\omega C_1 \text{ or } r_p = \omega^2 [L C_1 R_1 + r_p L_1 C_1]$$

$$\therefore\ \omega^2 = \frac{r_p}{L R_1 C_1 + r_p L_1 C_1} = \frac{1}{L_1 C_1 (1 + L R_1/r_p L_1)}$$

or $$\omega = \frac{1}{\sqrt{(L_1 C_1)}} \left[1 + \frac{L R_1}{r_p L_1}\right]^{-1/2} = \frac{1}{\sqrt{(L_1 C_1)}} \left[1 - \frac{L R_1}{2 r_p L_1}\right]$$

$$\therefore f = f_0 \left[1 - \frac{L R_1}{2 r_p L_1}\right], \text{ where } f_0 = \frac{1}{2\pi\sqrt{(L_1 C_1)}}. \qquad \text{...(13)}$$

The frequency generated in this way is slightly less than that of resonant frequency of the tuned circuit.

(B) Tuned Anode Oscillator. Fig 4.5 shows the circuit diagram of the tuned anode oscillator. It is essentially *a self grid excitation tuned class C amplifier*. It is a very common type of feedback oscillator. The +ve feedback in the circuit is possible due to the mutual inductance M between the antiresonant circuit of plate and coils of grid circuit. The grid leak resistor R_g and grid capacitor C_g combination is for the self excitation and the proper operatory grid bias.

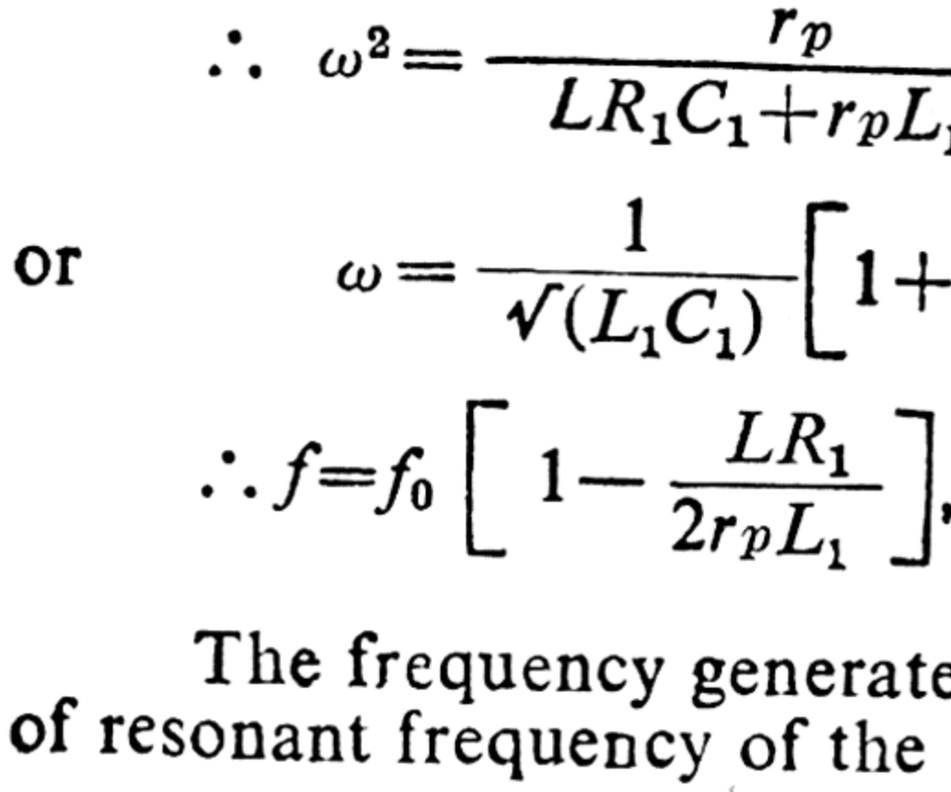


Fig. 4.5.

When the plate supply (E_{bb}) is on, the plate current through L increases slowly and an induced voltage ($M\ di_L/dt$) is established between grid and cathode. The polarity (due to winding, etc.) of

the two coils is such that with the increase in current through inductor (i_L), this induced voltage between grid and cathode also increases till the current reaches to its saturation value. There will then be no induced voltage. To attain normal value, the current i_L will increase and the induced voltage $M\, di_L/dt$ will become negative and the grid cathode voltage will go on decreasing unless M (di_L/dt) is again zero. The plate current will again increase to have normal value and increase upto saturation point. This type of cycle is repeated, with the frequency depending upon the inductance and capacitance values of the tank circuit.

Let an *ac* voltage e_s is applied at the input of the oscillator. This will be amplified by the tube and a current i_L will flow through inductance L. Due to mutual inductance M, an induced voltage $j\omega M i_L$ will be induced between grid and cathode of the tube. If this voltage is in phase and greater (even equal) to the applied voltage e_s, sustained oscillations will be maintained in the circuit. Since the tube makes a phase change of 180° and further the coil winding (similar to transformer) makes another 180° phase change hence the feedback will be applied in the same phase as the original voltage e_s. These are the conditions of positive feedback in a circuit.

The necessary conditions are derived by considering that the tube parameters μ, r_p and g_m are constants, there is no grid current and the wave form is sinusoidal. Fig. 4.6 shows the equivalent circuit diagram of the Fig. 4.5.

For the plate circuit

$$\mu e_s = r_p i + i_L(R + j\omega L) \quad \ldots(14)$$

For the tank circuit

$$i_C/j\omega C = i_L(R + j\omega L)$$

or $$i_C = i_L\, j\omega C(R + j\omega L).$$

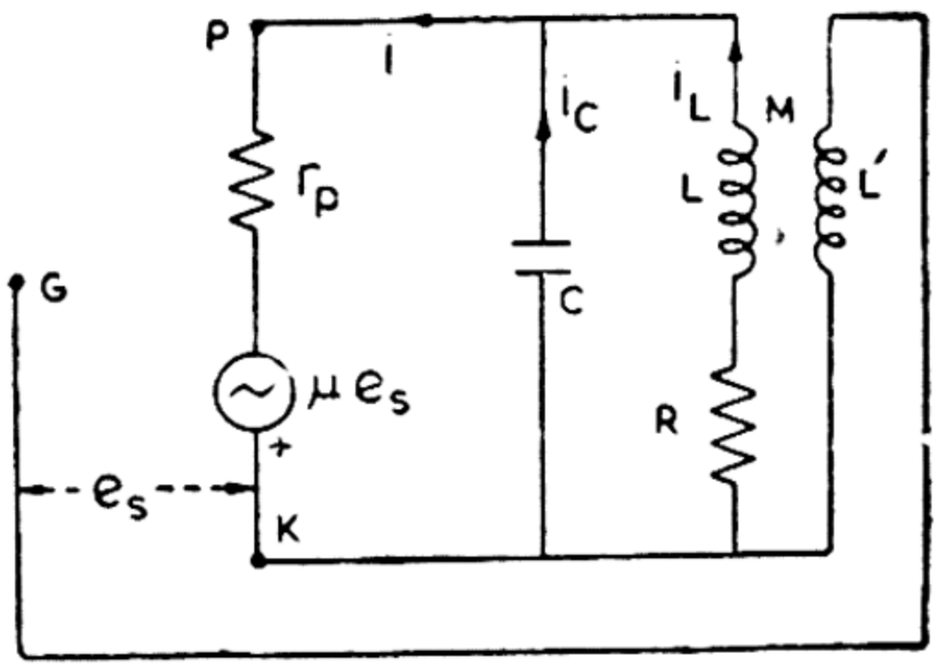


Fig. 4.6.

Voltage between grid and cathode

$$e_s = j\omega M i_L. \quad \ldots(15)$$

The total current in plate circuit

$$i = i_C + i_L.$$

$$\therefore\ \mu j\omega M i_L = r_p(i_L + i_C) + i_L(R + j\omega L)$$

$$= r_p i_C + i_L(r_p + R + j\omega L)$$

$$= r_p i_L j\omega C(R + j\omega L) + i_L(r_p + R + j\omega L)$$

or $$j\omega M\mu = j\omega C r_p(R + j\omega L) + (r_p + R + j\omega L). \quad \ldots(16)$$

Equating the real parts of this equation, we get

$$-\omega^2 LCr_p + r_p + R = 0 \text{ or } \omega^2 = (r_p + R)/r_p LC = (1 + R/r_p)/LC$$

$$\therefore\ \omega = \frac{1}{\sqrt{(LC)}}\left(1 + \frac{R}{r_p}\right)^{1/2}. \quad \ldots(17)$$

If the resonant frequency of tank circuit is defined as $\omega_0 = 1/\sqrt{(LC)}$, then for $R \ll r_p$ equation (17) can be written as

$$\omega = \omega_0(1 + R/2r_p) \text{ or } f = f_0(1 + R/2r_p). \quad \text{...(18)}$$

So the frequency of oscillator is slightly higher than the resonant frequency by a factor $(1 + R/2r_p)$. It is not generally desirable to change the frequency by changing R. It is also obvious from this equation that the tube plays insignificant role in determining the frequency of oscillator. Now by equating the imaginary parts of Eq. (16), we have

$$\omega M\mu = \omega C r_p R + \omega L,$$

or

$$M = (L + CRr_p)/\mu. \quad \text{...(19)}$$

This gives the minimum value of the mutual inductance between the plate circuit coil (tuned circuit) and grid circuit coil for oscillations.

$$M = L/\mu + CRr_p/\mu \text{ or } M - L/\mu = CR/g_m$$

or

$$g_m = \mu CR/(M\mu - L). \quad \text{...(20)}$$

This equation gives the value of g_m for sustained oscillations. If the value is less than this, the oscillations will die out.

(C) Hartley Oscillator. Hartley oscillator is one of the simplest types of oscillator circuit. The amplified energy in the plate circuit is fed back to the grid circuit by means of inductive coupling. Only a single coil is used, which is tapped at the middle. A portion L_1 of it is in plate circuit and L_2 is in grid circuit. The amount of feedback depends on the number of turns in these portions (L_2 and L_1). The alternating current in the plate section of the coil induces a voltage in the grid section. This voltage is amplified by the tube and is applied again to the plate section. As the plate and grid voltages are taken from opposite ends of the coil with respect to the common lead, hence are 180 degrees out of phase with each other.

Although the circuit arrangement can be of two types, (*i*) *series fed* and (*ii*) *parallel fed.* In order to avoid the danger of high plate vloltages appearing across tuning capacitor and the coil, the *paralle fed circuit* is generally used. In the parallel fed harmonic oscillator (Fig. 4.7) the plate circuit is divided intot two parallel branches, one to provide a path for dc and other for ac. The *rf* choke keeps the ac out of the dc circuit, and the blocking capacitor *Cb* keeps the dc out of the ac-circuit. The reactance of *Cb* at resonant frequency should be small in comparison with the reactance of the choke coil at the same frequency. It will allow most of the *rf* current through

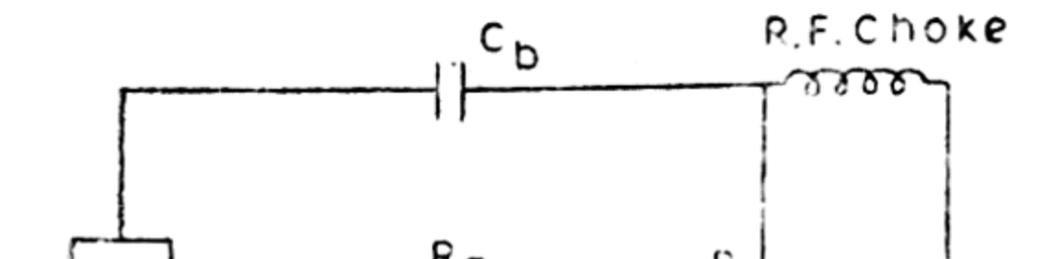


Fig. 4.7.

the plate section L_1. The frequency of oscillations is the resonance frequency of the parallel circuit formed by C and coil (L_1+L_2).

Neglecting the effects of Cb and *rf* choke, the equivalent circuit will be same as shown in Fig. 4.8. Applying the Kirchhoff's second law to meshes 1, 2, and 3.

$$\mu es = i_1(R_1+j\omega L_1)+j\omega M i_2+ipr_p$$
$$es = i_2(R_2+j\omega L_2)+j\omega M i_1$$
$$0 = i_1(R_1+j\omega L_1)+i_2(R_2+j\omega L_2)+j\omega M(i_1+i_2)+i_2/j\omega C,$$

where i_1 and i_2 be the currents in the L_1 and L_2 respectively and the plate current $i_p=i_1-i_2$. Solving these equations, we get

$$\frac{R_1+r_p+j\omega(L_1-\mu M)}{\mu R_2+r_p+j\omega(\mu L_2-M)} = -\frac{R_1+j\omega(L_1+M))}{R_2+j\omega(L_2+M)-j/\omega C}. \quad ...(21)$$

After solving and equating real parts

$$\mu = \frac{L_1+M}{L_2+M}+\frac{Cr_p(R_1+R_2)(L_1+L_2+2M}{(L_1+M)(L_2+M)}.$$

If the resistance of the coil is negligible

$$\mu = (L_1+M)/(L_2+M). \quad ...(22)$$

The circuit can thus be adjusted for oscillations by altering the tapping point on the coil. On equating imaginary parts of Eq. (21), we get

$$\omega^2[L_1R_2(1+\mu)+R_1L_2(1+\mu)+r_p(L_1+L_2+2M)] = (R_1+r_p)/C.$$

First two terms can be neglected in comparison to the third term on the LHS, hence

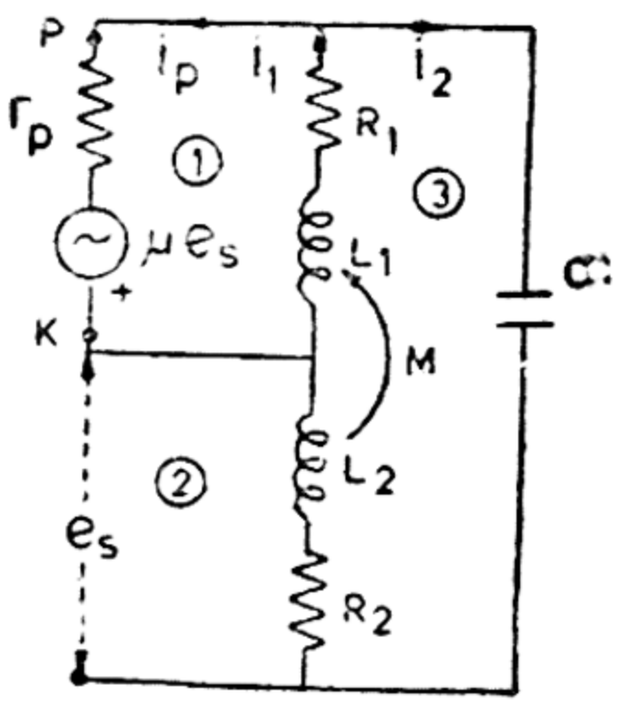


Fig. 4.8.

$$\omega^2 = (R_1+r_p)/Cr_p(L_1+L_2+2M).$$

or $$\omega = \left(\frac{R_1+r_p}{r_p}\right)^{1/2}\left[\frac{1}{C(L_1+L_2+2M)}\right]^{1/2} = \left(\frac{R_1+r_p}{r_p}\right)^{1/2}\frac{1}{(LC)^{1/2}}$$

where $L=L_1+L_2+2M$.

As resonant frequency $f_r = 1/2\pi\sqrt{(LC)}$

$$\therefore \quad f = f_r\,(1+R_1/r_p)^{1/2}. \quad ...(23)$$

The frequency generated in this way is slightly greater than that of resonant frequency of the tuned circuit.

(D) Colpitt's Oscillator. It is similar to that of Hartley oscillator, except that the cathode is connected to a tapping point on the capacitor instead of the coil.

In this oscillator, a grid leak resistor R_g is used to provide a conduction path between the grid and cathode, and a low reactance capacitor C to block the direct potential of anode from reaching the grid. C_1 and C_2 are the plate and grid tank capacitors, as shown in Fig. 4.9.

As the plate current starts to flow, the blocking condenser C starts to charge, which in turn charges the plate tank capacitor C_1. This produces an effect to the grid tank capacitor C_2 through the *r.f.* choke, causing the terminal connected to the grid to become more negative. This negative feedback will reduce the plate current and thus the energy stored in the plate tank capacitor. In this way the direction of the feed back voltage is reversed. This process continues and the sustained oscillations are produced.

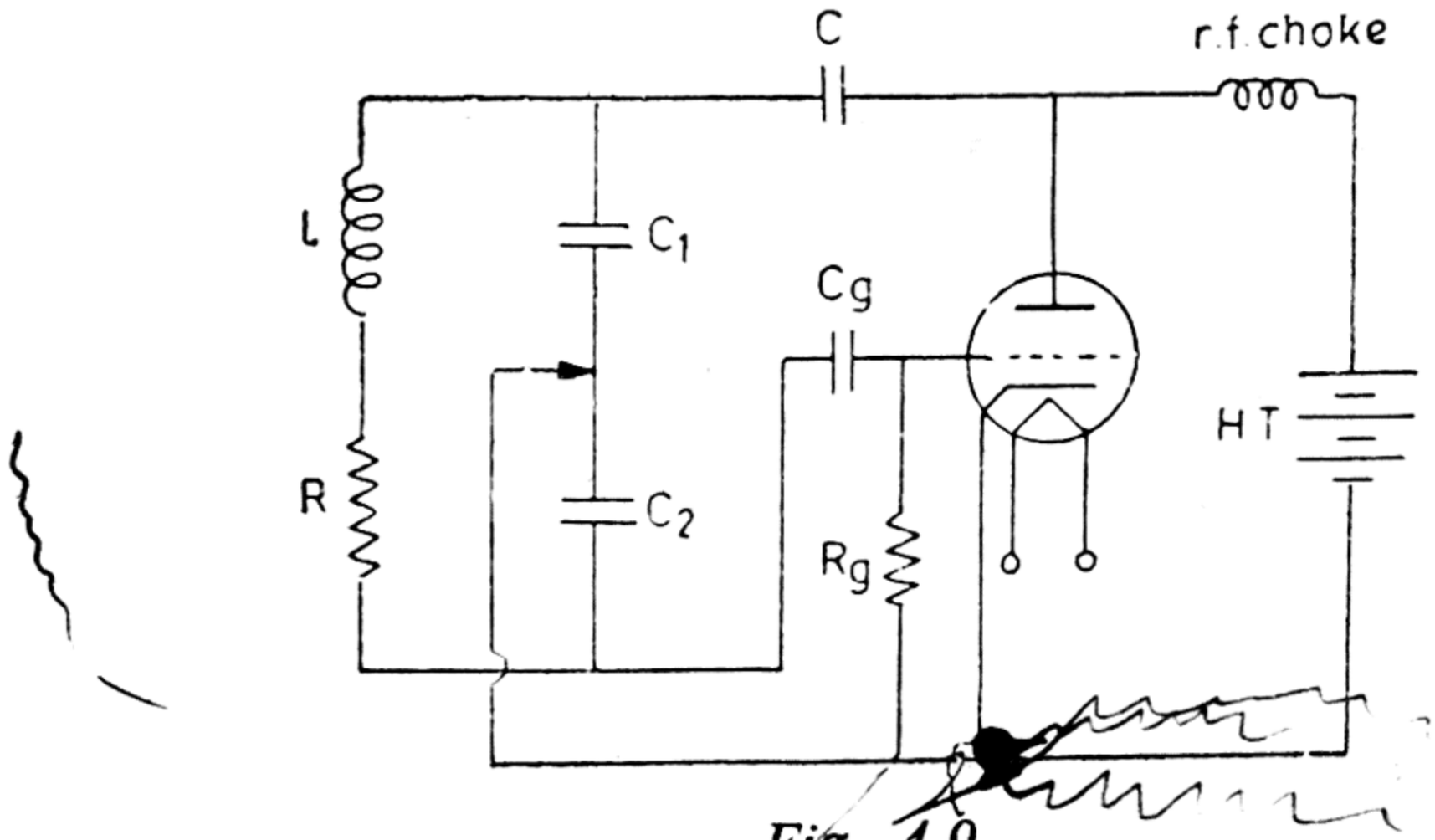


Fig. 4.9.

If the effect of *r.f.* choke, C, C_g and R_g are neglected, the equivalent circuit of Fig. 4.9 will be of the form as shown in Fig. 4.10. Application of Kirchhoff's second law to the meshes (1), (2) & (3) respectively gives

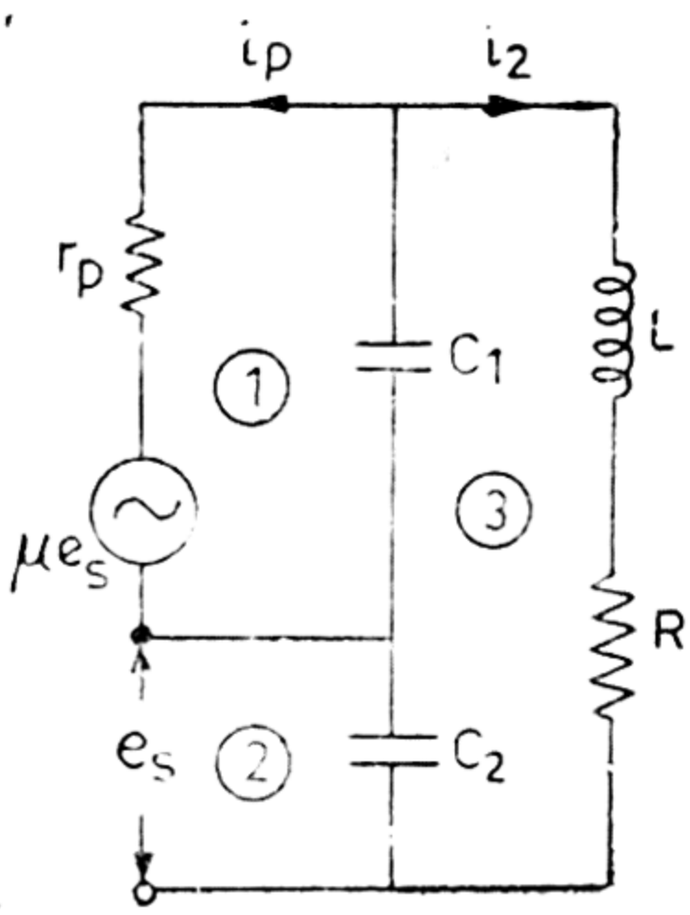


Fig. 4.10.

$$\mu e_s = i_p r_p + i_1/j\omega C_1 \quad \ldots(24)$$

$$e_s = i_2/j\omega C_2 \quad \ldots(25)$$

and $$0 = i_1/j\omega C_1 + i_2\,(R + j\omega L + 1/j\omega C_2) \quad \ldots(26)$$

Since $i_p = i_1 - i_2$. $\quad \ldots(27)$

The substitution of the value of e_s from Eq. (25) and i_p from Eq. (27) in Eq. (24) we get

$$\mu(i_2/j\omega C_2) = (i_1 - i_2)r_p + i_1/j\omega C_1$$

or $$\frac{i_2}{i_1} = \frac{r_p + 1/j\omega C_1}{r_p + \mu/j\omega C_2}. \quad \ldots(28)$$

From equation (26), we get

$$\frac{i_2}{i_1} = -\frac{1}{j\omega C_1\,(R + j\omega L + 1/j\omega C_2)}. \quad \ldots(29)$$

Equations (28) and (29) give

$$\frac{r_p+1/j\omega C_1}{r_p+\mu/j\omega C_2}=\frac{-1}{j\omega C_1(R+j\omega L+1/j\omega C_2)}$$

or $$(j\omega C_1 r_p+1)(R+j\omega L+1/j\omega C_2)+(r_p+\mu/j\omega C_2)=0 \qquad \ldots(30)$$

Equating the imaginary parts, we get

$$-r_p\omega^3 C_1C_2L+r_p\omega C_1+\omega C_2R+r_p\omega C_2=0$$

or $$\omega^2=\frac{1}{C_2L}+\frac{R}{r_pC_1L}+\frac{1}{C_1L}=\frac{C_1+C_2}{C_1C_2L}+\frac{R}{r_pC_1L}$$

$$=\frac{1}{C_1L}\left[\frac{C_1+C_2}{C_2}\right]\left[1+\left(\frac{C_2}{C_1+C_2}\right)\frac{R}{r_p}\right]$$

$\therefore$ $$\omega=\sqrt{\frac{C_1+C_2}{C_1C_2L}}\sqrt{1+\left(\frac{C_2}{C_1+C_2}\right)\frac{R}{r_p}}$$

or $$f=f_0\left[1+\left(\frac{C_2}{C_1+C_2}\right)\frac{R}{r_p}\right]^{1/2}, \qquad \ldots(31)$$

where $f_0=[(C_1+C_2)/C_1C_2L]^{1/2}/2\pi$.

Equating the real parts, we get

$$-Rr_p\omega^2C_1C_2-\omega^2LC_2+1+\mu=0.$$

or $$1+\mu=\omega^2[Rr_pC_1C_2+C_2L]$$

$$=\left[(C_1+C_2)\frac{Rr_p}{L}+\frac{C_1+C_2}{C_1}\right]\left[1+\left(\frac{C_2}{C_1+C_2}\right)\frac{R}{r_p}\right]$$

$$=(C_1+C_2)\left(\frac{Rr_p}{L}+\frac{1}{C_1}\right).$$

($\because$ $C_2R/(C_1+C_2)r_p<<1$)

or $$\mu=\frac{R\,r_p}{L}(C_1+C_2)+\frac{C_2}{C_1}. \qquad \ldots(32)$$

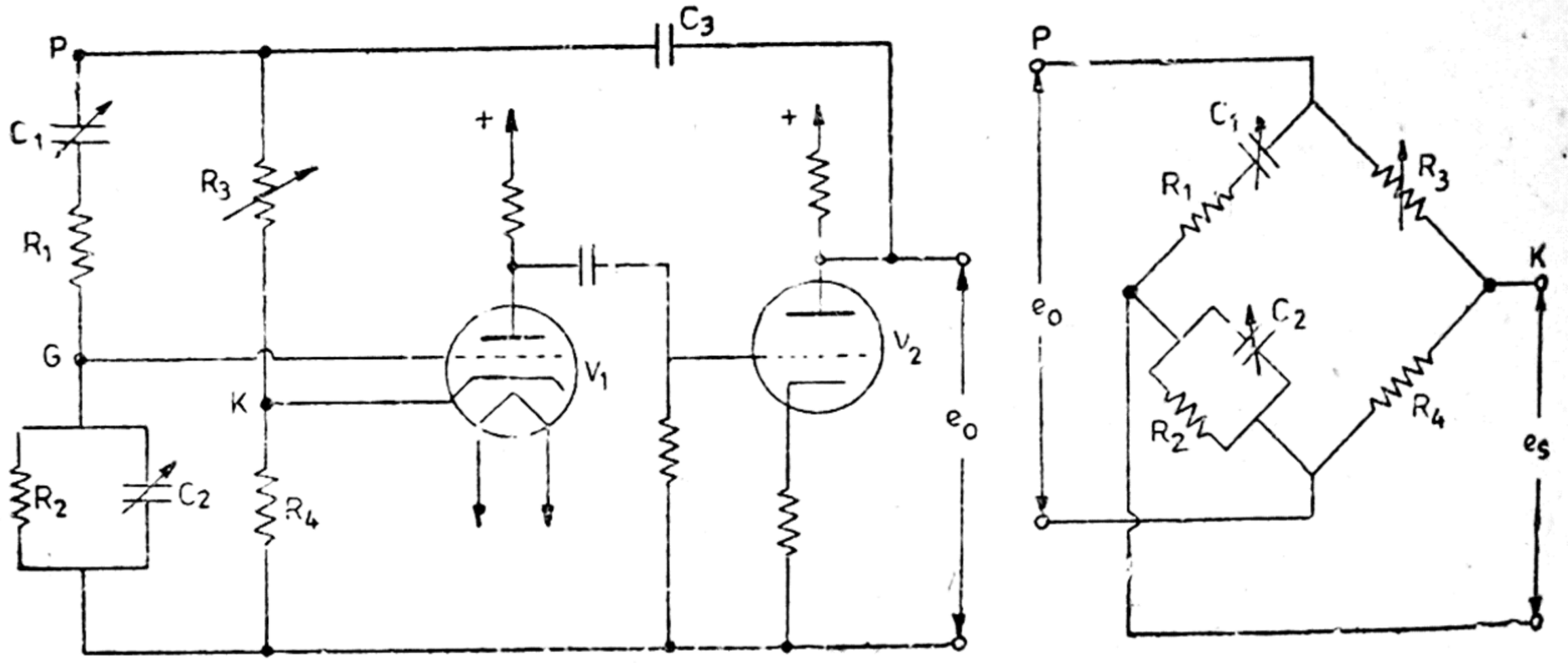
This is the condition for the circuit to be oscillatory. This can be obtained by the alteration of the tapping point on the capacitor.

(E) Wien Bridge Oscillator (RC-oscillator). Instead of *LC*-circuit, it is also possible to have an oscillator with *RC*-network, which alone provides the requisite phase change. Such circuits are called *RC-oscillators.* These have an advantage that they can be used for very low frequencies. The *RC*-oscillators are both sinusoidal and non-sinusoidal. The sinusoidal oscillators are of two types : (*i*) *phase shift oscillator* and (*ii*) *Wien bridge oscillator.* The non-sinusoidal oscillators are known as relaxation oscillators.

Wien bridge oscillator is a two-stage *RC*-coupled amplifier, in which a fraction of the output voltage is fed back to the input and the oscillations are produced when this feedback voltage is equal in magnitude and phase with the initial voltage at the input of the

first stage. The circuit diagram of this oscillator is shown in Fig. 4.11. Because of the form of the coupling network, this *RC*-oscillator is known as a Wien bridge oscillator.

In this circuit the output voltage e_0 of the two-stage *RC* amplifier is fed to the coupling network and its output *es* is fed as



(*a*) (*b*)

Fig. 4.11.

the input of the first valve V_1. The coupling network is composed of C_1 R_1 in series and C_2R_2 in parallel. The resistances R_3 and R_4 are used to stabilize the amplitude of the output. For a given R_4 and valve V_1, the bridge is balanced by adjusting R_3. At this condition, the grid-cathode potential (*es*) is zero and there will be no feedback. The system will thus not oscillate at the balance point. At the balance position, the condition for balance is given by

$$\frac{R_3}{R_4} = \frac{R_1+1/j\omega C_1}{R_2/(1+j\omega C_2R_2)} = \frac{(j\omega R_1C_1+1)}{j\omega C_1} \times \frac{1+j\omega C_2R_2}{R_2}$$

$$= \frac{R_1}{R_2} + \frac{C_2}{C_1} - j\left(\frac{1-\omega^2C_1C_2R_1R_2}{\omega C_1R_2}\right). \quad \text{...(33)}$$

Equating real and imaginary parts, we get

$$\frac{R_3}{R_4} = \frac{R_1}{R_2} + \frac{C_2}{C_1} \quad \text{and} \quad 1-\omega^2C_1C_2R_1R_2=0. \quad \text{...(34)}$$

If $R_1 = R_2 = R$ (say) and $C_1 = C_2 = C$ (say), then we have

$$R_3/R_4 = 2 \text{ and } \omega = \omega_0 = 1/RC. \quad \text{...(35)}$$

Under these conditions, the circuit will act as a frequency selective feedback rejection amplifier. The circuit may work as an oscillator if R_3/R_4 is chosen to be greater than 2.

The continuous variation of frequency of this oscillator is done with the help of the two variable capacitors C_1 and C_2. To stabilize the amplitude against range switching, the circuit is modified by replacing R_4 by a tungsten-filament lamp. The thermal time constant of the lamp must be large compared with the period of the oscillations, otherwise the amplitude control will not be adeq uate.

(F) Crystal Oscillator. In the crystals such as quartz tourmaline, etc., if mechanical stresses are applied on two opposite faces, the electrical charges appear on the pair of faces perpendicular to the former faces. If the stresses are changed from compression to tension, the electrical polarity changes sign. Conversely if the electrical charges are placed on two opposite faces of the crystal or the potential difference is applied across two opposite faces of the crystal the mechanical stress is produced and the result is the compression or extension across the other pair of opposite faces. This effect is known as *Piezo electric effect* and the crystal as *piezo-electric crystal.* Such crystals are utilized in *crystal oscillators*, which are frequently used for measurement of time to high degrees of precision.

The natural form of a quartz crystal is hexagonal in cross-section. It has pointed ends, the line joining these ends is called the *z*-axis or *optic axis.* In the hexagonal cross section there are three axes passing through its opposite corners. These axes are called *x*-axis or *electrical axes.* There are three axes called *y*-axis or mechanical axes perpendicular to these axes as shown in Fig. 4.12. We obtain slicing sections from the crystal.

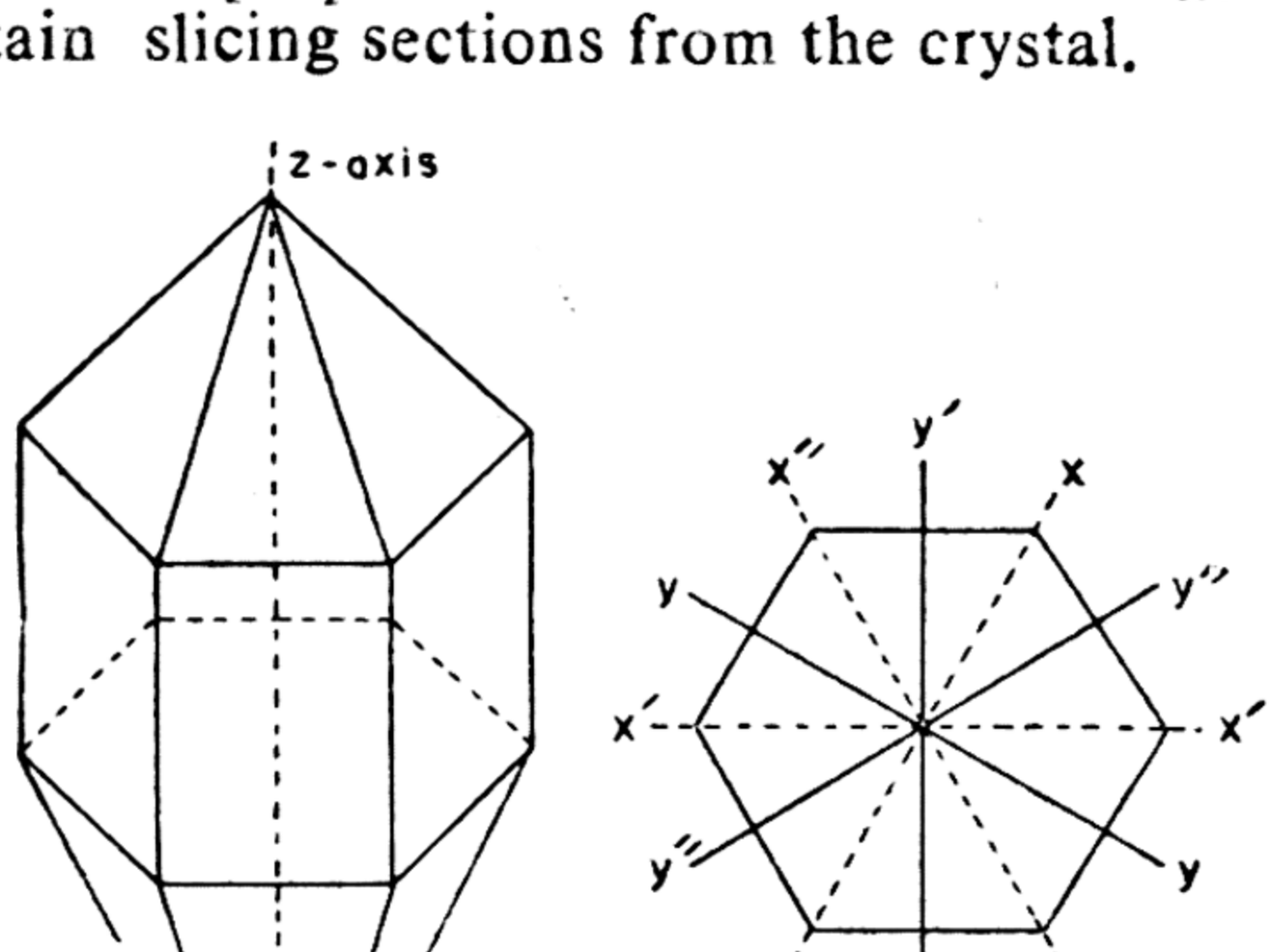


Fig. 4.12.

X-cut Crystals. If the crystal is cut by a plane parallel to *y*-axis and perpendicular to *x* axis. Its faces are either silvered or tin foiled to ensure good electrical contact. The field is applied along the *x*-axis and the crystal vibrates along the *y* axis. If the frequency of the applied emf is same as that the natural frequency of the crystal, the mechanical resonance will take place.

Similarly *Y*-cut crystal is obtained by a plane parallel to *x*-axis and perpendicular to *y*-axis. *X*— and *Y*— cuts are not used in the modern oscillators because of their large temperature coefficients. It is either positive or negative depending on the crystal

cut. A zero temperature coefficient crystal called *GT* cut may be obtained by proper orientation of the axes at which the crystal cut is made. The *GT* cut has constant frequency over the temperature range 0°—100°*C*.

Crystal Oscillator Circuit. The crystal is mounted horizontally by clamping between two metal plates or by two electrodes electroplated on its side faces. The mechanical vibrations in the crystal are analogous to the electrical vibrations in *L-C* circuit. The vibrating crystal may thus be replaced by an *equivalent electrical resonance circuit* (Fig. 4.13). The capacitance C_s represents the shunting capacitance which actually arises between two metal plates. Parameters L, C and R represent values analogous to the mechanical properties of the crystal, the mass, the elasticity and the damping respectively. The ratio of mass to elasticity is very high and is equivalent to a high L/C ratio. The mass to damping is also very high and is equivalent to a high Q value. The inductance of the quartz crystal is very high. It depends upon the cut and dimensions of the crystal slice. The crystal has relatively low value of R. The capacitances C and C_s are low and offer high impedances at low frequencies. At some frequency, known as series resonant frequency, the reactances offered by L and C are equal. If the frequency ω is increased, there is some frequency at which the combined reactance of L and C is inductive and is equal to the reactance offered by C_s. At this frequency the crystal acts as a *parallel resonant circuit* and the crystal vibrations are maximum.

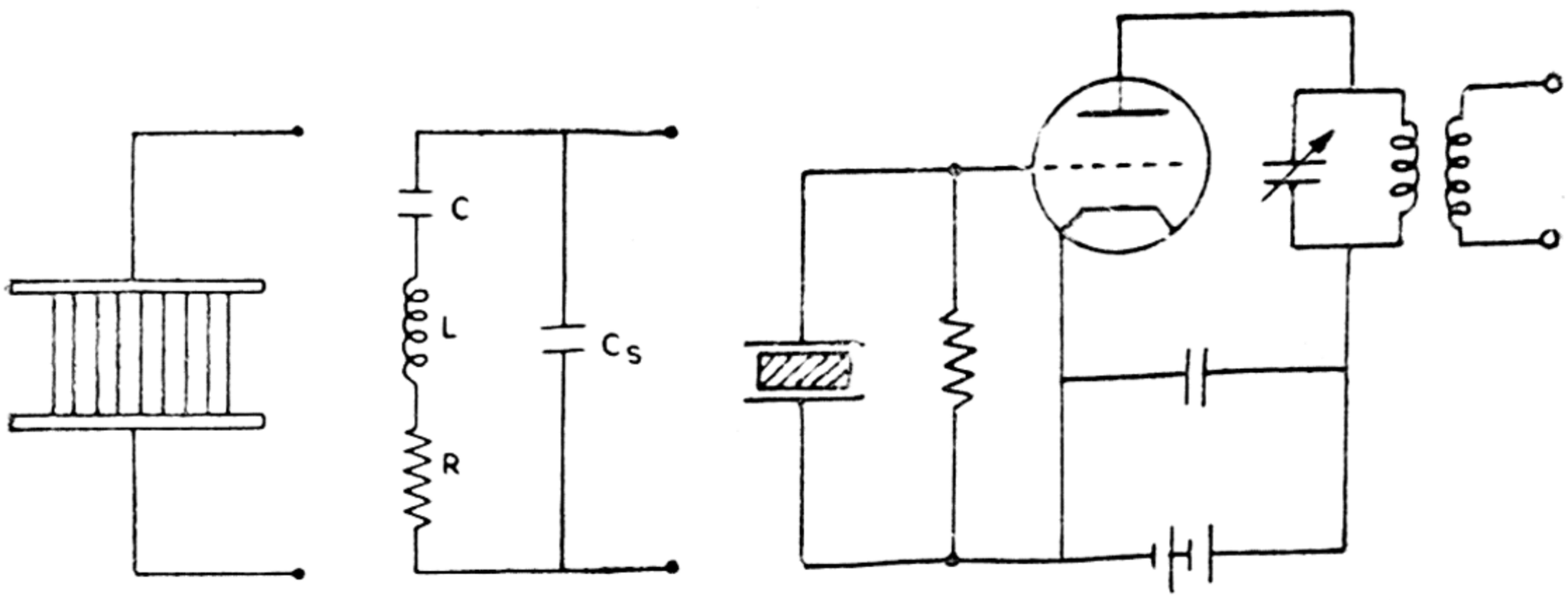


Fig. 4.13.

In a crystal oscillator quartz crystal is placed in place of resonant circuit. Several circuit forms are used. In a tuned plate crystal oscillator the crystal replaces the grid circuit from the tuned grid-tuned plate oscillator. The feed back takes place from the plate into the grid circuit through the internal plate-grid inter-electrode capacitance C_{pg} of the tube. This causes the crystal to vibratee These vibrations induce voltage on the faces of the crystal. This acts on the grid of the tube and controls the energy released in the plate circuit. The oscillation can occur only when the Miller effect resistance in the input circuit is negative, which is possible when the anode circuit is tuned on the inductive side of resonance.

(G) Relaxation Oscillators. The oscillators so far discussed are sinusoidal oscillators. The output signal was a sine wave in

these oscillators. There is another class of oscillators which produce non-sinusoidal waves, *e.g.*, triangular, rectangular wave, etc. These are called *relaxation oscillators*. Among the important relaxation oscillators are the *multi-vibrator* and the *blocking-oscillator*. The former provides a sensibly square wave in the output and the latter provides relatively narrow pulses.

The multi-vibrator can be considered as a two stage *RC*-coupled oscillator, with the output of the second stage coupled to the first tube. Depending upon the coupling, these are named as plate coupled or cathode coupled multi-vibrators. In this device the output of the second stage is in the phase with the input signal on the grid of the first stage. Thus the positive feedback takes place which causes oscillations.

In the blocking oscillator, the second stage of the multi-vibrator is replaced by a transformer, which is so connected that regenerative feedback results.

4.3. MODULATION

The need for modulation arose because of the human desire of sending usual audio or informational signals to long distances. These signals are not suited to direct radiation as intensity decreases with distance rapidly. High frequency level is, therefore, used for transmission as, (*i*) *transmission by radiation is possible and* (*ii*) *a number of messages having different energy levels can be transmitted simultaneously without interference*. For this message is superposed over a high frequency *rf*-voltage.

Modulation is the process by which some characteristics, *viz.*, amplitude, frequency or phase, of a high frequency wave is modified in accordance with the instantaneous value of some another wave, called *modulating wave*. The high frequency wave is known as '*carrier wave*'. It modifies according to the signal (*modulating wave*) and *modulated wave* is obtained by superposition. Usually the modulating (signal) frequency is considerably lower than the carrier frequency. Let the carrier voltage at any instant t is represented by the expression.

$$e_c = E_c \cos(\omega_c t + \theta), \qquad \ldots(36)$$

where E_c is the peak amplitude of the carrier voltage, ω_c $(=2\pi f_c)$ is angular frequency of carrier wave and θ is phase angle.

The modulation process is possible in following three ways :

(*i*) When the amplitude E_c is modified in accordance with the modulating wave, keeping θ and ω_c as constant, this process is known as *amplitude modulation*. It is most frequently adopted and will be discussed here.

(*ii*) When ω_c is modified in accordance with the modulating wave keeping E_c and θ as constant, this process is known as *frequency modulation*.

(*iii*) When θ is modified in accordance with the modulating wave keeping E_c and ω_c as constant, this process is known as *phase modulation*.

4.4. AMPLITUDE MODULATION

In amplitude modulation, the amplitude of the carrier voltage is modified in accordance with the instantaneous value of the modulating voltage. Let the modulating voltage (*signal*) be represented as

$$e_m = E_m \cos \omega_m t, \qquad \ldots(37)$$

where E_m is the amplitude of the signal of frequency $\omega_m (= 2\pi f_m)$. Let the carrier voltage (*rf*-wave) be expressed as

$$e_c = E_c \cos \omega c t. \qquad \ldots(38)$$

For convenience of calculation, phase angle θ in Eq. (36) is taken as zero, as it plays no part in this modulation process. The carrier frequency f_c is usually much greater than the signal frequency f_m and is chosen as larger as required for the transmission of the signal. The carrier wave amplitude modulated by a single signal component f_m can be expressed as

$$e = (E_c + K_a E_m \cos \omega_m t) \cos \omega c t, \qquad \ldots(39)$$

where K_a is proportionality factor, which determines the maximum variation in amplitude for a given signal voltage E_n and $K_a E_m \cos \omega_m t$ shows the change in carrier voltage amplitude.

$$\begin{aligned} e &= E_c \cos \omega c t = E_c [1 + (K_a E_m / E_c) \cos \omega_m t\ ] \cos \omega c t \\ &= E_c [1 + m_a \cos \omega_m t] \cos \omega c t = E_c \cos \omega c t + m_a E_c \cos \omega_m t \cos \omega c t \\ &= E_c \cos \omega c t + \tfrac{1}{2} m_a E_c \cos (\omega_c + \omega_m) t + \tfrac{1}{2} m_a E_c \cos (\omega_c - \omega_m) t, \qquad \ldots(40) \end{aligned}$$

where $m_a = K_a E_m / E_c$ is called *modulation index* or modulation factor or degree of modulation. So the modulation % will be $100 \times m_a$.

The Eq. (40) consists of three terms : (*i*) original carrier frequency term $E_c \cos \omega c t$, representing the carrier voltage, (*ii*) upper frequency term as $\frac{1}{2} m_a E_c \cos (\omega_c + \omega_m) t$ and (*iii*) lower frequency term $\frac{1}{2} m_a E_c \cos (\omega_c - \omega_m) t$. These upper and lower frequency terms represent respectively the *upper side and lower side bands*. These are located on either side of the carrier at a frequency interval of ω_m in their frequency spectrum. These are of same magnitude $\frac{1}{2} m_a E_c$. The carrier voltage component transmits no information while these side bands give complete information.

Wave form of Amplitude Modulated Voltage. Fig. 4.14(*a*) shows the waveform of unmodulated carrier voltage of radio frequency. Fig. 4.14(*b*) shows the waveform of sinusoidal modulating voltage or modulating signal of audio frequency. Fig. 4.14 (*c*) shows the waveform of amplitude modulated carrier voltage and that the frequency of the carrier remains unaltered but its amplitude varies according to the variation of modulating voltage. It is clear from Fig. 4.14(*c*) that

$$m_a E_c = K_a E_m = E_{max} - E_c,$$

or $$m_a=(E_{max}-E_c)/E_c. \quad \ldots(41)$$

Also $$m_a=(E_c-E_{min})/E_c. \quad \ldots(42)$$

Hence $$m_a=\frac{E_{max}-E_{min}}{E_{max}+E_{min}}. \quad \ldots(43)$$

$$\therefore \text{ Percentage of modulation}=\frac{E_{max}-E_{min}}{E_{max}+E_{min}}\times 100. \quad \ldots(44)$$

We have following cases of modulated waves :

(*i*) When $m_a=0$, Eq. (40) will show the pure carrier wave:

(*ii*) When $m_a=0.5$, percentage of modulation is 50%, *i.e.*, the extent to which carrier wave is being modulated is equal to the half of its amplitude.

(*iii*) When $m_a=1$, percentage of modulation is 100%, *i.e.*, the extent to which carrier wave is being modulated is equal to its amplitude or at the negative peak of e_m the e_c reduces to zero.

(*iv*) When $m_a>1$, the carrier wave is overmodulated, the Eq. (40) is not valid for such cases. Near the negative peaks of e_m signal the modulated wave will be absent which creates distortions.

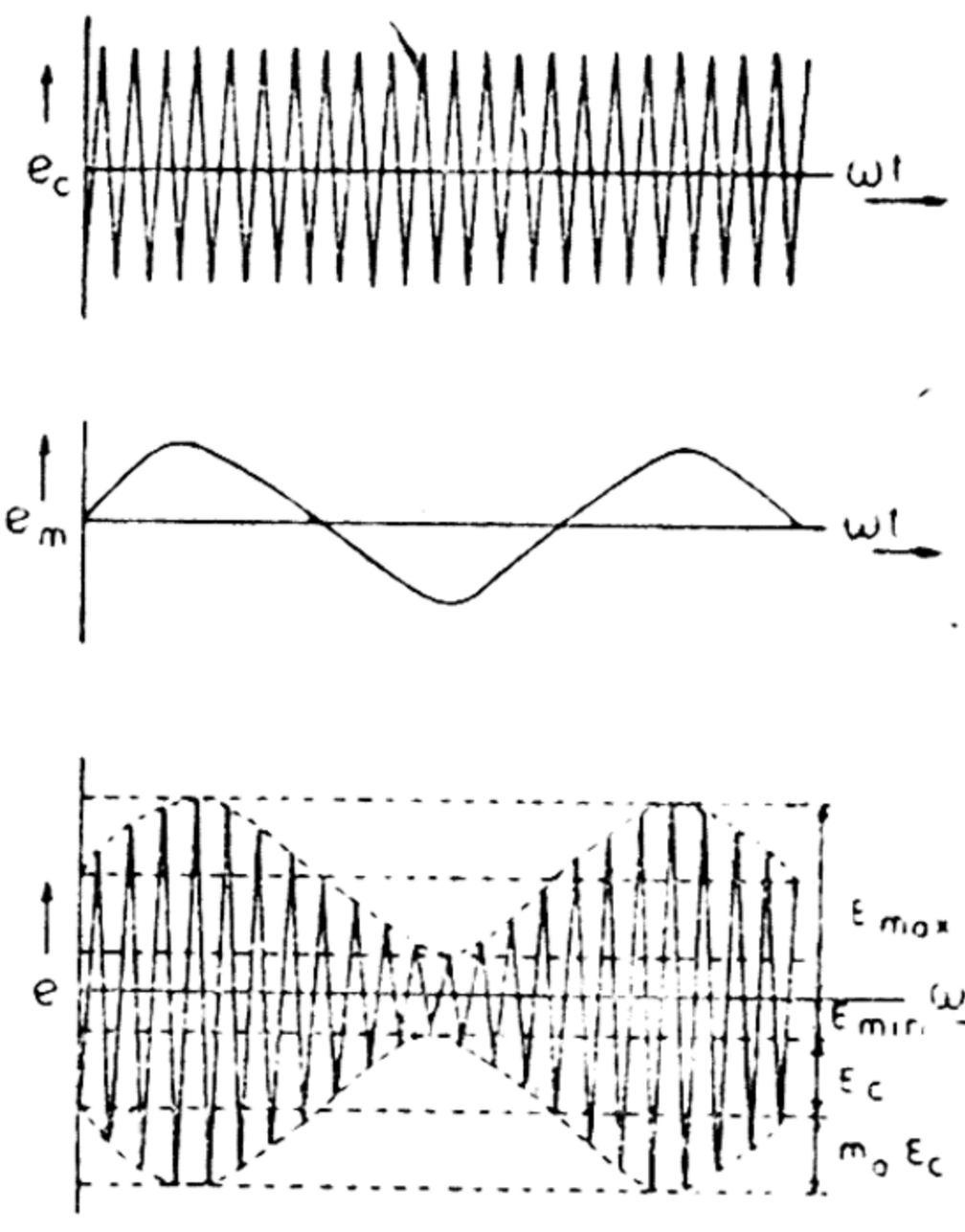


Fig. 4.14.

Energy Distribution. Eq. (40) shows three components of the modulated wave. These components have their *rms* values as $E_c/\sqrt{2}$, $m_aE_c/2\sqrt{2}$ and $m_aE_c/2\sqrt{2}$ corresponding to carrier component, upper side band and lower side band respectively.

$$\therefore \quad P_{av}=\frac{E_c^2}{2R}+\frac{m_a^2E_c^2}{8R}+\frac{m_a^2E_c^2}{8R}=\frac{E_c^2}{2R}\left[1+\frac{m_a^2}{2}\right] \quad \ldots(45)$$

As m_a^2 is always +ve, hence the power dissipated by the modulated wave is always greater than the power dissipated by the carrier wave only.

$$\frac{\text{Power associated with side bands}}{\text{Total power contained}}=\frac{m_a^2E_c^2/4R}{E_c^2(1+\frac{1}{2}m_a^2)/2R}$$

$$=m_a^2/(2+m_a^2). \quad \ldots(46)$$

For 100% modulation, $m_a=1$ and

$$P_{side\ band}/P_{total}=1/3.$$

Hence for 100% modulation only 1/3 of the total energy of the modulated wave is contained in the two side bands and rest 2/3 part is with the carrier component.

Production of Amplitude Modulation. There are two main categories of methods of amplitude modulation :—

(*a*) *Linear Modulation Methods*. In these methods linear portion of the tube characteristic is used.

(*b*) *Square Law Modulation*. In these methods the square law region (curved portion) of the tube characteristic is used.

Let us now discuss only following two methods of the linear modulation type.

(A) Linear Series Plate Modulation. A simple form of plate modulation circuit, utilizing linear class *C* tuned amplifier as modulated amplifier, is shown in Fig. 4.15. It is named linear series plate modulator as the modulating voltage e_m appears in series with the plate dc supply voltage E_{bb} of tuned modulated amplifier. Thus the total effective plate supply e_{bb} ($= E_{bb} + e_m$) varies as the modulating voltage e_m and the peak tank circuit varies as e_{bb} and hence as e_m. In the grid circuit the carrier frequency voltage signal e_s from R.F. oscillator is applied. Grid bias voltage E_{cc} is used in series with the signal. Thus the potential between grid and cathode of the tube becomes $e_c = E_{cc} + e_s$. As E_{cc} is fixed, hence grid voltage e_c varies as the carrier voltage e_s. The plate to cathode voltage $e_b = e_{bb} - e_L$.

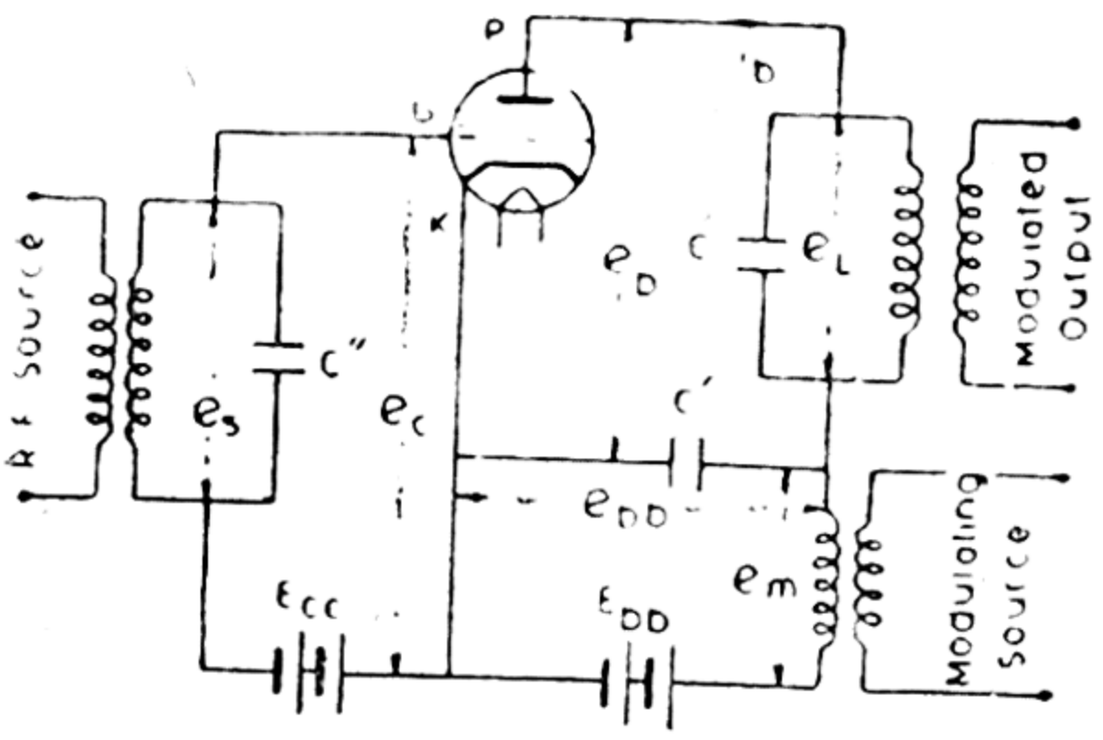


Fig. 4.15.

e_L, the instantaneous output voltage across the tank circuit,

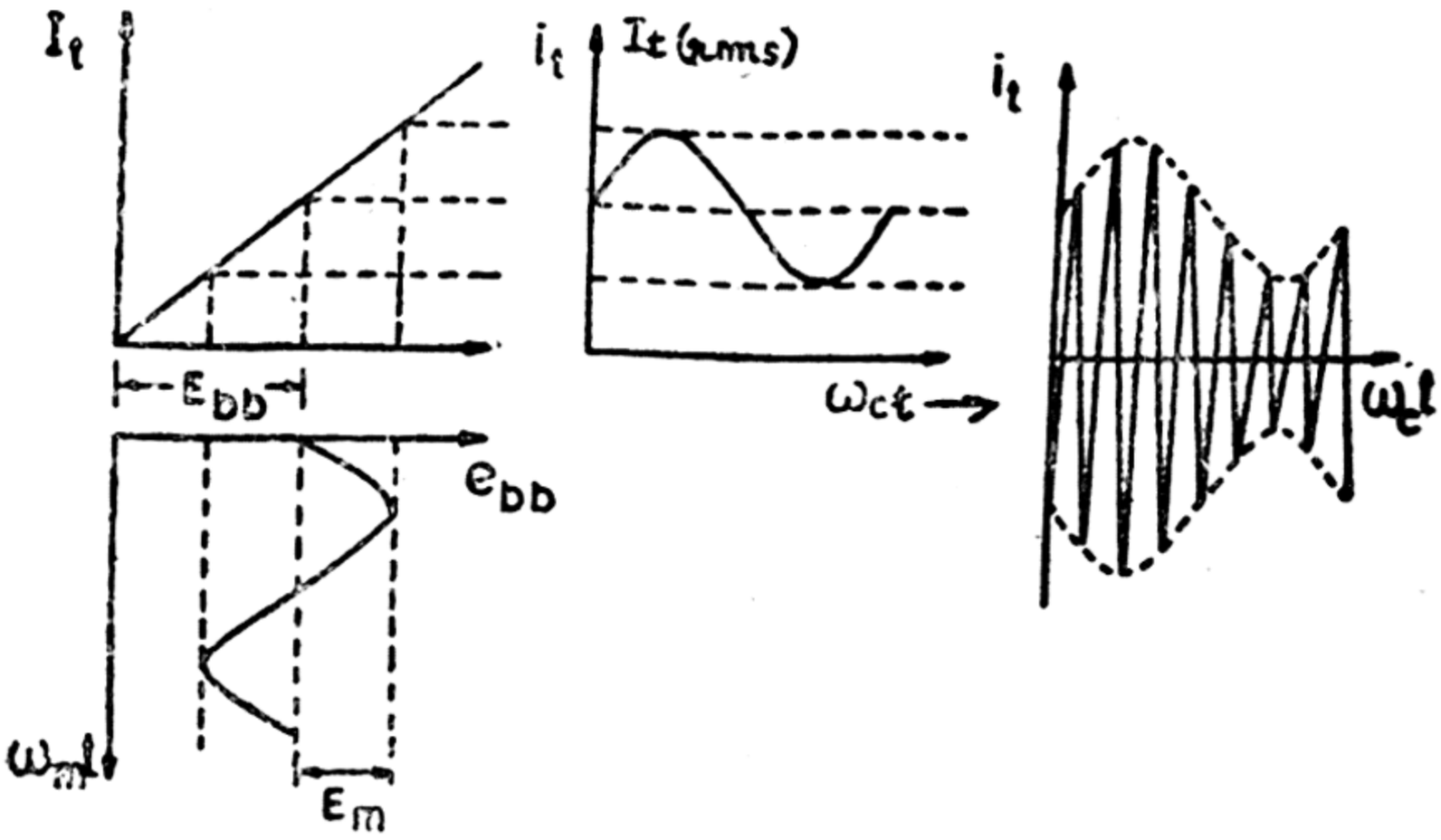


Fig. 4.16.

depends upon the modulating voltage e_m, carrier voltage e_s and the tube characteristic. When the modulating signal is having +ve half cycle the voltage which is already due to carrier signal across the tank circuit will increase and when it is having —ve half cycle, the voltage across the tank circuit will fall. Thus *rf* output voltage amplitude varies in accordance with the modulating signal. This results in linear plate modulation. The resulting wave form in the output is shown in Fig. 4.16.

To analyse the modulation process, we assume that $\omega_c \gg \omega_m$, the grid circuit potentials are

$$e_s = E_c \cos \omega_c t \text{ and } e_c = E_{cc} + E_c \cos \omega c t. \quad \ldots(47)$$

Modulating voltage $e_m = E_m \cos \omega_m t$. ...(48)

$$\therefore \quad e_{bb} = E_{bb} + E_m \cos \omega_m t = E_{bb}\,(1 + m_a \cos \omega_m t), \quad \ldots(49)$$

where $m_a = E_m / E_{bb}$.

Since rms tank current I_t and dc plate current I_b are related to the plate potential by linear relations as

$$I_t = K_t e_{bb}, \quad I_b = K_b e_{bb} \text{ and } I_{bb} = K_b E_{bb}. \quad \ldots(50)$$

Combining Eqs. (49) and (50), we get

$$I_t = K_t E_{bb}\,(1 + m_a \cos \omega_m t)$$

$$I_b = K_b E_{bb}\,(1 + m_a \cos \omega_m t) = I_{bb}\,(1 + m_a \cos \omega_m t).$$

Corresponding to the rms tank current the instantaneous value of tank current will be

$$i_t = \sqrt{2}\, I_t \sin \omega c t = \sqrt{2}\, K_t E_{bb}\,(1 + m_a \cos \omega_m t) \sin \omega_o t.$$

The rms value of output potential across the tank circuit

$$E_L = jX_L I_t = X_L I_t \underline{/90°}.$$

$\therefore$ Instantaneous value of output voltage $e_L = \sqrt{2} E_L \sin \omega c t$.

or $$e_L = \sqrt{2}\, X_L I_t \sin (\omega c t + 90°)$$

$$= \sqrt{2}\, X_L K_t E_{bb}\,(1 + m_a \cos \omega_m t) \cos \omega_c t. \quad \ldots(51)$$

At resonance $R_0 = L/RC$ and $X_L/R_0 = \omega_0 L/(L/CR) = \omega_0 CR = 1/Q$.

rms value of plate current at resonant frequency $I_p = E_L/R_0 = jX_L I_t/R_0$

$$= jI_t/Q = (I_t/Q)\underline{/90°},$$

where R_0 is the impedance of tank circuit at resonance,

$\therefore$ Instantaneous value of plate current $i_p = \sqrt{2}\, I_p \sin \omega_c t$

$$= \sqrt{2}\,(I_t/Q) \sin (\omega c t + 90°)$$

$$= \sqrt{2}\,(K_t E_{bb}/Q)\,(1 + m_a \cos \omega_m t) \cos \omega_o t. \quad \ldots(52)$$

The average power supplied by the dc power source

$$Pbb=\frac{1}{T}\int_0^T Ebb\ Ib\ dt=\frac{1}{T}\int_0^T Ebb\ Ibb\ (1+m_a\cos\omega_m t)\,dt$$
$$=EbbIbb=KbEbb^2. \qquad ...(53)$$

The power input by the modulating source

$$P_m=\frac{1}{T}\int_0^T e_mI_b\ dt=\frac{1}{T}\int_0^T E_m\cos\omega_m t\ Ibb(1+m_a\cos\omega_m t)\,dt$$
$$=\tfrac{1}{2}E_mIbb\ m_a=\tfrac{1}{2}Pbbm_a^2. \qquad ...(54)$$

The ac output power across the tank circuit

$$P_0=\frac{1}{T}\int_0^T e_{L}i_p\ dt=\frac{1}{T}\int_0^T R_0i_p^2\,dt$$
$$=\frac{1}{T}\int_0^T 2R_0\left(\frac{K_tEbb}{Q}\right)^2(1+m_a\cos\omega_m t)^2\cos^2\omega_c t\,dt$$
$$=R_0(K_tEbb/Q)^2\ (1+\tfrac{1}{4}m_a^2+\tfrac{1}{4}m_a^2)$$
$$=(R_0K_t^2\ Pbb/Q^2Kb)\ (1+\tfrac{1}{2}m_a^2). \qquad ...(55)$$

$$\therefore \text{ Plate circuit efficiency } \eta=\frac{P_0}{Pbb+P_m}=\frac{R_0K_t^2}{Q^2Kb}. \qquad ...(56)$$

This result shows that the plate circuit efficiency of the modulated amplifier is independent of the degree of modulation.

The power dissipated in the plate of the tube is given by the relation

$$P_p=Pbb+P_m-P_0$$
$$=Pbb\ (1-\eta)\ (1+\tfrac{1}{2}m_a^2). \qquad ...(57)$$

The plate modulated amplifier is used extensively in radio transmitters. It has the advantages : (*i*) distortion is small, (*ii*) it operates at high efficiency, and (*iii*) adjustments are easy. The disadvantages are : (*i*) large amount of modulating power is required, and (*ii*) costly due to heavy and bulky modulating equipment.

(B) Grid Bias Modulation. Fig. 4.17 shows the basic circuit of grid bias modulated amplifier. It consists of a class C amplifier in the grid circuit of which the modulating signal e_m, *rf* carrier signal e_s, and a fixed grid bias E_{cc} are in series. Due to the superposition of e_m and e_s modulated carrier wave is produced. The modulated carrier wave is further amplified by the valve. Thus the instantaneous tank current is a sinusoidal wave of frequency, equal to the frequency of carrier wave, and the amplitude

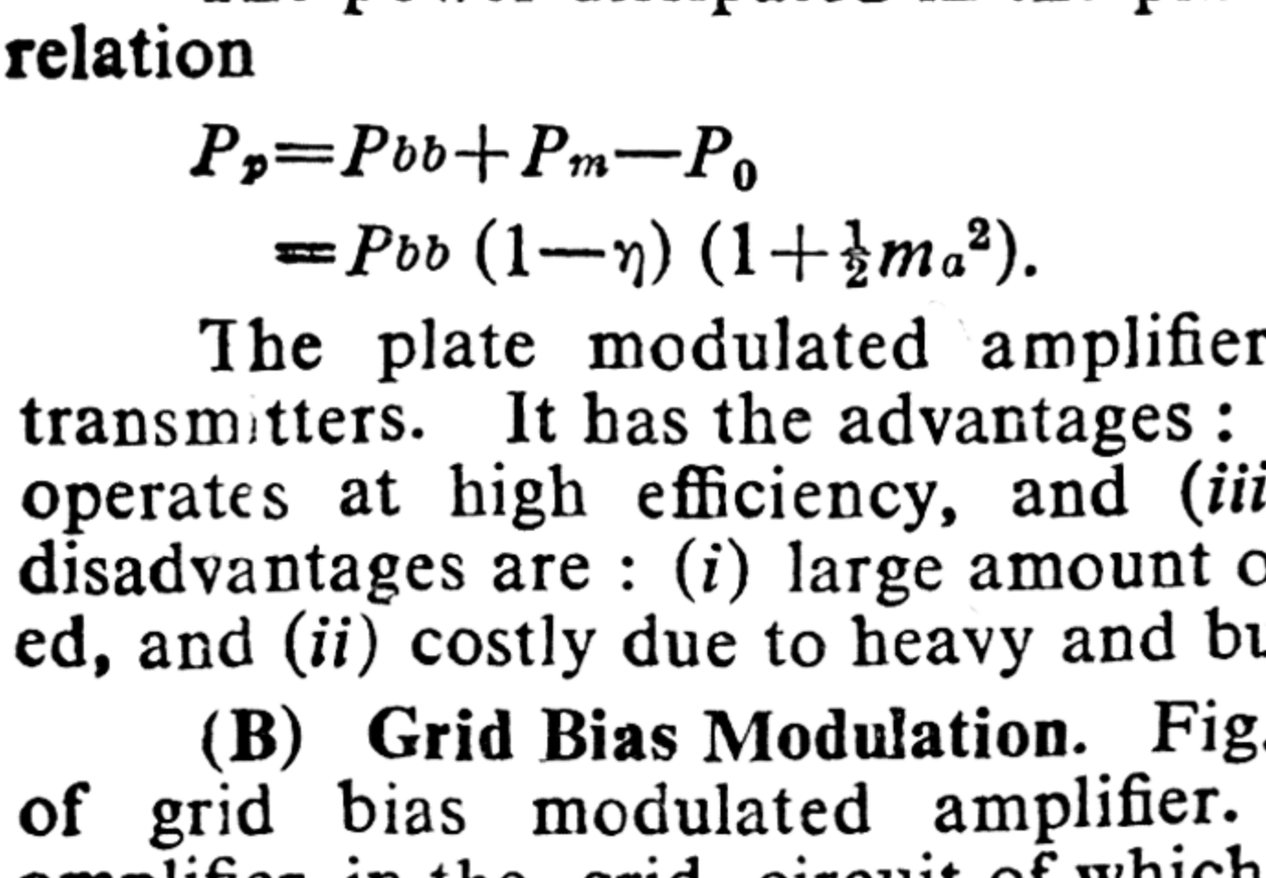


Fig. 4.17.

of which is not constant but varies as modulation frequency ω_m. Linear amplitude modulation is thus obtained.

The general character of the operation of grid-bias modulator is shown in Fig. 4.18. It is assumed that the modulation characteristic is linear and that $\omega_c \gg \omega_m$. The carrier signal is given by

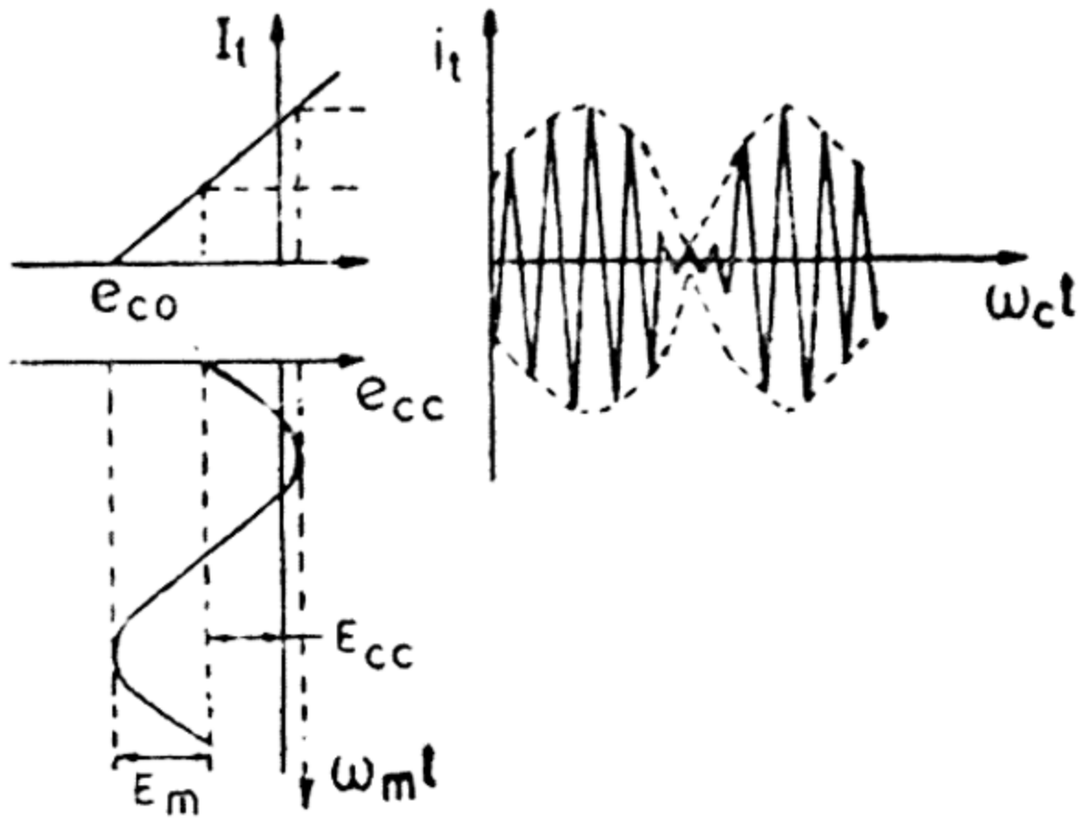


Fig. 4.18.

$$e_s = E_c \cos \omega_c t.$$

Modulating voltage $e_m = E_m \cos \omega_m t$.

$\therefore$ Total varying grid bias voltage

$$e_{cc} = e_m + E_{cc}$$
$$= E_m \cos \omega_m t + E_{cc}.$$

Total grid to cathode voltage $e_c = E_{cc} + E_m \cos \omega_m t + E_c \cos \omega_c t$. Over the linear range of operation, the rms value of tank current is given by the relation

$$I_t = K_t (e_{cc} - e_{c_0}), \qquad \ldots(58)$$

where e_{c_0} is the grid to cathode voltage for zero tank current. The instantaneous value of tank current corresponding to this rms value of tank current is given as

$$i_t = \sqrt{2}\, I_t \sin \omega_c t = \sqrt{2}\, K_t (e_{cc} - e_{c_0}) \sin \omega_c t$$
$$= \sqrt{2}\, K_t (E_m \cos \omega_m t + E_{cc} - e_{c_0}) \sin \omega_c t$$
$$= \sqrt{2}\, K_t (E_{cc} - e_{c_0}) [1 + \{E_m/(E_{cc} - e_{c_0})\} \cos \omega_m t] \sin \omega_c t$$
$$= I_t' [1 + m_a \cos \omega_m t] \sin \omega_c t, \qquad \ldots(59)$$

where m_a = modulation index $= E_m/(E_{cc} - e_{c_0})$, and $I_t' = \sqrt{2}\, K_t (E_{cc} - E_o)$

As the dc plate current varies with e_{cc}, hence on the parallel way one can write

$$I_b = K_b (e_{cc} - e_{c_0}). \qquad \ldots(60)$$

After solving similarly, we get

$$I_b = K_b (E_{cc} - e_{c_0})(1 + m_a \cos \omega_m t). \qquad \ldots(61)$$

Voltage across the tank circuit $\mathbf{E}_L = jX_L I_t$.

or $$\mathbf{E}_L = jX_L K_t (E_{cc} - e_{c_0})(1 + m_a \cos \omega_m t).$$

$\therefore$ Instantaneous potential across the tank circuit

$$e_L = \sqrt{2}\, X_L K_t (E_{cc} - e_{c_0})(1 + m_a \cos \omega_m t) \sin (\omega_c t + 90°)$$
$$= \sqrt{2}\, X_L K_t (E_{cc} - e_{c_0})(1 + m_a \cos \omega_m t) \cos \omega_c t \qquad \ldots(62)$$

The average power input to the plate circuit by the dc-plate supply over the modulating cycle is therefore given by

$$P_{bb}=\frac{1}{T}\int_0^T E_{bb}I_b\,dt=\frac{1}{T}\int_0^T E_{bb}K_b(E_{cc}-e_{c_0})(1+m_a\cos\omega_m t)\,dt$$

$$=E_{bb}K_b\,(E_{cc}-e_{c_0})=E_{bb}I_{bb}. \qquad \ldots(63)$$

This shows that the plate supply power is independent of the degree of modulation. The average power output at the tank circuit

$$P_0=\frac{1}{T}\int_0^T e_L\;i_p\,dt=\frac{1}{T}\int_0^T e_L.\,\frac{e_L}{R_0}\,dt$$

$$=\frac{1}{T}\int_0^T\frac{2}{R_0}X_L^2K_t^2\,(E_{cc}-e_{c_0})^2(1+m_a\cos\omega_m t)^2\cos^2\omega_c t\,dt$$

$$=\frac{R_0K_t^2}{Q^2}(E_{cc}-e_{c_0})^2\left[1+\frac{m_a^2}{2}\right]=RI_t^2\left[1+\frac{m_a^2}{2}\right], \qquad \ldots(64)$$

where R is the effective series resistance in the tank circuit.

$$\therefore \text{ Plate efficiency } \eta=\frac{P_0}{P_{bb}}=\frac{R_0K_t^2}{K_bE_{bb}Q^2}\;(E_{cc}-e_{c_0})\left[1+\frac{m_a^2}{2}\right] \qquad \ldots(65)$$

It shows that η increases as we increase m_a. The grid bias modulation has the advantages that only a small modulating power is required. It has disadvantages : (*i*) lower plate efficiency, (*ii*) higher amplitude distortion and (*iii*) more difficulty in adjustment.

4.5. DETECTION

The radio-frequency signal picked by the antenna of the receiver has the signal or intelligence. The process by which the signal is extracted from the modulated wave is known as *detection* or *demodulation*. Thus the detection is the inverse process of modulation.

We know that the process of modulation is one in which the signal frequencies ω_m are shifted upward on the frequency scale and the sidebands are symmetrically disposed about the carrier frequency ω_c. In the demodulation process, the signal is shifted downward on the frequency scale and thus returning it to its original frequency position. Therefore, we see that both these processes involve frequency shifting. Similar circuits are used in, certain cases for both processes. In the modulating process the carrier signal is generated in one channel, mixed with the audio signal and generated in a second channel. While in the demodulating process the required carrier wave is ordinarily contained in the receiving signal. Similar to the modulating process, the demodulation is also of three types : *amplitude*, *frequency* and *phase*

demodulations. The amplitude demodulation is further classified as : (*i*) Linear demodulation and (*ii*) the square law demodulation. Let us discuss below the diode detector only.

Diode Detector : Diode detectors are generally used for the detection of amplitude modulated waves, as they operate very satisfactorily. There are two important types of operation of a diode detector : (1) average detection and (2) envelope detection. The circuit for the former type is just a simple diode rectifier without a filter in the output. It yields an output the average value of which contains the modulating frequency. The circuit for the envelope detection is a simple diode rectifier with a capacitor filter, as shown in Fig 4.19. In this, the amplitude modulated (*AM*) signal is applied at the plate of diode through radio (high) frequency transformer. Diode, as half wave rectifier conducts only during the positive half cycle and thus eliminates the negative cycle of the *AM*-carrier waves. The positive cycle will thus appear across the output resistance *R*. The filter capacitor *C* across *R* is of sufficiently high value so it offers very small resistance for high frequency carrier waves and relatively high resistance for the modulating signal (low frequency). Therefore the carrier waves bypass or are short-circuited around *R*, while the modulating signal appears across *R*.

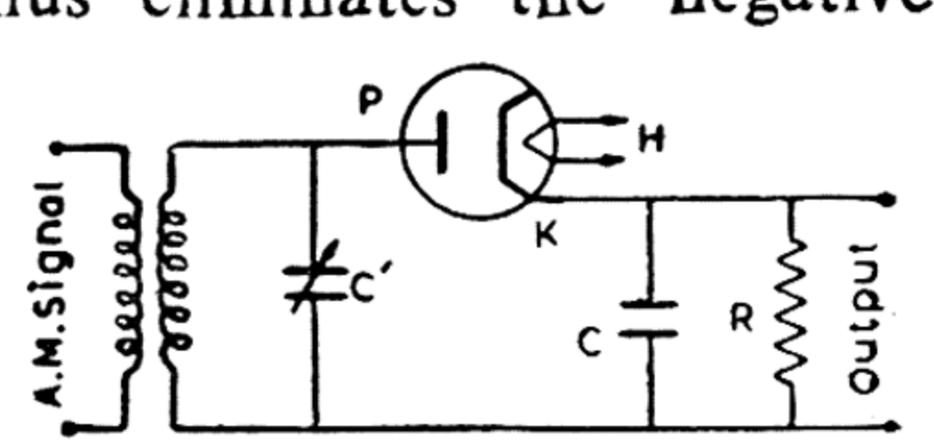


Fig. 4.19.

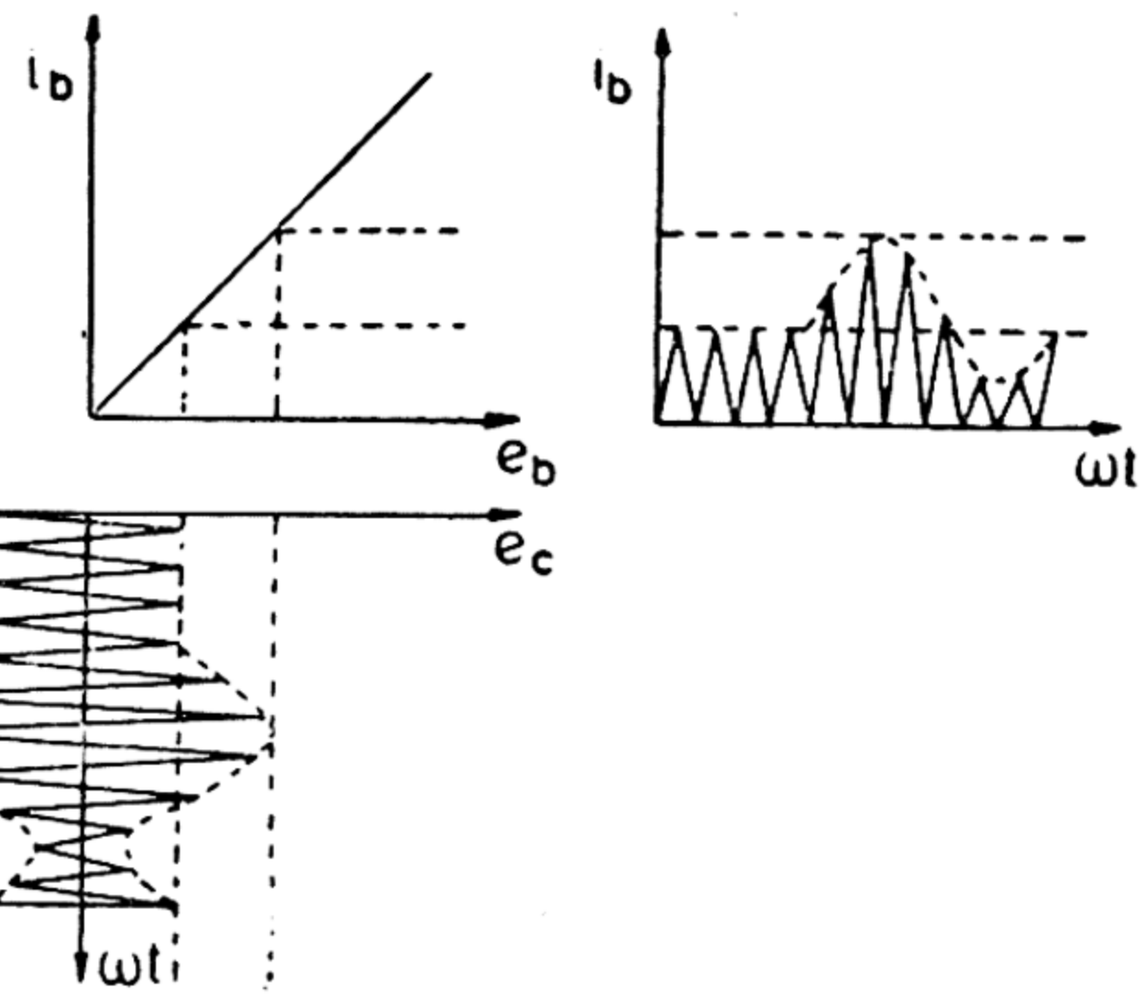


Fig. 4.20.

During the conduction period the condenser charges to the peak value of the applied voltage. Since time constant *CR* is high, so the voltage across *R* is almost maintained by the slow discharge of condenser through *R* during the successive pulses of positive cycle. The small change in the peak value of voltage is nothing but the modulating signal or the information we require. The

Fig. 4.20 shows the diode action and the variation of voltage across condenser in accordance with input amplitude modulating carrier voltage. It will be the same as across the output load *R*.

It is assumed that the dc potential across the diode load remains constant over the carrier frequency cycle and the static

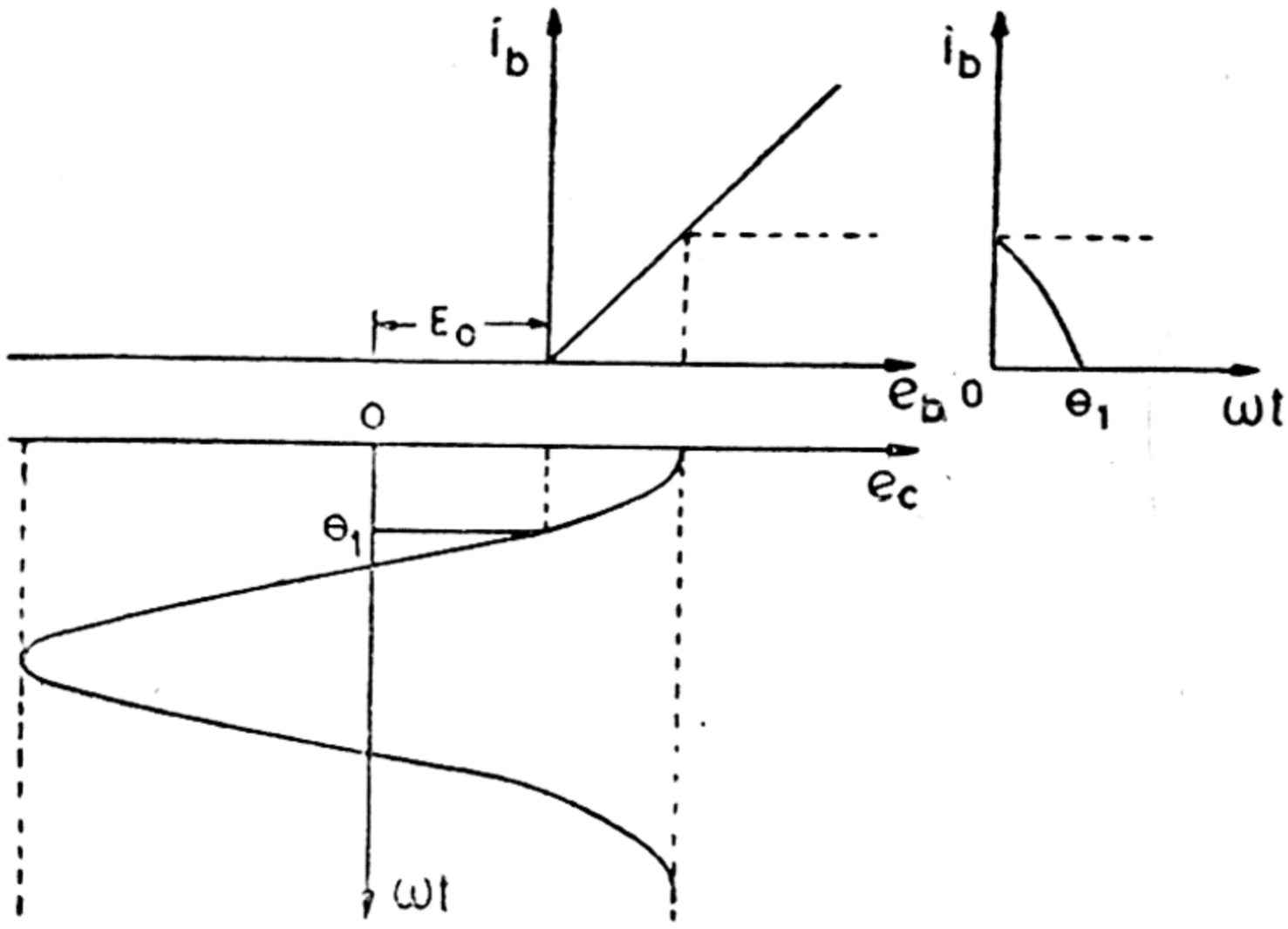


Fig. 4.21.

characteristic of the diode is linear. The operation of linear diode is shown in Fig. 4.21. An amplitude modulated wave $e_s = E_c (1 + m_a \cos \omega_m t) \cos \omega_c t$ is applied to the diode. The diode voltage will be biased by E_0 and the current is discontinuous. The method of analysis is follows :

$$e_b = Ec\ (1 + m_a \cos \omega_m t) \cos\ \omega ct - E_0. \qquad \ldots(66)$$

It may be more compactly written for one carrier cycle as

$$e_b = E' \cos \omega ct - E_0,$$

where the quantity E' is practically constant over a few cycles. The plate current will be of the form.

$$\begin{aligned} i_b = e_b / r_p = (E' \cos \omega ct - E_0)/r_p \qquad & e_b > 0 \\ = 0 \qquad & e_b < 0. \end{aligned} \qquad \ldots(67)$$

The shift from one equation to the other occurs at the angle θ_1, defined by

$$e_b = 0 \text{ or } E' \cos \omega ct_0 = E' \cos \theta_1 = E_0. \qquad \ldots(68)$$

The instantaneous current i_b will contain a number of harmonic components. The dc-component of i_b is given by

$$I_b = \frac{1}{2\pi} \int_0^{2\pi} i_b \, d\theta. \qquad \ldots(69)$$

Because of the symmetry, it will reduce to

$$\begin{aligned} I_b &= \frac{1}{\pi} \int_0^{\theta_1} i_b \, d\theta = \frac{1}{\pi r_p} \int_0^{\theta_1} (E' \cos \theta_1 - E_0)\, d\theta \\ &= (E' \sin \theta_1 - E_0\, \theta_1)/\pi r_p. \end{aligned} \qquad \ldots(70)$$

This becomes, by equation (68),

$$Ib = E' \ (\sin \theta_1 - \theta_1 \cos \theta_1)/\pi r_p. \qquad ...(71)$$

Since $E_0 = I_b R$, therefore the above two equations give

$$Ib = \frac{E_0}{R} = \frac{E' \cos\theta_1}{R} = \frac{E' \ (\sin \theta_1 - \theta_1 \cos \theta_1)}{\pi r_p}$$

or
$$\frac{r_p}{R} = \frac{1}{\pi} (\tan \theta_1 - \theta_1). \qquad .. (72)$$

This shows that there is a direct functional relationship between the operating angle θ_1 and the ratio of the plate resistance of the diode to the load resistance. The detection efficiency of the diode detector, which is defined as the ratio of the average value of the load potential E_0 to the peak *ac*-input E', is given as

$$\eta = \frac{E_0}{E'} = \frac{R}{\pi r_p} \ (\sin \theta_1 - \theta_1 \cos\theta_1). \qquad ...(73)$$

Since r_p/R is a constant of the circuit, the conduction angle θ_1 is also a constant for the circuit, Thus the θ_1 is independent of the degree of modulation.

For the average detector, *i.e.*, when the bypass or output capacitor is not used, the detection efficiency is given as

$$\eta = \frac{E_0}{E'} = \frac{R \ E_c \ (1 + m_a \cos \omega_m t)/\pi(R + r_p)}{E_c \ (1 + m_a \cos \omega_m t)}$$

$$= \frac{R/r_p}{\pi(1 + R/r_p)} . \qquad ...(74)$$

4.6. MICROPHONES

A microphone is an electro-acoustic transducer. It converts acoustic energy of sound vibrations into electrical energy which is in the form of audio frequency electrical vibrations. The microphones are very much used in telephones, broadcasting arrangements and electrical recording of sound. A good microphone must possess the following qualities : (1) *the same response over the entire range of audiofrequencies*, (2) *the linearity of response with sound pressure*, (3) *the adequate sensitivity to have a large electrical output*, (4) *free from background noise*, (5) *robust reliable and easy to maintain and handle.*

All microphones may be classified into three groups—

1. Pressure microphones. In these microphones the electrical output is proportional to and in phase with the acoustic pressure acting on one side of the diagram. The examples are the *carbon*

microphone, condenser type microphone, crystal and the *moving coil (dynamic) microphone.*

2. *Velocity microphones.*

3. *Hot wire microphones.*

Let us discuss these microphones in brief :

(*A*) *Carbon microphone.* As shown in Fig. 4.22 a small enclosure *E*, formed by a carbon block *A* and a plunger *P* is loosely packed with very fine carbon granules. It thus forms carbon button. The carbon granules are made from anthracite coal and are free from dirt and magnetic material. A thin metal or carbon diaphragm rigidly clamped at its edges is attached with the plunger *P*. The microphone is connected in series with a battery and the primary of a transformer.

When the sound waves from the speaker impinge on the diaphragm, they cause it to vibrate. As a consequence the plunger varies the pressure exerted on the carbon granules and hence the resistance across the carbon button. Hence the voltage across the secondary of the transformer is in accordance with the sound pressure exerted on the diaphragm.

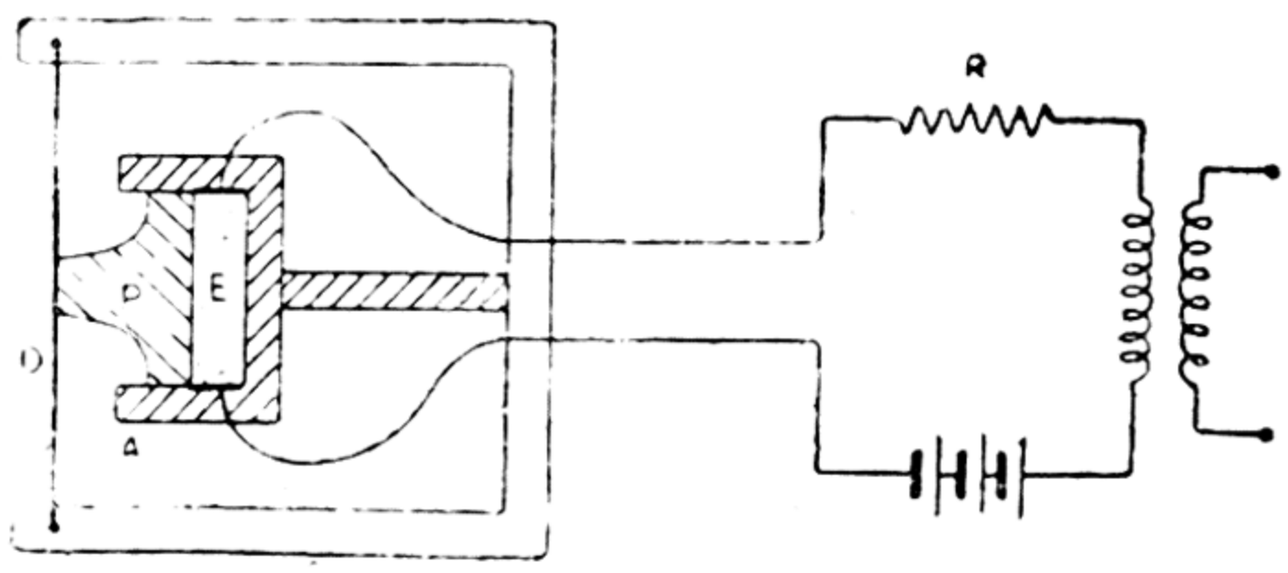


Fig. 4.22.

The main advantage of carbon-microphone is that it is non-directional in nature and gives a large output with good frequency response. On account of its large output, it is employed in telephone systems and radio work. Its disadvantages are : (1) the inconvenient need of a battery, (2) the steady background hiss resulting from random changes in carbon granules, (3) the uneven response at high frequencies and (4) the considerable harmonic distortion.

(*B*) *Condenser microphone.* It consists essentially of a thin stretched (nearly to its elastic limit) metal diaphragm *D* usually of steel or aluminium separated by a very small distance (~0.0025 cm) from a parallel rigid back plate *P*. The diaphragm and the plate form a condenser as shown in Fig. 4.23. A high d-c voltage (200-400 volts) is applied between *D* and *P* through a high resistance *R* (~20 megohms).

When the sound waves from the speaker impinge on the diaphragm, they cause it to vibrate. Hence the capacitance of the microphone changes. This change causes appreciable variations in voltages across the load *R*, which are fed to the grid of the amplifier through a blocking condenser.

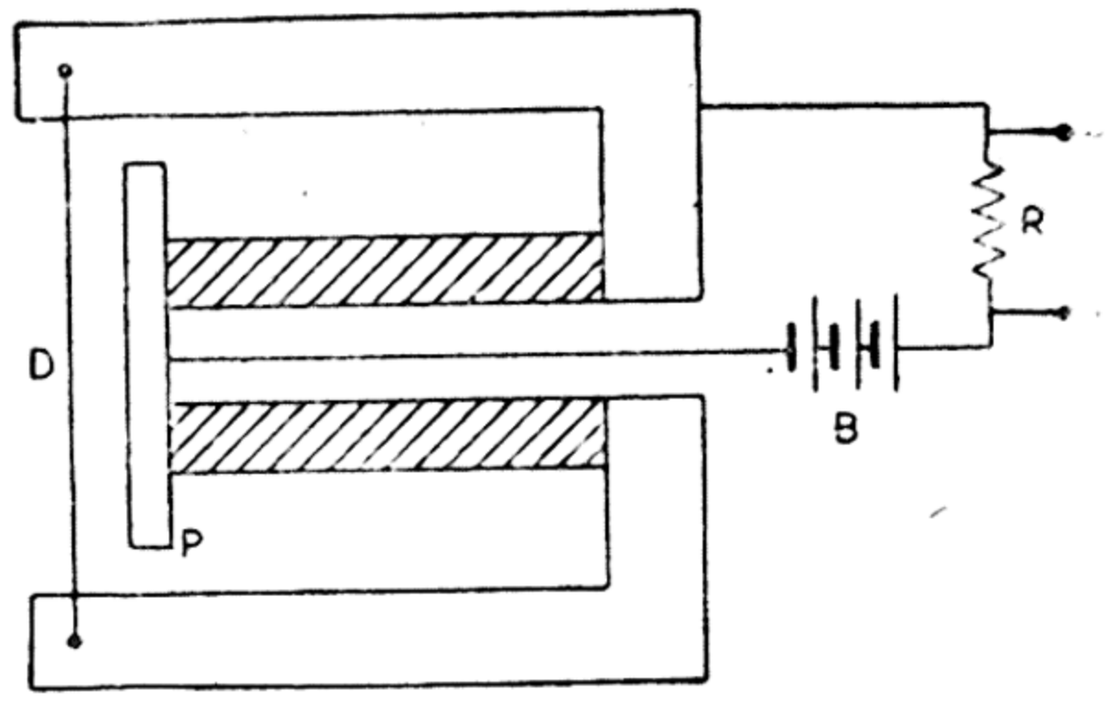
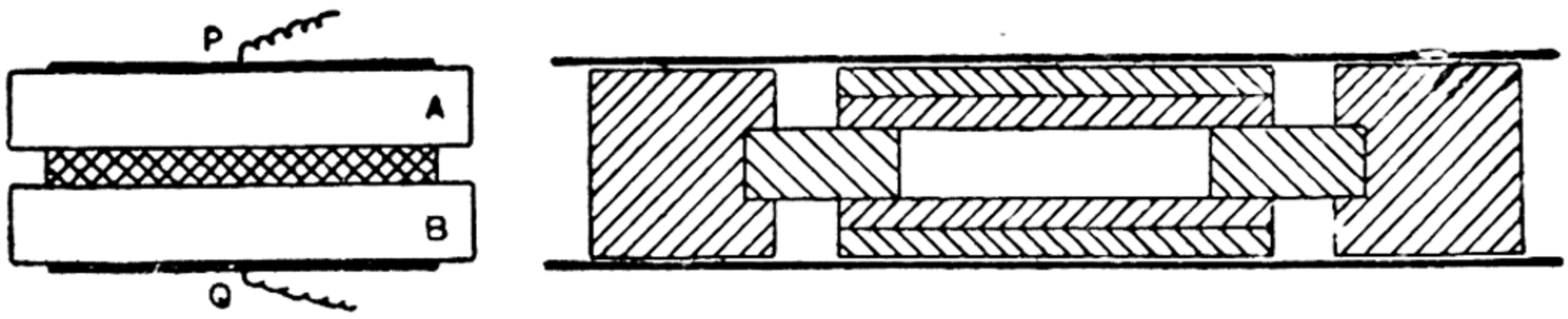


Fig. 4.23.

The condenser microphone possesses extremely uniform response over a major portion of the audio-frequency band and is almost free from background hiss. It has low sensitivity and thus requires at least one stage of preamplifier. Due to these reasons the condenser microphones are unsuitable for use in high quality sound system.

(*C*) *Crystal microphone.* A crystal microphone depends for its operation on *piezoelectric effect*, in which the mechanical



(*a*) (*b*)

Fig. 4.24.

stress causes electrical oscillations. Rochelle salt crystals exhibit this effect about a thousand times greater than that by quartz. There are two types of crystal microphones, *the diaphragm type* and the *sound cell type*. In the former type two very thin circular slices *A* and *B* of Rochelle salt crystals are cemented together with foil electrodes, as shown in Fig. 4.24 (*a*). This whole arrangement is called *bimorph plate*. When the sound waves from the speaker impinge on the faces of a bimorph plate it is subjected to pressure and tensile stress. This produces a mechanical bending, analogous to the bending of a beam supported at the two ends with a weight placed in the middle and thus causes a piezoelectric emf across two electrodes *P* and *Q*.

The sound cell type is the improved form of the above simple type. Its output is more and frequency response is more perfect. It consists of two similar bimorph plates mounted together with a small air gap between them. This air gap minimises resonance

effects. This whole system is mounted in a frame of insulating material and is made so small that its resonant frequency is above the audible range. When the sound waves from the speaker impinge on the crystal microphone, each bimorph plate vibrates like a diaphragm and as a consequence a varying piezo electric emf is generated across each bimorph plate. These are added by suitable connections and then fed to an amplifier.

A crystal microphone is less expensive practically nondirectional and is widely used in speech distribution with high fidelity. It has a high sensitivity audible frequency.

(*D*) *Moving coil electrodynamic microphone.* It depends for its operation on the generation of induced emf in a coil when it moves in a magnetic field. As shown in Fig. 4.25, it consists of a light diaphragm *D* with which a small coil *C* is rigidly attached. This coil is located in the annular space between the central pole piece and the peripheral pole peace of a powerful pot magnet *M*. To increase the stiffness and hence the resonant frequency of the diaphragm it is made *domes haped* at its middle. The coil consists of a large number of turns of fine wire and is called the *voice-coil*. A silk cloth *S* is attached to the pole pieces to provide requisite damping at the resonant frequency and to improve high frequency response.

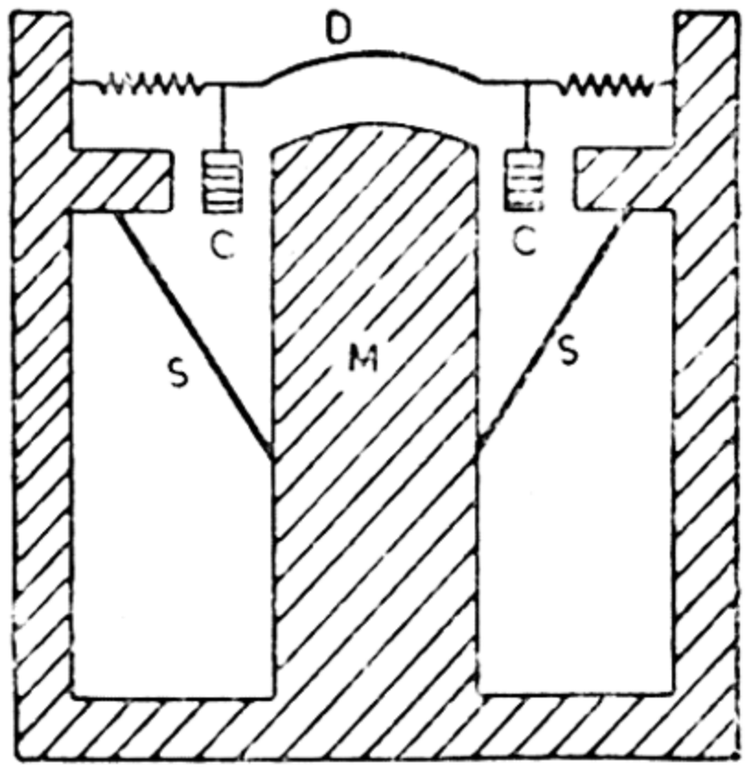


Fig. 4.25.

When the sound waves impinge on the diaphragm they cause it to vibrate. The voice coil attached to it thus moves in the radial field between the pole pieces of the magnet *M*. This causes the change in magnetic flux linked with the coil and as a consequence the induced emf is produced in the coil, which may be fed to the grid of the first valve of amplifier. Therefore, the induced emf is proportional to the acoustic pressure on the diaphragm.

A moving coil microphone has relatively good frequency response and high sensitivity. The microphones of this type are robust, highly non-directional and are widely used as good purpose high quality microphones.

(*E*) *Velocity-ribbon microphone.* It depends for its operation on the phenomenon of electromagnetic induction, like a moving coil microphone. It consists of a thin corrugated ribbon of aluminium suspended edge-wise between the pole pieces of a permanent magnet. The stiffness of the suspension system of the ribbon is made so small that its resonance frequency is well below the audible range. When the sound waves impinge on the ribbon, they cause it to vibrate. The magnetic flux linked with it thus changes.

As a consequence a varying induced emf is generated in the ribbon, which is proportional to the sound pressure and given by the relation.

$$\mathcal{E} = Blv,$$

where B is the magnetic flux density, l is the length of the ribbon and v its velocity.

A velocity-ribbon microphone has an excellent frequency response, a very low internal impedance and the advantage of simplicity. It is bidirectional in its response. If θ is the angle which the incident sound waves make with the normal then the pressure gradient and hence the response will vary as cos θ. Thus the response will be maximum for normal incident ($\theta = 0$ or $180°$) and will be zero for $\theta = 90°$ or $270°$.

(*F*) *Hot wire microphone.* It consists of a fine wire of platinum which is generally made red hot by an electric current and is mounted in the neck of a resonator. When the sound waves impinge on it, it cools down and its electrical resistance decreases. This resistance is measured by connecting the microphone in the unknown arm of the Wheatstone's bridge. If the incident sound waves have the frequency equal to the natural frequency of the resonator, this type of microphone will produce a marked effect. The hot wire microphones are generally used in sound analysis (in sound ranging and determination of sound velocity in air).

4.7. LOUD SPEAKERS

The audio frequency oscillations produced by the microphones are weak and hence are amplified. To obtain the sound energy back from these oscillations, loud-speakers are used. These loud-speakers must generate sound waves which are free from amplitude and frequency distortions and the efficiency should be the same at all frequencies. The loud-speakers are of the following types :

(A) **Fixed Coil Loud-speaker.** It consists of a permanent magnet *M*. To the ends of the magnet are attached soft iron pole pieces *PP* over which two field coils are wound. *A* diaphragm *D* of iron alloy is placed in front of the magnetic pole pieces. When the amplified audio frequency oscillatory current is fed into the field coils, the diaphragm is attracted towards the magnet with a varying force. As a consequence the diaphragm vibrates with the frequency of voice current and the sound reproduces.

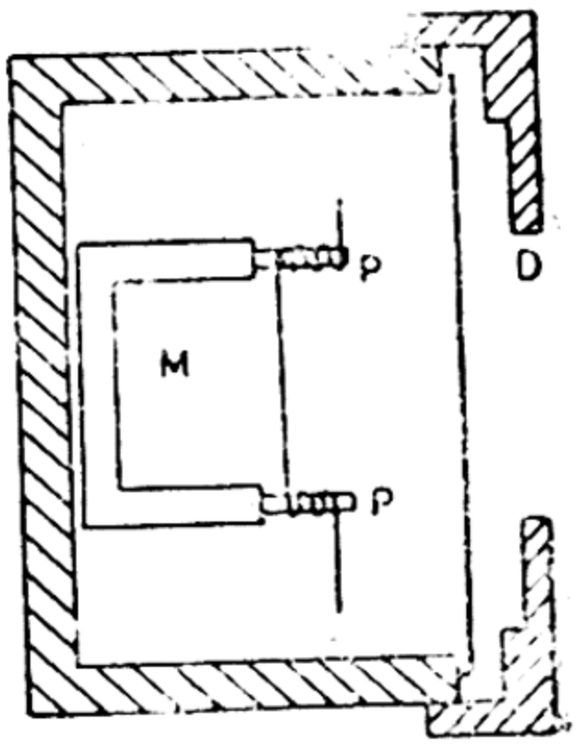


Fig. 4.26.

This type of loud-speakers is generally used in telephone receivers because of its feeble response at low frequencies and distortion of loud sounds.

(B) Dynamic Core Loud-speaker. It depends for its operation on the fact that a current carrying conductor, when placed in a magnetic field, experiences a mechanical force which is directly proportional to the current strength passing through this conductor. Its output is comparatively large and almost free from distortion.

It consists of a permanent pot magnet. In the annular gap between the pole pieces of which a circular coil *C*, called *voice coil*, is suspended. This coil is rigidly attached with a diaphragm of waxed paper or fibre which is in the form of a cone. The magnetic field is radial and is perpendicular to the coil winding. To increase the stiffness the cone is made dome-shaped *D* at the round apex and has circular corrugations at the rim. It is supported round its outer edge in a flexing mounting. The assemble of voice coil and the cone is free to move as one unit along the axis of the coil.

When the amplified current due to the sound waves is fed into the voice coil, the coil experiences a force in the direction perpendicular to both the magnetic field and the current directions. Its direction is determined by Fleming's left hand rule. As the current fluctuates, the mechanical force also fluctuates likewise. The result is that the coil moves to and fro along its own axis in the annular gap. The diaphragm therefore fluctuates and produces sound.

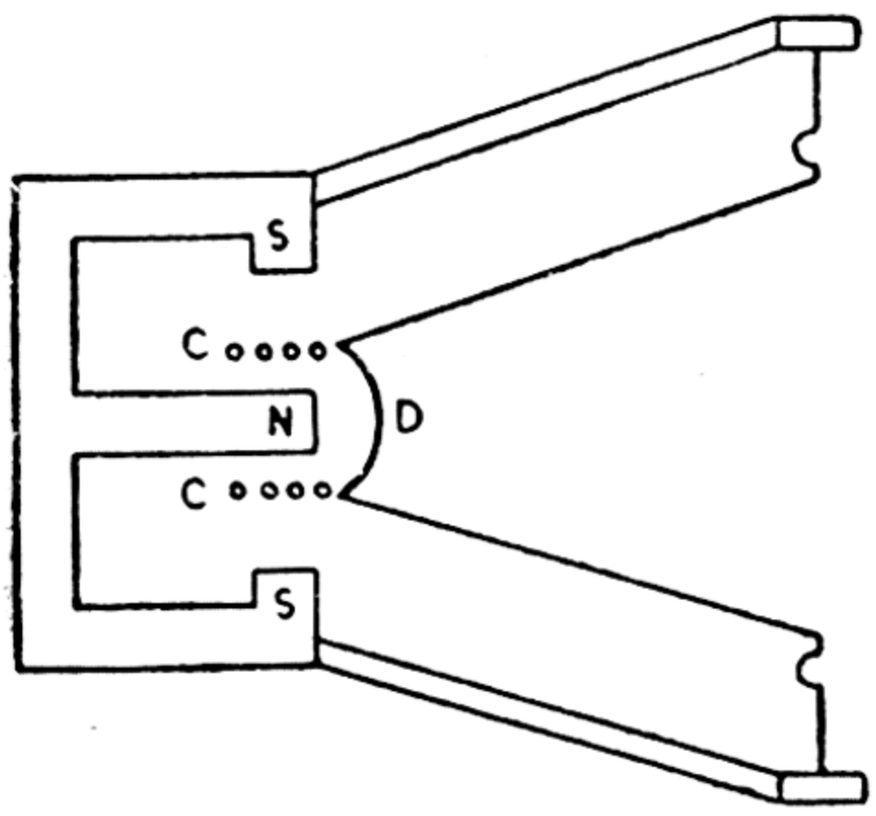


Fig. 4.27.

This type of loudspeakers is most commonly used these a days because of the small size and low cost. It has a fairly uniform frequency response. Its efficiency is very low hence to obtain sufficient sound output a large cone and considerable amplitude must be used.

(C) Horn-Type Loud-speaker. It consists of a moving coil driving arrangement and an exponential horn, as shown in Fig. 4.28. A light diaphragm *D* of duralumin is made in the form of a spherical shell and is placed within the voice coil *C*. These two are attached to a flexible annules. In front of the diaphragm is mounted an exponential horn in which the cross-sectional area increases exponentially with the distance from the throat *T*. A cylindrical plug *P* is placed in front of diaphragm to reduce the effective size of the air chamber, the area between the diaphragm and the throat of the horn. This will make the distance from each part of the diaphragm to the throat as nearly equal and

thus cause acoustic waves to arrive at the throat in the same phase.

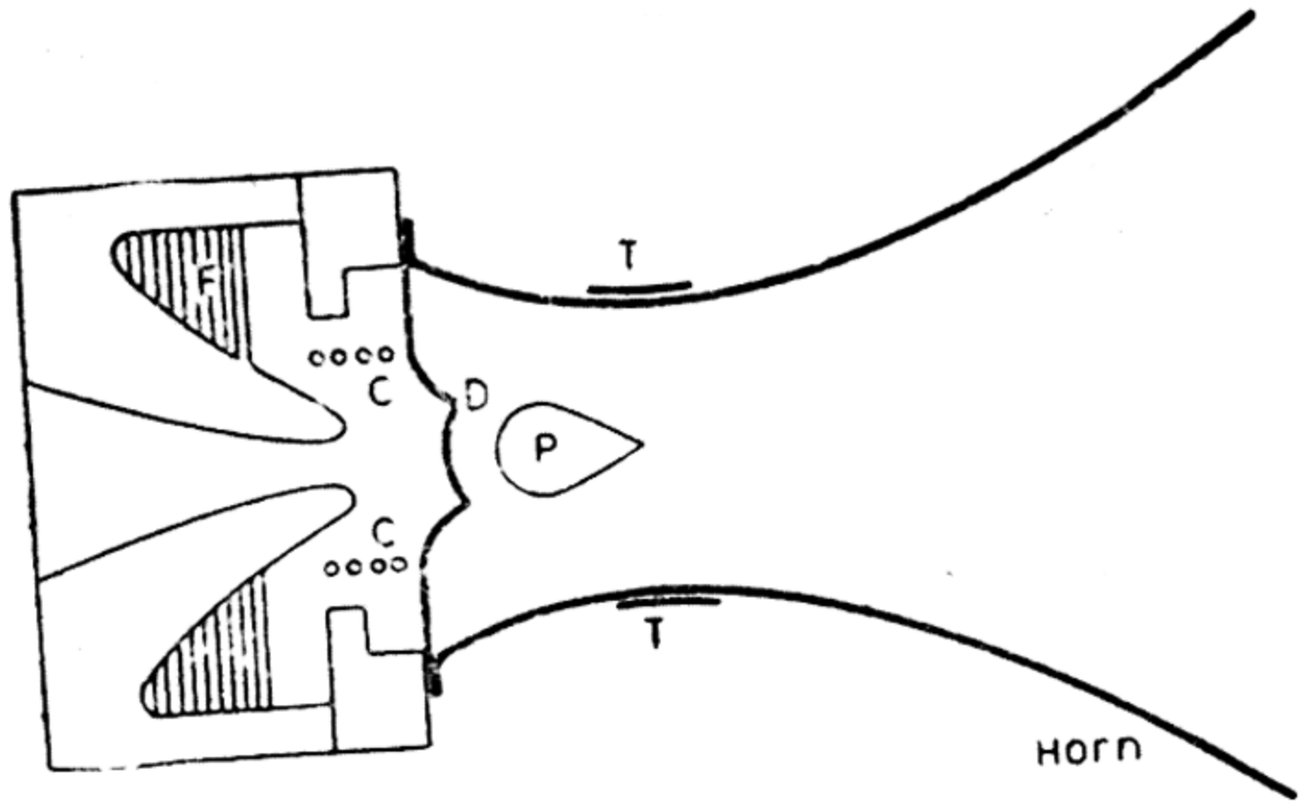


Fig. 4.28.

When the amplified current due to sound waves is fed into the voice coil, which is placed in the radial field of the electromagnet, the coil experiences a force and diaphragm starts vibrating. It produces sound at the throat of the horn, which travels along the horn without undue disturbance.

4.8. TRANSMITTERS

A radio transmitter produces radio frequency energy that is controlled by the intelligence to be transmitted. The transmitters are classified as the *frequency groups* :

(*i*) *VLF* (very low frequency ~10 – 30 kc/s),

(*ii*) *LF* (low frequency ~ 30 – 300 kc/s),

(*iii*) *MF* (medium frequency ~ 300 — 3000 kc/s),

(*iv*) *HF* (high frequency ~ 3 — 30 Mc/s),

(*v*) *VHF* (very high frequency ~ 30 — 300 Mc/s),

(*vi*) *UHF* (ultra high frequency ~ 300 — 3000 Mc/s),

(*vii*) *SHF* (super high frequency ~ 3000 — 30000 Mc/s).

According to the *use*, the transmitters are of two types :

(*a*) radio telegraph transmitter,

(*b*) radio telephone transmitter (broadcast).

The latter is fitted with modulation equipment while the former is fitted with keying system to change the information in telegraph code. According to the *methods of modulation*, radio telephone transmitters can be classified as :

(1) amplitude modulated,

(2) frequency modulated,

(3) phase modulated transmitters.

We shall discuss here only amplitude modulated transmitters.

Amplitude Modulated Transmitter. In these transmitters the energy control is by amplitude modulation. There are two channels in *AM* transmitter, as shown in block diagram 4.29.

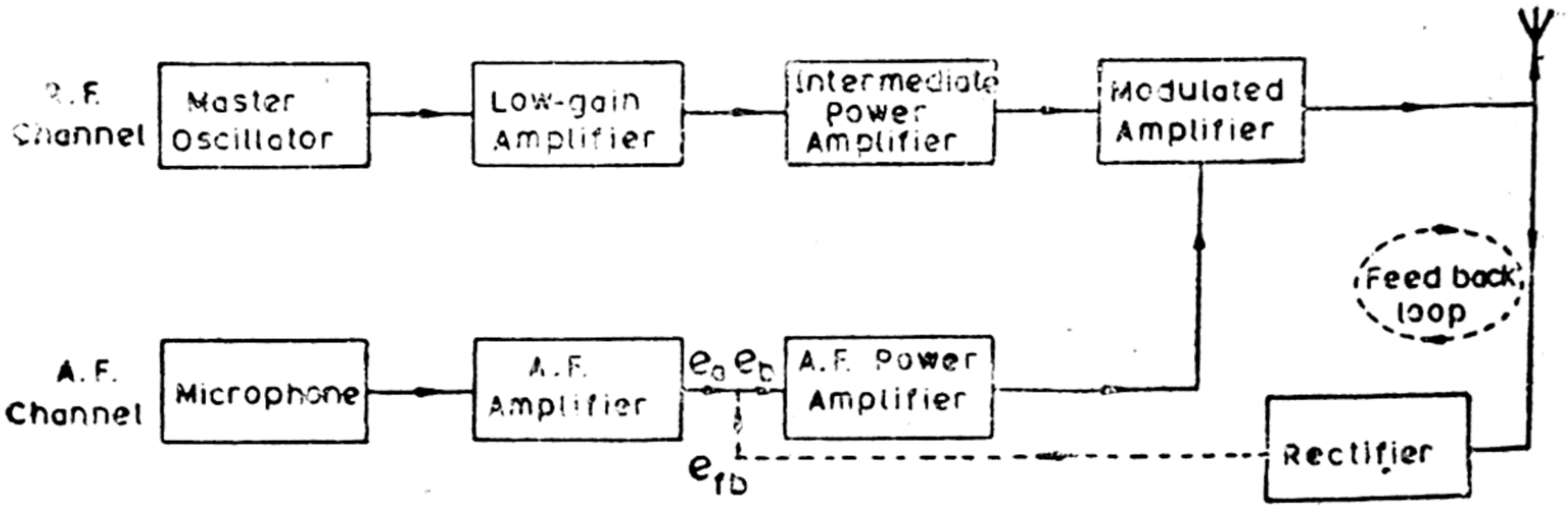


Fig. 4.29.

(1) **RF-Channel.** The radio frequency channel of the transmitter consists of the circuits wherein the carrier wave is generated, amplified and modulated. They may broadly be classified as :

(*a*) *Master oscillator.* The desired frequency is generated at low power level by a crystal controlled master oscillator. The temperature of the crystal is kept constant. A variable capacitor is used in the grid circuit for the minor adjustment in the operating frequency.

(*b*) *Buffer and Intemediate power amplifier.* Low gain (buffer) amplifier serves to isolate the oscillator from the succeeding stages. It is followed by an intermediate power amplifier which increases the power level of the carrier wave.

(*c*) *Frequency multiplier.* If the frequency of the oscillator is lower than the radiated frequency, the frequency multipliers or harmonic generators are introduced along with the intermediate power amplifier.

(*d*) *Final power amplifier.* The final power amplifier operates at highest power level to increase the power output of the transmitting system. When the modulating voltage is applied to this final *rf* stage, we call it *high level modulation.* In some cases modulated amplifier is followed by linear amplifiers and finally the energy is fed to the antenna for radiation.

(2) **AF-Channel.** This section generates and amplifies the modulating signal. The components are :—

(*a*) *Microphone.* It converts sound energy into electrical energy. The microphones generally used have already been discussed in the article 4.6.

(*b*) *Audio amplifier unit.* It consists of modulation and modulated amplifiers. The first stage is an audio power amplifier. The audio frequency power is then increased by power amplifier to modulate the carrier.

(*c*) *Circuits for improving performance—Negative feedback circuit* is used to improve the performance of an amplitude modulated transmitter. A portion of modulated output is rectified and fed in opposite phase to the input of modulation amplifier. This reduces amplitude, frequency and phase distortions, which may otherwise occur in different stages of R.F. channel. A limiting amplifier is used to minimise the possibility of over-modulation. Volume compressor is used to raise the modulation.

(*d*) *Power supplies.* Radio-transmitters are classified as *direct current operated* and *alternating current operated.* In the former, direct current is used throughout filament, bias and power supply. In the latter, the a-c is used directly in filament and through rectifiers in plate and grid circuits.

4.9. RECEIVERS

The device used to convert the transmitted radio waves into informational signals are called receivers. There are various types of receivers, but the fundamental performance of all receivers is the same.

Basic Functions of Receivers. The basic functions of a radio receiver are : (*i*) reception of radio signal at the antenna, (*ii*) selection of the desired R.F. signal (the radio wave from a particular station) and rejection of all others, (*iii*) amplification of this R.F. signal, (*iv*) demodulation of the R.F. signal to get back the

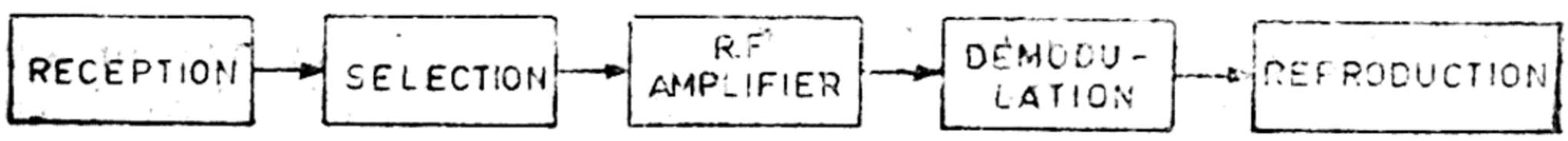


Fig. 4.30.

original modulating voltage and (*v*) reproduction of the audio-frequency signal by feeding this modulating voltage to an indicator (a loudspeaker in the case of radio broadcast). The simple basic radio receiver block diagram is shown in Fig. 4.30.

Parameters of Radio Receivers. The performance of a radio receiver is determined by means of certain characteristics or parameters, *e.g.*, sensitivity, selectivity, fidelity, stability, noise figure and signal to noise ratio.

(*i*) *Sensitivity.* It determines the ability to respond to weak signal voltages, *i.e.*, the minimum strength of an input signal necessary for producing a standard output signal.

(*ii*) *Selectivity.* It determines the extent to which the receiver is capable of differentiating between the desired signal and the unwanted signals.

(*iii*) *Fidelity.* It is a measure of receiver's ability to reproduce the intelligence of the signal. For a good fidelity, the receiver should have a reasonable large and uniform frequency response over almost the entire audio frequency band.

(*iv*) *Stability.* It is a measure of receiver's ability to deliver a constant output power for a given period of time when amplitude and frequency of the input signal remain constant. Instability in the receiver arises due to the variations in the output voltage of power supply unit, temperature changes and the mechanical imperfectness.

(*v*) *Noise figure.* It is a measure of the extent to which the noise appearing in the receiver output in the absence of a signal is greater than the noise present in a perfect receiver generating the minimum possible noise.

(*vi*) *Signal to noise ratio.* It is the ratio of the signal power output to the noise power output. It should be high for the satisfactory reception.

Classification of Radio Receivers. Radio receivers are classified in many ways. Based on modulation process they are classified as :

(1) *Amplitude modulated (AM) receivers.* These are meant for the programmes radiated from amplitude modulation broadcast transmitters at long, medium or short wave bands.

(2) *Frequency modulated (FM) receivers.* These are meant for the programmes radiated from frequency modulation broadcast transmitters operating usually in the very high frequency (*VHF*) range.

(3) *Television receivers.* These are designed to receive TV broadcast on *VHF* and *UHF* (ultra high frequency) range.

(4) *Radar receivers.* These are designed to receive radar signals.

4.10. AMPLITUDE MODULATED RECEIVERS

A.M. receiver may broadly be put into two categories: (*i*) straight receivers, which operate in a straight forward manner without any frequency conversion and (*ii*) superheterodyne receivers, which first convert the incoming *R.F.* signals into standard intermediate frequency before detection. The former are not commonly used these days, but useful in understanding the receivers.

(A) Straight receivers. Let us discuss simple circuits of straight receivers.

In the *single tube straight receiver with grid leak detector*, vacuum tube V_1 along with $R_g - C_g$ combination and by pass condenser C acts as the grid leak detector. The bypass capacitor bypasses the *R.F.* components allowing only the audio frequency components to flow through the headphone. The circuit diagram with the four essential functions of the receiver is shown in Fig. 4.31.

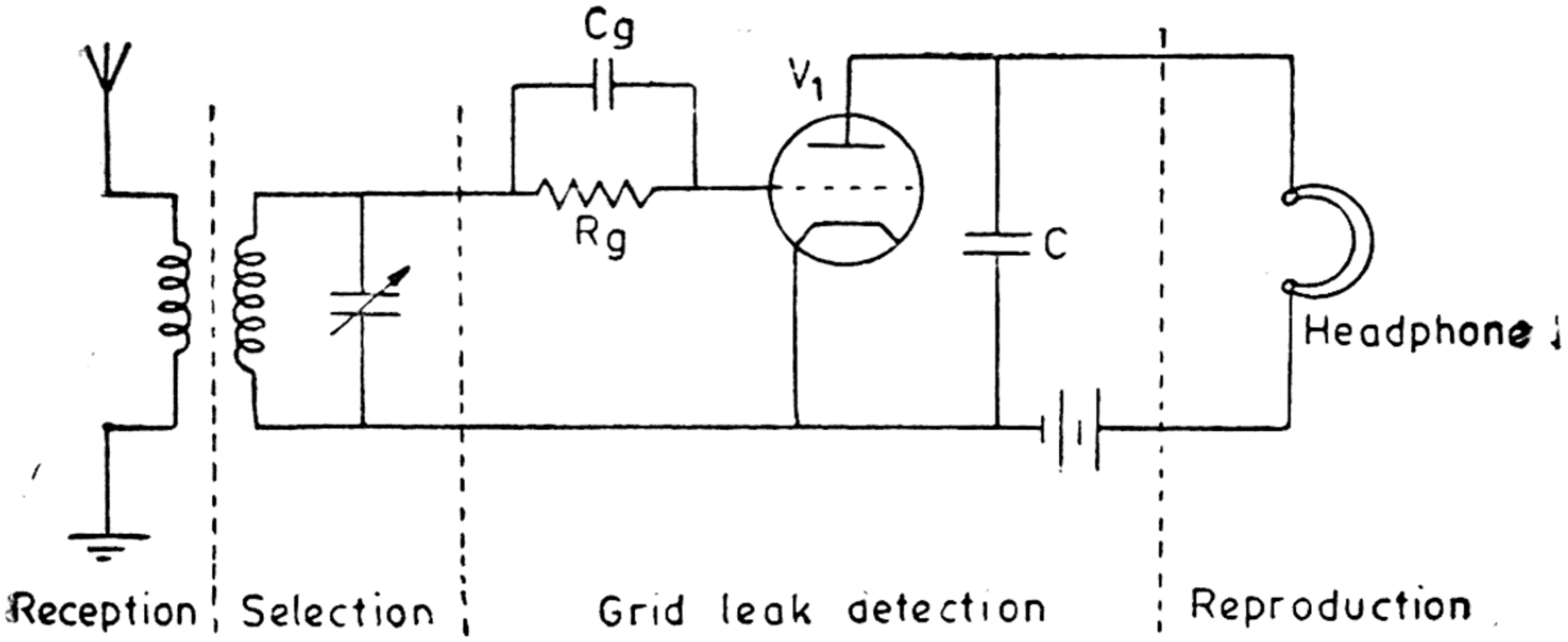


Fig. 4.31.

Sensitivity of the single tube straight receiver with grid leak detector is poor and may be improved by using *one or more stages of audio frequency amplification.* The *A.F.* amplifier stages may either be *RC*-coupled or transformer coupled. The first stage provides voltage amplification and the second stage provides power amplifications. These stages are in between detector and the loudspeaker. The a.f. output power is large enough to drive a loudspeaker.

In the *single tube straight receiver with regenerative detector*, vacuum tube V_1 along with R_g-C_g combination and bypass condenser C acts as the grid leak detector. The voltage gain of the circuit is considerably increased by the regenerative feedback or positive feedback through coil L_f which is inductively coupled to the coil L of the tuned circuit. The receiver thus has high gain, and a tendency of sustained oscillations.

Sensitivity of the single tube straight receiver with regenerative receiver may be improved by using one or more stages of A.F. amplification. The output is now large enough to drive a loudspeaker.

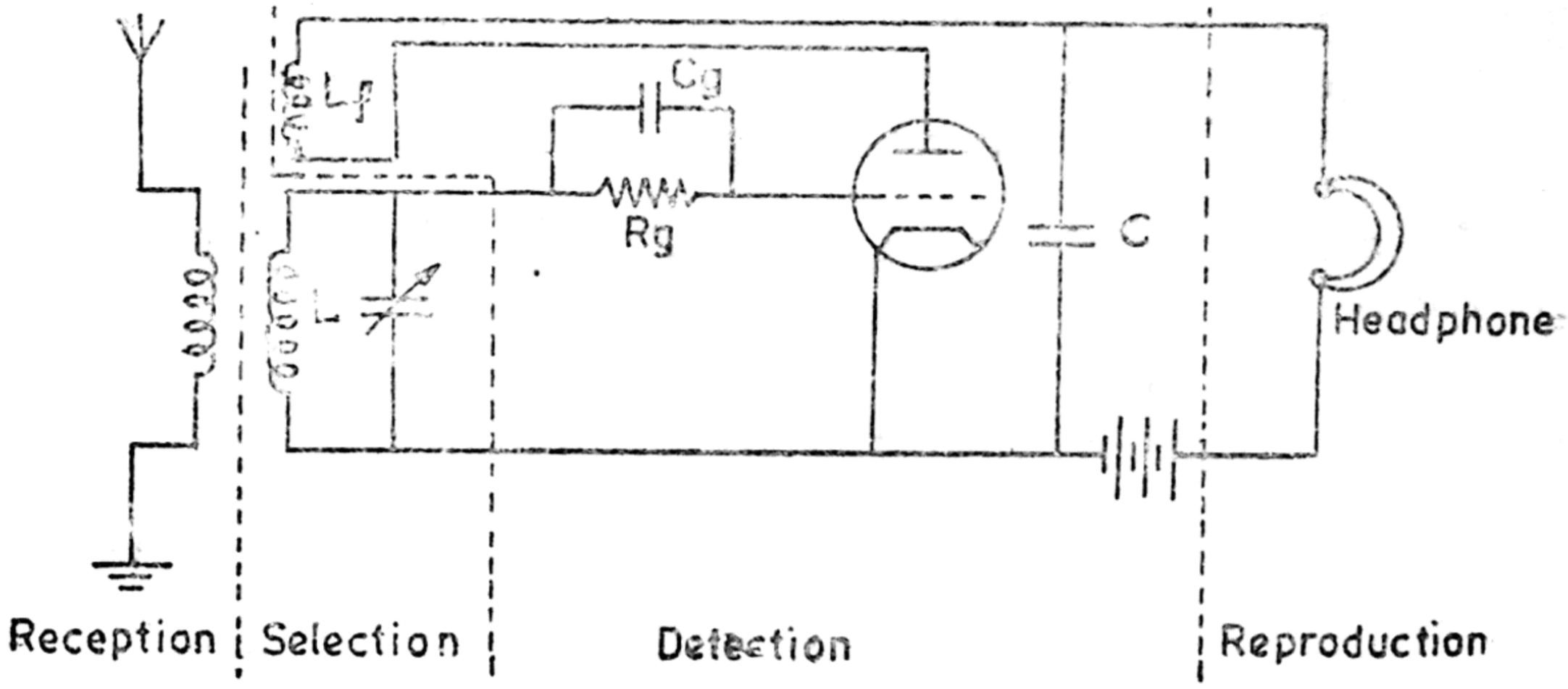


Fig. 4.32.

T.R.F. (*Tuned radio frequency*) *receivers.* The overall performance of the above discussed straight receivers is poor as the background noises are also amplified to the same degree as the signal. This defect (poor selectivity) may be removed by adding one or more additional tuning and R.F. amplifier stages. The tuning circuits are usually adjusted simultaneously by a single gang capacitor *tuning control. These tuning stages produce* amplification of the received signal in addition to improve selectivity. The R.F. amplifiers has a very narrow band width and thus amplifies only a narrow band of frequencies. Receivers that use R.F. amplifiers to provide the frequency selectivity and amplification are known as *tuned radio frequency receivers* or T.R.F. receivers. The selectivity increases as the number of tuned stages increases. The block diagram of a typical 5-valve T.R.F. receiver is shown in Fig. 4.33.

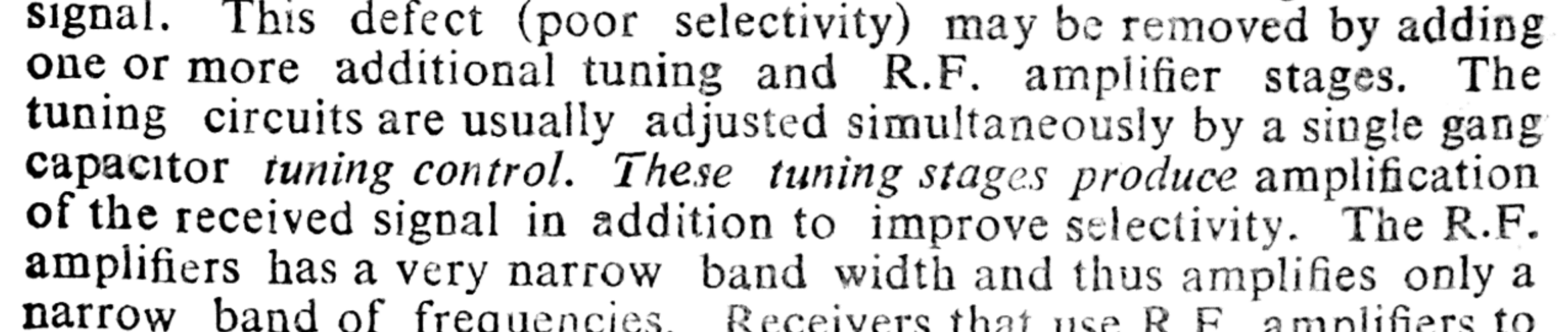


Fig. 4.33.

This has two tuned R.F. amplifier stages followed by a detector and two stages of *A.F.* amplifiers. The R.F. amplifiers work as a frequency selective network, detector rectifies and removes the R.F. component of the signal, the first stage A.F. amplifier provides voltage amplification and the second stage provides the power amplification. In this way the amplitude of the audio signal is increased to a value sufficient to drive a set of headphones or a loudspeaker. The volume control is made by varying the strength of the signal in the primary of the antenna coil.

The T.R.F. receiver with one or two tuned R.F. amplifier stages has enough selectivity. The use of too many tuning stages may increase the selectivity but at the cost of fidelity. The another disadvantage of the T.R.F. receiver is that the amplification or gain of this receiver is not constant over the tuning range of the receiver, as the selectivity decreases with the increase in frequency.

(B) Superheterodyne Receiver. In superheterodyne receiver the signal voltage is combined with a local oscillator voltage and converted into a new signal. The new signal has the same modulation but a lower carrier frequency, called *intermediate frequency* (IF). It is then amplified and detected to produce original modulation.

These receivers are better than the tuned radio frequency receivers because of the improved selectivity, improved signal to noise ratio, suitable operation and higher stage gain.

The block diagram of a superheterodyne receiver is shown in Fig. 4.34. Let us discuss different stages in brief.

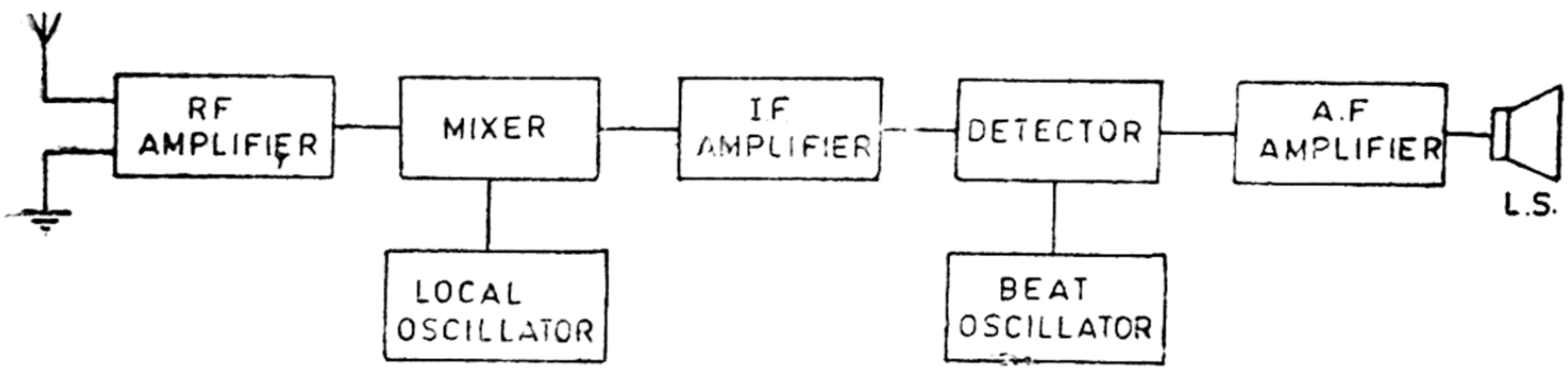


Fig. 4.34.

(1) *R.F. Amplifier.* The radio-frequency amplifier uses one or more amplifiers operating at the frequency of the incoming signal to raise the voltage level. It is used for eliminating certain forms of interference common to this receiver and thus called the preselection stage. In general pentode tube is used and a combination of R and C provides cathode biasing voltage. The output of this stage is coupled to converter (or mixer) stage by means of a transformer.

(2) *Mixer.* The incoming *R-F* signal f_s is mixed with slightly lower or higher frequency f_0 from the local oscillator. As the band of frequencies generally comprises an amplitude modulated carrier with side bands (*i.e.*, frequencies f_s, $f_s - f_m$ and $f_s + f_m$), hence on mixing the sum and difference frequencies are formed.

It is done with the help of pentagrid converter. Grids 1 and 2 and cathode work as oscillator components. Grid 2 acts as anode

for local oscillator. Radio frequency signal after preselection stage is applied to the grid 4 by means of *R-F* transformer. When the high tension is switched on, the local oscillations are set up in the oscillatory circuit which send oscillatory electron stream towards grid 3. The *R-F* signal applied to grid 4 further modulates this electron stream. The plate current of the pentagrid will then contain frequencies f_s, f_0, f_s+f_0, $f_s+f_0-f_m$, $f_s+f_0+f_m$, f_s-f_0, $f_s-f_0+f_m$, $f_s-f_0-f_m$ and their harmonics. The sum and difference terms will have amplitude proportional to the product of the amplitudes of individual frequency component. All these frequencies are not of interest. Only f_s-f_0, $f_s-f_0+f_m$ and $f_s-f_0-f_m$ are of our interest. The frequency f_0 is so chosen that $f_s-f_0=f_k$ where f_k is the desired constant frequency. It is generally intermediate in value between f_s and f_m and hence called the *intermediate frequency*.

(3) *I-F Amplifier.* The anode of the converter is connected with a tuned transformer. Since f_k is usually lower than f_s, the tuned circuits in the intermediate amplifier may be made more selective than possible at signal frequency. To have a very good response of the intermediate-frequency amplifier, the doubly-tuned transformers are used to couple the amplifier circuit at input and output both. The selectivity of the receiving system may be increased by using large number of tuned circuits each of the improved selectivity. This stage provides better gain and high selectivity.

(4) *Detector.* The output of the *I-F* amplifier is fed to the detector circuit through the secondary windings of its transformer. The modulation frequency f_m is thus derived.

For the signals such as telegraph transmission which is broken into dots and dashes, the output is only the d.c. To make them audible a separate *beat frequency oscillator* is used. Its frequency is f_y, which differs from f_k by a few hundred or few thousand cycles. Since f_k is the same for all signals, hence f_y may be fixed and preset to keep this difference and to make the dots and dashes audible.

(5) *A-F Amplifier.* The output of the detector stage is fed to the A-F amplifier through R-C coupling. The final stage of the receiver circuit, the loud-speaker thus converts etectrical energy into sound waves.

(6) *Power supply unit.* A full wave rectifier is used to provide the necessary power to the different stages of the receiver.

The superheterodyne circuits are now very much used for radio receivers on account of easy design for selectivity and high gains.

The main difficulty of this receiver circuit is the presence of image frequencies, or the possibility of reception of two signals at one control setting. For instant, two signals 545 *kc* and 1455 *kc* can be received with $f_0 = 1000$ *kc* and $f_k = 455$ *kc*. These image responses can be suppressed by introducing additional selectivity at the signal frequency to lower the strength of the undesired signal.

It is desirable to use as large intermediate frequency as possible for a good signal selectivity. But the increase of this frequency decreases the adjacent channel selectivity. To have a better compromise the value of the intermediate frequency has been selected as 465 *kHz*. For the excellent signal selectivity and the excellent adjacent channel selectivity, a *double superheterodyne receiver* or *triple detection receiver* is used. In such a receiver, the two mixer circuits are used. The signal is first reduced by first frequency converter to first intermediate frequency of high value of the order of 2 to 3 *MHz*. The second frequency converter then hydrodynes this signal to second intermediate frequency signal to provide adequate adjacent channel selectivity. The first converter provides high image signal selectivity. This receiver is named double super heterodyne receiver because the frequency heterodyning is done twice. As the frequency conversion is basically a detection, in this receiver all three detectors are used and hence the name *triple detection receiver* is given to this receiver. The block diagram of double superheterodyne receiver may be in the following order :

Receiving antenna→R.F. amplifier stages→first mixer with first local oscillator→first I.F. amplifier→second mixer with second local oscillator→second I.F. amplifier→detector→audio amplifier→loudspeaker.

4.11. FREQUENCY MODULATION RECEIVERS

These days, *F.M.* receivers find extensive use in broadcast services. These are basi cally same as the A.M. receivers. The F. M. receivers are superior to A.M. receivers on account of following reasons :

(*i*) The carrier frequencies are much higher and can be varied over a wider range.

(*ii*) The bandwidth is wider to accomodate the wide range of the signal. The F.M. stations operate on assigned frequencies ranging from 88 to 108 MHz.

(*iii*) The adjacent channel selectivity is greater.

(*iv*) The output of the receiver is practically free from external noise.

(*v*) The image frequency interference is rarer, as the intermediate frequency value is very high (~10.7 MHz) and its double

1.4 MHz) exceeds the complete span of *F.M.* band (88—108 MHz).

(*vi*) The signals of two stations operating on the same frequency can be separated, provided the interfering signal is not greater than 50% of the desired signal.

(*vii*) The intermediate frequency (I.F.) amplifier may contain one or more limiter stages, which limit the I.F. voltage to a predetermined level and thus remove all amplitude variations which may be incidently caused due to changes in the transmission path.

(*viii*) Frequency-sensitive device (F.M. demodulator) to reproduce the intelligence is used in place of A.M. detector.

(*ix*) De-emphasis circuit is connected in between F.M. demodulator and A.F. amplifier to remove the pre-emphasis inserted at the transmitter.

(*x*) For good fidelity, two or more loudspeakers are used, each reproducing a limited range of frequencies.

The block diagram of a superheterodyne F.M. receiver is shown in Fig. 4.35. Except the frequency differences, the F.M.

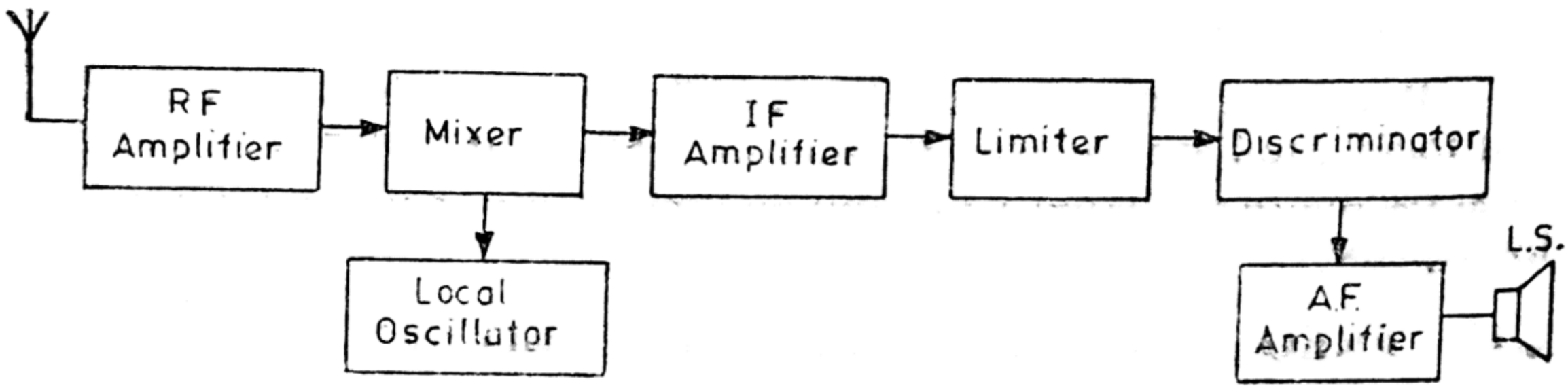


Fig. 4.35.

receiver follows the design of the A.M. type until the limiter is reached. There the varying amplitude signals are reduced to a common level and the signal passed to the F.M. demodulator (discriminator). The output of the discriminator is the original signal, which is further amplified and applied to the loudspeaker.

Exercises

Example 1. *In a tuned grid oscillator, the tuned circuit consists of a coil inductance $L=150\ \mu H$ and resistance $10\ \Omega$ in shunt with a tuning capacitor of capacitance 300 pf. If the vacuum tube used has $\mu=15$ and $r_p=10\ k\Omega$, calculate the maximum value of mutual inductance M between the grid and plate coils necessary for starting the oscillations.*

We know that $g_m=\mu RLC/M(L\mu-M)$.

Since $L\mu > M$, hence $g_m = \frac{\mu}{r_p} = \frac{\mu RLC}{ML\mu} = \frac{RC}{M}$.

or $$M = \frac{RCr_p}{\mu} = \frac{10 \times 300 \times 10^{-12} \times 10 \times 10^3}{15}$$

$$= 2 \times 10^{-6} \text{ henry.}$$

Example 2. *A tuned anode oscillator has a tube of amplification factor $\mu = 30$ and dynamical plate resistance $r_p = 30\ k\Omega$. The anode circuit resonant frequency is 400×10^3 radians/sec and inductance $L = 40$ milli henry. If the frequency of oscillation is 0.05% above the resonant frequency, calculate the minimum value of mutual inductance between the grid and plate coil to sustain continuous oscillations. Assume effect of inter-electrode capacitances to be negligible.*

Angular frequency of oscillation $\omega = \omega_0(1 + R/2r_p)$.

$$\therefore \quad \frac{\omega}{\omega_0} = 1.0005 = 1 + \frac{\mu}{2r_p} = 1 + \frac{R}{2 \times 30{,}000}$$

or $$R = 30\Omega.$$

Mutual conductance $g_m = \frac{\mu}{r_p} = \frac{30}{30{,}000} = 10^{-3}$ mho.

Tuning capacitance $C = \frac{1}{\omega_0^2 L} = \frac{1}{(4 \times 10^5)^2 \times 40 \times 10^{-3}}$

$$= 156.2\ pf.$$

The minimum value of mutual inductance M to sustain oscillation is given by

$$M = \frac{L + CRr_p}{\mu} = \frac{40 \times 10^{-3} + 156.2 \times 10^{-12} \times 30 \times 30 \times 10^8}{30}$$

$$= 1.338 \times 10^{-3} \text{ henry.}$$

Example 3. *A coil in a hartley oscillator has two sections of inductances of 90 mH and 30 mH. If the latter portion is in the grid circuit, the tuning capacitance is of value 100 pf and μ of the valve is 21. Calculate the critical inductance for maintaining oscillations and the oscillator frequency.*

We know that for a harmonic oscillator

$\mu = (L_1 + M)/(L_2 + M)$, or $21 = (90 + M)/(30 + M)$

or $$M = -27\ mH.$$

The oscillator frequency $f = \frac{1}{2\pi\,[C(L_1 + L_2 - 2M)]^{1/2}}$

$$= \frac{1}{2 \times 3.14\,[100 \times 10^{-12}(90 + 30 + 2 \times 27) \times 10^{-3}]^{1/2}}$$

$$= 38\ kc/sec.$$

Example 4. *In a Colpitts oscillator, the values of the inductance and capacitances in the tuned circuit are : $L=40$ mH, $C_1=100$ pf and $C_2=500$ pf. Find the frequency of oscillation. If the output voltage is 100 V, find the feedback voltage. What should be the minimum gain if the frequency is changed by changing L alone? What should be the value of C_1 for a gain of 10? What will be the frequency now?*

In a Colpitts oscillator, a series combination of C_1 and C_2 is in parallel with inductance L and the frequency of oscillation is given by

$$f=\frac{1}{2\pi\sqrt{LC_{eq}}}=\frac{1}{2\pi}\ \frac{1}{\sqrt{LC_1C_2/(C_1+C_2)}}.$$

Substitution of the numerical values gives

$$f=\frac{1}{2\pi\sqrt{40\times10^{-3}\times100\times10^{-12}\times500\times10^{-12}/(100\times10^{-12}+500\times10^{-12})}}$$

$=87.2\ k\ Hz.$

Since the output voltage is across C_1 and is proportional to X_{C_1} and the feedback voltage is across C_2 and proportional to X_{C_2}.

$$\therefore\qquad \frac{V_0}{V_{fb}}=\frac{X_{C_1}}{X_{C_2}}=\frac{1/\omega C_1}{1/\omega C_2}=\frac{C_2}{C_1}.$$

or $\qquad V_{fb}=V_0C_1/C_2=100\times100\times10^{-12}/500\times10^{-12}=20$ volts.

Since the gain depends upon C_1 and C_2 only, and is independent of L. Therefore the gain

$$=500\times10^{-12}/100\times10^{-12}=5.$$

For the gain equal to 10, the value of C_1 is given as

$$C_2/C_1=10 \text{ or } C_1=C_2/10=50\ pf.$$

In this case the frequency

$$f=\frac{1}{2\pi\sqrt{40\times10^{-3}\times50\times10^{-12}\times500\times10^{-12}/(50\times10^{-12}+500\times10^{-12})}}$$

$=118.1\ kHz.$

Example 5. *In a linear series plate modulated amplifier, the plate supply voltage is 300 volts and the dc plate current under unmodulated condition is 20 amp. The sinusoidal modulating voltage has amplitude of 150 volts. The unmodulated output carrier power is 4.5 k.W. Calculate (i) the modulation index. (ii) carrier power, (iii) plate circuit efficiency and (iv) plate dissipation under unmodulated and modulated conditions.*

Modulation index $m_a = \frac{E_m}{E_{bb}} = \frac{150}{300} = \frac{1}{2}$.

d.c. power from plate supply source $P_{bb} = E_{bb}.\ I_{bb} = 300 \times 20$
$= 6000$ watts.

As a.c. output carrier power is 4.5 $kW = 4500$ watts, hence the plate circuit efficiency $\eta_p = (4500/6000) \times 100 = 75\%$.

Under modulated condition, the carrier power $P = P_{ac}\,[1 + \frac{1}{2} m_a^2]$
$= 4500[1 + \frac{1}{2}(0.5)^2] = 5062.5$ watts.

Therefore the plate dissipation under unmodulated condition

$P_{pc} = P_{bb} - P_{ac} = 6000 - 4500 = 1500$ watts

and in under modulated condition $= P_{pc}\ [1 + \frac{1}{2}\ m_a^2]$
$= 1687.5$ watts.

Example 6. *An AM broadcast station transmits 10 kW of carrier power. If the modulation index is 0.7, find the power in each side band, the total transmitted power and the percentage of total power to the carrier.*

The total power in the modulated wave = Power content of the carrier + Power content of the upper and lower side bands, *i.e.*,

$$P_T = P_C + P_{USB} + P_{LSB}$$

Since $P_{USB} = P_{LSB} = \frac{m_a^2}{4} P_C$, hence

$$P_T = P_C(1 + \tfrac{1}{2} m_a^2).$$

In this problem, $m_a = 0.7$ and $P_C = 10kW$, hence the power in each side band

$$P_{USB} = P_{LSB} = \tfrac{1}{4} m_a^2 P_C = \tfrac{1}{4} \times (0.7)^2 \times 10 \times 10^3$$
$$= 1.225\ kW.$$

$\therefore$ $$P_T = 10 \times 10^3\ [1 + \tfrac{1}{2} \times (0.7)^2\ ] = 12.45\ kW$$

$$P_C/P_T = (1 + \tfrac{1}{2} m_a^2)^{-1} = 0.803.$$

$\therefore$ % of the total power to the carrier = 80.3%.

Example 7. *An audio signal which is given by 25 sin $(2\pi\ 1000t)$ modulates a carrier given by 75 sin $(2\pi\ 1{,}50{,}000t)$. (a) Calculate the modulation factor and percentage modulation. (b) What is the frequency of the audio signal and that of the carrier. (c) What frequencies would show up in spectrum analysis of modulated wave ?*

In this problem, the peak value of the audio signal=25 and that of carrier signal is 75. Therefore

$$\text{modulation factor } m_a = \frac{25}{75} = \frac{1}{3}.$$

$$\therefore \quad \text{Percentage modulation} = m_a \times 100\% = 33.33\%.$$

Frequency of the audio signal $f_a = 1000$

and the frequency of carrier signal $f_a = 1{,}50{,}000$.

Therefore the frequencies that show up in the spectrum of a modulated wave are

$$f_c = 1{,}50{,}000 \ Hz$$

$$f_c + f_a = 1{,}50{,}000 + 1{,}000 = 1{,}51{,}000 \ Hz$$

$$f_c - f_a = 1{,}50{,}000 - 1{,}000 = 1{,}49{,}000 \ Hz.$$

Example 8. *An AM transmitter is to deliver a total power of 100 watts to an antenna. The modulation is to be 100%. Determine the power contained of the carrier frequency.*

The power in modulated wave is related with the power in carrier wave as

$$P_m = P_c \ (1 + \tfrac{1}{2} m_a^2).$$

In the present problem $m_a = 100\% = 1$, $P_m = P_T = 100$ watts.

$$\therefore \qquad 100 = P_c \left(1 + \tfrac{1}{2} \right) \text{ or } P_c = 100 \times \frac{2}{3} = 66.66 \text{ watts.}$$

Example 9. *A tank circuit of the oscillator in a radio receiver has values $L = 80 \mu H$, $C = 350$ pf, $Q_{eff} = 20$ and $\mathcal{E} = 10$ volts. Find (a) the value of C for the resonance frequency of 10^6 cycles/sec, (b) the power supplied to the tank circuit, and (c) the energy stored per cycle in the tank circuit.*

The tank circuit should be adjusted so that its resonant frequency corresponds to that of the oscillator. The L and C should be related with the resonance frequency as

$$f_r = 1/2\pi(LC)^{1/2} \text{ or } C = 1/4\pi^2 f_r^2 L.$$

$$\therefore \quad C = \frac{1}{4 \times 3.14^2 \times 10^{12} \times 80 \times 10^{-6}} = 316 \mu\mu f.$$

The impedance of the tank circuit at resonance

$$Z = Q_{eff} X_L = Q_{eff} 2\pi f_r L = 20 \times 2 \times 3.14 \times 10^6 \times 80 \times 10^{-1}$$

$$= 10053 \text{ ohms.}$$

Hence the power supplied to the tank circuit

$P=\mathcal{E}I=\mathcal{E}^2/Z=10^2/10053=9.95\times10^{-3}$ watt.

Energy stored per cycle in the tank circuit

$W=C\mathcal{E}^2=316\times10^{-12}\times10^2=3.16\times10^{-8}$ watt/cycle.

Example 10. *An AM-receiver is tuned to 1000 kHz. Determine the local oscillator frequency and the image frequency.*

For amplitude modulated receiver the intermediate frequency is 455 *kHz*. Since the local oscillator is tuned with the RF amplifier so that its frequency always differs from the signal frequency by an amount equal to the receiver's intermediate frequency. Thus the local oscillator frequency

$f_0=fs+455=1000+455=1455$ *kHz*.

Since the image frequency=signal frequency+2× intermediate frequency.

$\therefore$ image frequency$=1000+2\times455=1910$ *kHz*.

Oral Questions

1. What is the function of a vacuum tube in an oscillator circuit.
2. What is meant by negative resistance ?
3. How may feedback oscillator circuit be classified ?
4. What is the principle of magnetostriction oscillator ?
5. What are the basic requirements of an oscillator ?
6. What is meant by self-biasing of the oscillator ?
7. What is Barkhausen criterion ?
8. What are the differences between tuned anode and tuned grid oscillators ?
9. What is the principle of Hartley oscillator ?
10. What is meant by modulation ?
11. How may the modulation process be classified.
12. What are the relative merits and demerits of grid bias modulation over linear series plate modulation ?
13. What is meant by detection ?
14. How may diode be used as a detector ?
15. How may the transmitters be classified ?
16. What is meant by receivers ? How are they classified ?
17. What is the value of intermediate frequency for AM receiver ?
18. What is the value of image frequency for FM receiver ?

Problems

1. In a vaccum tube oscillator, using a vacuum tube having $\mu=100$ and $r_p=50\ k\Omega$, consists a tuned circuit of resonance frequency 500 kilo hertz. If the inductance of the tuned circuit is 1 milli henry and the effective Q-value of anode circuit is 10, calculate (*i*) the frequency of oscillation and (*ii*) minimum value of mutual inductance between plate and grid coils required for sustained oscillations.

(50.053 *kHz*, 25.4×10^{-7} *henry*)

2. In a tuned grid oscillator the tuned circuit consists of coil of inductance $L=200\ \mu H$, resistance $R=15\Omega$ and shunt tuning capacitor of capacitance 400 *pf*. If the minimum value of mutual inductance between grid and plate coils necessary to sustained oscillations is $4\mu H$. Calculate the dynamic plate resistance of the vacuum tube used, of which amplification factor is 20.

(10*k*Ω)

3. A parallel fed Hartley oscillator circuit uses an RF-choke of 0.5 *m*H and a blocking capacitor of 0.005 *μf*. Calculate the reactance of the choke and of capacitance at 2.2 mega cycles.

(34540Ω, 14.5Ω)

4. In a Hartley oscillator, the two sections of the coil have inductance of 4 *m*H and 1 *m*H. The capacitor has capacitance of 500 *pf*. Find the frequency of oscillation neglecting mutual inductance. What fraction of the output is fed back ?

(100 *kHz*, ~0.25)

5. In a Wien-bridge type of oscillator frequency selective network employs resistors and capacitors. What would be value of R when frequency of 1 *kHz* is required with C=0.01 *μf* ? What must be the value of R_3, if $R_4=14\ k\Omega$.

(15.9 *k*Ω, 3 *k*Ω)

6. A certain X-cut crystal whose resonant frequency is 4 *MHz* has an inductance of 0.3 henry and effective series resistance is 10,000Ω. What is the *Q*-value of the crystal ?

(753.6)

7. In a linear series plate modulated amplifier, the plate supply voltage is 300 volts, the amplitude of modulating voltage is 120 volts. The *dc* plate current under modulated condition is 8.5 amp and the unmodulated carrier output power is 1.8 kilo watts. Calculate (i) modulation index, (ii) plate circuit efficiency, (iii) modulated carrier power and (iv) plate dissipation under unmodulated and modulated conditions.

(0.4, 70.6%. 750, 810 *watts*)

8. An AM transmitter is to deliver a total power of 200 watts to an antenna. Calculate (i) the power contained at the carrier frequency and (ii) the power contained within each sidebands, if the per cent of modulation is given as 50% ?

(177.8 *watts*, 1.1 *watts*)

9. A 2 *kHz* audio signal having an amplitude of 15 volts amplitude modulates a 100 *kHz* carrier signal which has a peak value of 25 volts when not modulated. (i) Calculate the modulation factor and percentage modulation, (ii) What frequencies appear in a spectrum analysis ? What trigonometric equations for carrier and audio signal.

(0.6, 60%, 100 *kHz*, 120 *kHz*, 93 *kHz*, 25 sin (2π 100,000 *t*), 15 sin (2π 2000 *t*)

10. The *r.m.s* value of antenna current of an AM transmitter is 4 amp when only the carrier is sent. It increases to 4.46 amp, if it is amplitude modulated. Calculate the percentage modulation. Determine the antenna current when the modulation index is 0.9.

(69.8%, 15.8 *amp*)

11. The tank circuit of the oscillator in a certain radio receiver uses $L=80\ \mu H$, $C=316\ \mu f$, $f_r=1$ *M* c/sec and $E_t=10$ volts. Calculate (a) the plate current required to produce the desired voltage at the tank circuit, (b) R_{eq} and (c) the ratio of the energy stored per cycle to the energy dissipated per cycle ?

12. An FM receiver is tuned to 107.5 *MHz*. Find the local oscillator frequency and the corresponding image frequency.

(118.2 *MHz*, 128 *MHz*)

5

Semi-Conductor Devices

5.1. ELECTRONS IN SOLIDS

The operation of semiconductor devices depends upon the motion of charge carriers within them. The study of electron energies in atoms, molecules and solids is thus desired. The classical mechanics and electromagnetic theory failed to explain the atomic spectra. Quantum mechanics was used by Bohr to explain the energy level diagram for the hydrogen atom. Bohr showed that the electron in an atom can exist in a series of levels, called states, represented by a number '*n*' known as **principal quantum number.** Electrons with a common *n* form a *shell.* Electrons in a shell differ in angular momentum, magnetic orientation or electron spin and form subshells. The quantum numbers *l* and *m* are used for this purpose. The **orbital angular momentum quantum number** '*l*' is associated with the ellipticity of the orbit of the electron, *i.e.*, the angular distribution of the electron. The **magnetic quantum number** '*m*' is associated with the magnetic orientation.

Energy Bands in Solids—Two atoms coming nearer to each other form a suitable bond. When the atoms are in such close proximity that the forces surrounding an atom are influenced by another atom the energy levels of each atom are disturbed. This leads to splitting of energy levels into closely spaced states. A similar situation exists when many atoms come together to form a solid. The N atoms in proximity will split the outer most levels into *N* levels. When these levels are too close, we call them a band of energy levels.

In the crystals, there is an array of atoms. The inner orbits continue to be attached to specific atoms while outer levels are shared by the neighbouring atoms and thus form bands. The one dimensional energy level diagram for a crystal containing several atoms is shown in Fig. 5.1. The band consisting of valence electrons is called the **valence band.** It is shared by the crystal as a whole. At 0°K, when the electrons are in their lowest energy

states, there will be unfilled higher energy levels adjacent to the valence levels. These excitation levels of the atoms form a band,

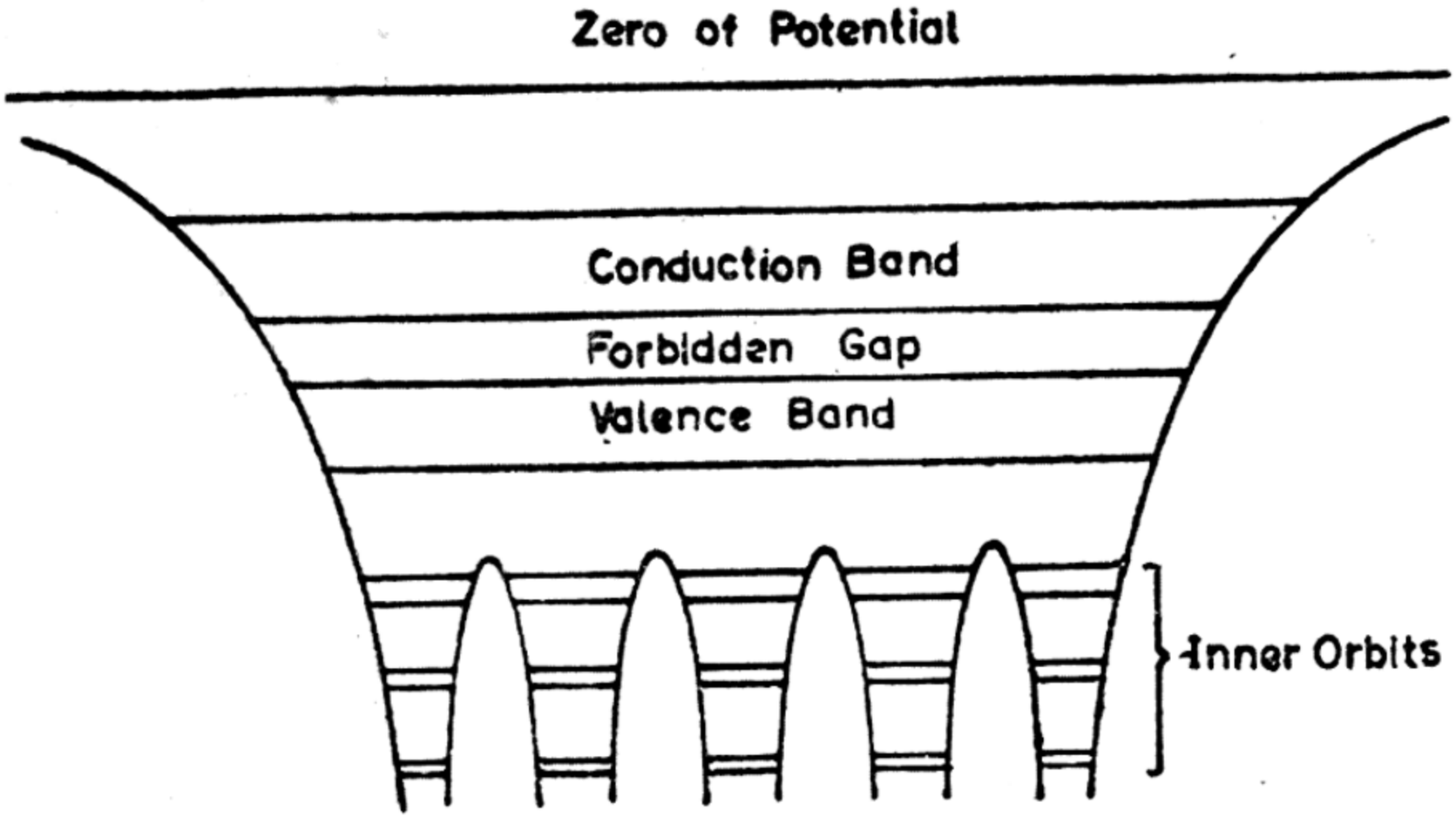


Fig. 5.1.

known as **conduction band.** This band corresponds to the first excited states and is normally almost empty. The gap between the valence and conduction bands is called the forbidden gap or **forbidden-energy band,** because electrons may not exist in the crystal with these energies. As the chemical properties of atoms are determined by the valence electrons, the electrical properties of the solid are determined by these energy bands.

The energy levels for a substance as a function of atomic spacing is shown in Fig. 5.2 (*a*), we see that the outermost (highest)

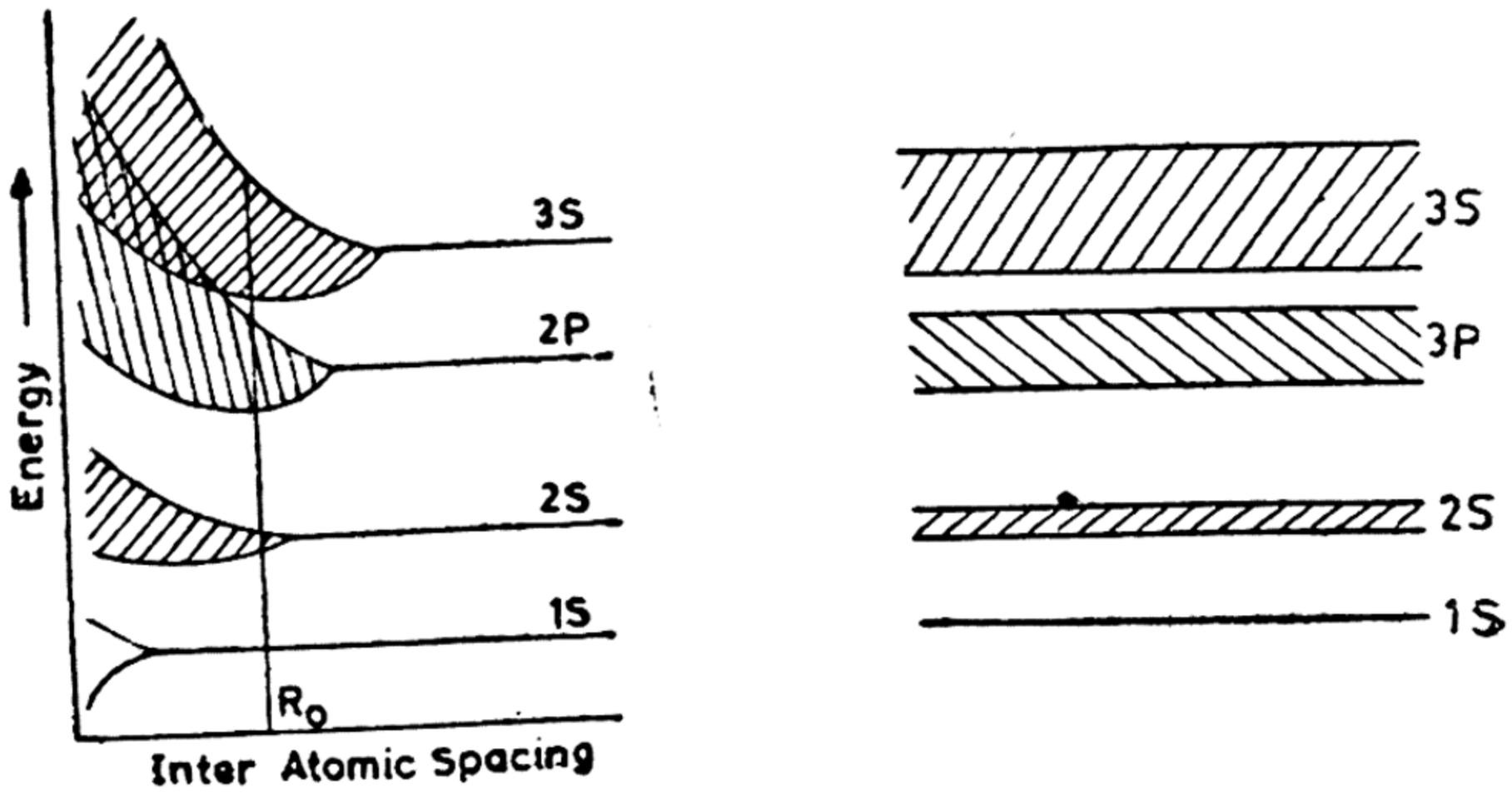


Fig. 5.2.

energy levels split first and the inner shells are successively affected, because the outer shells come under the influence of neighbouring atoms most readily as the spacing between atoms is decreased.

The splitting in the inner shells is extremely small. At a particular value of lattice spacing R_0, the band structure for the crystal may be drawn as shown in Fig. 5.2 (*b*).

5.2. CRYSTALLINE SOLIDS

We shall first consider the crystal structure, since the electronic properties of solids are intimately related to their lattice structures. There are two primary kinds of forces involved in the formation of solids : electrostatic attraction and quantum mechanical force. We concentrate our attention on crystalline solids, which have perfect enough structures to be readily described and useful in semi-conductor devices.

Crystals are formed of atoms in a well-defined regular array called a *crystal lattice*. The following types of crystal structures are of particular importance :

(*a*) **Simple cubic**—This lattice contains atoms lying at the corners of a cube. The example is *cesium chloride.*

(*b*) **Face-centered cubic**—This is simply a cubic array with an additional atom centered in each of the six faces of the cube. The example is *sodium chloride.*

(*c*) **Body-centered cubic**—This is a simple cubic array with an additional atom at the centre of the cube. Some metals have this structure.

(*d*) **Diamond or Tetrahedral**—In this lattice, each atom has four nearest neighbours, bound by sharing of electrons. This type of binding is called covalent or tetrahedral binding. The examples are *silicon and germanium*, the important crystals used in semiconductor electronics.

(*e*) **Zinc blend**—This structure is like the diamond except that it is made up of atoms of two kinds. The binding is usually partially covalent and partially ionic. *Cadmium sulphide* is the example.

Three types of binding, the ionic, covalent and metallic, are important. In the ionic binding the outershell electron of the alkali is transferred to the halide atom and the attraction is thus purely electrostatic one. The four elements carbon, silicon, germanium and gray tin form covalent crystals, having crystal structure tetrahedral or diamond. The attractive force is due to the sharing of electrons between the half filled outer shell and its four nearest neighbours. The metallic binding results from the sharing of electrons among atoms. The electrons in metallic bound crystals (cohesive crystals) move freely from one atom to the next and behave like an ordinary gas.

Metals, Insulators and Semi-conductors—The solids are often classified as metals, insulators and semi-conductors on the basis of electrical conduction in them. In metals the valence electrons are free at room temperature and able to take part in electrical conduction. As a result of this the metals are good conductors having resistivity in the range 10^{-8} to 10^{-6} ohm-m. In insulators, the valence electrons are all rigidly bound even at high temperatures. The insulator materials (quartz, ceramics, plastics, etc.) have resistivities at 300°*K* in the range 10^{7} to 10^{18} ohm-m. (In the semi-conductors, the valence electrons are strongly coupled by covalent bonds. At 0°*K*, no free electron is available for electrical conduction. As the temperature of the semi-conductor crystal is raised, thermal energy creates vibration of the atoms about their lattice position. A few of the electron may acquire sufficient energy to break their covalent bond. These electrons contribute electrical conduction.)

At 300°*K*, the semi-conductor materials have resistivities in the range 0.4 to 2500 ohm-m. These resistivities of the crystals may be explained by **band theory** as follows :

In monovalent metals, at 0°*K* all electrons lie in their lowest energy levels, represented by filled levels of valence energy band. The higher unfilled levels, represented as conduction band, are just above the valence band. In an insulator the energy difference between the top of the filled valence levels and the lowest level of the conduction band, known as gap energy E_G, is very large. The electron can not thus bridge the forbidden energy region and remains in the valence band. In a semi-conductor the filled valence and unfilled conduction bands are separated by a very narrow forbidden energy gap. The electrons lying near the top of the filled

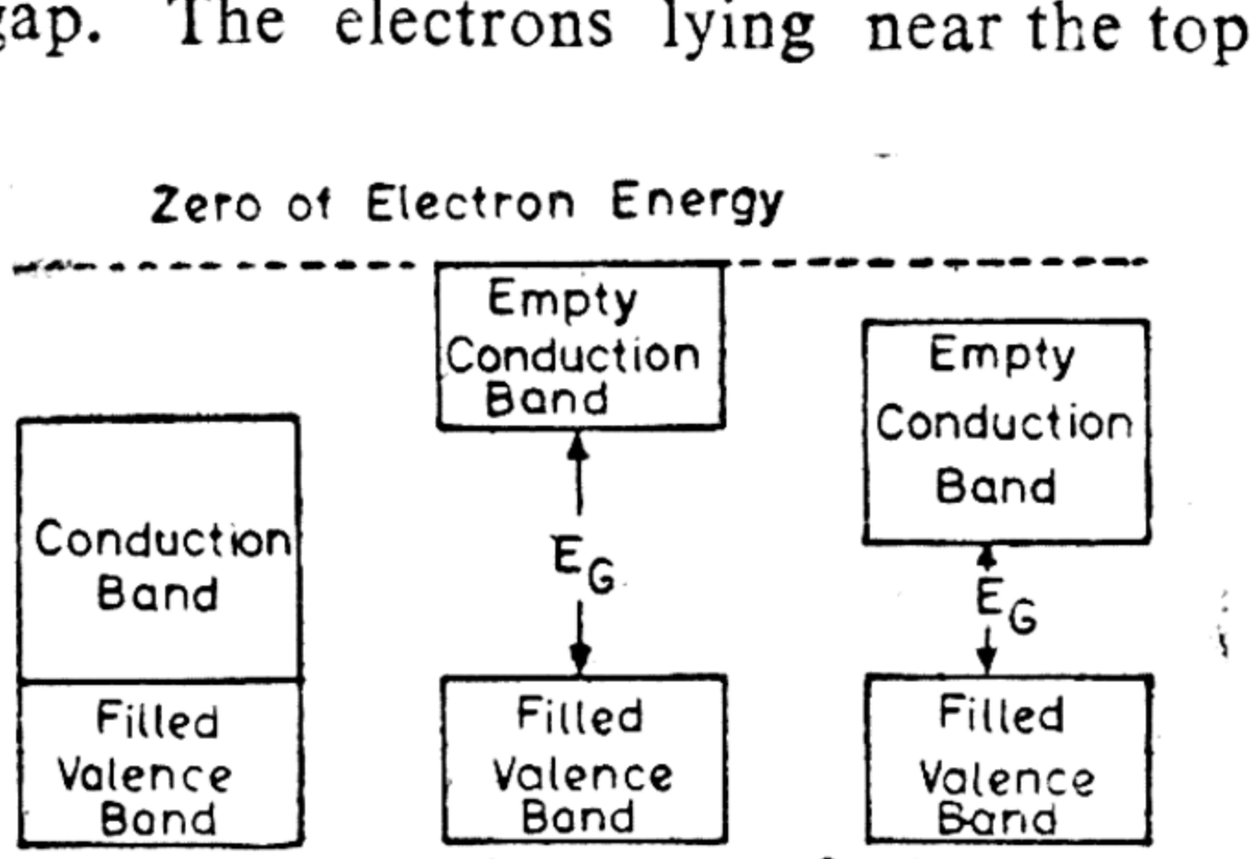


Fig. 5.3.

band can be excited to the unfilled band after the absorption of a relatively small amount of energy. The vacant electron sites or holes are left in the valence band of levels. The valence band and conduction band and their relative spacings are shown in Fig. 5.3.

5.3. SEMI-CONDUCTORS

Mainly the semi-conductors are of two types : (*i*) *Intrinsic*, pure or natural semi-conductors and (*ii*) *Extrinsic*, impure or artificial semi-conductors.

(A) Intrinsic Semi-Conductors. Silicon and Germanium are intrinsic or pure semi-conductors. Their crystals have cubic lattice structure similar to diamond. They are tetravalent (member of fourth group of periodic table) and consists 14 and 32 electrons (atomic numbers) respectively. The electron distribution for silicon in different shells is 2, 8 and 4. Thus the inner shells are completely filled and the outermost shell consisting four electrons (valency electrons) is an incomplete shell. The electron distribution for germanium is 2, 8, 18 and 4. Similar to the silicon, its outermost shell is incomplete with its four valency electrons. So the physical and chemical properties of silicon and germanium are similar to ea h other.

In the crystal lattice, each of germanium or silicon is bound to its four nearest neighbours by shearing four of its valence electrons by four valence-bonds. Each of the bond consists two valence electrons one from each atom. Each atom is thus surrounded by a complete and stable shell of eight electrons. Due to this configuration the free electrons are not available at absolute zero for conduction and the semi-conductor behaves as an insulator. With the increase in temperature the kinetic energy of electrons increases, few of the valency bonds break up and electrons become free, so due to this supply of thermal energy the random motion of electrons takes place. As soon as electron leaves the valence bond an electron vacancy is created there in the valency bond. This empty space left behind by the electron in the crystal structure is commonly known as *hole*. So an *electron-hole* pair is produced, the number will increase with the increase in temperature. Both electron and the hole take part in the electrical conduction, the hole acts as a positive charge (*equal to electron*) carrier. The process of conduction by means of electron-hole pair due to thermal energy is known as *intrinsic conduction*. Under the effect of external electric field or thermal agitation the electron from a nearby bond may fill up a hole and in turn leaves a new hole in the position previously occupied by the electron. So in this way the hole is not destroyed but its location is shifted. When a free electron encounters a hole it re establishes the broken covalent bond, the excess of energy is released in the form of electromagnetic radiation (heat or

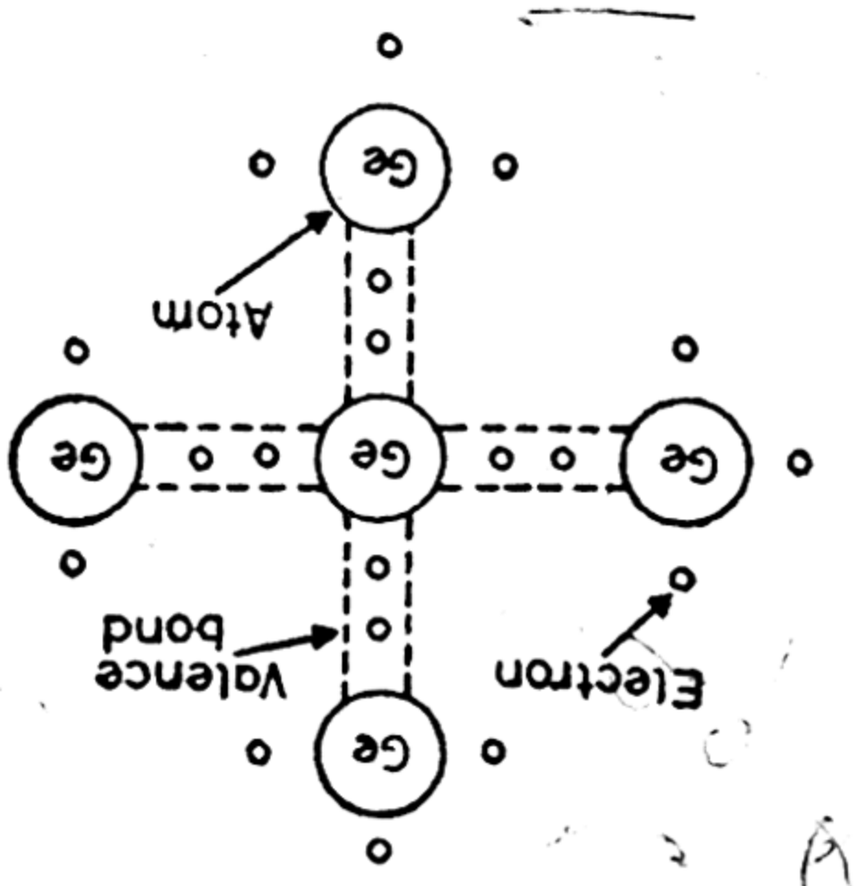


Fig. 5.4.

light). This released energy may be further used to break the another covalent bond and thus to create new electron-hole pairs.

When a sufficient thermal energy ($>E_g$, the gap energy) is absored by the electrons occupying states at the top of valence band, it is possible to transfer energy to the empty states of conduction band. Such electrons thus contribute to conductivity and the crystal is called *intrinsic semi-conductor.*

The difference between an insulator and an intrinsic semi-conductor can be described by a function

$$f(E)=1/[e^{(E-E_f)e/kT}+1], \qquad ...(1)$$

known as Fermi distribution function. E_f is known as Fermi energy. Here it is given in eV. At the absolute zero of temperature, $f(E)=1$ for $E<E_f$ and $f(E)=0$ for $E>E_f$. As the electron states in valence band are occupied at $T=0$, hence E_f must be greater than the energy of the uppermost state in the valence band. Similarly, as the conduction band is completely empty at $T=0$, hence E_f must be less than the lowest level of conduction band.

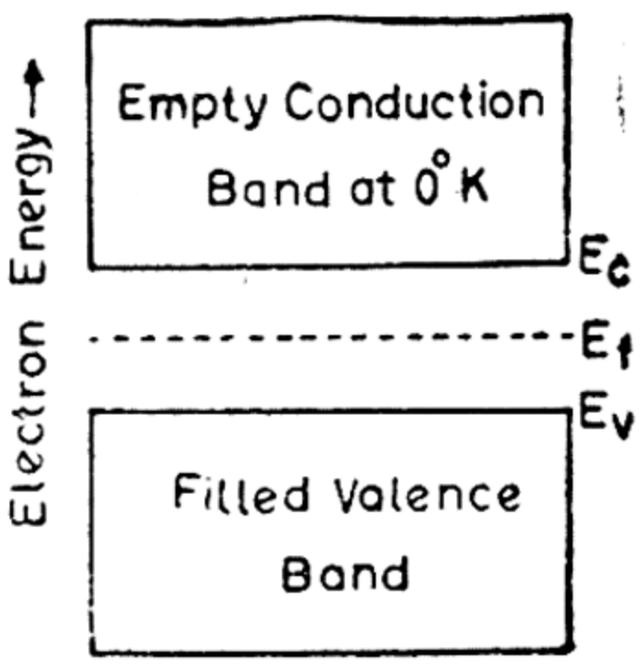


Fig. 5.5.

According to the Fermi-Dirac energy distribution the number of electrons occupying energy states between E and $E+dE$ in a unit volume is given by

$$N(E)\,dE = Z(E)\,f(E)\,dE, \qquad ...(2)$$

where $Z(E)=CE^{1/2}$ and $C=4\pi\,(2\,m_e^*)^{3/2}\,h^{-3}$. Here m_e^* is the effective mass of an electron in the semi-conductor.

The number of free electrons per unit volume of a metal at 0° K for $E<E_f$ [*i e.*, $f(E)=1$] is given by

$$N=\int N(E)\,dE = C\int_0^{E_f} E^{1/2}dE$$

$$\text{or} \qquad E_f=\frac{h^2}{2m_e^*}\left(\frac{3N}{8\pi}\right)^{2/3}=3.64\times10^{-19}\,N^{2/3}\,eV. \qquad ...(3)$$

In an intrinsic semi-conductor at a temperature well above 0°*K*, there are thermal electrons in the conduction band and equal number of holes in the valence band. Shockley has suggested that the number of energy states in the conduction band varies as

$$Z(E)=C(E-E_c)^{1/2}, \qquad ...(4)$$

where E_c is the energy at the bottom of the conduction band.

Thus the density of thermal electrons in the conduction band will be given by

$$n_e = \int_{Ec}^{\infty} C(E-E_c)^{1/2}\ e^{-(E-E_f)e/kT}\ dE. \qquad ...(5)$$

The upper limit of the integral is put as infinity for convenience in integration, however, the thermal electrons will remain in the lower energy levels of the conduction band.

Let $x = (E-E_c)\ e/kT$, then

$$n_e = C\left(\frac{kT}{e}\right)^{3/2} e^{-(E_c-E_f)e/kT} \int_0^{\infty} x^{1/2}\ e^{-x} dx.$$

$$= 2\left(\frac{2\pi m_e^* k}{h^2}\right)^{3/2} T^{3/2}\ \ e^{-(E_c-E_f)e/kT}$$

$$= N_c\ \ e^{-(E_c-E_f)e/kT}. \qquad ...(6)$$

Similarly the density of holes states near the top of the valence band will be given by

$$n_h = \int_0^{E_v} Z(E)\ [\ 1-f(E)]\ \ dE. \qquad ...(7)$$

Shockley has suggested the number of energy states in the valence band as

$$Z(E) = C(E_v-E)^{1/2}. \qquad ...(8)$$

Since E_f lies in the forbidden gap, therefore, will be large with respect to any E in the valence band. Thus

$$1-f(E) \simeq e^{(E-E_f)e/kT}. \qquad ...(9)$$

$$\therefore \qquad n_h = \int_0^{E_v} C(E_v-E)^{1/2}\ e^{(E-E_f)e/kT}\ dE$$

$$= 2\left(\frac{2\pi m_h^* k}{h^2}\right)^{3/2} T^{3/2}\ e^{-(E_f-E_v)\ e/kT}$$

$$= N_v\ e^{-(E_f-E_v)\ e/kT}. \qquad ...(10)$$

Here m_h^* is the effective hole mass, which is nearly equal to m_e^* Therefore $N_c \simeq N_v$. Since in an intrinsic semi-conductors, $n_e = n_h$, therefore, with these conditions, we get

$$E_c - E_f = E_f - E_v \qquad \text{or} \qquad E_f = \tfrac{1}{2}(E_v + E_c), \qquad ...(11)$$

i.e., the Fermi energy lies half way between the top of the valence band and the bottom of the conduction band in an intrinsic semi-conductor.

(B) Extrinsic Semi-Conductors. At normal temperature the intrinsic conductivity of a pure semi conductor is very small and is of little importance The conductivity of the intrinsic (pure or natural) semi-conductors can be increased by encorporating (called *doping*) a very small percentage of selected impurities in the host crystals (*Ge* or *Si*). The resulting material is known as *impurity* or *extrinsic semi-conductor*. A very small amount nearly one atom of impurity in 10 million atoms of pure semi-conductor changes the concentration of hole and electron to a large extent. These impurities are either of fifth group [phosphorous *P*, antimony *Sb*, o arsenic *As*] or of third group [*indium In, boron B, aluminium Al o gallium Ga*].

These impurity atoms have size of the same order as that o germanium atoms. Each of the impurity atom thus replaces th germanium atom. As the percentage of impurity is very small, normally each impurity atom is surrounded by 10^8 germanium atoms. If the impurity atom is pentavalent, having five valence electrons, out of the five, only four valence electrons of the impurity atom combine with the four nearest germanium atoms by covalent bonds. The remaining fifth electron remains loosely bound with the parent impurity. Only a very small amount of energy is sufficient to excite this electron. This liberated electron is then free to move in the crystal lattice and thus works as conduction electron. This energy, also known as ionisation energy, is very small (0.01 *eV* for germanium with arsenic impurity) in comparison to the energy required

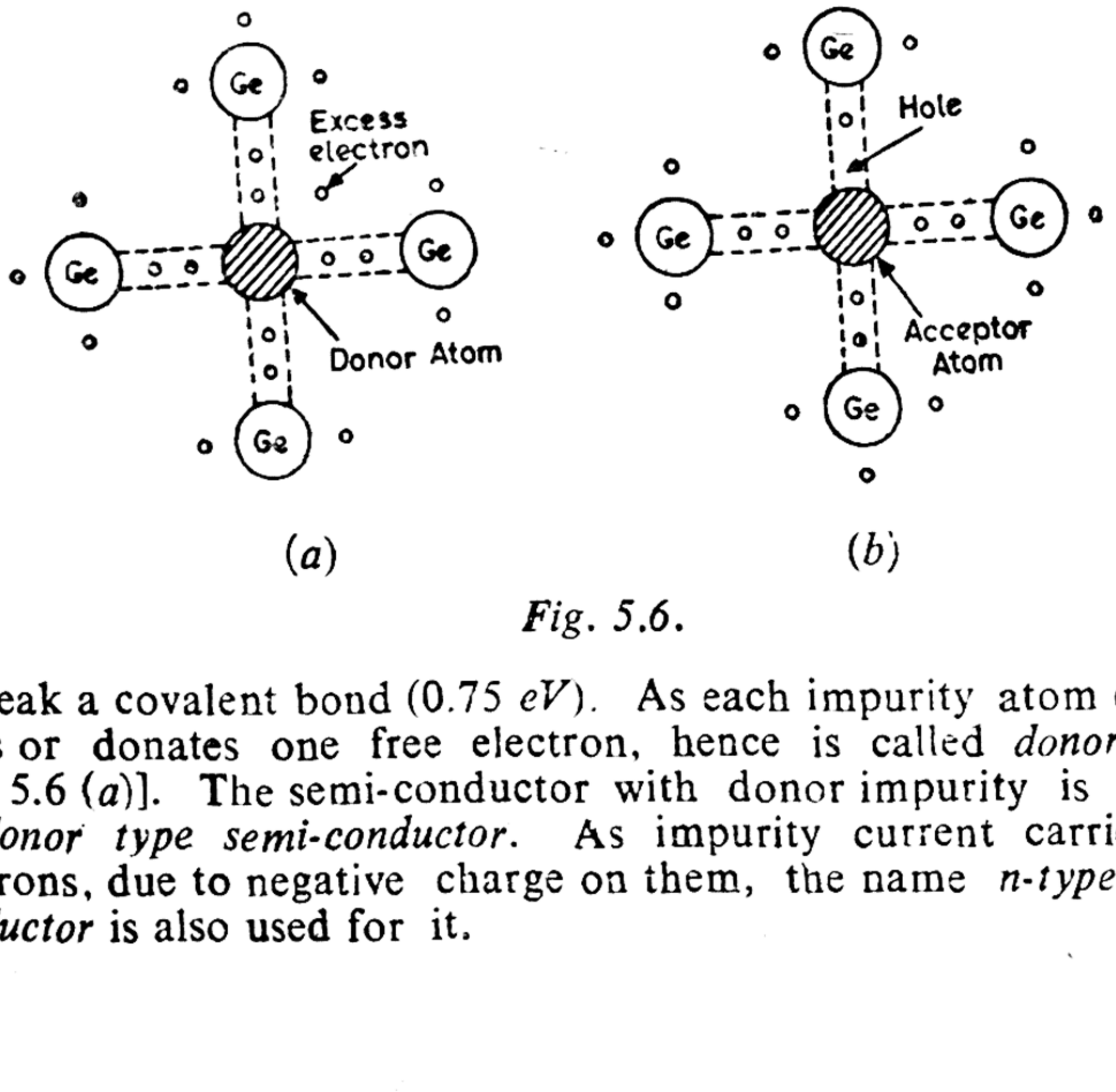


Fig. 5.6.

to break a covalent bond (0.75 *eV*). As each impurity atom contributes or donates one free electron, hence is called *donor atom* [Fig. 5.6 (*a*)]. The semi-conductor with donor impurity is known as *donor type semi-conductor*. As impurity current carriers are electrons, due to negative charge on them, the name *n-type semiconductor* is also used for it.

If the impurity atom is trivalent, having three valence electrons, the three electrons of this impurity atom form covalent bonds with the three neighbouring germanium atoms. Thus there is a lack of one electron to complete covalent bonds between this impurity atom and surrounding four atoms of germanium. This lack of electron is known as hole. When the impurity atom has captured or accepted one electron from a neighbouring germanium atom, a hole is created in that atom. This hole moves from one atom to another with in the crystal lattice due to thermal effects. In other words the incomplete covalent bond propagates from one atom to another due to thermal agitation. The holes act as mobile positive charge carriers. As each impurity atom accepts one electron from a neighbouring germanium atom, hence it is known as acceptor atom. The semi-conductor consisting such type of impurities is known as *acceptor type semi-conductor*. As charge carriers are holes, which are positively charged, hence the semi-conductor is also named as positively or *p-type semi-conductor*, Fig. 5.6(*b*).

In extrinsic semi-conductors, the impurity atoms reduce the energy gap E_g. In an *n*-type semi-conductor, the energy levels of donors (E_d) are isolated states located below but near (0.01 *eV* in *Ge*) the bottom of the conduction band. At $T=0°\ K$, the valence band is filled with electrons, the donor levels are filled, the donors are not ionised and the conduction band is empty. The Fermi level thus lies half way between the donor level and the bottom of the conduction band. As the temperature increases the donor electrons transfer, leaving most donor atoms activated. Thus, the intrinsic charges become important. At a high temperature the *n*-type conduction is almost entirely due to extrinsically supplied electrons from the donors. Similar to the case of intrinsic semi-conductor, the distribution of electrons and holes is given by

Density of electrons in the conduction band

$$n_e = N_c\, f(E_c). \qquad \ldots(12)$$

Density of holes in the valence band

$$n_h = N_v[1 - f(E_v)]. \qquad \ldots(13)$$

Density of filled donors

$$n_d = N_d\, f(E_d). \qquad \ldots(14)$$

Density of filled accepters

$$n_a = N_a f(E_a). \qquad \ldots(15)$$

The condition for the electrical neutrality of the material gives

Positive charge $n_h + (N_a - n_d)$ = negative charge $n_e + n_a$.

For the *n*-type semi-conductor $n_a=0$ and $n_h=0$.

$$\therefore \quad n_e=N_d-n_d=N_d[1-f(E_d)],$$

or $$N_c f(E_c)=N_d[1-f(E_d)]$$

$$N_c e^{-(E_c-E_f)\,e/kT} \simeq N_d\, e^{(E_d-E_f)\,e/kT}. \qquad \ldots(16)$$

Thus in the *n*-type semi-conductor,

$$E_f=\frac{1}{2}(E_c+E_d)-\frac{kT}{2e}\log_e\frac{N_c}{N_d}. \qquad \ldots(17)$$

At $T=0°\ K$, $E_f=\frac{1}{2}(E_c+E_d)$, *i.e.*, the Fermi level will lie half way between the donor level and the bottom of the conduction band. The Fermi level drops as T increases.

Similarly, the Fermi level in *p*-type semi-conductor is given by

$$E_f=\frac{1}{2}(E_a+E_v)+\frac{kT}{2e}\log_e\frac{N_v}{N_a}. \qquad \ldots(18)$$

At $T=0°\ K$, $E_f=\frac{1}{2}(E_a+E_v)$, *i.e.*, the Fermi level will lie half way between the top of the valence band and the acceptor level. The Fermi level rises as T increases.

Thus we see that at $T=0°K$, the Fermi level lies half way between the donor level and the bottom of the conduction band in *n*-type material. In *p*-type material at absolute zero it lies half way between the acceptor level and the top of the valence band. As the temperature increases, the Fermi level falls off in the former case

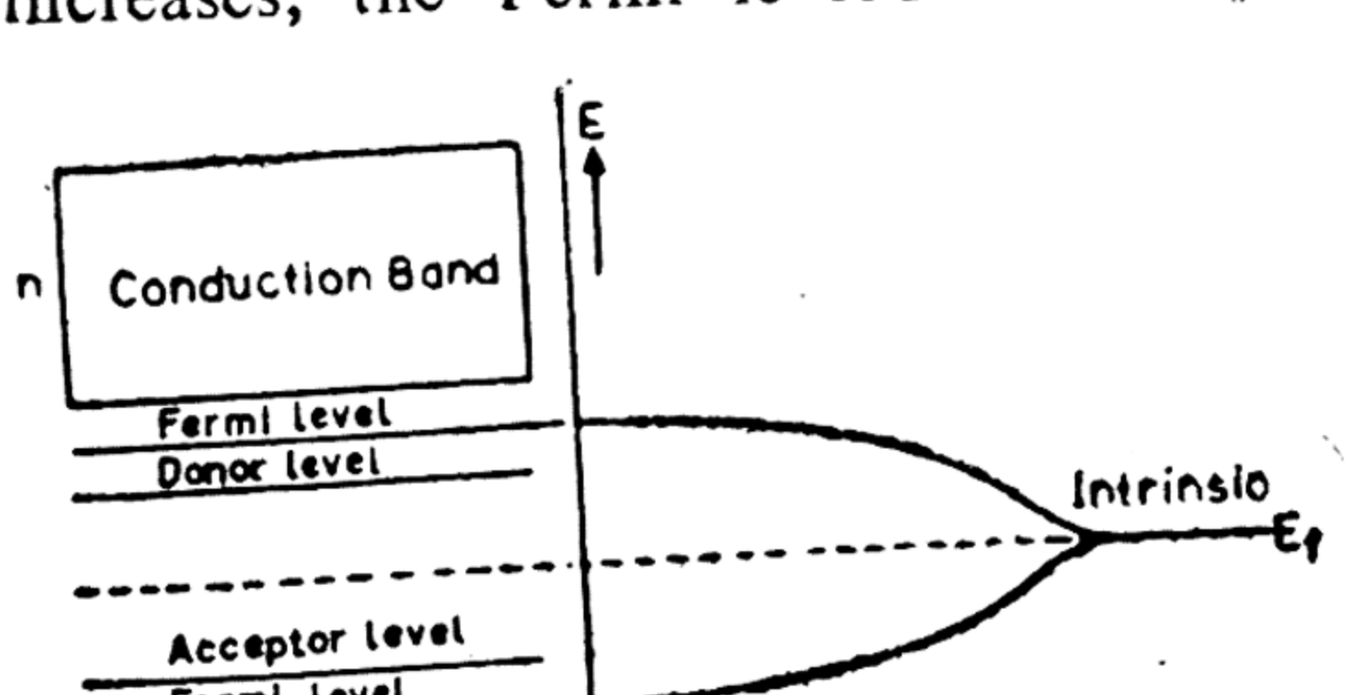


Fig. 5.7.

and rises up in the latter case. At a sufficiently high temperature the Fermi level in either material reaches the half gap level as in an intrinsic semi-conductor [Fig. 5.7].

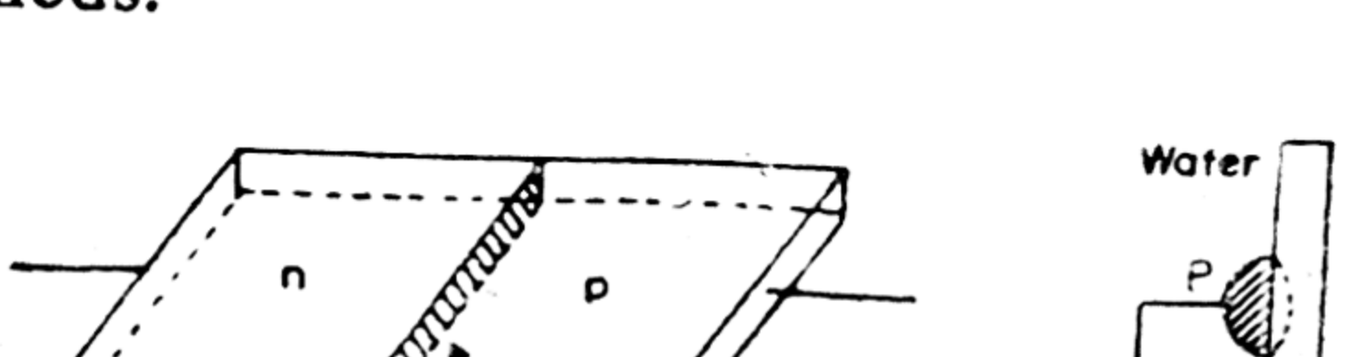
TABLE 5.1
Semi-Conductor Constants

Constants	Germanium	Silicon
Atomic number	32	14
Atoms/m^3	4.4×10^{28}	5.0×10^{28}
Dielectric constant	15.8	11.7
Gap energy E_G, eV, $0°K$.	0.75	1.15
n_i, m^{-3}, $300°K$	2.4×10^{19}	1.5×10^{16}
μ_e, m^2/volt-sec	0.39	0.13
μ_h, m^2/volt-sec	0.19	0.05
density gm/cm^3	5.33	2.33

5.4. SEMI-CONDUCTOR DIODES

Similar to vacuum and gas diodes the semi-conductor diodes are two terminal devices. The conduction current consists of both electron and hole flow, similar to the flow of electrons and ions in gas diodes. The types of diodes, widely used in circuits, are : *junction diodes, point contact diodes, zener diodes and tunnel diodes.* Let us discuss each of these diode types.

(A) Junction Diode : A junction diode consists of *n*-type semi-conductor and a *p*-type semi-conductor joined together through a junction. There are a large variety of techniques used to produce *p-n* junctions within a single crystal of germanium or silicon. The most popular ones are : The grown, the alloy, and the diffused-junction methods.



(*a*) (*b*)

Fig. 5.8.

The grown junction is formed during the crystal growing process. A crystal of say *n*-type of impurity is first grown. In the middle of this process sufficient impurity of the opposite kind (*p*-type) is added. This impurity is such that after developing, the next region behaves as *p*-type. In this way a continuous crystal is formed which is partly of *n*-type with a junction in the middle [Fig. 5.8 (*a*)].

The alloy or fused junctions are commonly used. In this process a piece of material tor base is selected and a small piece of material of the opposite impurity type (indium on *n*-type germanium) is placed on the surface. The wafer is heated, the temperature is raised (~500°C) for a minute or so. The indium melts and dissolves the germanium to form a liquid solution or alloy. The junction cools and a recrystallized germanium layer highly doped with indium and hence a strongly *p*-type is formed [Fig 5.8 (*b*)].

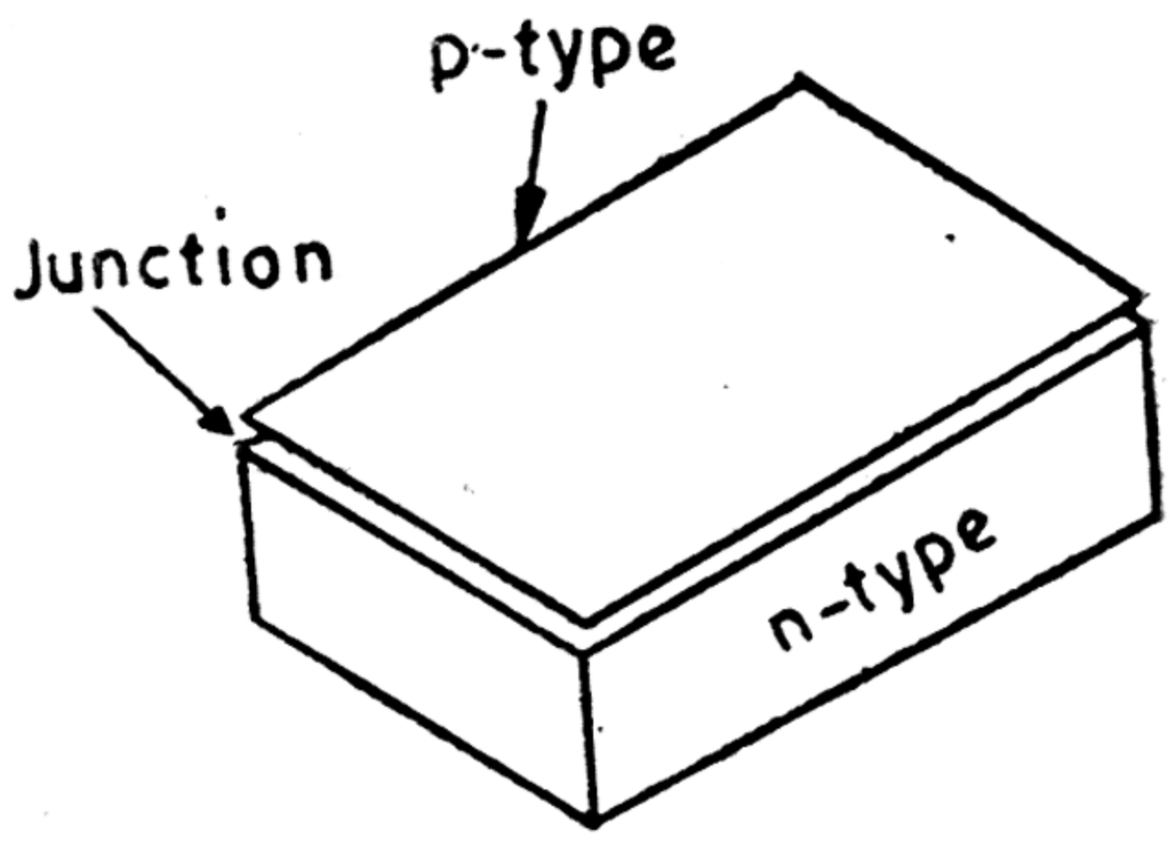


Fig. 5.8 (c)

In the diffused-junction method, a slab of material is heated to a high temperature and exposed to a gaseous impurity. This impurity diffuses slowly into the surface of the material. The impurity concentration is highest at the surface and decreases exponentially inward. The surface thus becomes doped. This technique is especially useful in making large and uniform junctions such as solar cells.

Before the *p-n* junction is formed, in the *n*-type germanium there are large number of free electrons and few intrinsic holes and in *p*-type there are large number of holes and a few intrinsic electrons. The excess of electrons or holes are due to impurity and a few intrinsic holes or electrons are due to thermal energy of the electrons. When they are joined and junction is formed, the electric charge diffuses across the common junction boundary under the thermal motion. Since there are excess of high energy conduction electrons in *n*-type, the diffusion of electrons takes place on the *p*-type side and thus a deficiency of electron will exist on *n*-type side. This creates a potential difference, called as *barrier potential* (0.5 volt for *Ge* and 0.8 volt for *Si*) across the region on both sides of the contact, known as *depletion region*, with *p* type side negative with respect to *n*-type. This potential difference or *barrier potential* provides a *drift current*. Since an isolated junction has to be neutral no current can flow. The drift (due to barrier potential) and diffusion (due to charge diffusion) currents must be equal and opposite. The width of the depletion region, called *depletion layer*, is generally of the order of 10^{-8} to 10^{-6} meter. The region is said to contain a space charge, hence is also called *space charge region* or the *transition region*. Using energy level diagram one can explain depletion layer and *n-p* junction. As shown in Fig. 5.9 (*a*) and (*b*), the energies at the band edges E_c and E_v are the same in both-*p* and *n*-type of semi conductors. After formation of junction, the Fermi level is constant throughout at equilibrium Fig. (*c*).

The situation is analogous to the manner in which the water levels adjust in two tanks connected with each other.

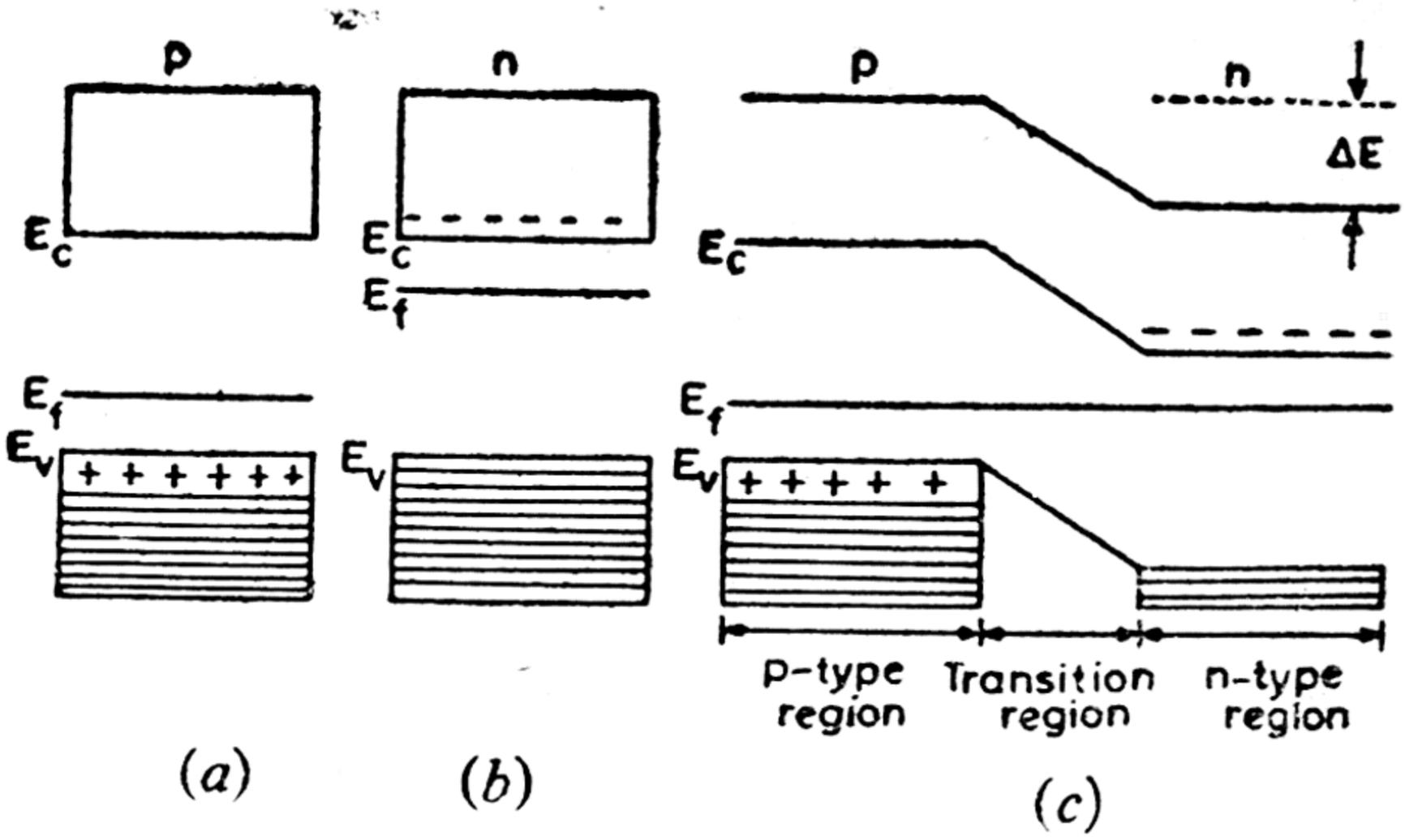


Fig. 5.9.

For the diode characteristic, the external voltage to a junction diode can be applied in the following two manners.

(1) *Forward bias.* When the positive terminal of the battery is connected to *p*-type end and negative to *n*-type end of the junction diode then it is said to be under *forward bias condition.* The positive and negative voltages of battery repel respectively the holes (*majority carriers*) and electrons towards the junction simultaneously. The barrier potential at junction offers small resistance in the flow of these carriers from moving across the depletion region but the applied bias (*potential difference*) makes their flow possible. Under the influence of the electric field holes move to the right across the junction and electrons to the left. In the region of *p-n* junction electrons and holes combine, thus ceasing to exist as mobile charge carriers. For each electron-hole combination that takes place near the junction, a covalent bond near the positive

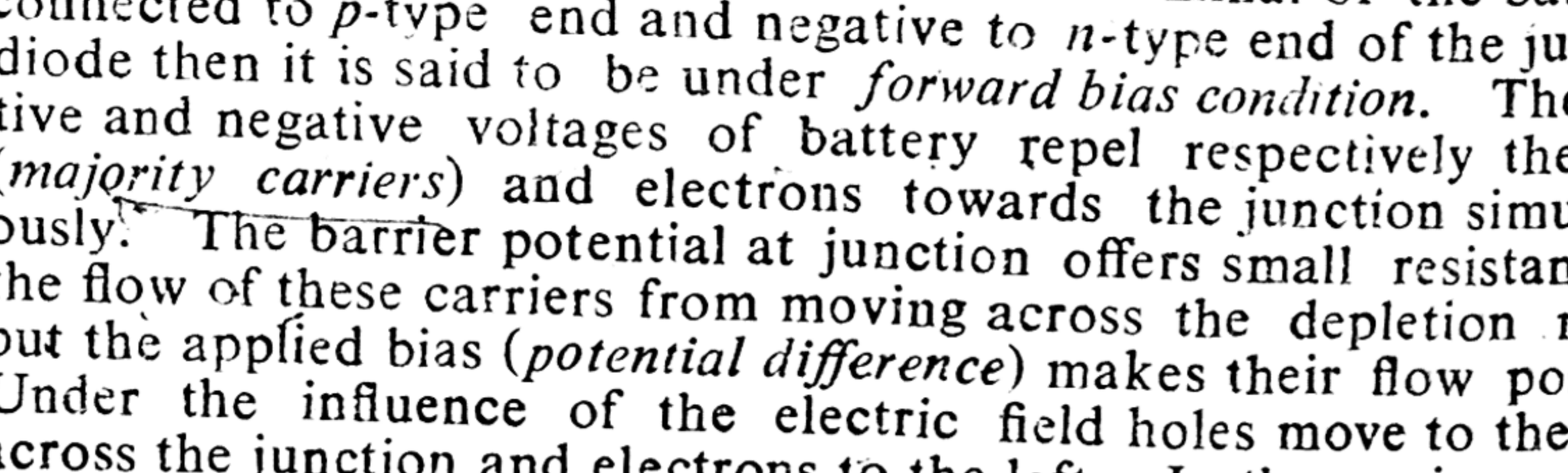


Fig. 5.10.

terminal of the battery breaks down, and an electron is liberated and enters the positive terminal. This creates a new hole which

moves to the right of the *p-n* junction. At the right end of the diode, more electrons arrive from negative battery terminal and enters the *n*-region to replace the electron lost by combination with the holes at the junction. The electrons move towards the junction at the left, where they again combine with holes arriving there. Thus a large current flows through the junction. The arrangement is shown in Fig. 5.10 (*a*).

(2) *Reverse bias.* If the terminals of battery are so connected that the *p*-type end is at negative and *n*-type end is at positive terminal, the semi-conductor diode is said to be under *reverse bias* condition, Fig. 5.10(*b*). The majority carriers, holes, are now attracted to the negative voltage terminal, and electrons are attracted to the positive voltage terminal, both displaced away from the junction. The depletion region increases, barrier layer is increased, the majority carriers are not able to cross the barrier potential. Since the junction offers very high resistance, therefore, the flow of current is almost stopped. But still a few micro-ampere reverse current passes through the junction. It is due to the electron-hole pair (minority carriers) produced by the thermal effect even in depletion region. Some covalent bonds always break down because of the normal thermal energy of the crystal molecules. The electrons in *p*-type material are more, right across the junction while the holes in the *n*-type materials are more, to the left across the junction. Thus a small electron hole combination current is maintained by these, so called *minority carriers.* This is called as *reverse saturated current.* If the reverse bias is made very high, the covalent bonds near the junction break down and a large number of electron-hole pairs will be liberated, the reverse current then increases abruptly to relatively large value, this is known as *break-down.*

The effect on the band model of applying reverse bias (V volts) is to increase the energy barrier from ΔE to $\Delta E + eV$. Thus virtually no electrons can cross over from the *n* type region to the *p*-type region. The potential barrier limits the flow of holes from the *p*-type to the *n*-type region (as holes prefer to go up hill). The energy barrier is decreased to $\Delta E - eV$ for a forward bias of V volts. It increases both the electron and hole currents.

A *p-n* junction diode is connected with a low voltage battery, using a potential divider in series with a *mA*. The positive end of the battery is connected with *p*-region end and negative of the battery is connected with *n*-region end of the diode. The junction offers a very small resistance, in forward direction. By increasing the voltage (V) across the ends of diode, the corresponding current is recorded. The variation of diode current is plotted with the applied voltage. A voltage of about 0.1 volt is sufficient to cross the potential barrier of the junction, after that the current increases rapidly. It is an exponential shape of the curve. By reversing the

direction of the applied voltage a small-current due to minority carriers will flow in the circuit (of the order of micro ampere). This current depends on the temperature of the diode. The junction offers a high resistance in the reverse direction. When the reverse voltage is very high, then the break down comes and the current is abruptly increased. The resistance property, low in forward bias condition and high in reverse bias condition, is just similar to unidirectional conduction of current as of the conventional diode valve. Thus a junction diode can be used as a *rectifier*.

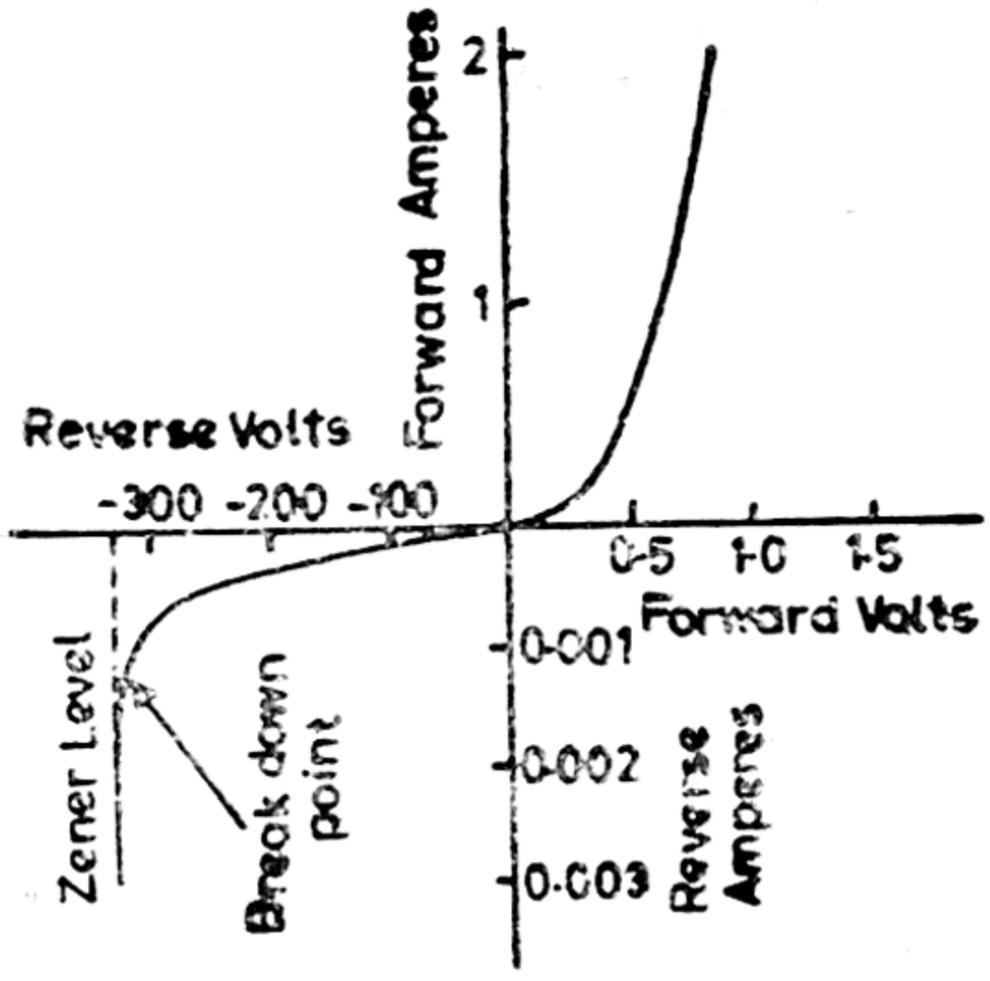


Fig. 5.11.

(B) Point-Contact Diode. Early diodes were of the point-contact type. The point-contact diode consists of a small wafer *S* of semi-conducting crystal. This wafer is a slice, either from a single crystal or may be poly-crystalline, of area few millimeters square and thickness fraction of a millimeter. The crystal is soldered to a metallic base *M* so that an external contact can be made. This base contact (*A*) is ohmic. The point contact *B* is made on the other side of the crystal by pressing a pointed wire *W*, known as *whisker or cat's whisker*, of one to five mils in diameter phosphor bronze. The current pulse by ohmic heating melts the crystal in the vicinity of the point contact and thus forms a *p*-region around the contact if the semi-conductor is of *n*-type.

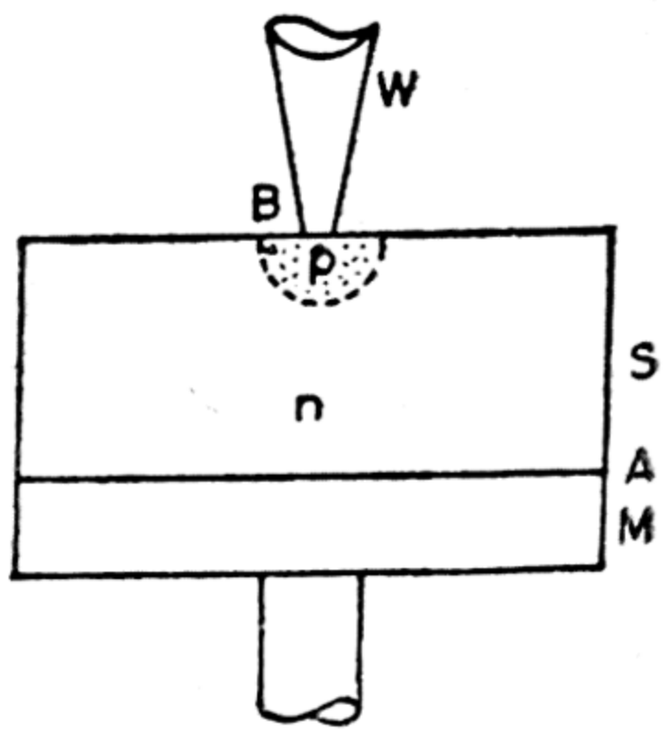


Fig. 5.12.

The characteristics of point contact diode depend on the crystal structure and its surface, whisker composition and its contact area and pressure, etc. The general shape of *v-i* curve is the same as for the junction type diode but with lower possible currents. The point contact diode has higher forward resistance, higher reverse current and smaller rectification ratio than does the junction diode. The point contact diodes are widely used in high frequency applications such as microwave detectors and pulse circuits.

(C) Zener Diode: There are two distinct mechanisms which can be used to explain electric break down : *Zener breakdown* and *avalanche breakdown*. The former is evident in the case of very narrow junctions and the latter in wider junctions. The breakdown

voltage decreases as the temperature increases in the former case and increases with the temperature in the latter case. The zener breakdown may be explained as under :

In a very thin junction, the electric field is large with small applied reverse bias voltages and an appreciable number of electrons may pass right through the potential barrier formed by the forbidden band. As the temperature increases, it causes the forbidden band to become narrower and thus the probability of zener emission increases. Therefore the Zener breakdown occurs at a lower voltage.

In the practical junction diodes the reverse current increases rapidly at large reverse voltages, known as breakdown level or *Zener level* (Fig. 5.11). Diodes with high reverse resistance and low power ratings are destroyed by reverse breakdown. The units, designed to operate in the reverse breakdown mode continuously without being damaged are called *Zener diodes.* Such diodes are useful in circuits where constant voltages are required and current demand is vary large. The limits of reverse current are determined from the rated power and Zener-breakdown voltage of the particular diode. The Zener breakdown voltage V_z is determined by the properties of the *p*- and *n*- regions at the *p-n* junction. The strongly doped materials have low V_z, whereas the weakly doped materials have high V_z. The Zener diodes are manufactured to offer V_z ranging from 2 to 200 volts.

(D) Tunnel Diodes. In certain junction diodes it has been found that the current voltage characteristics are of peculiar type, *current decreases with the increase of voltage.* Such diodes were discovered by L. Esaki in 1957 and were known as *Esaki diodes.* The mechanism of such diodes is purely quantum mechanical in nature (the *tunneling process*). For this reason these diodes are now called the *tunnel diodes.*

In the tunnel diode a strongly doped *n*-region is allowed with a group III material to form a *p-n* junction in which the *p*-material is also strongly doped. The tunnel diodes are made of highly conductivity semi-conductors, because of the high electron mobility and reasonable gap energy. The strong doping has two effects. (*i*) It makes the depletion region very narrow and thus decreases the thickness of the potential barrier to about $100A°$. (*ii*) It affects the band structure. The donor and acceptor energy levels broaden and merge with the conduction and valence bands. The Fermi level also moves up into the conduction band in *n*-type material and moves into the valence band in the *p*-type material.

Let us draw the energy level diagram of the strongly doped *p-n* junction. For zero bias, the energy level situation is as shown in Fig. 5.13 (*a*). The net tunneling is zero because the probability of electrons going from states in the conduction band on the *n*-side to states in the valence band on the *p*-side is equal to that in the opposite direction. When a reverse bias is applied to the

diode [Fig. 5.13 (*b*)], the conduction electrons of the *n*-region cannot move into the *p*-region, but the electrons from the valence band of *p*-region can tunnel and enter the conduction band of the *n*-region because of the small width of the potential barrier. Thus the *tunnel diode conducts in the reverse direction* [part *a b* of Fig. 5.13 (*e*)]. When a forward bias is applied to the diode it begins to conduct because of tunneling of electrons from the conduction band of *n*-region into the valence band of the *p*-region. As the forward bias voltage is increased [Fig. 5.13(*c*)], the total current is the sum of the tunnel current and the normal forward current. The tunnel current is maximum [point *c* in Fig. 5.13 (*e*)] when the bottom of the conduction band in the *n*-region coincides with Fermi level in the *p*-region. Here the electrons in the conduction band find empty states just across the barrier and the tunneling is maximum. The electrons in the valence band find filled states just in the left and can not tunnel.

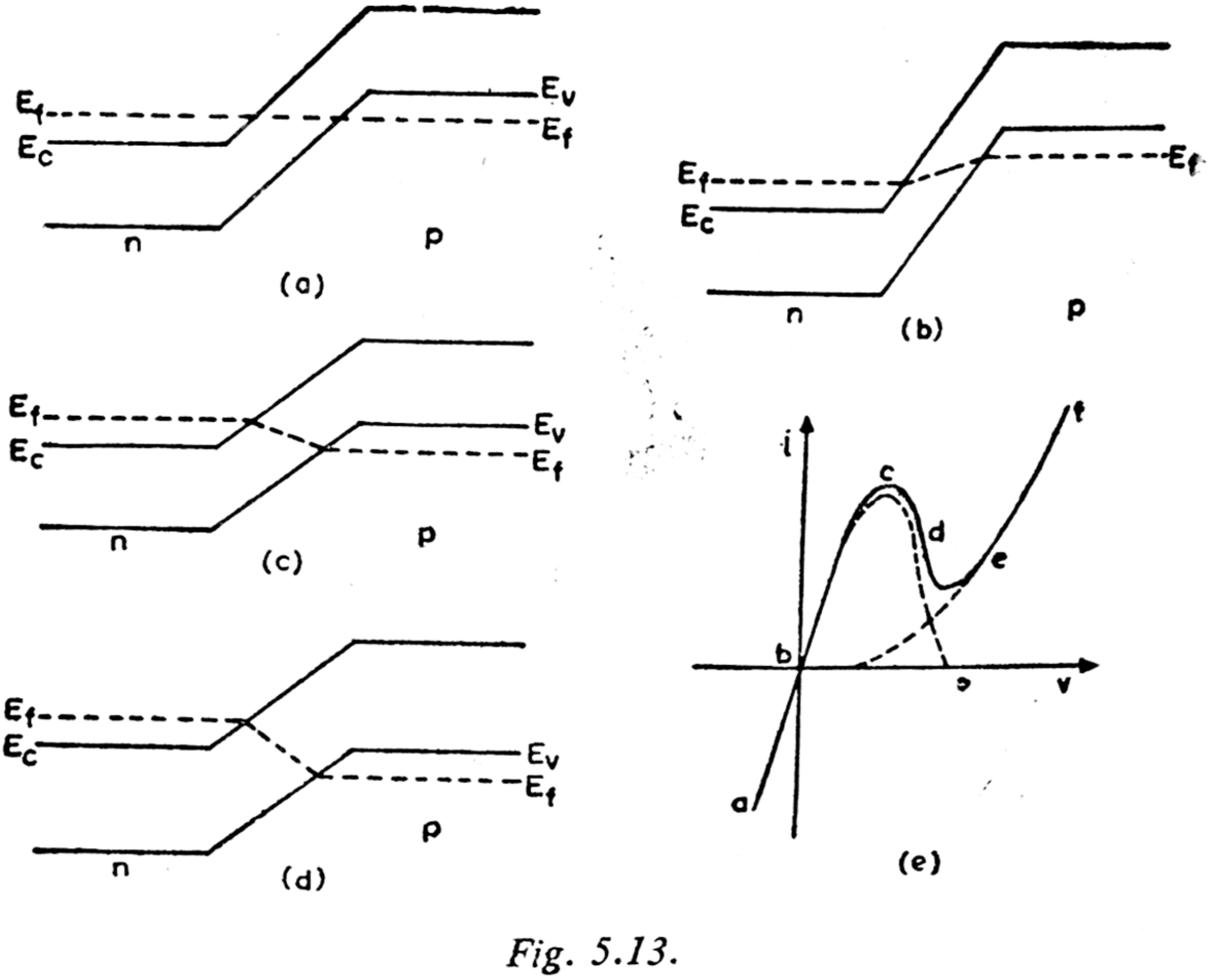


Fig. 5.13.

As the forward bias voltage is further increased [Fig. 5.13 (*d*)], most of the conduction band electrons face the forbidden energy gap across the barrier and part of them see available energy levels. Thus the tunneling current is smaller than above [point *d* in Fig. 5.13 (*e*)]. If the voltage is further increased all the conduction band electrons face the forbidden energy gap and the tunneling ceases. The current is simply the normal forward current following the normal diode law, as shown by the portion *ef* of the curve Fig. 5.13 (*e*).

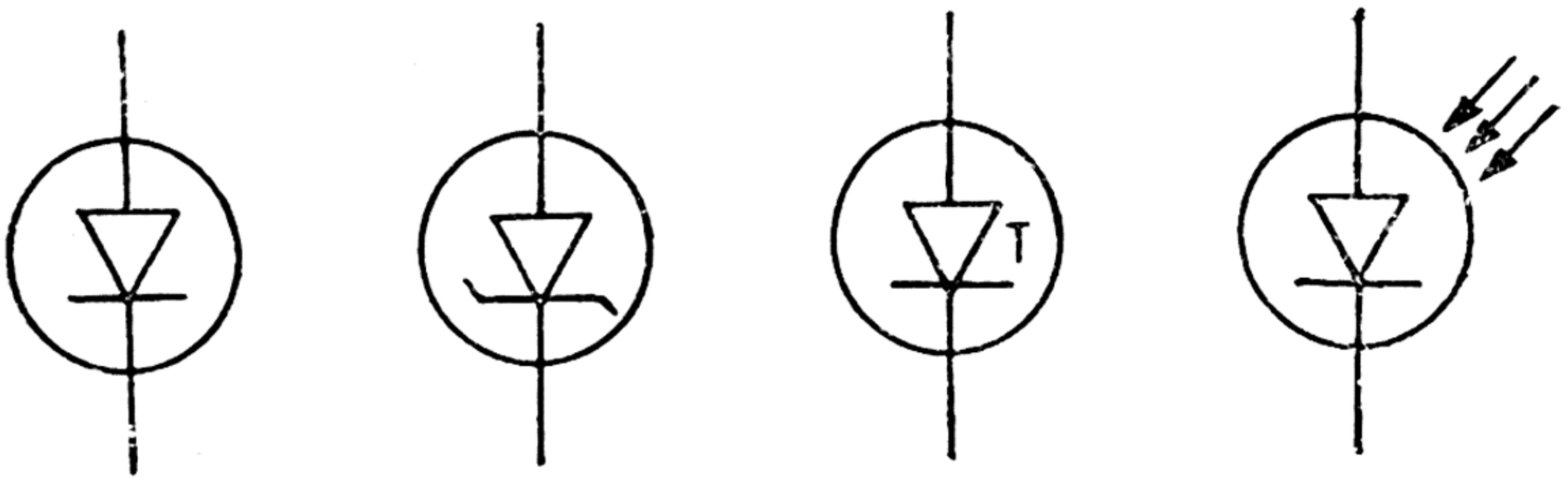
(*E*) Photodiode—The semi-conductors are not only sensitive to heat energy, but also light sensitive. A photodiode consists of a single *p-n* junction, made from light sensitive semi-conductor. When the light photons of proper frequency reach the transition region, each photon carries an energy $E=h\nu$, the valence electrons absorb this energy. If the photon energy exceeds the forbidden gap energy (eE_0), the valence electron will jump to conduction band leaving a hole in the valence band. In this way electron-hole pairs are produced. The number of such pairs produced depend on the intensity of the incident photons. If a reverse bias is applied the carriers are swept out of the transition region as a *photocurrent*. Corresponding to the zero light reaching the junction, the reverse saturation current exists. It is called *dark current*.

The maximum wavelength corresponding to the minimum photon energy which will create a pair is

$$\lambda_{max} = \frac{c}{\nu_0} = \frac{hc}{eE_0} \text{ meter.} \qquad \ldots(19)$$

It has a value of 1.1 μm for pure silicon and 1.7 μm for germanium. These values lie in the near infrared region.

These different types of diodes are used in electronic circuits and are generally represented by the following symbols.



(*a*) Junction or Point contact (*b*) Zener (*c*) Tunnel diode (*d*) Photodiode

Fig. 5.14.

The Fig 5.15 shows the circuit diagram of full wave junction diode rectifier with filter (Π-type). The analysis of this is almost similar to diode valve.

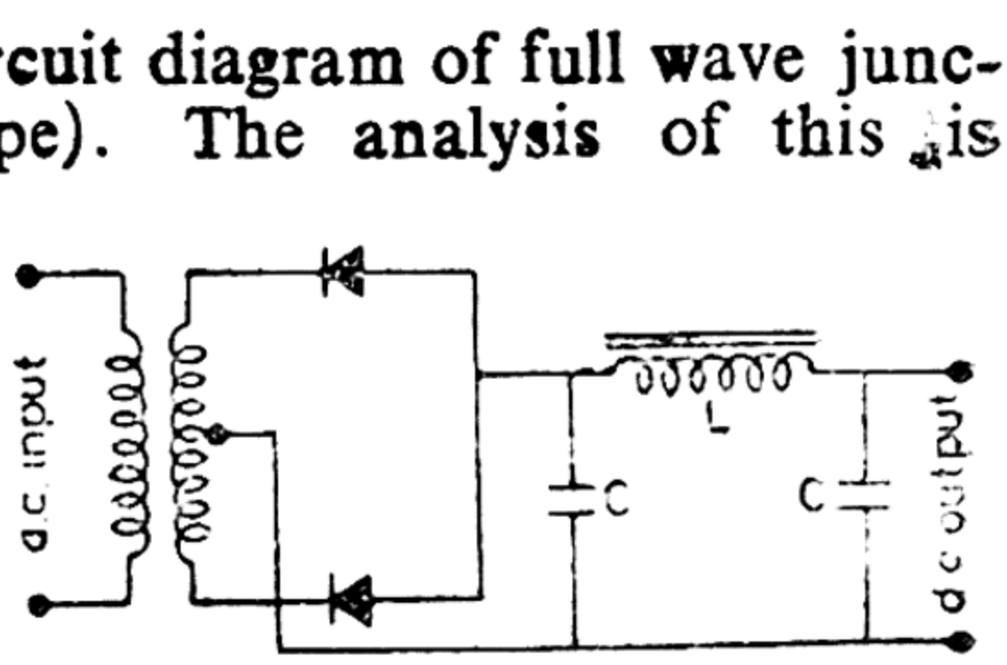


Fig. 5.15.

Semi-conductor rectifiers are the selenium, silicon and germanium diodes. The selenium diode rectifier is a junction diode consisting of two sections of *n*-type and *p*-type selenium joined together. For radio frequency waves it has a forward resistance of the order of 1Ω and a reverse resistance as high as $10^8\Omega$. These rectifiers have advantages : (1) high efficiency, (2) small size, (3) long life, (4)

low forward voltage drop, (5) low heat radiation and (6) wide range of current ratings.

5.5. CONDUCTION IN SEMI-CONDUCTORS

This occurs through two processes : (1) charge drift under an applied electric field, and (2) diffusion of charge from a region of high charge density to a region of low charge density.

The conduction electrons and holes move in random motion in a crystal. When an electric field E is applied, the holes try to move along the field direction and the electrons in opposite direction. If n_e is the electron density, v_e the net mean free velocity of electron, the current density due to electron motion

$$j_e = -n_e e v_e.$$

Similarly for the holes motion with n_h and v_h as their density and velocity respectively, we have

$$j_h = n_h e v_h.$$

Since v_e is oppositely directed to v_h, hence the total current density

$$j = e\,(n_e v_e + n_h v_h). \qquad \ldots(20)$$

As the charge mobility $\mu_e = v_e/\mathcal{E}$ and $\mu_h = v_h/\mathcal{E}$ hence the conductivity or current density per unit of field intensity

$$\sigma = j/\mathcal{E} = e\,(n_e \mu_e + n_h \mu_h). \qquad \ldots(21)$$

The resistivity is high at very low temperatures and falls as the temperature rises.

In an in homogeneous field free region of a semi-conductor the electron will diffuse by thermal agitation in the direction from high electron density to low electron density. The current density due to electron diffusion depends on the electron density gradient dn_e/dx and is given as

$$j_e = eD_e\,(dn_e/dx), \qquad \ldots(22)$$

where D_e is the electron diffusion constant. Similarly for holes we have

$$j_h = -e\,D_h(dn_h/dx). \qquad \ldots(23)$$

In semi-conductors the drift and diffusion actions may be simultaneously present. Thus the expressions for total electron and hole current densities become

$$j_h = e\mu_h n_h \mathcal{E} - eD_h\,(dn_h/dx)\ amp/m^2. \qquad \ldots(24)$$

$$j_e = e\mu_e n_e \mathcal{E} + eD_e\,(dn_e/dx)\ amp/m^2. \qquad \ldots(25)$$

In each Eq. first term on the right is a drift component and the second term is due to diffusion. The first term depends on the

electric field. At lower voltage differences (<0.25 volt) the diffusion term predominates and at the greater potentials (>0.25 volt), the drift current predominates. At voltages (~0.25 volt) both types of current exist.

Thermistor. The conductivity of germanium (silicon) increases to about 6 (8) percent per degree increase in temperature. Such a large change in conductivity limits the use of semi-conductor devices in some circuits. However this property is very much useful in thermometry, in the measurement of microwave frequency power, as a thermal relay and in devices controlled by changes in temperatures. The semi-conductor used in this manner is called *thermistor.* As silicon and germanium are too sensitive to impurities. hence are not used as thermistors. The commercial thermistors consist of sintered mixtures of such oxides as NiO, Mn_2O_3 and Co_2O_3.

5.6. THE *pn* DIODE EQUATION

We can derive a quantitative expression for the diode current from the barrier potential and the n and p charge densities.

The Einstein relation for the diffusion constant D and the charge mobility μ states that

$$D_e/\mu_e = D_h/\mu_h = kT/e, \qquad \ldots(26)$$

where k is the Boltzmann constant. The quantity kT/e represents the mean thermal energy of the particles and is often called the voltage equivalent of temperature. In case of no bias, the current must be zero. Thus the expression for electron flow [Eq. (25)] gives

$$\frac{dn_e}{n_e} = -\frac{\mu_e}{D_e}\mathcal{E}dx = -\frac{e}{kT}\mathcal{E}dx. \qquad \ldots(27)$$

This expression may be integrated across the junction, from x_1 to x_2 the transition region

$$\int_p^n \frac{dn_e}{ne} = \frac{e}{kT}\int_{x_1}^{x_2}(-\mathcal{E})\,dx. \qquad \ldots(28)$$

The integration in the right side leads to the value of the barrier potential V_B with n positive to p. The above relation thus reduces to

$$(n_e)_n/(n_e)_p = e^{eV_B/kT}. \qquad \ldots(29)$$

This gives the ratio of electron densities at the junction (depletion region face) in the n-region to that in the p-region.

Similarly, we can get

$$(n_h)_p/(n_h)_n = e^{eV_B/kT}. \qquad \ldots(30)$$

Equations (29) and (30) are known as the Boltzmann equations. If a small potential V is applied to the junction with the positive terminal to the p-type, *i.e.*, as a forward bias. The junction barrier potential reduces to $V_B - V$. The mobile charge from the source goes to the depletion region (electrons added to n and holes to p). The current far from the junction is due to majority carriers. The diffusion at the junction is by minority carriers predominantly holes in the n-region. Thus under the forward bias condition, the relation (30) for the hole density on the right face of the junction becomes

$$(n_h)_n + \Delta(n_h)_n = (n_h)_p\, e^{-e(V_B - V)/kT} = [(n_h)_p\, e^{-eV_B/kT}]\, e^{eV/kT}$$

$$\therefore \Delta(n_h)_n = (n_h)_p\, e^{-eV_B/kT}\, (e^{eV/kT} - 1). \qquad \ldots(31)$$

Similarly the electron density at the left face of the junction is given as

$$(n_e)_p + \Delta(n_e)_p = (n_e)_n\, e^{-e(V_B - V)/kT} = [(n_e)_n\, e^{-eV_B/kT}]\, e^{eV/kT}$$

$$\therefore \Delta(n_e)_p = (n_e)_n\, e^{-eV_B/kT}\, (e^{eV/kT} - 1). \qquad \ldots(32)$$

Using equation (20) and assuming the junction area S, we get the total current across the junction as

$$i = jS = Se[\Delta(n_e)_p\, v_e + \Delta(n_h)_n\, v_h]$$

$$= Se[(n_e)_n\, v_e + (n_h)_p\, v_h]\, e^{-eV_B/kT}\, (e^{eV/kT} - 1) \qquad \ldots(33)$$

$$= I_s(e^{eV/kT} - 1). \qquad \ldots(34)$$

This is known as the diode equation (plotted in Fig. 5.11). This equation shows that for V positive (or forward bias), the current increases rapidly and for V negative (or reverse bias) the current decreases to a limiting value $-I_s$. This current is called the *reverse bias saturation current*. The ratio of forward to reverse current at a given applied voltage is called *rectification ratio*.

5.7. TRANSISTORS

Transistors are in general two-junction, three-junction, three-terminal devices whose primary function is amplification. The first three electrode semi-conductor control device was named the *transistor*, as a contraction of the words *transfer resistor*. There is a large variety of transistor-like devices. They are superior than the vacuum tubes as they have (1) long operating lifetime, (2) small physical size, (3) good operation at low voltages and (4) do not require heating power. The characteristics of transistors depend upon the operating temperature to a great extent, therefore the transistor circuits are relatively cool. The transistor circuits can

perform the processes of electrical signal amplification, generation, waveshaping and switching more efficiently than the vacuum tubes do.

The first transistor discovered by Bardeen and Brattain in 1947 was called a *type A transistor*. It is now known *as point contact transistor*. Such transistors were limited in their power handling capacity, difficult to fabricate and electrically noisy. Two years after Shockley suggested *junction transistors*. These devices operate with both holes and electrons and are therefore called *bipolar junction transistor* (BJT). The transistors with three terminals are called transistor triode. The junction transistors are more stable both mechanically and electronically, can handle large amounts of power, can be fabricated economically, generate little noise and are operable to the kilomegacycle region.

Field-effect transistors (FET) have been successfully fabricated for commercial use since 1961, exhibit certain characteristics superior to those of the junction type. These are like the vacuum tetrode. One can expect that there will be at least as many transistor types as there are vacuum tube types.

Let us first discuss the junction transistors in details.

5.8. JUNCTION TRANSISTOR

It is the development of junction diode. The diode is analogous to diode valve, the *transistor is analogous to triode valve*. The silicon or germanium transistor can be grown with two *p*-regions separated by a thin *n*-region (nearly 0.002 cm) or with two *n*-regions separated by a thin *p*-region. These are named respectively as *p-n-p* type and *n-p-n* type transistors. The common layer (middle one) is known as *base* and the other two as *emitter and collector*. These three serve the purpose of three electrodes. The emitter junction injects the charge carriers into the base through which these carriers reach the collector junction. The base current thus controls the collector current, similar to the grid voltage controlling the plate current in a triode valve. In most of the transistors, the leads corresponding to emitter, base and collector lie in the same plane and the central lead corresponds to base. The spacing between emitter lead and the base lead is generally less than the spacing between collector lead and the base lead. Another identification of collector lead is the dot placed on the body of the transistor close to the collector. The centre lead is the base lead in both the cases. When the leads are spaced around the circumference of a circle, the centre lead is the base lead, left is collector lead and right is emitter lead when viewed looking into the base of the transistor. The Fig 5.16 shows both the junction transistors, their symbols for circuit and biasing processes. The emitter (E)-base (B) is forwardly biased and base (B)-collector (C) circuit is reversed biased. Since the E-B circuit is forwardly biased

the barrier potential is lower, the circuit will offer a small resistance. So the majority carriers (either holes in *p-n-p* type or electron in *n-p-n* type) with sufficient energy from emitter will enter the thin base region. The number of majority carriers crossing the barrier will depend on the applied forward bias. Since the base layer is

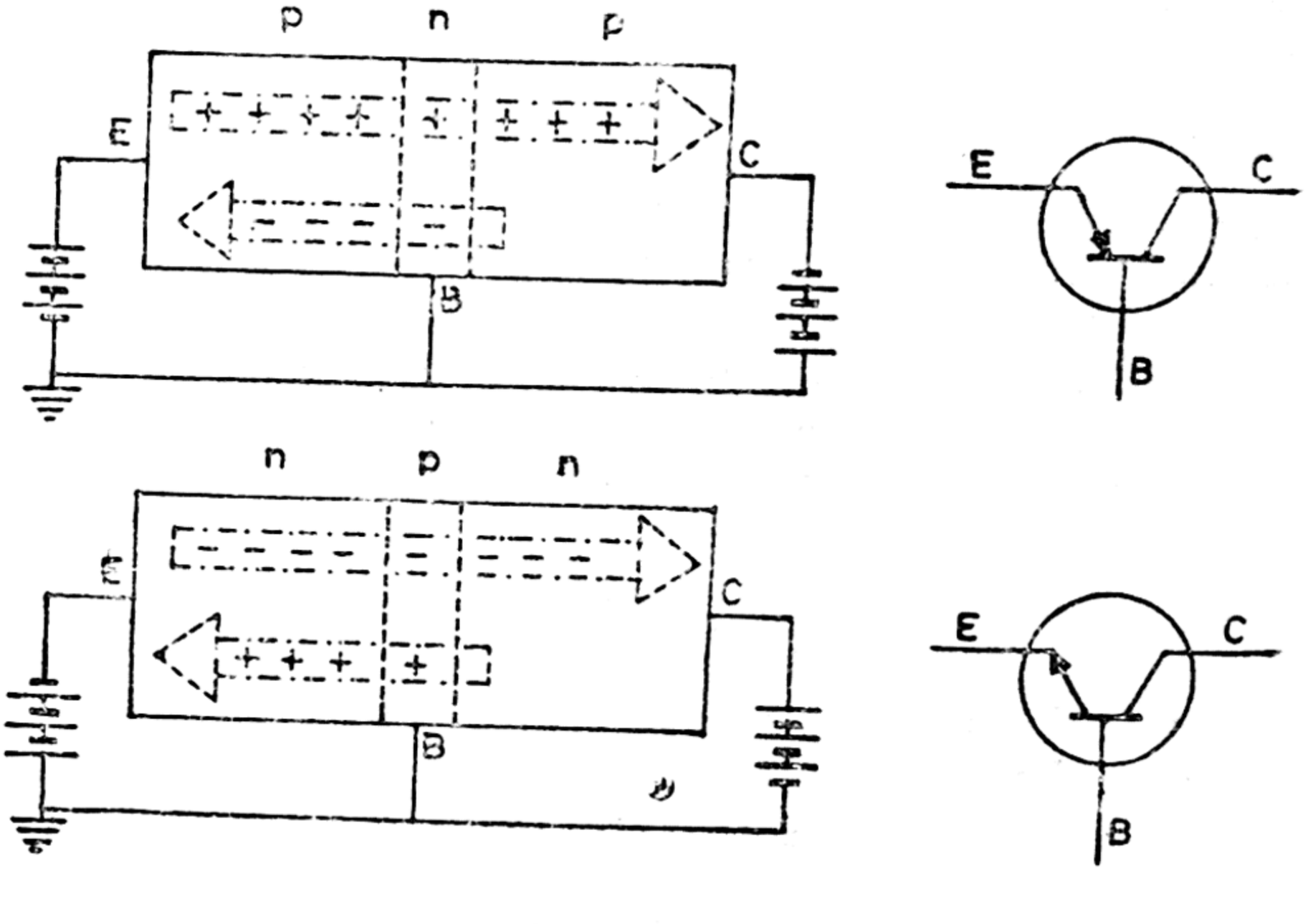


Fig. 5.16.

very thin so there is a small possibility of majority carriers from *E*-region to combine the majority carriers of base region. So most of the majority carriers from *E*-region pass into the collector region through the base region. The reverse biasing attracts the majority carriers from *E*-region. These majority carriers constitute current in the collector circuit. No doubt a small current do exist in *E-B* circuit.

If an time varying voltage or alternating signal is applied (input) between E and B, the number of majority carriers crossing over will also vary with input or time. So varying current will be produced in the collector output. Thus in transistor the output is controlled by input.

Transistor Parameters. There are two important parameters for transistor circuits :

(*i*) The *collector to emitter short circuit current amplification factor* (α)—It is the negative of the ratio of change in collector current (Δi_C) to a change in emitter current (Δi_E), keeping collector to base voltage (v_{CB}) constant, *i.e.*,

$$\alpha = -(\Delta i_C / \Delta i_E)_{v_{CB}=constant}. \qquad ...(35)$$

It is also written as α_{FB} or h_{FB}.

Since a few majority carriers of emitter combine with majority carriers of the base region while passing to collector region via base, so the variation of collector current will be less than the variation in emitter current. Thus the value of α will always be less than 1 (nearly 0.95 to 0.99). This value of $\alpha(<1)$ does not mean that transistor cannot be used as an amplifier because the majority carriers injected under forward bias in emitter-base circuit having low impedance (nearly zero to 1000 ohms) are extracted at the reverse biased collector base circuit having much higher impedance (more than mega ohms).

(*ii*) ***The collector to base short circuit current amplification factor*** (β)—It is the ratio of the change in collector current (Δic) to a given change in the base current (Δi_B) keeping collector to emitter voltage v_{CE} constant, *i.e.*,

$$\beta=(\Delta ic/\Delta i_B)_{v_{CE}=constant}. \qquad \text{...(36)}$$

Since the change in base current $\Delta i_B=-\Delta i_E-\Delta ic$

$$=-(1-\alpha)\,\Delta i_E.$$

$$\therefore \quad \beta=\frac{\Delta i_C}{\Delta i_B}=\frac{-\alpha\Delta i_E}{-(1-\alpha)\,\Delta i_E}=\frac{\alpha}{1-\alpha}. \qquad \text{...(37)}$$

The value of the parameter β is generally greater than one and may be as high as several hundred. This parameter is also written as α_{FE} or h_{FE}.

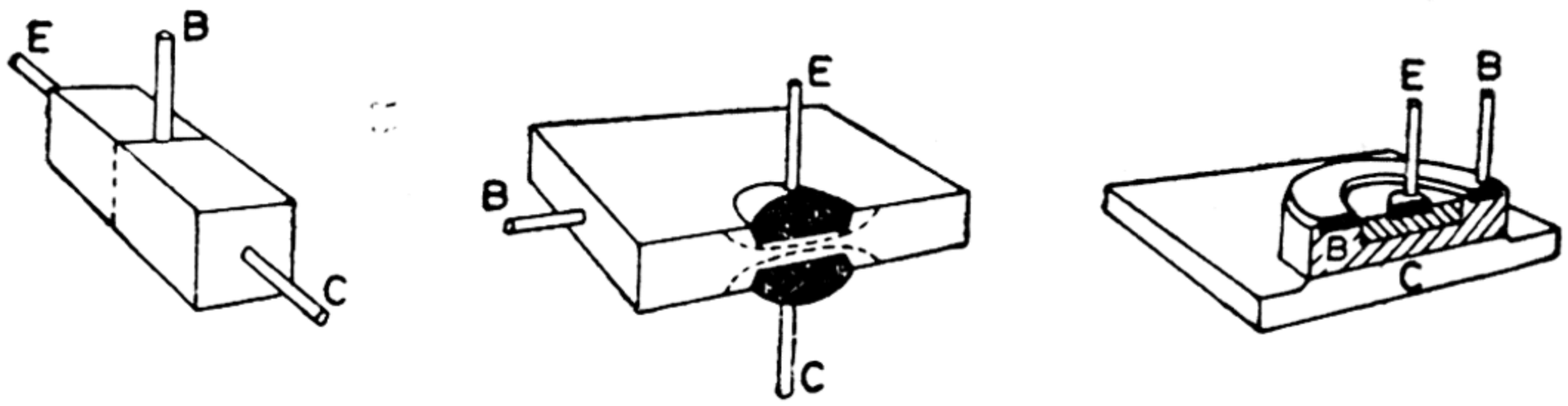
Transistor Manufacturing. There is a large variety of ways to construct junction transistors. The methods generally used at present time lead to *grown junction, rate grown junction, fused alloy junction, mesa junction and epitaxial mesa junction.*

(*a*) *Grown-junction transistors.* The process of their fabrication involves three main steps : (*i*) growing the bulk crystal, (*ii*) slicing it into individual bars and (*iii*) attaching leads and mounting. The grown *n-p-n* junction [Fig. 5.17 (*a*)] is produced during the pulling of a large crystal from a purified semi-conductor melt (*Si* or *Ge*) which has been weakly doped with *n*-impurity. In an appropriate time in the growth process, a *p*-type impurity is added in the melt to produce a *p*-type base.

After a short interval the melt is once again doped with *n*-type impurity and a second *n*-region is formed. The crystal is then sliced into small rectangular bars. After chemical etching and cleaning, the connection leads are soldered on the ends and on the central base zone. The assembly is mounted in its housing.

(*b*) *Rate-grown-junction transistors.* Such type of transistors are fabricated by a suitable doped semi-conductor melt having both *p*- and *n*-impurities present. The impurity materials used for this

purpose are those whose relative solubility in the solid or liquid depends on the rate of crystal growth. The *p*-materials such as *Ga* and *In* tend to enter the solid at a rate independent of the rate of crystal growth. Therefore at a slow rate of growth a *p*-region is formed. The *n*-material such as *Sb* tends to enter the solid at a faster rate with a rapid rate of crystal growth. Therefore at a rapid rate of crystal growth *n*-region is formed. By cycling the rate of growth *p* and *n*-regions are formed.



(*a*) (*b*) (*c*)

Fig . 5.17.

(*c*) *Fused-alloy-junction transistors.* Fused alloy transistors [Fig. 5.17 (*b*)] are fabricated by placing a tiny pellet of impurity on either side of a small wafer of germanium and by heating the whole to a temperature above the melting point of the impurity material but below that of *Ge*. For *p-n-p* transistors, the pellet is of *p*-material (*e.g.*, *In*). Thus on heating, indium melts and diffuses slightly into the wafer (*Ge*) and a *p*-layer is formed. The depth of impurity diffusion and the thickness of the base region between emitter and collector is controlled by time and temperature.

(*d*) *Mesa-junction transistors.* The mesa transistor derives its name from its geometry. In such transistors the emitter and base regions rise above the collector block like *mesas in the desert.* Such a structure is formed by diffusion and etching. In *n-p-n* mesa transistor, a *p*-type layer is produced by diffusion on an *n*-type silicon (or *Ge*) which acts as a collector. The depth of the diffused base layer can be accurately controlled as this process takes place in several hours. A small area is exposed to an *n*-diffusant by masking on the base layer and the emitter is formed by diffusion into the *p*-region. The material is etched down to the original collector level, leaving tiny base and emitter contact areas standing above the surroundings, as shown in Fig. 5.17 (*c*).

(*e*) *Epitaxial mesa junction transistors.* The name epitaxy is for the oriented intergrowth of the two solid materials. The growth occurs from a gaseous compound at elevated temperature. In the construction of such a transistor, a *Si* or *Ge* crystal is placed in a vapour consisting of the same atoms as those in the crystal. The crystal is kept at a low temperature and the atoms from the vapour affix themselves to the crystal. The crystal therefore grows. A new layer so grown on a crystal is called epitaxial layer.

In the epitaxial-mesa transistor a weakly doped *n*-type epitaxial layer is grown on a strongly doped *n*-type block which gives collector characteristics. A base layer may be diffused through a mask into epitaxial layer by usual methods. It is followed by diffusion of the emitter onto the base through a second mask. This technique produces transistors with very low collector saturation resistance, high voltage breakdown, reduced power losses within the transistor, increased upper frequency limits and smaller switching time.

Transistor Circuit Forms. The basic transistor has three electrodes, the *emitter, base and collector.* In terms of four-terminal device, one electrode is always common out of three to both input and output circuits. There are three possible circuit arrange-

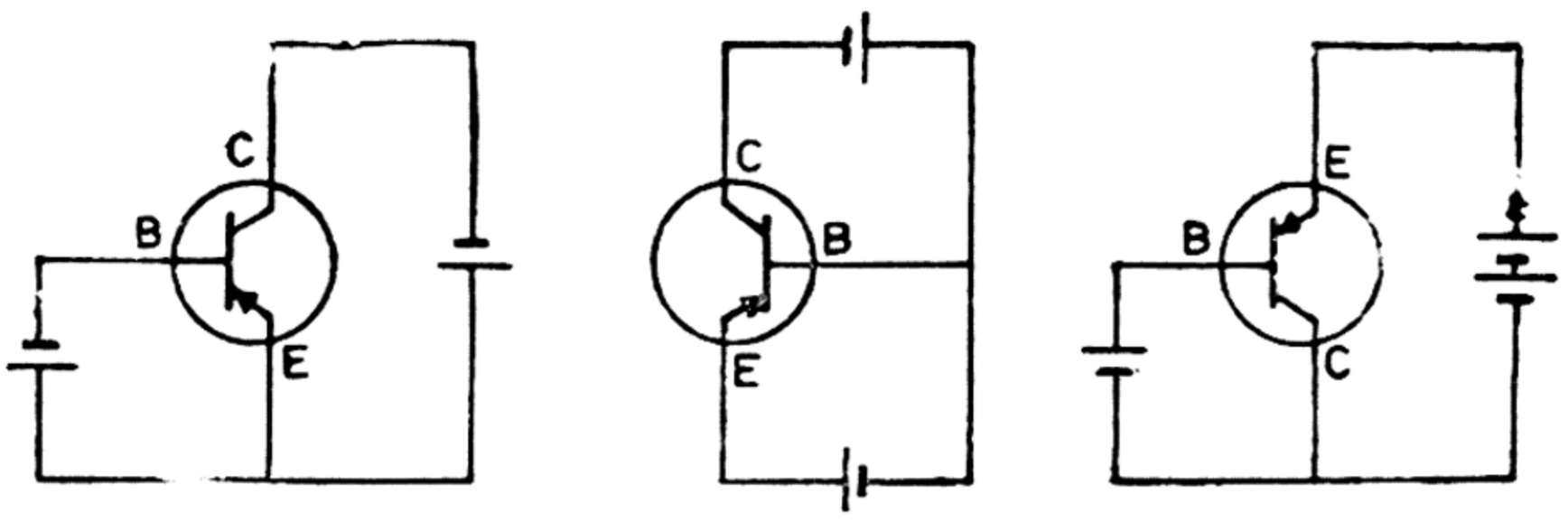


Fig. 5.18.

ments corresponding to three possible choices of common electrode. These are *common emitter, common base* and *common collector* circuits. For each connection in *p-n-p* transistor, the emitter terminal is made more positive than the base and the collector terminal is made more negative than the base. Since in a *p-n-p* transistor base current flows away from the base terminal hence for an *n-p-n* transistor it must flow into the base. Thus the polarities of emitter and collectors will be opposite in *n-p-n* transistor than there in *p-n-p* transistor. Supply potential connections in a *p-n-p* transistor in above mentioned circuit forms are as shown in Fig. 5.18.

5.9. GRAPHICAL CHARACTERISTICS

A characteristic curve relates two or more variables in a system. In the case of the transistor we are interested in electrical quantities which appear at the terminals, *i.e.*, the currents and potentials in the input and output circuits. The input as well as output characteristics are required. Two sets of curves are necessary to specify completely transistor operation.

The most often employed connection is the common emitter which resembles with the *grounded cathode vacuum triode.* The common emitter (*p-n-p*) transistor along with supply potentials and

indicating devices is shown in Fig. 5.19. For such a transistor, collector characteristics and static base characteristics may be

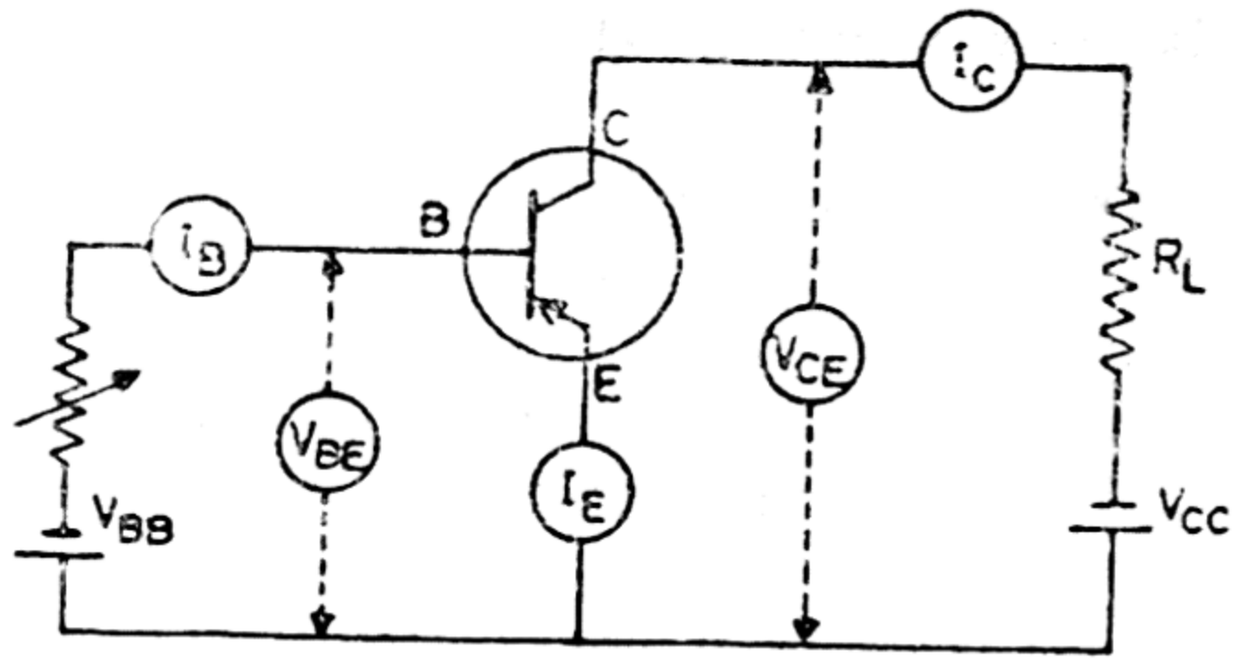


Fig. 5.19.

obtained from the observed data. The mathematical relations corresponding to these characteristics are

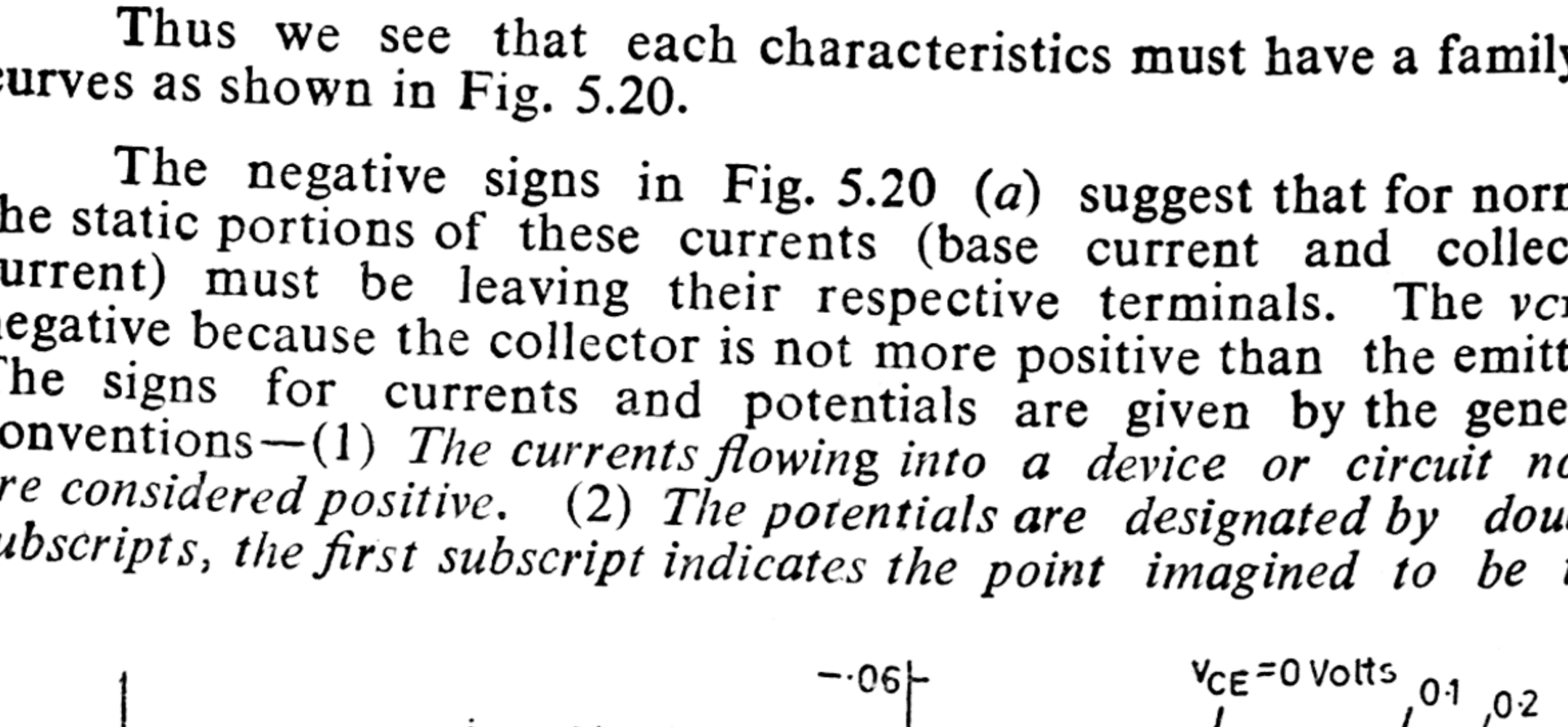
$$i_C = f(i_B, v_{CE})$$

and

$$i_B = f(v_{BE}, v_{CE}). \qquad \ldots(38)$$

Thus we see that each characteristics must have a family of curves as shown in Fig. 5.20.

The negative signs in Fig. 5.20 (*a*) suggest that for normal the static portions of these currents (base current and collector current) must be leaving their respective terminals. The v_{CE} is negative because the collector is not more positive than the emitter. The signs for currents and potentials are given by the general conventions—(1) *The currents flowing into a device or circuit node are considered positive.* (2) *The potentials are designated by double subscripts, the first subscript indicates the point imagined to be the*



(*a*) (*b*)

Fig. 5.20.

more positive. The graphs shown in Fig. 5.20 (*a*) and 5.20 (*b*) are commonly known as the *static collector characteristics* and the *static base characteristics* respectively. The slopes of curves shown

in Fig. 5.20 (*a*) represent $1/r_0$, where r_0 is the *output resistance*. The slope $\Delta ic/\Delta\ v_{CE}$ is minimum at the point known as point of inflection. The Fig. 5.20 (*b*) represents the input characteristics. These curves are roughly exponential in shape. The slope of these curves yields the small signal input impedance. The transfer and transconductance curves for a transistor may be deduced directly from the curves of Fig. 5.20. The ic versus i_B curve is known as *current transfer* and the ic versus v_{BE} curve is known as *transconductance curve* for a transistor. The transconductance curve gives the dependence of the output current on the input voltage. The slope of the curve $\Delta ic/\Delta v_{BE}$ is a conductance relating input and output variables. It is therefore called a *transconductance*, g_m.

There are three operating regions, as shown by the output curves [Fig. 5.20 (*a*)]. (*i*) The *cut-off region*, in which the transistor operates below the $i_B=0$ line and near the abcissa or in other words the emitter and collector are both reverse biased. This is considered to be the *off* condition of the transistor. (*ii*) The *saturation region*, in which the transistor operates on the saturation line near the ordinate with a minimum value of v_{CE}. In this region the emitter is forward biased and the collector has a very small reverse bias or small forward bias. Thus the collector is not able to collect all the carriers injected by the emitter. The saturation is the *on* condition of the transistor. Often used is a *saturation resistance* R_{CE}, which is defined as the reciprocal slope of the saturation line. This is low for most of the transistors. (*iii*) The *normal or active region*, in which the transistor operates in the central region. The curves are relatively uniform in spacing and slope. This region is suited to linear operation of the transistor as an amplifier. In this region the emitter is forward biased and the collector is in reverse bias.

Static or dc Load Line. The current voltage relations existing in the collector circuit may be written as

$$v_{CE}=V_{CC}-icR_L. \qquad ...(39)$$

Since the transistor and load are in series, the current must be the same for both. Since V_{CC} as a source voltage is constant, hence the above equation is the linear equation of form $x=c+my$ with x intercept V_{CC}, slope of $-R_L$ and y—intercept $ic=V_{CC}/R_L$, Drawing such a line on the output characteristics gives a *d-c load line*. For a particular load R_L, *d-c* load line is thus a straight line drawn between the intercepts V_{CC}/R_L and V_{CC}. It gives us all the possible combinations of current and voltage v_{CE} which may exist in the series combinations of load and transistor.

5.10. EQUIVALENT CIRCUITS

A large number of different equivalent circuits have been proposed for the transistor. In the case of vacuum tubes para-

meters μ, g_m and r_p are used. In the transistors several types of parameters are generally used as both the common base and common emitter connections are utilized. The transistor may be treated as an active three terminal network. In such a network there will always be a common ground or reference terminal and the three-terminal network is equivalent to a four-terminal or two-terminal pair network (Fig. 5.21). We shall specify only v_1, i_1 and v_2, i_2 as the input and output variables. The operation of a three-terminal network may be specified completely with two sets of equations, each involving one dependent and two independent variables. Any two of the four variables may be selected as the independent variables.

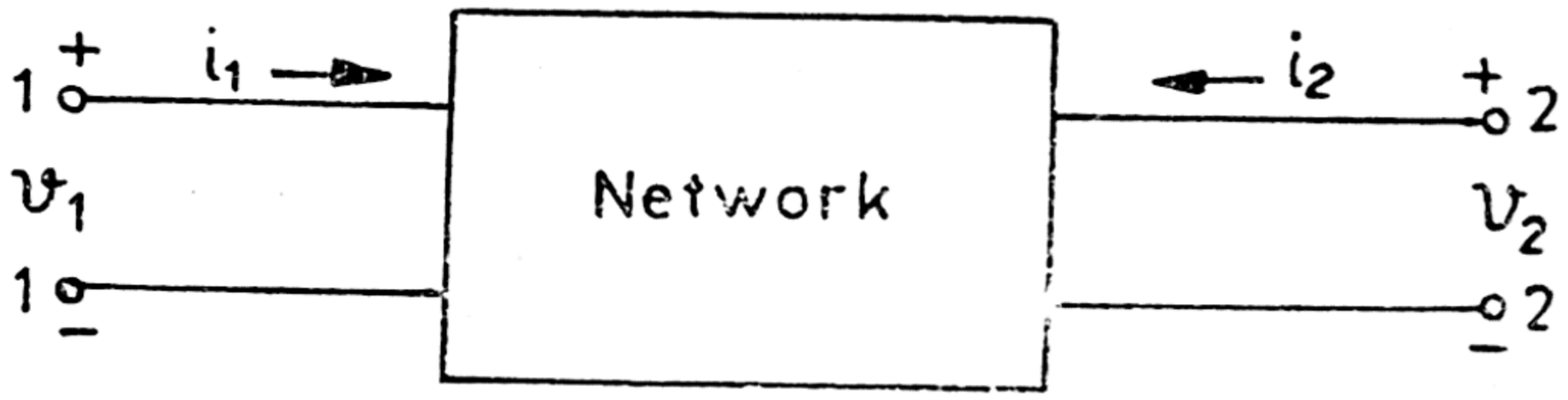


Fig. 5.21.

(*a*) Making the choice of i_1 and i_2 as independent variables implies the general relations

$$v_1=f(i_1, i_2) \text{ and } v_2=f(i_1, i_2).$$

These may be written as

$$dv_1=(\partial v_1/\partial i_1)\, di_1+(\partial v_1/\partial i_2)di_2$$

$$dv_2=(\partial v_2/\partial i_1)\, di_1+(\partial v_2/\partial i_2)di_2.$$

For sufficiently small *a-c* signals these equations may be written as

$$\mathbf{V}_1=z_{11}\mathbf{I}_1+z_{12}\mathbf{I}_2$$

$$\mathbf{V}_2=z_{21}\mathbf{I}_1+z_{22}\mathbf{I}_2,$$

or

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}=\begin{bmatrix}z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}. \qquad \ldots(40)$$

Here z parameters have been defined as the slopes of the $v-i$ relations and are known as open circuit *impedance parameters.* Using $\mathbf{I}_1=0$ and $\mathbf{I}_2=0$, in equations (40), we have

$z_{11}=\partial v_1/\partial i_1=$ Open-circuit input impedance $(\mathbf{I}_2=0)$

$z_{12}=\partial v_1/\partial i_2=$ Open-circuit reverse transfer impedance $(\mathbf{I}_1=0)$

$z_{21}=\partial v_2/\partial i_1=$ Open-circuit forward transfer impedance $(\mathbf{I}_2=0)$

$z_{22}=\partial v_2/\partial i_2=$ Open-circuit output impedance $(\mathbf{I}_1=0)$

(*b*) If the choice of v_1 and v_2 as the independent variables is made, then the general relations become

$$di_1=(\partial i_1/\partial v_1)dv_1+(\partial i_1/\partial v_2)dv_2$$
$$di_2=(\partial i_2/\partial v_1)dv_1+(\partial i_2/\partial v_2)dv_2.$$

For sufficiently small *a-c* signals where the partial derivatives are constant, we have

$$\mathbf{I}_1=y_{11}\,\mathbf{V}_1+y_{12}\,\mathbf{V}_2$$
$$\mathbf{I}_2=y_{21}\,\mathbf{V}_1+y_{22}\,\mathbf{V}_2,$$

or
$$\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}=\begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}. \qquad \ldots(41)$$

Here *y*—parameters are known as short circuit *admittance parameters*. Using $\mathbf{V}_1=0$ and $\mathbf{V}_2=0$ in equation (41), we have

$y_{11}=\partial i_1/\partial v_1=$ Short-circuit input admittance $(\mathbf{V}_2=0)$

$y_{12}=\partial i_1/\partial v_2=$ Short-circuit reverse transfer admittance $(\mathbf{V}_1=0)$

$y_{21}=\partial i_2/\partial v_1=$ Short-circuit forward transfer admittance $(\mathbf{V}_2=0)$

$y_{22}=\partial i_2/\partial v_2=$ Short-circuit output admittance $(\mathbf{V}_1=0)$

(*c*) The third choice of i_1 and v_2 as independent variables leads to a set of network parameters of considerable value in transistor circuit analysis. It gives

$$v_1=f(i_1, v_2),\ i_2=f(i_1, v_2).$$
$$dv_1=(\partial v_1/\partial i_1)di_1+(\partial v_1/\partial v_2)dv_2$$

and
$$di_2=(\partial i_2/\partial i_1)di_1+(\partial i_2/\partial v_2)dv_2.$$

For sufficiently small *a-c* signals where the partial derivatives become constants, we have

$$\mathbf{V}_1=h_{11}\,\mathbf{I}_1+h_{12}\,\mathbf{V}_2$$
$$\mathbf{I}_2=h_{21}\,\mathbf{I}_1+h_{22}\,\mathbf{V}_2.$$

or
$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{I}_2\end{bmatrix}=\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{V}_2\end{bmatrix}. \qquad \ldots(42)$$

Here *h*·parameters are known as *hybrid parameters*, as they are mixed parameters compared to purely impedance and admittance parameters. These are defined under either open or short-circuit conditions. We have $\mathbf{V}_2=0$ for a short circuit and $\mathbf{I}_1=0$ for an open circuit. Thus we have two sets of *h*-parameters, as defined below :

$h_{11}=\partial v_1/\partial i_1=$ Short-circuit input impedance $(\mathbf{V}_2=0)$

$h_{12}=\partial v_1/\partial v_2=$ Open-circuit reverse voltage gain $(\mathbf{I}_1=0)$

$h_{21}=\partial i_2/\partial i_1=$Short-circuit forward current gain $(V_2=0)$

$h_{22}=\partial i_2/\partial v_2=$Open-circuit output admittance $(I_1=0)$

Here we see that one parameter is an impedance, one an admittance and two are dimensionless ratios.

These three sets of parameters have been found useful in electronic circuit analysis. They can be measured under the specified conditions or evaluated from the characteristics. If one set of parameters are given, other two can be calculated using equations (40)—(42). One can find

$$z_{11}=h_{11}-h_{12}h_{21}/h_{22}\ ;\ z_{12}=h_{12}/h_{22},$$

$$z_{21}=-h_{21}/h_{22}\ ;\ z_{22}=1/h_{22}. \qquad ...(43)$$

For each set of equations represented in matrix form we may write down an equivalent circuit. The equivalent circuit that satisfies equation (40) is shown in Fig. 5.22(*a*) and has been called the *z-equivalent.* Each of the *z*-parameters have the dimensions of impedance and the quantities $z_{12}\mathbf{I}_2$ and $z_{21}\mathbf{I}_1$ are dependent voltage sources. The *z* parameters are not particularly only for transistor studies.

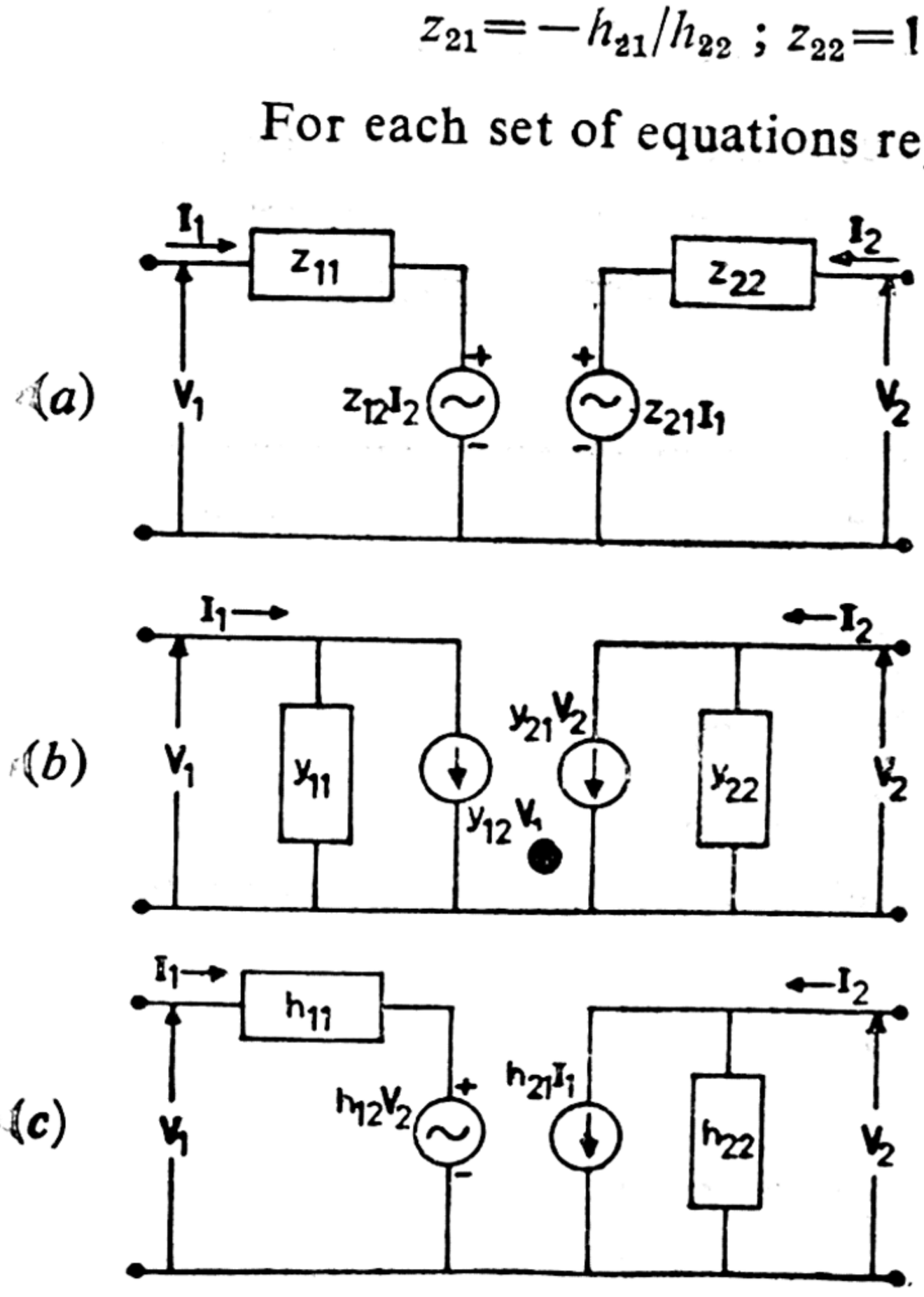


Fig. 5.22.

The equivalent circuit that satisfies Eq. (41) is shown in Fig. 5.22 (*b*) and is known as the *y-equivalent.* Each of the *y*-parameters have the dimensions of admittance. $y_{12}\mathbf{V}_1$ and $y_{21}\mathbf{V}_2$ are the two dependent current generators. The *y*-parameters are primarily applicable at high frequencies and are often employed to describe a transistor at a particular frequency.

The equivalent circuit that satisfies Eq. (42) is shown in Fig. 5.22(*c*) and is known as the *hybrid* (*or h*) *parameter equivalent.* Since the defining equations must obey Kirchhoff's laws, h_{11} must be an impedance, h_{22} an admittance while h_{12} and h_{21} are dimensionless. For low-frequency analysis of junction transistors h_{11} and h_{22} will be resistive. The hybrid π-circuit is very useful

for common emitter circuit analysis and at high frequencies. The hybrid parameters have become popular because (1) they are convenient to use in circuit analysis, (2) They correspond to actual values and (3) are relatively easy to measure in the laboratory.

The standard transistor nomenclature tells us the following parameter symbols :

$$h_{11}=h_i,\ h_{12}=h_r,\ h_{21}=h_f,\ \text{and}\ h_{22}=h_o. \qquad \ldots(43)$$

A second subscript is also added to the h-parameters to designate the orientation. It is b, e or c depending upon the common terminal. Thus h_{ib}, h_{rb}, h_{fb} and h_{ob} describe common base transistor while h_{ie}, h_{re}, h_{fe}, and h_{oe} are for common emitter connected transistors. The equivalent circuits for three standard configurations (common base, common emitter, common collector) do not differ because they all must satisfy the defining equations.

For each type of transistor a complete set of 12 parameters is required. Transistor manufacturers usually publish the h-parameters of a transistor in the common-emitter form. If the data are given in common-base or common-collector form or required in these cases, the conversion relations are required. The approximate relations among h-parameters are

$$h_{ie}=\frac{h_{ib}}{1+h_{fb}},\quad h_{re}=\frac{h_{ib}\,h_{ob}-h_{rb}\,h_{fb}-h_{rb}}{1+h_{fb}},\quad h_{fe}=\frac{-h_{fb}}{1+h_{fb}},$$

$$h_{oe}=\frac{h_{ob}}{1+h_{fb}};\quad h_{ib}=\frac{h_{ie}}{1+h_{fe}},\quad h_{rb}=\frac{h_{ie}\,h_{oe}-h_{re}\,h_{fe}-h_{re}}{1+h_{fe}},$$

$$h_{fb}=\frac{-h_{fe}}{1+h_{fe}},\quad h_{ob}=\frac{h_{oe}}{1+h_{fe}},$$

$$h_{ic}=h_{ie},\ h_{rc}=1,\ h_{fc}=-(1+h_{fe}),\ h_{oc}=h_{oe}. \qquad \ldots(44)$$

The typical h-parameters for a transistor for different forms are given in Table 5.2.

TABLE 5.2

Typical h-parameters for a Transistor

Parameter	*CE*	*CB*	*CC*
h_i (ohms)	1000	20	1000
h_r (ratio)	2.5×10^{-4}	3×10^{-4}	1
h_f (ratio)	50	−0.98	−51
h_o (mhos)	2.5×10^{-5}	0.5×10^{-6}	2.5×10^{-5}

5.11. COMMON-EMITTER AMPLIFIER

Consider a single stage common-emitter amplifier circuit. We determine the input resistance, the voltage gain, current gain and power gain with the aid of the equivalent circuit with the load resistance R_L and without the source resistance R_s. The output resistance is determined with R_s in circuit and R_L removed. The amplifier circuit, equivalent circuit with R_L, without R_s and equivalent circuit with R_s, without R_L are shown in Fig. 5.23 (*a*), (*b*) and (*c*) respectively.

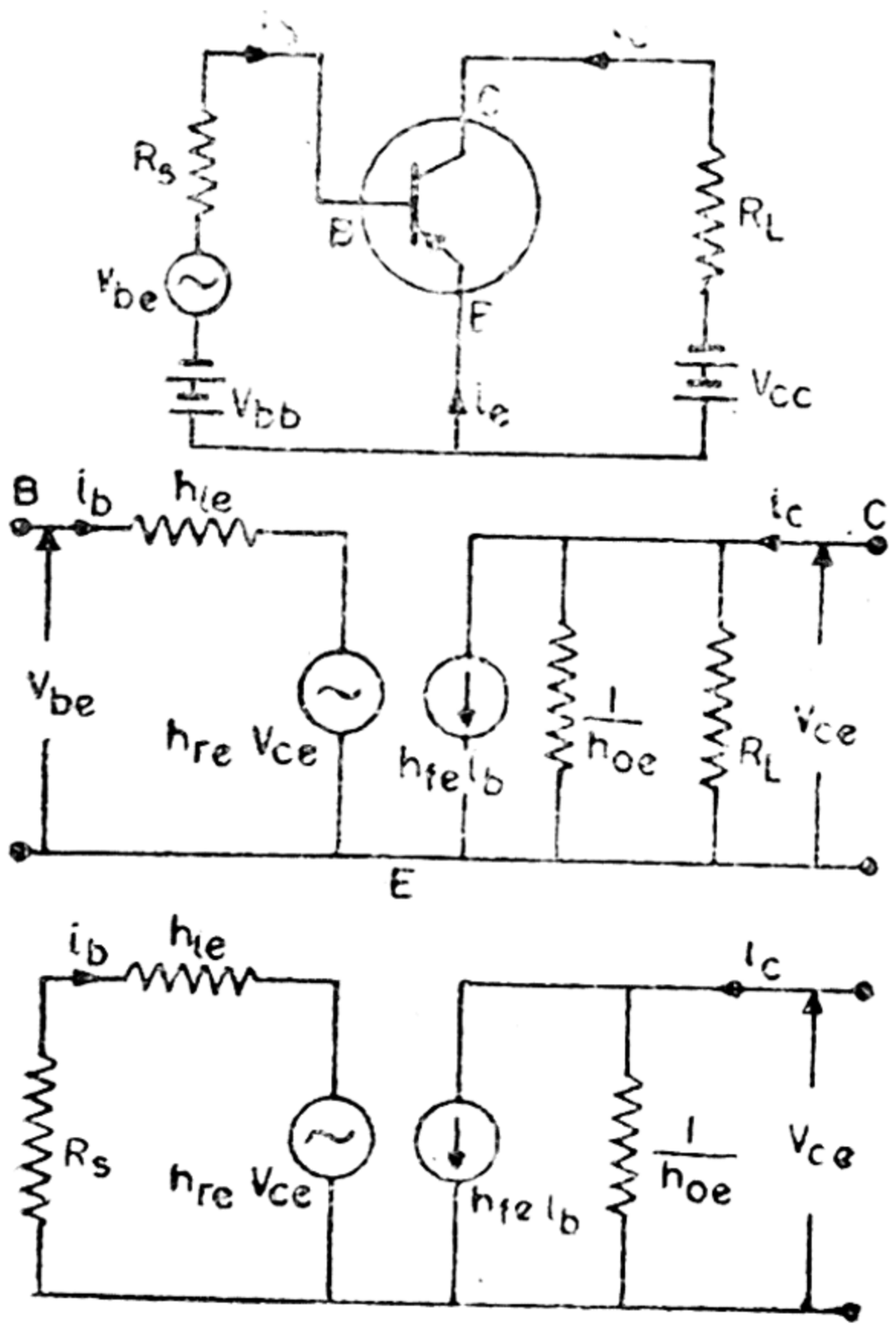


Fig. 5.23.

The equations for the input side and output of the circuit of Fig 5.21. (*b*) are respectively

$$v_{be} = h_{ie} i_b + h_{re} v_{ce} \qquad \ldots(45)$$

$$0 = h_{fe} i_b + h_{oe} v_{ce} + v_{ce}/R_L. \qquad \ldots(46)$$

Solving Eq. (46), we get

$$v_{ce} = \frac{-h_{fe} i_b}{h_{oe} + (1/R_L)}. \qquad \ldots(47)$$

Inserting this value in Eq. (45), we get

$$v_{be} = \left[h_{ie} - \frac{h_{re} h_{fe}}{h_{oe} + (1/R_L)} \right] i_b. \qquad \ldots(48)$$

$\therefore$ Input impedance Z_{in} or resistance $R_{in} = \dfrac{v_{be}}{i_b}$

$$= \frac{h_{ie} h_{oe} - h_{re} h_{fe} + h_{ie}/R_L}{h_{oe} + (1/R_L)}$$

$$= (R_L H + h_{ie})/(R_L h_{oe} + 1), \qquad \ldots(49)$$

where $H = h_{ie} h_{oe} - h_{re} h_{fe}$.

Here we see that the input resistance depends on load resistance, which does not occur in vacuum tube circuits. The voltage gain is defined as

$$A_v = \frac{v_{ce}}{v_{be}} = \frac{-h_{fe}/[h_{oe} + 1/R_L]}{h_{ie} - h_{re} h_{fe}/[h_{oe} + 1/R_L]}$$

$$= \frac{-h_{fe} R_L}{R_L H + h_{ie}}. \qquad \ldots(50$$

Since $v_{ce} = -i_e R_L$, hence the current gain

$$A_i = \frac{i_e}{i_b} = \frac{h_{fe}}{R_L h_{oe} + 1} \quad \ldots(51)$$

The power gain is the product of the magnitudes of Av and A_i. Thus

$$A_p = |A_v| \; |A_i| = h_{fe}^2 R_L / (R_L H + h_{ie})(R_L h_{oe} + 1) \quad \ldots(52)$$

From the equivalent circuit 5.21 (*c*), we get

$$0 = i_b(Rs + h_{ie}) = h_{re} v_{ce} \quad \ldots(53)$$

and

$$i_c = h_{fe} i_b + h_{oe} v_{ce}. \quad \ldots(54)$$

Substituting the value of i_b from Eq. (53) into Eq. (54) and solving, we get the output impedence Z_{out} or resistance R_{out} as

$$R_{out} = \frac{v_{ce}}{ic} = \frac{Rs + h_{ie}}{H + h_{oe} Rs} \quad \ldots(55)$$

Here we see that the output resistance is determined in part by the source resistance Rs.

Similarly we may develop expressions for the common base and common collector amplifier circuits.

Substituting the values of the corresponding hybrid parameter from Table 5.1 in the above relations, we can summarise the values of the quantities (input impedence, output impedence, voltage gain, current and power gain) for the different forms of the transistor. The comparison is given in Table 5.3.

TABLE 5.3

Comparison of the three basic transistor amplifier configurations

Quantity	*CE*	*CB*	*CC*
Z_{in}	Medium (1K)	Low (22Ω)	High (144K)
Z_{out}	High (45K)	High (1.72M)	Low (80Ω)
A_v	High (−131)*	High (131)	1 (0.99)
A_i	High (−46)	1 (0.98)	High (47)
A_p	High (6026)	High (128)	Medium (47)

* A negative sign indicates a phase change of π at the output.

Here we see that the common emitter amplifier is best used for voltage and power amplifications. The common base configuration is used for matching a low impedence source to a high impedence load, *i.e.*, as a step up impedence transformer. The common collector configuration is used for matching a high impedence

source to a low impedence load. *i.e.*, as a stepdown impedence transformer. The comman collector amplifier is also called emitter follower or cathode follower.

5.12. THE FIELD EFFECT TRANSISTOR

The *field effect transistor* (FET) is a different device proposed by Shockley in 1948. Its operation depends on electric fields across the conducting region. Since it uses only one type of charge carriers in the conducting region, it is said to be *unipolar*. The usual transistor uses both positive and negative carriers and is said to be *bipolar*. The FET differs from the bipolar type in several ways : (*i*) It has a very high input resistance of many megohms, (*ii*) It suffers less from random variation of current or electrical noise and is most useful with small input signals.

In an idealized picture a bar of *n*-type silicon has an ohmic contact at either end. The upper contact is known as the *drain* and the lower contact as the *source*. A current is established in the bar by an applied voltage (0—30 volts) with negative terminal to the source electrode. The electrons are thus supplied to the source electrode and are collected at the drain electrode. On one side of the bar we diffuse in some *p*-type material, so as to make a *p-n* junction. This junction is called *gate*. It is held at a negative voltage with respect to the source. If the *p-n* junction at the gate is asymmetrical, with the *p*-type material much more heavily doped than the *n*-type, the depletion region will be developed largely in the *n*-type material, as shown by double hatched region in the diagram. If the voltage V_{DS} is increased, the *p-n* junction at the gate becomes more reverse biased and the depletion region spreads out in the *n*-region, as shown by the single hatched region. The depletion region is thicker towards drain side of the gate as the reverse voltage at this side is larger than at the source side.

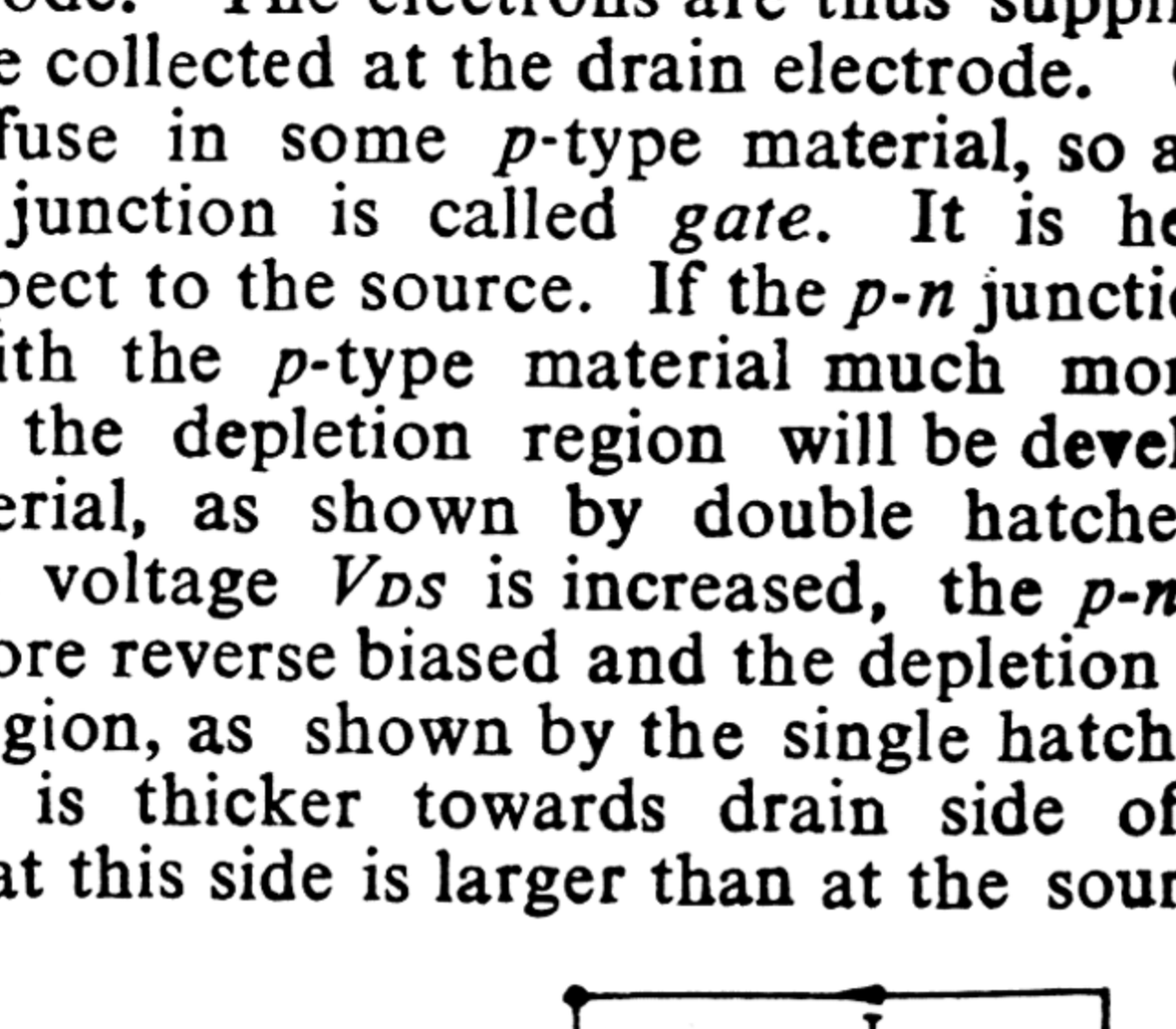


Fig. 5.24.

There are two basic types of FETs. (*i*) The junction FET, in which the grid is isolated from the conducting channel by a reverse biased junction. (*ii*) The insulated gate FET, in which the

gate is separated from the rest of the device by an insulating layer of metallic oxide. For this reason the insulated gate FET is often called MOSFET (metallic oxide semiconductor) transistor. Another form of insulated gate FET is obtained by vacuum deposition of material and is thus known as TFT (thin film transistor). The MOS design of FET is widely used in switching circuits and in complex circuit arrays. The several circuit symbols used to identify FET are shown in Fig 5.25.

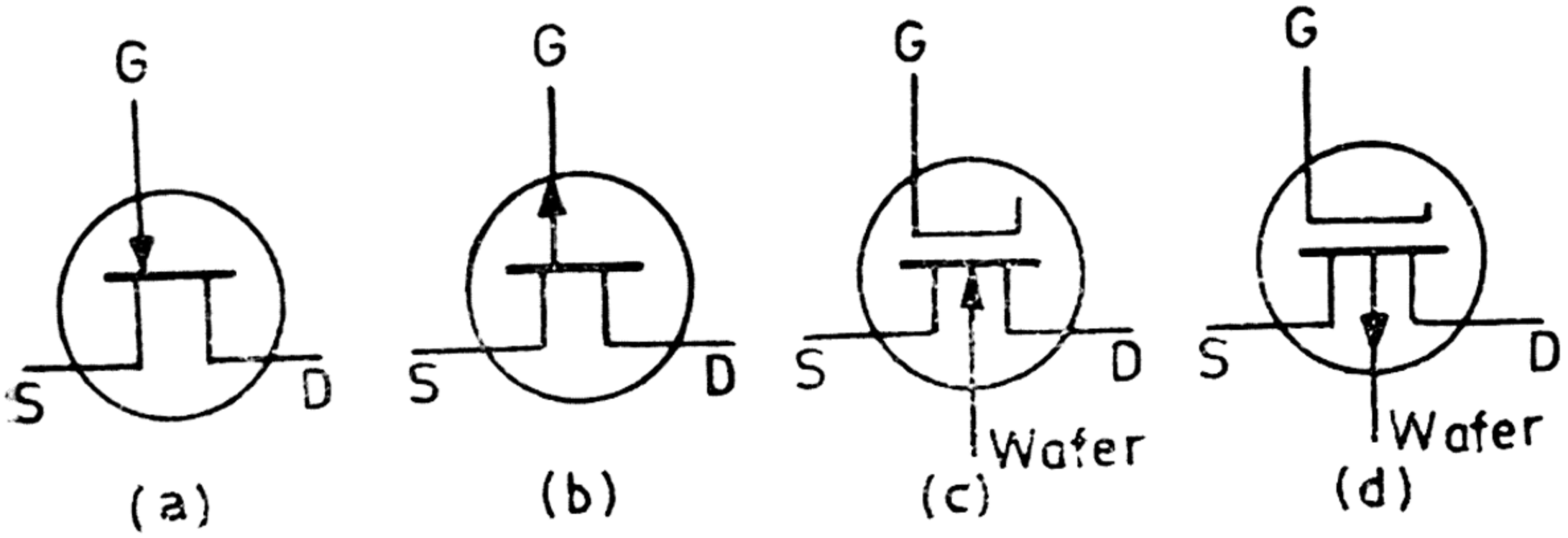


Fig. 5.25.

Fig. 5.25. JFET : (a) n-channel, (b) p-channel, MOSFET : (c) n-channel, (d) p-channel.

Two salient features should be noted. (*i*) The FETs are voltage controlled devices, the gate voltage is responsible for the basic mechanism. (*ii*) These are majority carrier devices, *i.e.*, the electrons carry the current in *n*-channel devices, while the bipolar transistor are minority carrier devices. Although the majority carrier devices are much less sensitive to nuclear radiations than minority carrier devices, the FETs are important in certain space and nuclear physics applications.

FET parameters—The FET characteristics can be obtained by investigating how the current flowing between drain and source, the drain current I_D, varies with the drain to source voltage V_{DS} for a fixed value of V_{GS}. Let V_{GS} be initially fixed at zero. At low V_{DS} voltages the depletion regions are thin and the current increases with voltage as shown by the curve *OA* (Fig. 5.26). As V_{DS} is further increased, the *p-n* junction at the gate becomes more reverse biased, the current is now forced to flow in a channel which becomes increasingly contracted when the channel becomes very small, an increase in current due to the increase in V_{DS} is balanced by the reduction in current due to channel narrowing, and the characteristic curve becomes practically horizontal (*AB* part of the curve). The FET is said to be in the *pinch-off* region. The abrupt rise in current

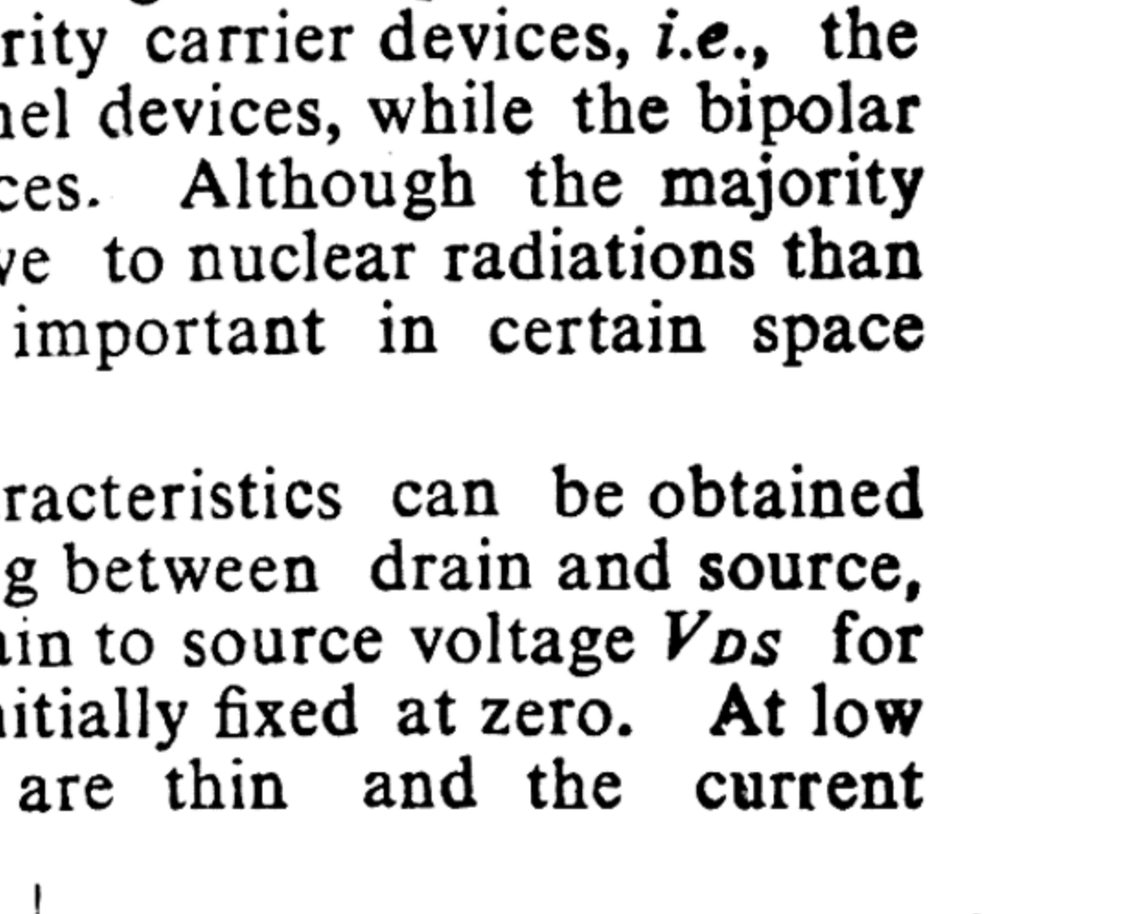


Fig. 5.26.

(*BC*) is due to an avalanche voltage breakdown in the depletion region between gate and drain. If V_{GS} is fixed at some small negative voltage and V_{DS} is gradually increased as the depletion layer is initially larger, the slope of the curve will be initially smaller and the curve will turn over and the pitch off will occur at a lower current I_D. The family of curves are shown in Fig. 5.26.

It is clear from above curves that the drain current I_D depends on V_{DS} and V_{GS}, therefore we may write

$$dI_D=(\partial I_D/\partial V_{DS})V_{GS}\, dV_{DS}+(\partial I_D/\partial V_{GS})V_{DS}\, dV_{GS}. \qquad ...(56)$$

Its comparison with the curves (Fig. 5.26) shows that the quantity $(\partial I_D/\partial V_{DS})$ gives the slope of the characteristic at any point say *P* for the particular value of V_{GS}, while $(\partial I_D/\partial V_{GS})$ gives the spacing of the curves near the point *P* for the particular value of V_{DS} at *P*. These quantities are not constant. If we restrict ourselves to small variations around a specified operating point, we may write above equation as

$$i_d=g_d\, v_{ds}+g_m\, v_{gs}, \qquad ...(57)$$

where we have written $\partial I_D/\partial V_{DS}$ as g_d (the drain or output conductance) and $\partial I_D/\partial V_{GS}$ as g_m (the mutual conductance). Both these quantities have dimensions ohms^{-1}. Generally g_d is replaced by $1/r_d$, where r_d is called drain resistance. Thus we have

$$i_d=v_{ds}/r_d+g_m\, v_{gs}. \qquad ...(58)$$

This type of equation also describes the operation of a triode or pentode valve. Similar to the case of triode valve we may define the amplification factor, μ for the FET as

$$\mu=(\partial V_{DS}/\partial V_{GS})_{I_D} \qquad ...(59)$$

Thus equation (56) gives

$$\mu=g_m/g_d=g_m\, r_d. \qquad ...(60)$$

5.13. THE UNIJUNCTION TRANSISTOR

Another useful solid state device is the *unijunction transistor*.

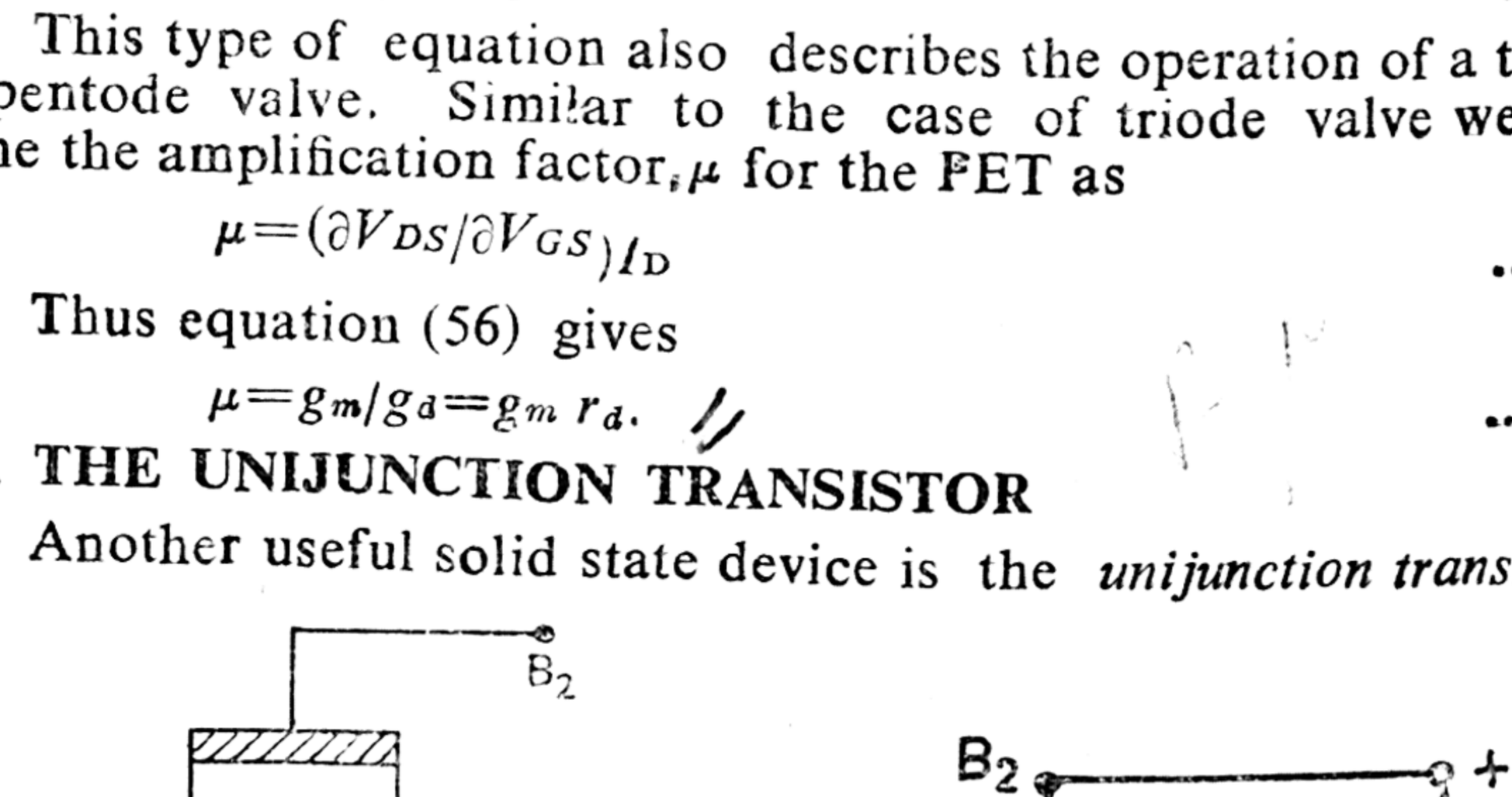


Fig. 5.27.

It, sometimes referred to as a double base diode, consists of a bar of *n*-silicon, which has two ohmic contacts (bases B_1 and B_2 at its two ends and a *p*-emitter junction *E* located on the side of the bar in the middle. The resistance between the bases, the *interbase resistance* R_{BB}, is of the order of 5 to 10*k* Ω. Base B_2 is maintained positive at a potential V_{BB} with respect to B_1. The emitter will be at some potential $\eta\ V_{BB}$. The fraction η is known as the *intrinsic stand-off ratio*, generally having a value 0.5—0.75. It shows that the emitter is somewhat closer to base 2 than base 1.

If the input to the emitter V_E is less than $\eta\ V_{BB}$, the emitter junction is reverse-biased and only a low level of emitted current passes through the junction. If V_E is more than $\eta\ V_{BB}$, the emitter is forward-biased as at V_p. Due to it the holes are injected into the bar, repelled by base B_2 and moving to base B_1. This increases the conductivity of the emitter-B_1 portion of the bar which results in a reduced voltage drop V_E, required to support a given current level I_E. The process, in which the current increases the conductivity and thus the drop in voltage V_E, continues until we reach the valley with coordinates (V_V, I_V). This region (*B* to *C* in Fig. 5.28) is known as a region of negative resistance. For values

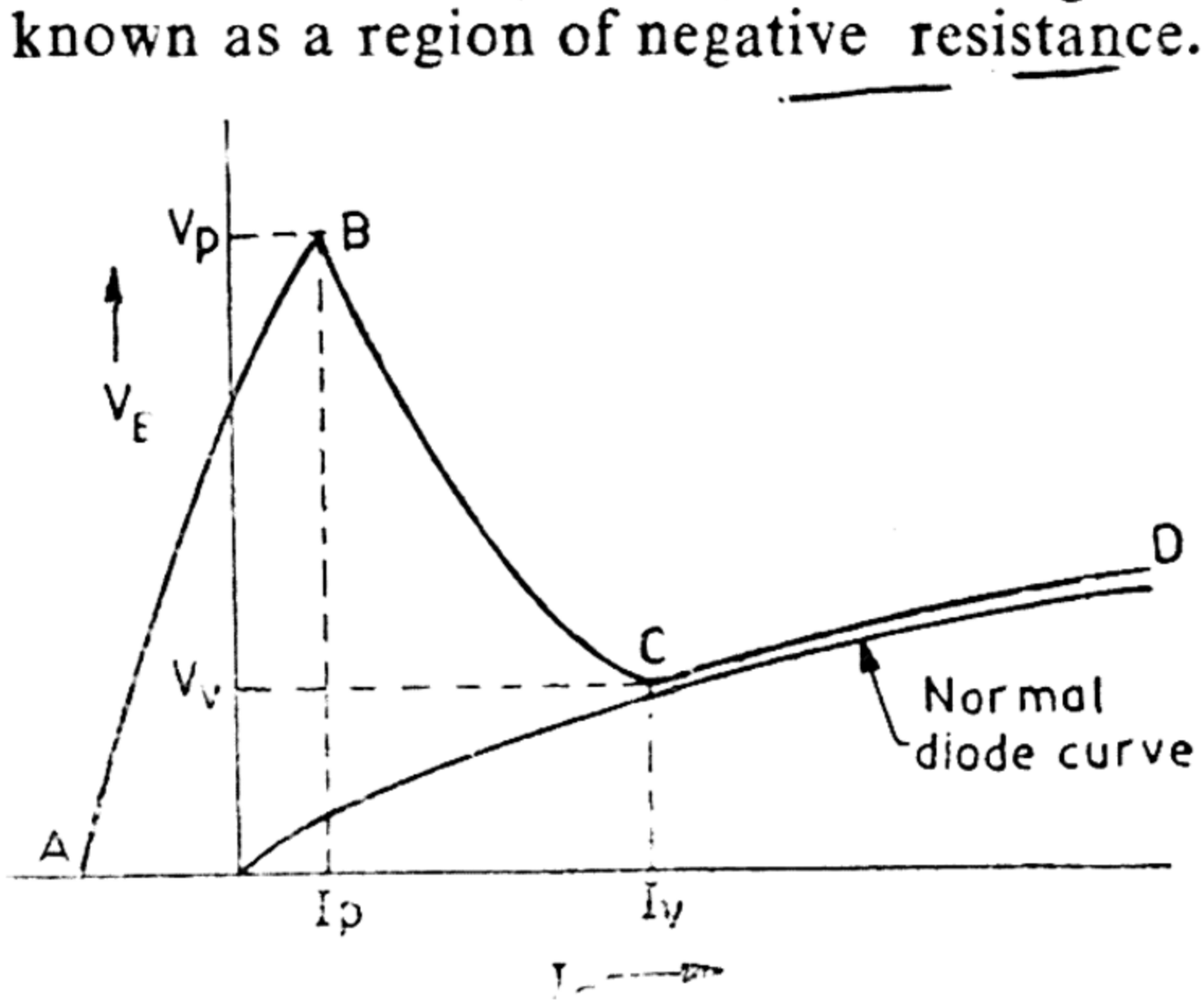


Fig. 5.28.

of emitter current greater than I_V, the voltage is found to follow the normal forward-biased *p-n* junction diode curve (curve from *C* to *D*).

The curve between the peak and valley points, *i.e.*, negative resistance region of the characteristic, is useful in the generation of time varying waveforms (pulse and timing circuits) and for amplification.

5.14. INTEGRATED CIRCUITS (IC)

The *microelectronics* is the branch of electronics in which we describe the technology that is used to produce complete electronic circuits or circuit modules of extremely small size. It has many advantages over the conventional circuits, such as improved uniformity and reliability, decreased weight and power consumption

and lower total cost. *Integrated* or *microcircuit* technology is a part of microelectronics. The integrated circuit (*IC*) or microcircuit is a device that will perform the function of a complete circuit, in which the circuit elements (both active and passive) are included or integrated into a single solid wafer of semiconductor material. The techniques of epitaxial growth, oxidation, diffusion through masks and metal deposition have made possible the production of complete electronic circuits on a small silicon chip.

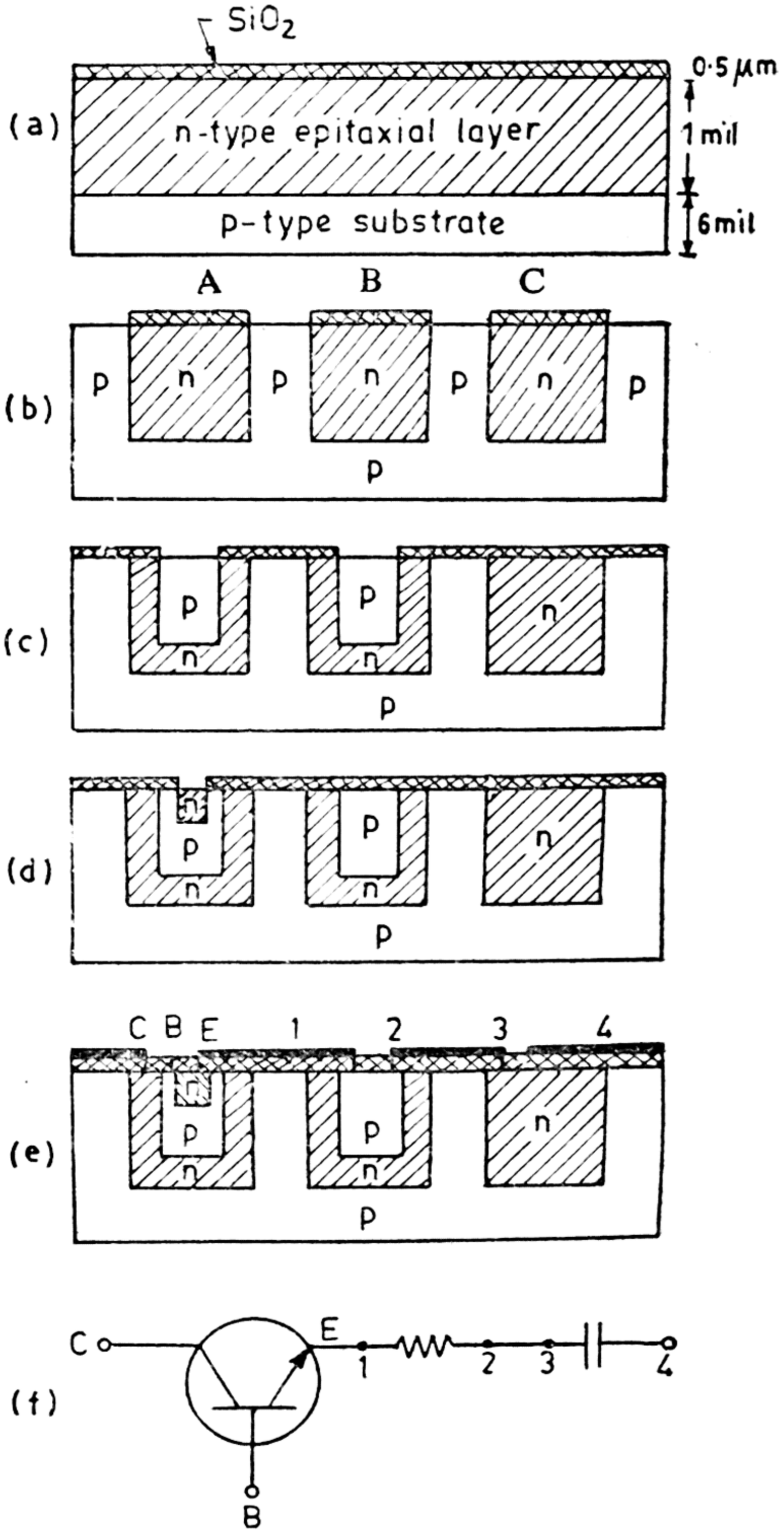


Fig. 5.29.

The steps in production of an integrated circuit in a chip are shown in Fig. 5.29. Starting with the *p* silicon wafer of

thickness about 6 mils (150 μm), an *n* impurity is deposited on the *p*-type substrate. This *n*-type layer of silicon is of 1 mil thickness and is called the *n*-type epitaxial layer. A thin layer of SiO_2 is formed over this layer by exposure to oxygen at high temperature [Fig. (*a*)]. To construct a particular circuit [Fig. 5.29 (*f*)], silicon dioxide is removed at different places by means of photolithographic etching process. The remaining portion of SiO_2 layer serves as a mask for the diffusion of impurities. The *p*-type impurity is allowed to diffuse through the channel opened (etched region) and the *n*-type epitaxial layer down to *p*-type substrate. In this way we have the shaded *n*-type regions, called *isolated islands* [Fig. (*b*)]. All active and passive components are formed from these isolated regions. A SiO_2 layer is again formed over the wafer and the photolithographic process is used again to create openings. Through these the *p*-type impurity is diffused into the pockets of the *n*-type regions to form base for transistors, or anode for diodes or resistors [Fig (*c*)]. Let us call these *p*-type regions as *A*, *B* and *C*.

A SiO_2 layer is again formed over the wafer and the photolithographic process is used to create opening in the *p*-type region at *A*. Through this opening *n*-type impurity is diffused to form emitter for transistor (or cathode for diode) [Fig. (*d*)]. We thus have transistor at *A* and a resistor at *B*.

It is now necessary to make interconnections among the various elements on the silicon wafer. SiO_2 layer is again formed over the wafer. The photolithographic process is used to create openings at the points where the electrical connections are to be made. A thin coating of aluminium is made over the entire wafer by placing it in a high vacuum chamber in which aluminium is being evaporated or boiled. This *Al* layer is masked and selectively etched to form the pattern of connections [Fig. (*e*)]. At region *C*, the *Al* forms the upper plate of the capacitor with silicon dioxide as a dielectric and *n* area as the underplate. This is known as a metal-oxide-silicon (*MOS*) capacitor. One can develop capacitor of capacitance as high as 300 pf/mm^2. This chip shown in Fig. (*e*) is equivalent to an electrical circuit shown in Fig. (*f*).

The resistance of the integrated circuit known as integrated resistor is the bulk resistivity of a *p*-diffused in the isolated *n*-type region. Let the resistivity of the diffused *p*-type impurity be ρ. Let the diffused region be in the form of a sheet with length *l*, width *w* and thickness *t*. Then the resistance *R* of the resistor will be given by

$$R = \rho l/A = \rho l/wt \text{ ohms.}$$

For a square of any size (*i.e.*, $l = w$), we have

$$R_s = \rho/t \text{ ohms.}$$

Thus we see that for the square sheet, the resistance depends on the thickness. If the length and width are not equal, then we have

$$R=\rho l/wt=R_s\ l/w \text{ ohms.} \qquad ...(61)$$

The resistors up to 20 $k\Omega$ can be constructed by using several strips in series in a grid form. For lower resistances ($\sim$ ohms) the n-type emitter diffusion is used. These resistors have accuracy of the order of 5 to 10%.

Exercises

Example 1. *Calculate the donor concentration in a sample of n-germanium having a resistivity of 0.15 ohm-m at 300°K?*

For n material,

The resistivity $1/\sigma=1/n_e e\mu_e=0.15$.

From table 5.1, we find $\mu_e=0.39$. Therefore the donor concentration

$$N_D\simeq n_e=\frac{\sigma}{e\mu_e}=\frac{(1/0.15)}{1.6\times10^{-19}\times0.39}$$
$$=1.07\times10^{20} \text{ electrons}/m^3.$$

The hole concentration $n_h=n_i^2/N_D$.

The table 5.1 gives $n_i=2.4\times10^{19}m^{-3}$. Therefore

$$n_h=(2.4\times10^{19})^2/1.07\times10^{20}=5.39\times10^{18} \text{ holes}/m^3.$$

Example 2. *Determine the rectification ratio for a germanium pn-junction at 0.13 V and $i_s=25\times10^{-6}$ amp, $T=300°K$.*

We know that the rectification ratio of a p-n junction is defined as the ratio of forward to reverse current at a given applied voltage. The diode equation is given as

$$i=i_s(e^{Ve/kT}-1).$$

For the forward direction of current, V is positive. It is given as 0.13 V, whereas $kT/e=0.026$ V. Thus we have

$$i\simeq i_s e^{Ve/kT}=i_s e^{0\cdot13/0\cdot026}=i_s e^5.$$

For the reverse direction, V is negative. The exponential term quickly becomes negligible with respect to unity. Thus the reverse current reduces to reverse saturation current.

$\therefore$ The rectification ratio $r=e^5=148.4$.

Example 3. *A germanium diode with $i_0=10\mu A$ is to be used. Calculate the forward voltage at which the diode current is 100 mA. What will be the current for reverse bias of 100 volts?*

The diode current is defined as

$$i = i_s\,(e^{eV/kT} - 1).$$

In the present problem the saturation current $i_s = i_o = 10\mu A$, $i = 100\ mA$ and $kT/e = 26\ mA$ at room temperature.

$$\therefore \quad 100 \times 10^{-3} = 10 \times 10^{-6}[e^{V/0\cdot026} - 1]$$

$$\text{or} \quad V = 0.026 \log_e [10^4 - 1] = 0.026 \times 2.3026 \times 3.99$$

$$= 0.23 \text{ volt.}$$

For the second part of the problem, we have $V = -100$ volts.

$$\therefore \quad i = 10 \times 10^{-6}[e^{-100/0\cdot026} - 1]$$

$$= -10 \times 10^{-6} A = -10\mu A.$$

Example 4. *A sample of germanium has intrinsic electron density of* 2.5×10^{19} *electrons/*m^3 *at 300°K. It is made p-material by addition of indium at the rate of one indium atom per* 4×10^8 *germanium atoms. Calculate the concentration of charge carriers.*

We know that the number of germanium atoms per cubic meter is 4.4×10^{28}. Thus the density of acceptor atoms

$$N_A = \frac{4.4 \times 10^{28}}{4 \times 10^8} = 1.1 \times 10^{20} \text{ atoms}/m^3.$$

Assuming the donor density to be zero, we have the density of majority carriers

$$p_h = N_A + \frac{n_i^2}{N_A} = 1.10 \times 10^{20} + \frac{(2.5 \times 10^{19})^2}{1.10 \times 10^{20}}$$

$$= 1.157 \times 10^{20} \text{ atoms}/m^3.$$

and the density of minority carriers

$$p_e = \frac{n_i^2}{N_A} = \frac{(2.5 \times 10^{19})^2}{1.10 \times 10^{20}} = 5.68 \times 10^{18} \text{ atoms}/m^3.$$

Thus we see that the addition of *p*-impurity reduces the density of minority charges to about one-fifth of the value it would have in intrinsic material.

Example 5. *A given transistor exhibits a short-circuit current amplification of 0.995 when used in a common base circuit. Calculate its short-circuit current gain when used as (a) common-emitter and (b) common collector amplifier.*

The short-circuit current amplification of the transistor in the common-base configuration is denoted by a Greek letter α and defined as

$$\alpha = \underset{\Delta i_E \to 0}{\text{Limit}} \left(\frac{\Delta i_C}{\Delta i_E} \right)_{V_{CB} = \text{const.}}$$

It is also known as short-circuit current gain.

For a transistor in the common-emitter circuit, the short-circuit current gain

$$\beta = \underset{\Delta i_B \to 0}{\text{Limit}} \left(\frac{\Delta i_C}{\Delta i_B} \right)_{v_{CE}=\text{const.}}$$

Since $i_E = i_C + i_B$, therefore, we get

$$\beta = \frac{\alpha}{1-\alpha} = \frac{0.995}{1-0.995}$$
$$= 199.$$

Similarly for a transistor in the common collector circuit the short circuit current gain

$$= \underset{\Delta i_B \to 0}{\text{Limit}} \left(\frac{\Delta i_E}{\Delta i_B} \right) = 1 + \left(\frac{\Delta i_C}{\Delta i_B} \right)$$
$$= 1 + \beta = 1 + 199 = 200.$$

Example 6. *The current gain of a transistor in a common emitter circuit is 49. Calculate its common base current gain. Find the base current when the emitter current is 3 mA.*

The current gain in the common-emitter circuit β is related with the gain in the common-base circuit α as

$$\beta = \alpha/(1-\alpha) \quad \text{or} \quad \alpha = \beta/(1+\beta).$$

Given $\beta = 49$, we get

$$\alpha = 49/(1+49) = 0.98.$$

Since $\alpha = I_C/I_E$, therefore

Collector current $I_C = \alpha I_E = 0.98 \times 3\ mA.$

As $\beta = I_C/I_B.$

$\therefore$ Base current $I_B = I_C/\beta = (0.98 \times 3/49)\ mA$
$= 0.06\ mA = 60 \mu A.$

Example 7. *A transistor in a circuit as a common emitter stage has the short circuit current gain 100 and $I_{CBO} = 5\mu A$. If the measurement of the collector current yields $I_c = 1mA$ with zero load resistance, calculate I_E, I_B, α, and I_{CEO} under these conditions.*

We know that I_{CBO} is the collector base current with emitter open and I_{EBO} is the emitter base current with collector open. Using the relationship describing the magnitudes of the terminal currents

$$I_E = I_C + I_B \qquad \text{...}(i)$$

and the relations for α and β, we get

$$\alpha = \left(\frac{\Delta i_C}{\Delta i_E} \right)_{v_{CB}=\text{const}} = \frac{I_C - I_{CBO}}{I_E}. \qquad \text{...}(ii)$$

or $I_C = aI_E + I_{CBO}$ at a specific value of V_{CB}.

$\therefore$ $I_C = a[I_C + I_B] + I_{CBO}$

or $$I_C = \left[\frac{aI_B}{1-a} + \frac{I_{CBO}}{1-a} \right] = \beta I_B + (\beta+1) I_{CBO}.$$

When $I_B = 0$, the resulting collector current is I_{CEO}. Therefore

$$I_{CEO} = (\beta+1) I_{CBO} \quad \ldots(iii)$$

$$\therefore \quad I_C = \beta I_B + I_{CEO}. \quad \ldots(iv)$$

In the given problem $\beta = 100$, $I_{CBO} = 5\ \mu A$ and $I_C = 1\ mA$. Thus equation (*iii*) gives

$$I_{CEO} = (100+1)\ 5 = 505\ \mu A.$$

Since $$\alpha = \frac{\beta}{\beta+1} = \frac{100}{100+1} = 0.99.$$

Equation (*ii*) gives

$$I_E = \frac{I_C - I_{CBO}}{\alpha} = \frac{1\ mA - 5\mu A}{0.99} = 1.005\ mA.$$

Thus equation (*i*) gives

$$I_B = I_E - I_C = 1.005 - 1 = 0.005\ mA = 5\mu A.$$

Oral Questions –

1. Define the following terms : (*a*) insulator, (*b*) conductor, (*c*) semi-conductor.
2. What is meant by (*a*) intrinsic and (*b*) extrinsic semi-conductors ?
3. What is the difference between the band structure of a semi-conductor and of a metal ?
4. Define (*a*) donor, (*a*) acceptor impurities.
5. Define the following terms : (*a*) valence electron, (*b*) valence band, (*c*) conduction band, and (*d*) forbidden zone.
6. What is meant by (*a*) p-type and (*b*) n-type semi-conductors ?
7. What is the effect on a semi-conductor of adding (*a*) a donor impurity, (*b*) an acceptor impurity ?
8. What is the effect on donor and acceptor levels when the semi-conductor is heated up ?
9. How can you prepare junction diodes ?
10. What is meant by (*a*) reverse bias, (*b*) forward bias ?
11. Compare the forward and reverse current static volt-amp characteristics of a semi-conductor diode.
12. What is meant by (*a*) Zener load, (*b*) Zener diode ?
13. What are the processes for the conduction in semi-conductor ?
14. How can one identify the leads of a triode transistor ?
15. How can one bias *pnp* and *npn* transistors ?

16. What are the transistor parameters ?

17. Compare the vacuum tube amplifier circuits with the transistor amplifiers with (1) common base, (2) common emitter, and (3) common collector.

18. What is meant by Hybrid parameters ?

19. Explain why a large reverse current exists in tunnel diode.

20. What is the difference between JFET and MOSFET .

Problems :

1. Determine the resistivity of intrinsic germanium with 3×10^{13} per cm^3 electrons and holes. *(36 ohm cm)*

2. The specimen of germanium has conductivity of 0.01 mho/cm. Calculate the density of electron-hole pairs. Given that $\mu_e = 3800\ cm^2/V$ sec and $\mu_h = 1800\ cm^2/V$ sec. *(1.12×10^{13} pairs)*

3. What will be the ratio of forward to reverse current, if the magnitude of bias is 0.05 V at room temperature. *(−8.4)*

4. A transistor has emitter current of 1 mA for the collector current 0.98 mA. Calculate the base current and the gain in the common base circuit. What will happen to the gain if the slight change in base current changes the emitter and collector currents to 1.05 mA and 1.028 mA respectively.
(0.02 mA, 0.98, 0.96)

5. An impurity of phosphorus is added to a pure silicon sample in the amount of 10^{13} atoms/cm^3. If all donor atoms are activated, calculate the resistivity at 20°C. *(5.22 ohm-m)*

6. Germanium has one arsenic atom added per 10^8 atoms of Ge. How many holes and electrons are present for conduction at 300°K ?
($1.42 \times 10^{18}/m^3$, $4.41 \times 10^{20}/m^3$)

7. A sample of transistor has $\triangle i_c = 1.8\ mA$ for $\triangle i_E = 1.89\ mA$. What change in i_B will produce an equivalent change in i_c ?

8. A sample of germanium has intrinsic resistivity of 0.54 ohm-m. If an impurity concentration of one antimony atom per 10^8 germanium atoms is added. Find the Fermi energy level after this addition of impurity.
(falls 0.186 eV)

6

Electronic Devices

6.1. SWEEP GENERATORS

The sweep circuits are very much used in cathode ray oscilloscopes. The proper selection of the form of sweep is most important. Many of the more common circuits employ the changing potential across a capacitor. The potential across the capacitor without initial charge is

$$e_c = \frac{1}{C}\int_o^t i\,dt. \qquad \ldots(1)$$

If one can supply a constant changing current i, then e_c will be a linear function of time t. Means must also be found for discharging the capacitor, when a maximum potential is reached. A pentode can be used to supply a constant charging current, as the anode current of pentode is independent of anode voltage over a wide range. This circuit is occasionally employed as there are difficulties in supplying potentials above ground.

More usually the simple sweep circuits use the rise of potential when capacitor is charged through a series resistance R. When some potential is reached it is necessary to discharge the capacitor. The potential across the capacity varies as

$$e = \mathcal{E}\,(1-e^{t/RC})$$
$$= \mathcal{E}t/RC = Kt. \qquad (\text{If } t/RC << 1) \qquad \ldots(2)$$

Thus for a reasonable linearity t_1/RC must be small, here t_1 is the time of the sweep. To provide sufficient sweep voltage for deflection of the cathode ray beam, this small output is to be amplified. The frequency of sweep $f(=1/t_1)$ is controlled by variation of RC.

One of the earliest capacitive saw tooth generators utilizes a glow discharge tube as the switch in the circuit.

Thyratron Sweep Generators—Two shortcomings of the simple glow tube sweep generator are (*i*) the requirement of high potential

to obtain linear saw tooth waves and (*ii*) the small amplitude of the output wave. The use of a thyratron as a switch allows reasonable flexibility in control of the period of oscillation and also in the amplitude of the output wave.

The basic circuit is shown in Fig. 6.1 (*a*). When switch *S* is closed, the power is supplied to the circuit and *C* begins to charge through resistance R_3. The voltage developed across *C* is +ve with respect to ground and is applied to the plate of the thyratron. If it

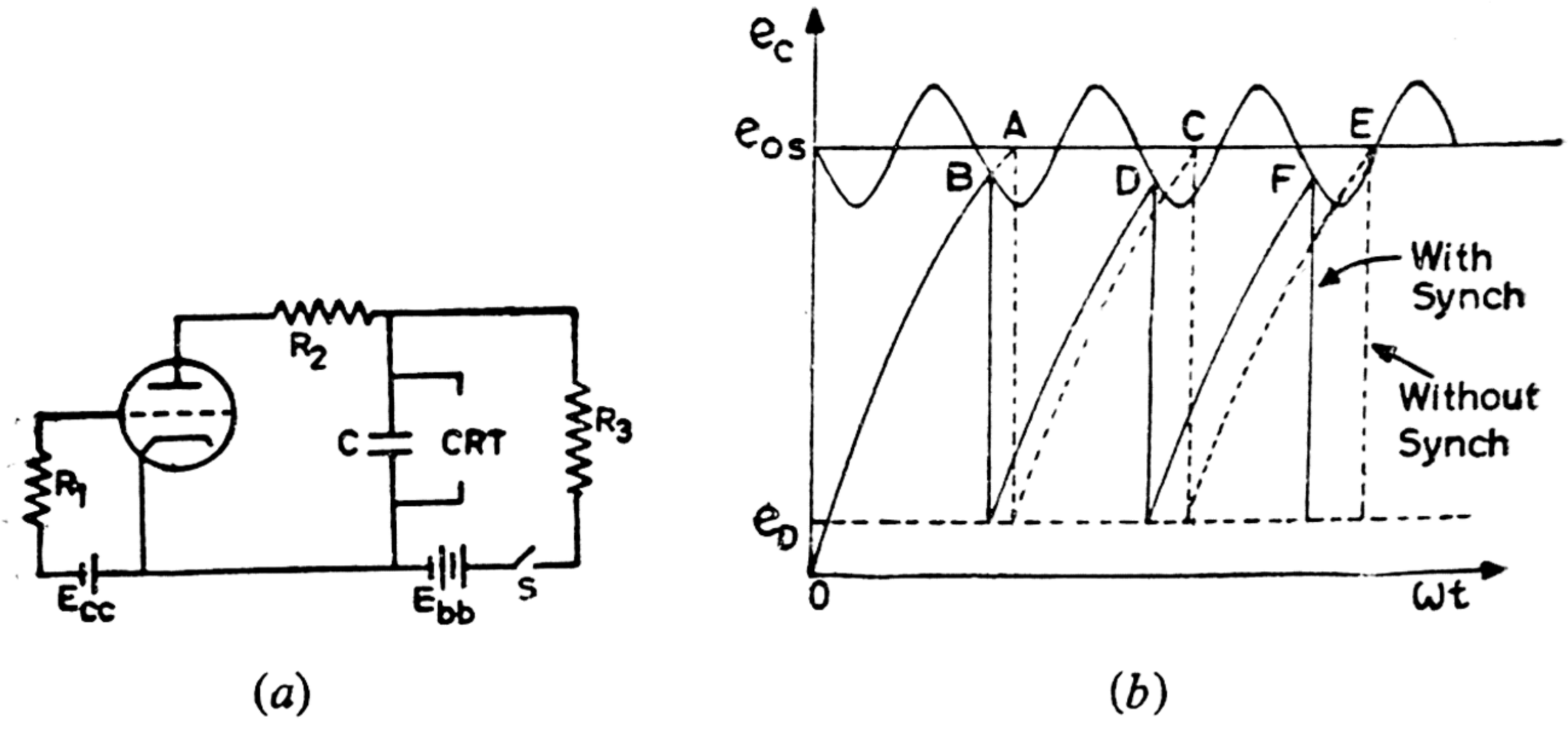


Fig. 6.1.

reaches the firing potential of the tube, which depends upon the grid bias, the tube starts to conduct and capacitor *C* discharges very quickly through the tube and the series resistance R_2. The tube stops conducting, when the potential of the capacitor falls below the firing potential of the tube. The tube again begins to charge and the cycle repeats itself. The resistance R_2 is made as small as possible in consistent with the tube current rating in order that the capacitor may discharge very quickly. The R_3 must be large enough to prevent the supply voltage from providing the tube with enough current to maintain ionization. The output wave appears across *C* and is a sawtooth voltage of frequency depending on plate voltage, grid bias, R_3 and *C*.

The thyratron saw tooth waves are not very stable in frequency but can be synchronised with a given frequency by injecting a small potential of the desired frequency into the grid. The synchronizing voltage causes the ionizing voltage of the thyratron to vary sinusoidally with time. If the frequency generated by the free running saw tooth wave is slightly greater than the period of the synchronizing signal, the effect is to cause the firing potential e_F of the circuit to vary in accordance with this varying grid potential. If firing occurs at the points *A*, *C*, etc., in the normal case, at some time during the synchronizing cycle the firing potential e_F will be low (below the zero

level of synchronizing potential e_{os}) and the tube will fire at point *B*. On the next cycle the circuit is fired at point *D* instead of *C*. The period is thus reduced from *AC* to *BD* and the circuit is now synchronised to the frequency of the injected potential. The circuit may be synchronized to a submultiple or a multiple of the synchronizing frequency. In Fig. 6.1 (*b*) the continuous and dotted lines represent the voltage across the capacitor with synchronism and without synchronism respectively. Here e_D represents the de-ionizing potential.

6.2. CATHODE-RAY OSCILLOGRAPH

Cathode-ray oscillograph (C.R.O.) consists of a cathode-ray tube (C.R.T.), sweep generator and other circuits. A block

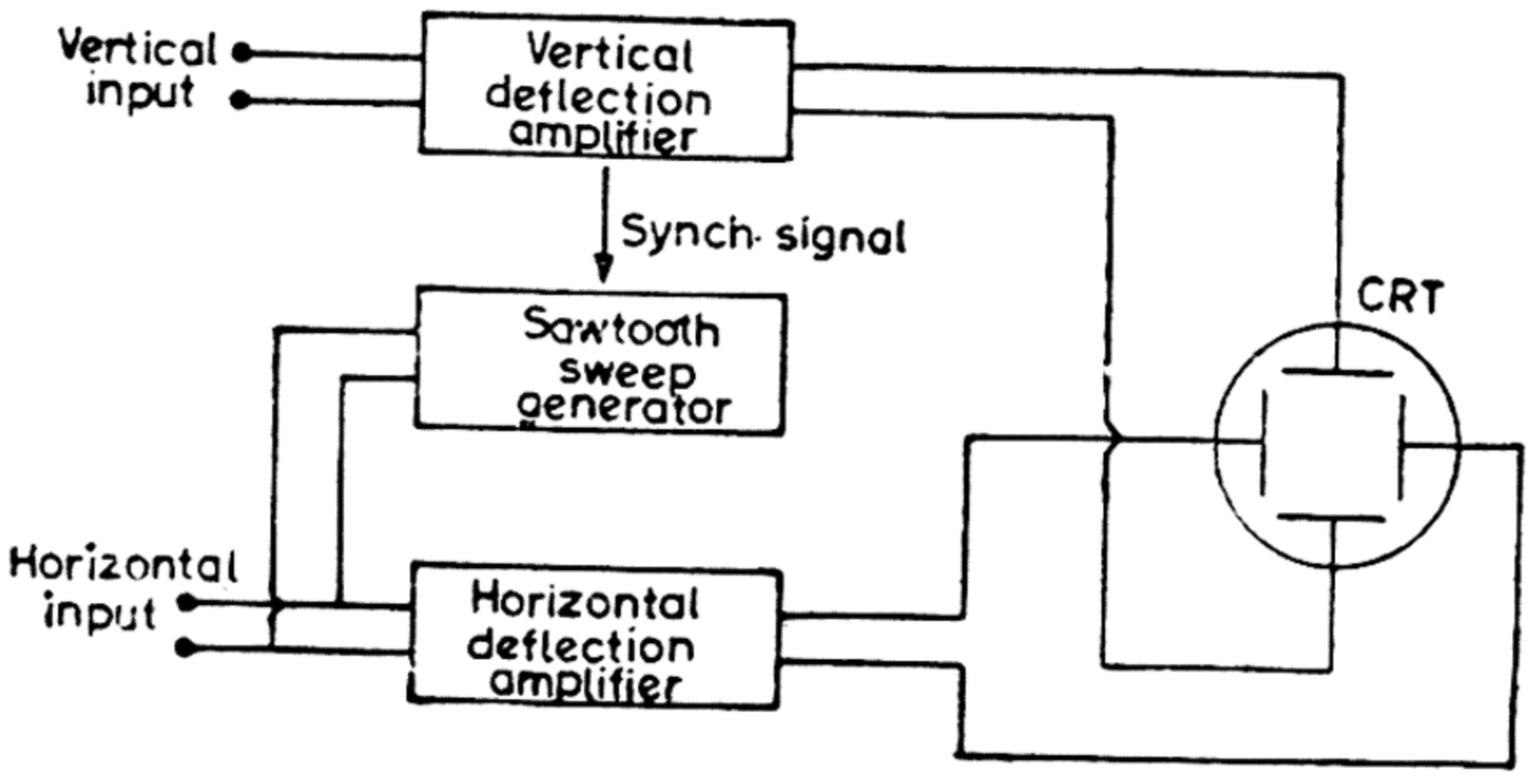


Fig. 6.2.

diagram is shown in Fig. 6.2. Let us discuss individual components.

1. Cathode-ray tube. It consists of an evacuated (10^{-5} to 10^{-6} mm of Hg) space in which are an electron gun, a set of deflection plane electrodes and a fluorescent screen.

(*a*) *Electron Gun.* It contains a heated cathode coated with oxides of rare earth elements, which emit electrons readily. The

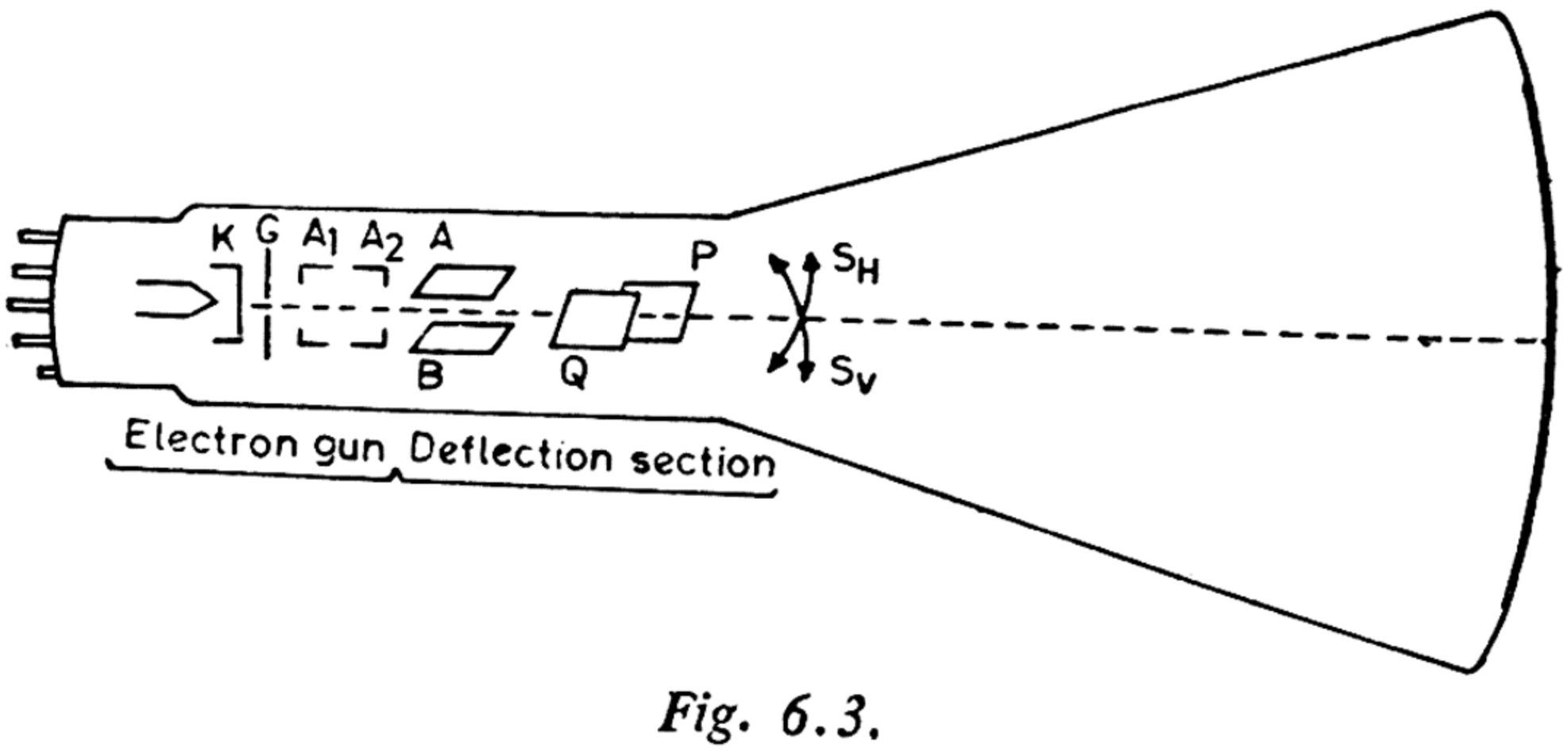


Fig. 6.3.

cathode K is partially surrounded with a metal grid. The voltage is applied so that the grid is negative with respect to cathode. The rate of electron flow through the grid can be regulated with this voltage. To focus and accelerate the electrons, passing through a hole in grid, two anodes A_1 and A_2 are used. These anodes are cylindrical with narrow openings and are at high potentials with respect to the cathode. The electrostatic field that exists between the anodes provides the necessary focussing* of the electron beam and the system of anodes is termed as *electron lens*. The diaphragm cuts off electrons which passed through the outer edge of second anode and were not properly focussed.

The correct focus, on the fluorescent screen S, can be obtained by varying the voltage on one anode with respect to the other. The *focus control* is the front panel adjustment on the oscilloscope for this purpose. The number of electrons, therefore, the brightness of the spot, can be controlled by the grid potential which may be done by the *intensity control*, another front panel adjustment on the oscilloscope.

(*b*) *Deflection Plates.* The electron beam then passes through the first set of deflection plates (*A* and *B*) that are mounted in horizontal direction and produce electric field in vertical direction when a voltage is applied across them. Due to this field the electron beam will be deflected and will provide a vertical deflection on the fluorescent screen. The electron beam then passes between two

*The electron beam converges when passes through a metal cylinder charged to a negative potential. Thus the negatively charged metal cylinder acts as a convex lens which converges the beam of light. On the other hand if the cylinder is given a positive potential, the electron beam gets diverged. If there are two cylinders A and B charged to different potentials, the resultant field shown by dotted lines in Fig. 6.4 (*c*) converges the electrons to a point.

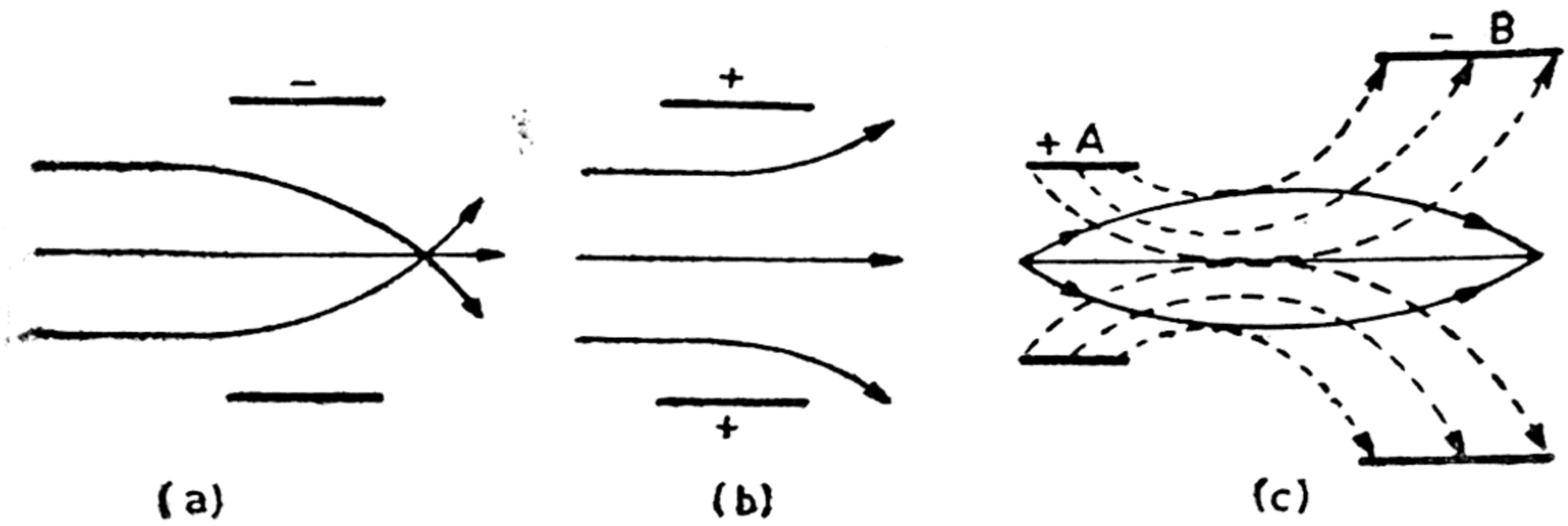


Fig. 6.4.

The lines of force may be spreaded out by using the cylinder B larger. The focal length of such a lens can be changed by changing the potentials. The electrostatic focusing will be discussed in the article 6.5.

plates mounted perpendicular to the first set of plates (P and Q) and a potential difference between these plates causes a horizontal deflection of the beam. Therefore, the position of the electron beam and thus of bright light spot on the screen depends on the voltages across these two sets of plates.

Electrostatic Deflection. Let us now discuss the motion of electron in electrostatic field (Fig. 6.5). If the mass of an electron is m, its charge e, the potential difference between cathode and last anode V, the electrostatic field between deflecting plates E, and l and L are respectively the axial length of the deflecting plates and the distance from the centre of the deflecting plates to the screen, then the velocity of the electron at point O is given by

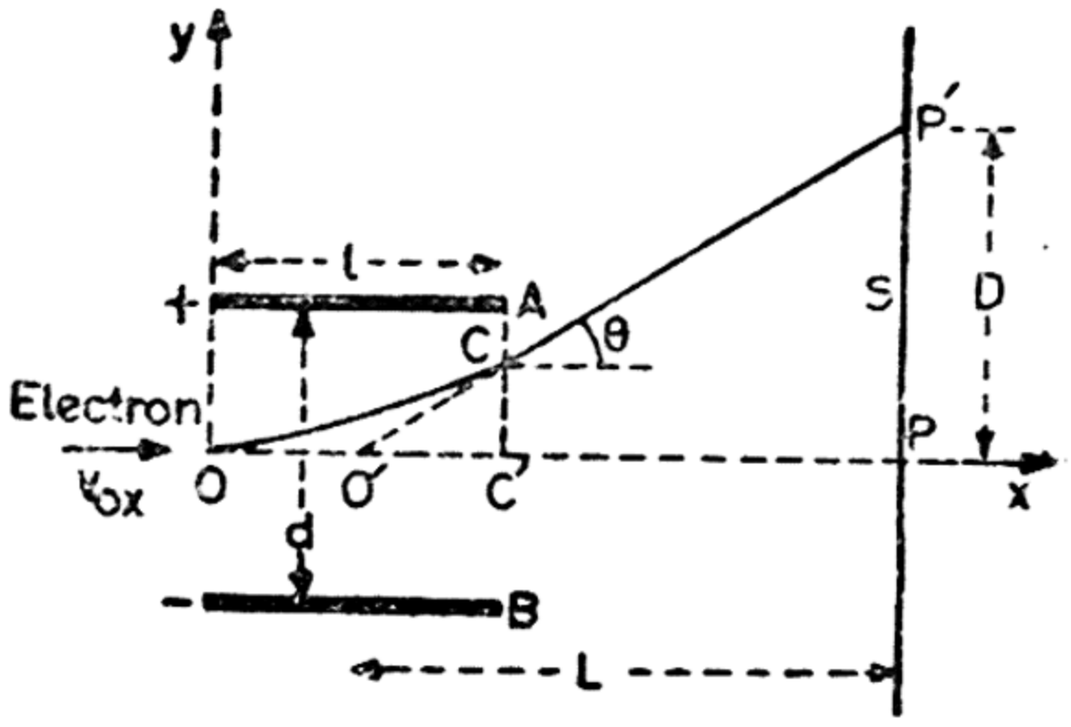


Fig. 6.5.

$$eV=\tfrac{1}{2}mv_{0x}^2,\ v_{0y}=v_{0z}=0. \quad \ldots (3)$$

The acceleration in the electrostatic region is given by

$$ma_y=eE,\ a_z=a_x=0. \quad \ldots (4)$$

If E is assumed constant with time, this Eq. leads to

$$v_y=\int_0^t a_y\,dt=\frac{e}{m}Et, \quad \ldots (5)$$

and that $v_x=v_{0x}$, the initial velocity. The integration of Eq. (5) gives

$$y=\tfrac{1}{2}\frac{e}{m}Et^2=\tfrac{1}{2}\frac{e}{m}E\cdot\left(\frac{x}{v_{0x}}\right)^2=\frac{eE}{2mv_{0x}^2}x^2. \quad \ldots (6)$$

This shows that the path of the electron between the deflecting plates is a parabola in the x-y plane. When the electron leaves the point C, it will travel in straight line toward the screen. This straight line is tangent to the parabola at this point. Hence

$$\tan\theta=\left(\frac{dy}{dx}\right)_{x=l}=\frac{eE}{mv_{0x}^2}l. \quad \ldots (7)$$

Since $\tan\theta=\dfrac{CC'}{O'C'}$, or $O'C'=\dfrac{(y)_{x=l}}{\tan\theta}=\dfrac{l}{2}$,

hence point O' is at the centre of the deflecting plates.

$$\therefore\quad D=L\tan\theta=eELl/mv_{0x}^2=ELl/2V. \quad \ldots (8)$$

The length l and spacing d are limited by the beam striking the plates for large deflections. We can compromise only but can

not decrease more V as it should be high for maximum spot brightness.

In the above analysis we have assumed E as constant, neglected initial velocity of electrons. However the expression does give proper dependence on these parameters.

Magnetic Deflection. In some cases magnetic field is used instead of electrostatic field. A transverse magnetic field is produced by the current carrying coils mounted on the sides of the tube (Fig. 6.6). When the electrons enter the region of magnetic field (which is normal to the paper), a force Bev acts on them. As this force is $\perp$ to v, hence no work is done on the electrons. Therefore the speed of electrons remain unchanged due to magnetic field and the path must be a circle of radius R, given by

$$mv^2/R=Bev \quad \text{or} \quad R=mv/eB. \qquad ...(9)$$

The corresponding angular velocity

$$\omega=v/R=eB/m. \qquad ...(10)$$

This shows that the faster moving particles will traverse large circles in the same time that a slower particle traverses a smaller circle. The segment OA of a circle whose centre is at B is given by

$$OA=R \sin \theta \quad \text{or} \quad \theta=l/R \text{ (Approx.)} \qquad ...(11)$$

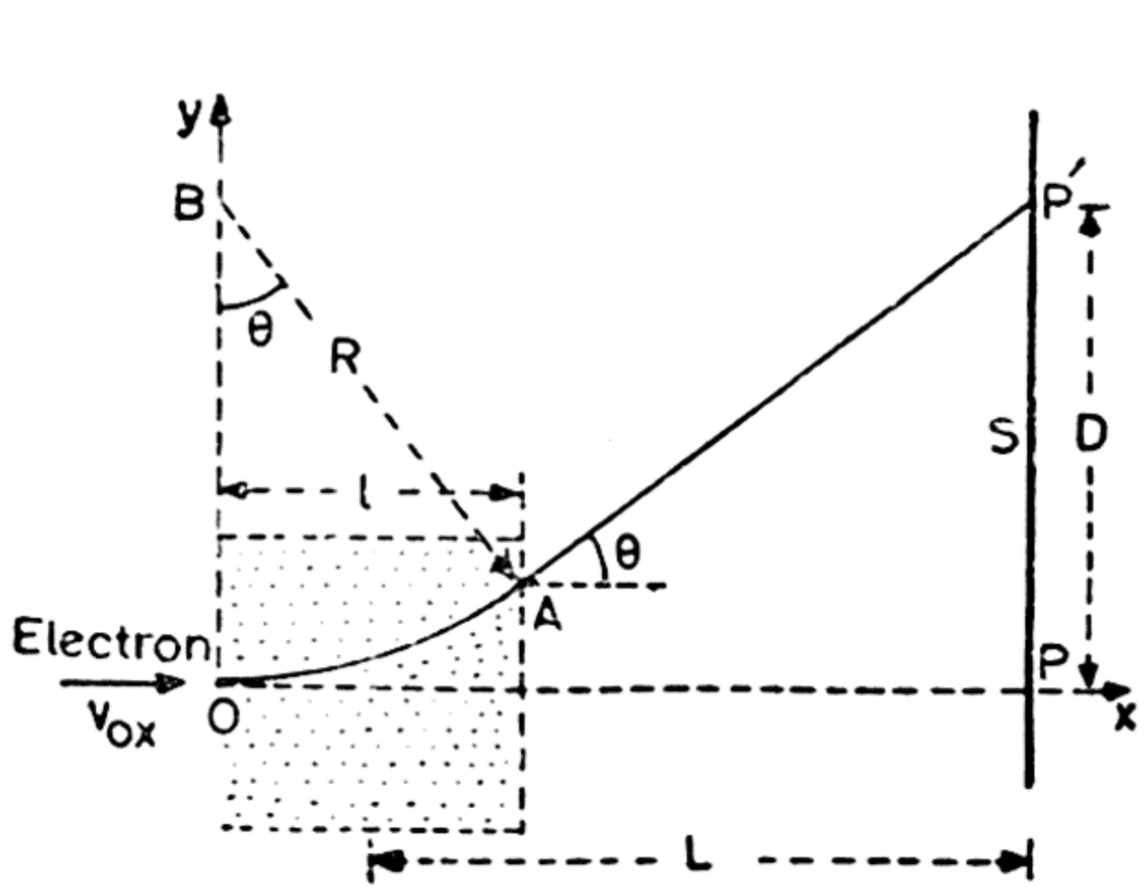


Fig. 6.6.

Distance L is measured from the point of intersection of a line tangent to the final deflection of the beam and the line of the original velocity, and is approximately from the mid point of the field.

$$\therefore \tan \theta=D/L. \qquad ...(12)$$

or $$D=L \tan \theta=L\theta=Ll/R$$
$$=Lle\ B/mv.$$

But $$v=v_{ox}=(2eV/m)^{1/2}$$

$$\therefore \quad D=LlB\ (e/2mV)^{1/2} \qquad ...(13)$$

Thus the deflection on the screen is proportional to B and inversely proportional to square root of the accelerating potential V. Two fields at right angles are used to produce deflections in the x-and y-directions. Magnetic focussing is usually better than with an electrostatic focusing system. It provides a high intensity beam and the minimum distortion of the spot. The Magnetic tube is less complicated than the electrostatic type and is of smallar length. However, the power required is considerably greater and the induction of the coils leads to a difficulty when deflections

are very rapid. Thus the magnetic deflection is limited essentially to constant frequency service, such as that found in television or radar.

(*c*) *Fluorescent Screen.* The beam of electrons, after deflection, travels along the *C.R.*-tube until it impinges on the screen *S*, which is a coating of fluorescent materials on the end face of the tube. For normal laboratory observation, *zinc orthosilicate* is used. When electron strikes the screen, the part of the energy is transferred to the atoms of the screen material and is emitted as light, the remainder is dissipated as heat in the screen. The spot is of a yellow green colour for the above fluorescent. *The spot should not be left stationary on the screen because of the danger of overheating the screen material.*

2. Sweep Generator. For most applications of the oscilloscope, the light spot is moved horizontally across the face of the tube at a uniform rate. It provides a uniform time scale against which another voltage can be plotted by applying it across the vertical deflection plates. It is often stated that the electron beam is swept across the screen at a uniform rate. The oscillator circuit that produces sweep wave is called the *sweep generator.* Good linearity of the sweep is a major requirement for a reliable oscilloscope. At the end of the sweep the electron beam is moved back to the starting point by lowering the sweep voltage back to the zero. The sweep generator must oscillate at exactly the same frequency as the repetition rate of the signal or some submultiple thereof. so that the repetitive displays can be superimposed. This type of sweep generator is called a *synchronized sweep* and is found in many oscilloscopes. Coarse and fine adjustments to vary the sweep rate are on the front panel.

3. Other components. Besides the C.R. tube and sweep generator, other circuits, *e.g.*, very high frequency and dc amplifiers are also available in some oscilloscopes. Power supplies, which are essential for the operation, are not shown in block diagram.

Calibration of the Oscilloscope. The position of zero signal deflection, as set by the VERT CEN control, should not change when the vertical attenuator and gain controls are varied. If the zero does shift, an adjustment of the vertical dc balance control is required. The dc balance of the horizontal amplifier is done similarly with the horizontal sweep disconnected.

Uses of the C.R.O. (*i*) To determine the amplitude of a variable quantity such as an oscillatory current or voltage, (*ii*) To study the waveform of a periodic disturbance, (*iii*) To study hysteresis curves, (*iv*) To compare frequencies by Lissajous figures, (*v*) To study phase difference between voltage and current in any circuit, (*vi*) To compare input and output voltages of an amplifier. (*vii*) To plot response curves of high and low frequeney circuits, (*viii*) To

study transformer magnetisation, vibrations in machinery, echotones due to reverberation of building, (*ix*) To study defects in generators, motors and radio valves, (*x*) To study heart beats and nervous actions, (*xi*) To measure a very small intervals of time, (*xii*) It is very useful in television and radars.

6.3. TELEVISION

The television is the art of transmission and reproduction of moving figures at a distance. The light intensities from the illuminated object are converted in to the electrical fluctuations. These vibrations called *video signals* after amplitude modulation are transmitted by the antenna. At the receiver, this signal after demodulation is applied to a device which reproduces the transmitted picture. The first part is known *as iconoscope* and the latter the *kinescope.*

1. Iconoscope. The iconoscope at the transmitting station is fundamentally a combination of C.R.O. and a photocell (Fig. 6.7). The object *O* to be televised is focussed by means of a lens *L* on the photo mosaic plate *P*, which consists of a large number of tiny globules *G* of caesium-silver compound isolated from each other and deposited on mica sheet *M*. The back side of the sheet is coated with a metal film *F* and is connected with a load resistor *R*, across which signal voltage is devoloped. As caesium is a photosensitive material, each individual globule acts as a tiny photocell and emits electrons in proportion to the amount of light received from the

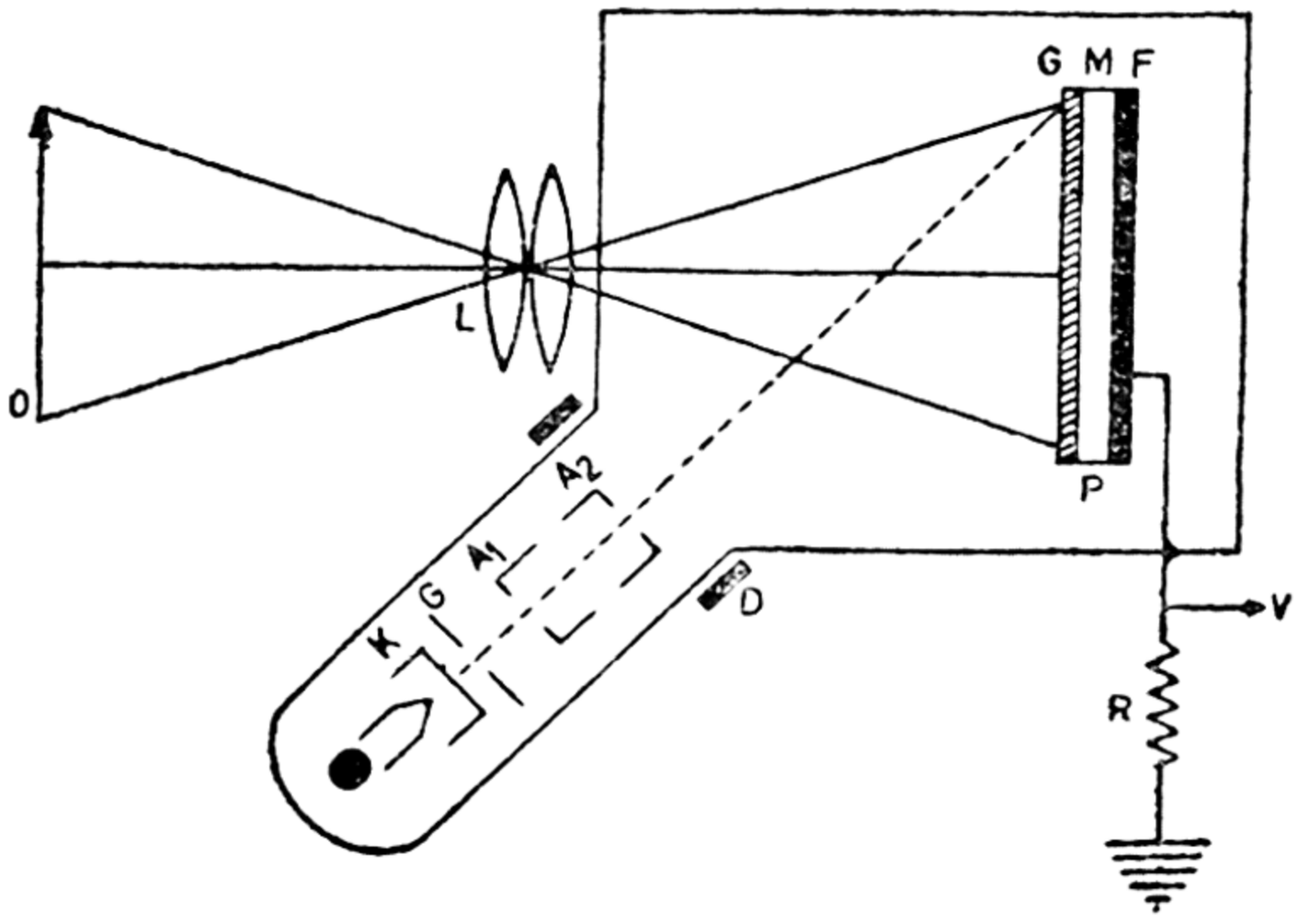


Fig. 6.7.

particular portion of the object. As a result of the emission of electrons each elementary condenser, constituted by the globule and the signal plate (coated side of mica sheet), will become positively charged in accordance with brightness of the portion of the object. The electrical image stored on the mosaic cannot be transmitted as a whole but the individual picture elements are scanned one at a time by discharging the globule capacitors, in ordinary sequence.

The electron beam from the electron gun is made to fall on the mosaic and is deflected by the deflection section *D*. This consists of two deflection coil pairs, the vertical deflection coils and horizontal deflection coils. The current in one coil causes the electron beam to move from side to side across the mosaic about 1000 times a second, while the current in an another coil causes the beam to move up and down about 50 times a second. Thus the electron beam is made to scan very rapidly the whole mosaic in about 1/30 sec. The scanning is from left to right and top to bottom, one line at a time. When the beam impinges on a particular globule, it replaces its positive charge and induces a positive charge on the metallic surface. These changes of electric charge produce corresponding minute electric impulses, video signal *V*, across the load resistor *R*. The signals are amplified and then modulated and passed to transmitter so that they can be radiated.

2. Kinescope. At the receiving station, the signals received by the antenna and are subjected to superheterodyne receiver (S.R.) to separate both audio and video signals. The detector is used to convert the *FM* sound (*IF*) signals into sound signal which is finally fed through *AF* amplifiers to speaker. The video signals are then fed into a kinescope, *i.e.*, a cathode ray television receiver, similar to the transmission camera with the exception that the electron beam is focussed on a fluorescent screen instead of the mosaic screen. To reconstruct the picture the electron beam must scan the receiving screen in exactly the same speed as the mosaic screen in the transmission camera is scanned. It is done by two pairs of deflecting coils, one of high frequency for the "*line*" scanning and the other of low frequency for the "*frame*" scanning. To get the original picture the scanning beam of the receiver must be kept synchroniz-

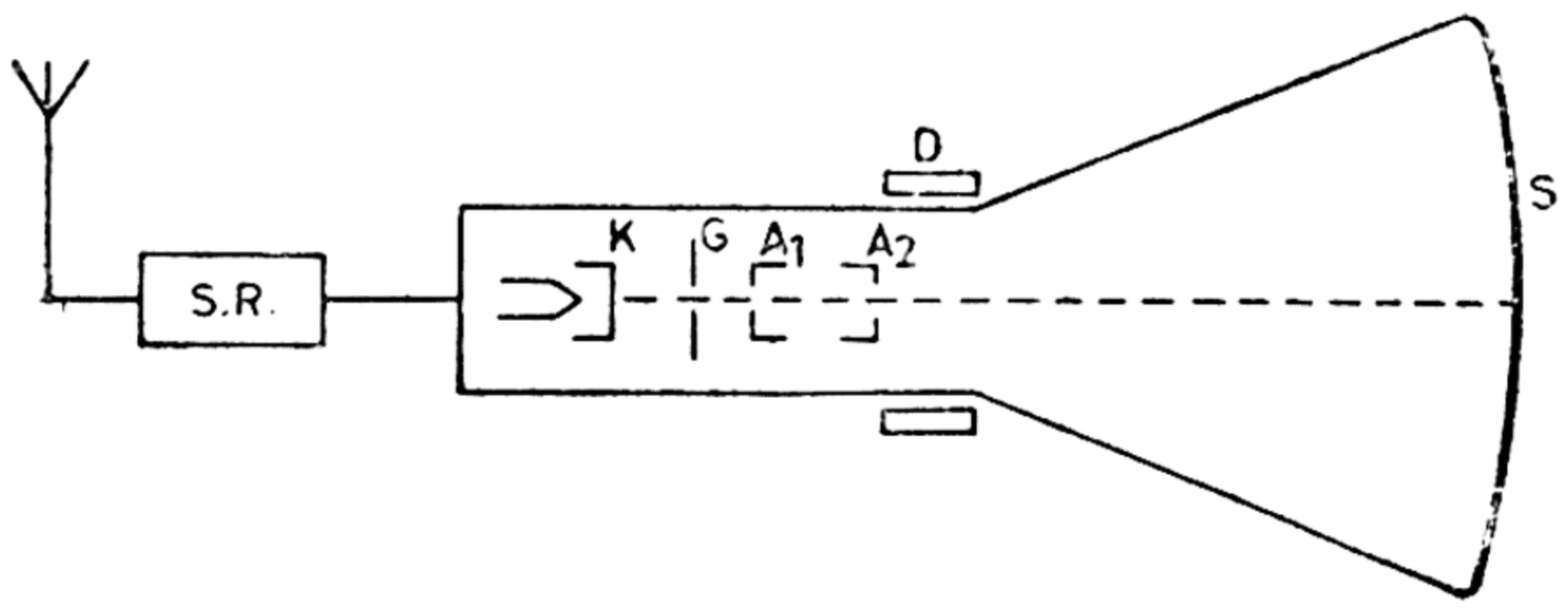


Fig. 6.8.

ed with that of the transmitter. The variation in fluorescence emitted from the screen corresponds to the video signal and thus to the object. Thus the object is reproduced on the flourescent screen of the receiver.

Televisions are very useful now a days not only to view parades, matches, dramas and pictures but also to hear and view the ectures and lessons sitting at home.

Colour Television. In the colour television the transmission system consists of camera with *vidicon camera tubes.* Light from the scene enters the camera through the camera lens *L* and is split into three beams as shown in Fig. 6.9. It is done by four mirrors, 1 and 4 total reflecting and 2 and 3 are beam splitting (partially reflective and partially transmissive). Blue, green and red filters (5, 6 and 7 respectively) transmit only blue light to the blue vidicon, only green light to the green vidicon and only red light to the red vidicon. The images corresponding to blue, green and red in the scene are formed on the targets of the blue, green and red vidicon respectively.

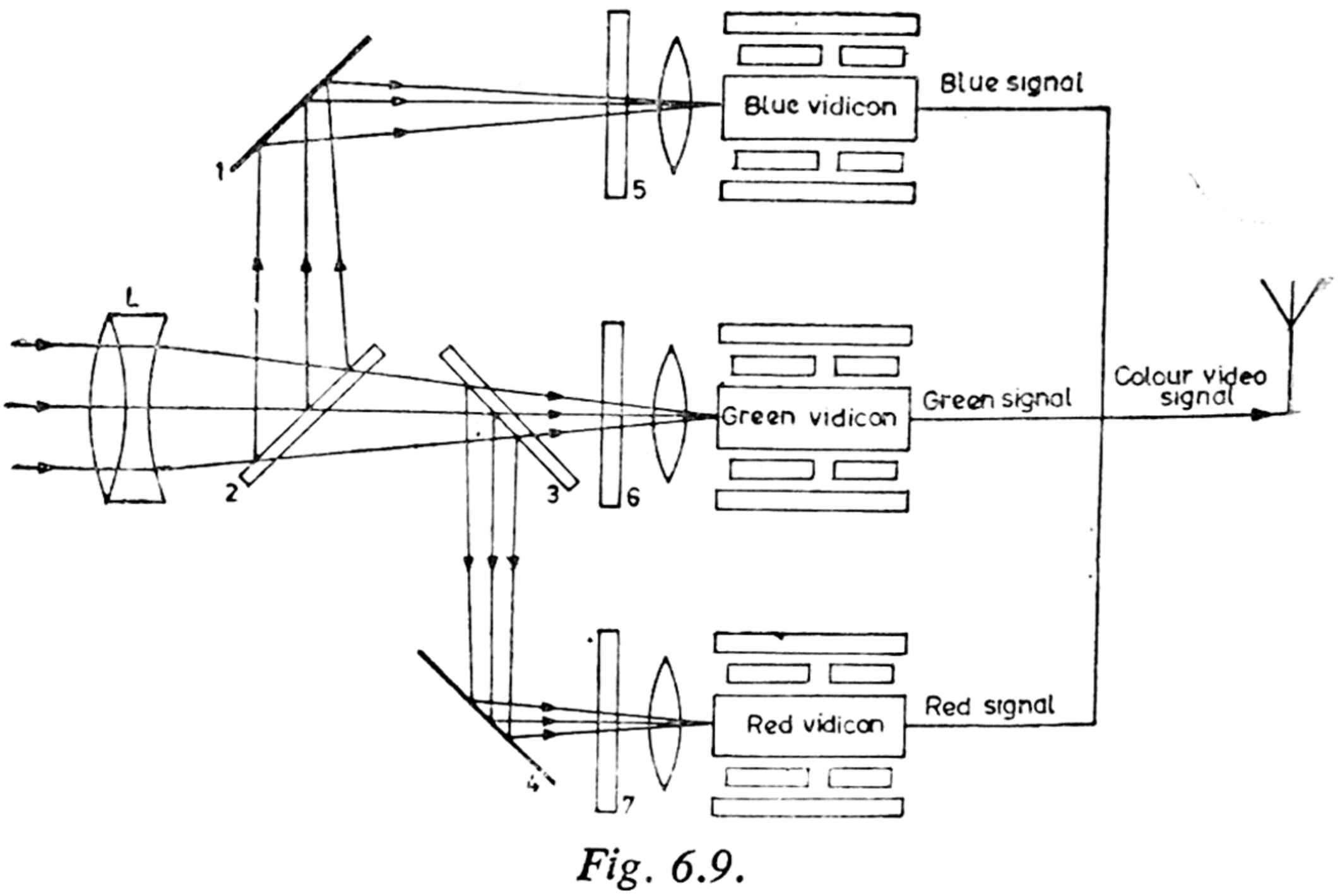


Fig. 6.9.

The video signals by the vidicons are generated. This generation is based upon the principle of photo conductivity. A thin photo conductive target is deposited on a transparent conductive film on the inner surface of the tube face. When an optical image is produced on the target, the resistivity at each point decreases according to the image intensity at that point.

The blue, green and red video signals generated in the blue, green and red vidicons respectively are received by the receiver separated and fed to the *colour kinescope.* The colour kinescope may be regarded as three ordinary kinescopes one for blue, one for green and one for red. The gun section consists of three independent electron guns one for each colour. The deflection section consists of a single yoke, which provides simultaneous deflection in all three beams. The phosphor screen of the image section consists of an array of dots which may be subdivided into triangular groups of three dots, *B*, *G* and *R*. The *B* dot fluoresces blue when excited by an electron beam, likewise for the green and red. Thus the colour scene is reproduced on the screen of the colour kinescope.

6.4. RADAR

The term *Radar* is the abbreviation for "*radio detection and ranging*." It is not a single instrument but includes various techniques. A short burst or pulse of electromagnetic radiation is transmitted from a highly directional antenna, when this transmitted radiation hits a distant target, a part of the radiation is reflected back and an echo pulse is picked by a receiver. The time interval between transmitted and echo pulses provides an accurate measure of the target distance. The function of radar system is very much precise, rapid and unaffected by darkness, clouds, fog, etc. Thus the basic principle of radar is vary similar to that of sight. We switch on a light in the dark and we see an object because the light waves are reflected from it and return to our eye. The distance measurement is similar to that by means of scund echoes.

Basic arrangements :—(1) Very powerful and very brief electromagnetic pulses are supplied by the *micro-wave oscillator* and are supplied to the modulator.

(2) The resulting radio frequency pulses are transmitted by the use of wave guides and are radiated by a highly *directional antenna* to confine the radiated pulses in a definite range of direction.

(3) Since the same antenna is used for reception of radio pulse echo, hence a *duplexer* is used to isolate the receiving unit from strong transmitting pulses at the time of transmission to avoid the damage to switch and the receiver set in the intervals equal to the duration between pulse echoes.

(4) The received *RF* pulses are allowed to pass through *superheterodyne receiver* to get video signals.

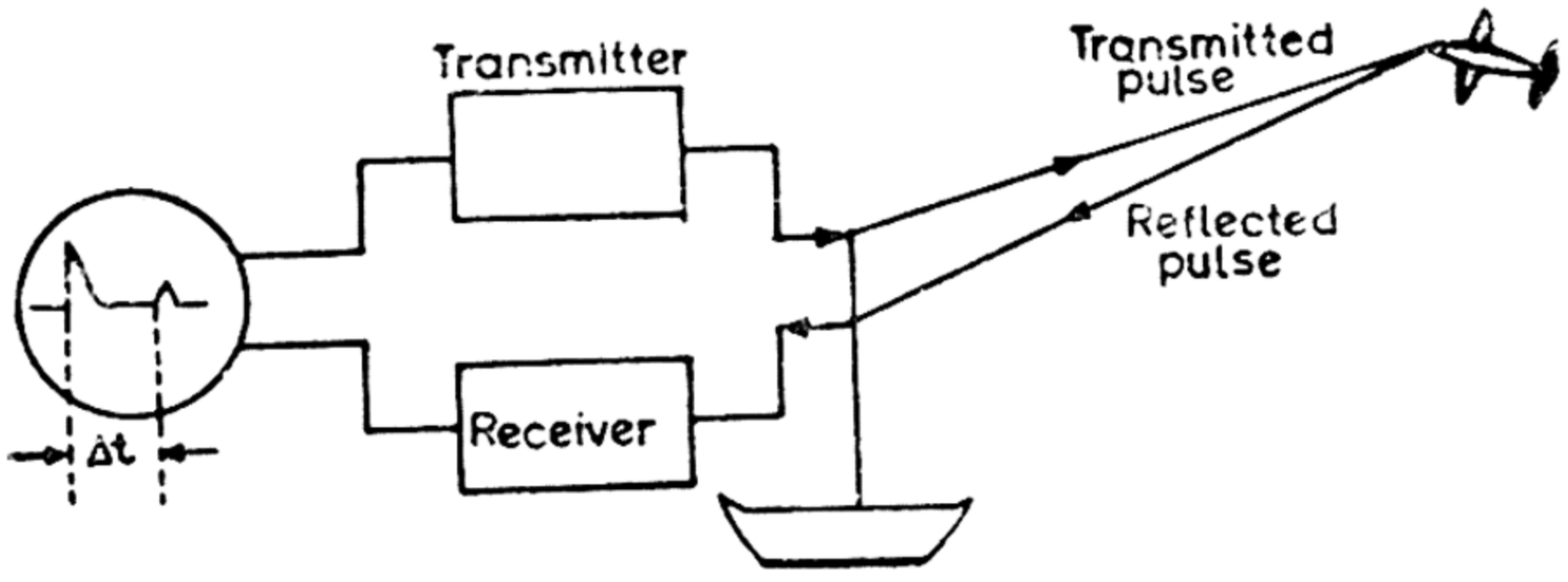


Fig. 6.10.

(5) The signals are then fed to the indicator unit, which is a *cathode ray tube*. A time base circuit of desired number of microseconds is used to synchronise exactly the echo pulses with the transmitted pulses. From the curve on fluorescent screen, the position of the object that sends out the echoes can be determined.

(6) In searching the surrounding surface for targets, the antenna is given a *rotatory motion*.

Operating Characteristics. The operation of radar system depends upon the following factors :

(1) *Choice of operating frequency.* High frequency ($>10^9$Hz) gives high resolution, requires antenna of small size but increases noise and decreases power generated.

(2) *Pulse duration.* For good range resolution, pulse durations of 1μ-sec or less are employed.

(3) *Pulse repetition frequency.* The time spent in the path between radar and object may not exceed the interval between transmitted pulses. For larger range we require low pulse repetition rate. Usually, it is from 350 to 10,000 cycles.

(4) *Transmitted power output.* The transmitted power is determined by the maximum distance over which it is desired to receive target information. It is generally more than one mega watt.

(5) *Maximum Range.* It depends upon the energy of the transmitted pulses and the sensitivity of the receiving system.

Uses of Radar—The radar is a very sensitive instrument to find under all weather conditions, the positions of mountains, icebergs in sea, shore lines, buried metal, oil and ores. It is also useful in safe landing of aircrafts and to detect any aircraft in atmosphere even in poor visibility due to heavy fog. The uses of radar may broadly be discussed as :—

(*i*) *Civilian uses*—Radar has been widely used for navigation. Air-traffic-control-radar monitors, the air traffic radar with *GCA* (ground control of approach) system is used to guide aircraft to a safe landing in the foul weather. The radars are equipped with altimeters to determine the height of the aircraft above the ground. The radar is also used by ships for navigation.

(*ii*) *Military uses*—In military, the radar is used for surveillance and for the control of weapons. The radars detect and locate hostile targets for the purpose of proper military action. *DEW* (distant early warning) radars are used for the detection of aircraft, *BMEWS* (ballistic missile early warning system) radars for detecting and tracking of ballistic missiles, *AEW* (airborne early warning) radars and *AI* (airborne interception) radars are used to guide a fighter aircrafts to its target. Shipboard surveillance radars are used for *NAVY* purposes.

(*iii*) *Scientific uses*—Radars have been used as measuring tools in meteorology, microwave spectroscopy, radio-astronomy, solar energy. The radar can be used to guide space vehicles, satellites and for exploration of interplanetary space.

6.5. ELECTRON MICROSCOPE

It is a device to magnify minute objects similar to optical microscopes. A beam of electrons is employed instead of light rays because :

(1) Electrons have a wave nature, similar to light rays but of much shorter wavelength.

(2) Electrons can be focussed by electric and magnetic field, very much like light rays which are focussed by glass lenses.

We know that for the useful magnification high resolution and clear perceptibility are the two important conditions. If the perceptibility decreases with increase in resolution, the image would get blurred and no useful purpose is served.

The *resolving power* of the electron microscope can be greatly increased without decreasing perceptibility. The resolving power of a microscope depends on the wavelength of light and the numerical aperture. The smaller the wavelength of light used the greater will be the resolving power, but the smaller the aperture the smaller will be resolving power. The wavelength of the electrons accelerated by a potential difference of about 60,000 volts and of mass 9×10^{-28} kg is of the order of 10^{-12} metre, *i.e.*, 10^5 times smaller than that of visible light. Hence the resolving power of the electron microscope should be 10^5 times greater than that of optical microscope. The decrease of resolving power due to smaller aperture is not very large.

The clear *perceptibility* under high resolution is secured by a special focussing scheme of the electron beam. The greater the concentration of electrons in the beam that strikes the minute object, the greater will be the number of scattered electrons, hence the perceptibility. Two types of focussing are in use.

1. Electrostatic focussing. An electron lens, consisting of a succession of anodes at increasing potentials, is used to converge and to get focussed the electrons flying off in different directions. The electrons from a source *O* are accelerated successively in the regions and are focussed at some point *I* on the axis similar to the light rays

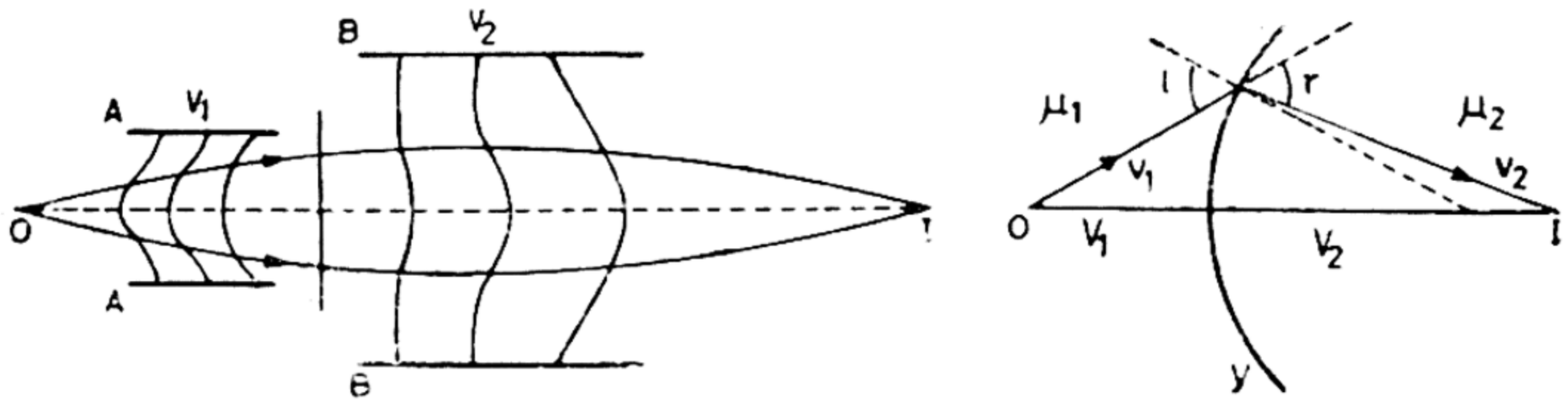


Fig. 6.11.

that are focussed through a lens. The focal length of the electron lens can be changed at will by altering the potential of the cylinders. This can be shown by the Fig. 6.11.

If v_1 and v_2 be the velocities of the electron in voltage regions V_1 and V_2, respectively. If e is the charge on the electron and m its mass, then we can write $\frac{1}{2}mv_1^2=V_1e$ and $\frac{1}{2}mv_2^2=V_2e$,

or $$v_1/v_2=(V_1/V_2)^{1/2}. \quad ...(14)$$

If the velocities are resolved in two components, parallel and perpendicular to the equipotential surface XY, then it is clear that there is no change in the components ‖ to the surface XY, hence

$$v_1 \sin i=v_2 \sin r \text{ or } v_2/v_1=\sin i / \sin r=(V_2/V_1)^{1/2}. \quad ...(15)$$

Thus for $V_2> V_1$, $i> r$, *hence the beam is deflected toward the normal, similar to refraction of light.*

2. Magnetic Focussing. Magnetic focussing is chiefly used in the electron microscope, when a very fast beam of electrons is required.

Consider an electron at point O inside a uniform magnetic field **B** moving with a velocity v and making an angle θ with the direction of B. Its component $v \cos \theta$ along **B** will not be affected and will proceed in a straight line, while the perpendicular component $v \sin \theta$ will describe acircle. Hence, the resultant motionis a helix. The circular motion is governed by the relation

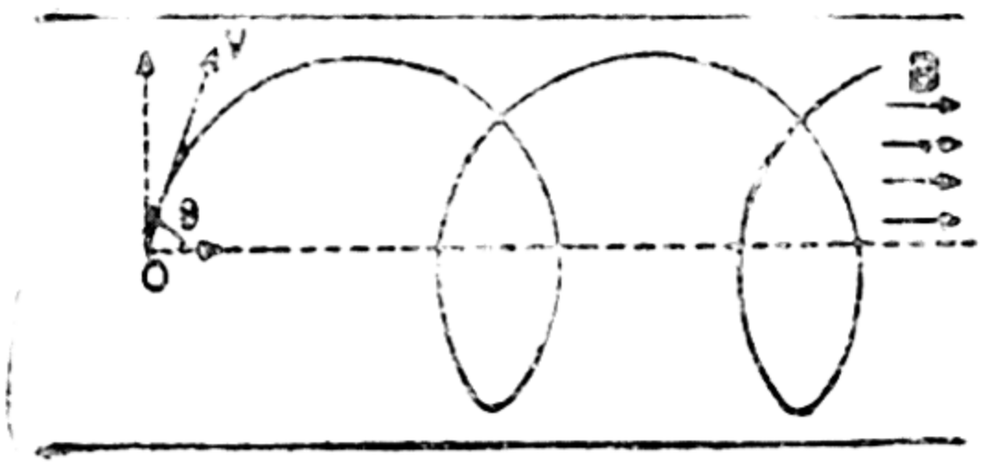


Fig. 6.12.

$$Bev \sin \theta=m (v \sin \theta)^2/R,$$

or the time taken by the electron to describe the circle

$$t=\frac{2\pi R}{v \sin \theta}=\frac{2\pi m}{Be}. \quad ...(16)$$

Hence, the pitch of the helix $L=OP=v \cos \theta \times t$

$$=2\pi vm \cos \theta/Be. \quad ...(17)$$

It is clear from the expression for t that this time does not depend on θ, hence all the electrons diverging from point O in any direction are all focussed on the same point P.

The distance OP depends upon **B**, hence the focal length of this magnetic lens can be altered by changing the value of **B**. The magnetic field represents the refractive medium which acts upon the electron beam as the glass does on a light beam.

There are several types of magnetic electron lens. Gabor in 1927 suggested that the magnetic field could be concentrated within a shorter distance along the axis of the solenoid coil if the latter was encased in an iron shield which reduced stray magnetic fields considerably. For the large magnifications, minimum focal length should be used. The schematic diagram of magnetic electron

lens is shown in Fig. 6.13. *AB* is the section of the coil of an electromagnet. The coilis surrounded by a shoft iron shield except for the gap *pq*. The magnetic field is strong across the gap between *p* and *q* and is symmetrical about the axis *OI* and thus focusses the electron beam at *I*.

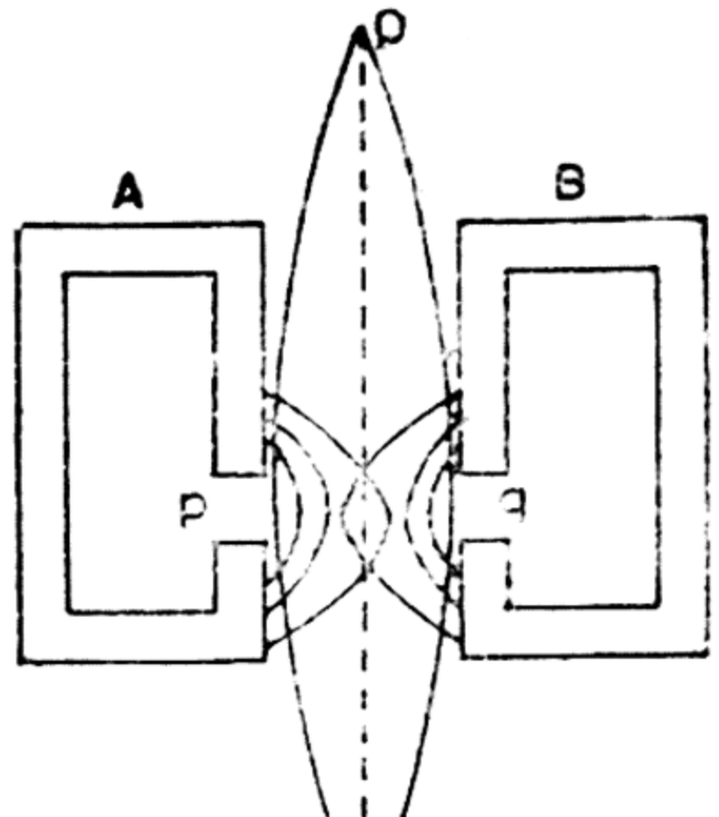


Fig. 6.13.

Electron microscopes are very useful in many fields of modern research in industry, medicine and study of atomic structure. The difficulty that the electrons destroy the object can be removed by different methods, one is to soak the specimen in a solution of osmium salts.

6.6. MASERS AND LASERS

The term *MASER* is an abbreviation for "*microwave amplification by stimulated emission of radiation*". Similarly *LASER* is an abbreviation for "*light amplification by stimulated emission of radiation*". For the former the emitted radiation is at microwave frequencies, while the latter produces light in the infra-red or visible red spectrum. Since in the laser the maser principle has been extended to optical frequencies, therefore, also termed as *optical maser.*

(A) Basic principle of maser operation. Let us first understand the terms : *coherence, stimulated emission, population inversion* and *pumping*.

(*i*) *Coherence.* The short wave transmitter should be powerful, pure, precise and free from noise. Therefore, the radiated power should be concentrated in a narrow frequency band which is known as *coherent radiation.* The sun radiations and radiations from light sources are incoherent as they cover a wide range of frequencies. For long range communication, intense and noise free (*i.e.*, coherent) radiation is required, which may be used to shoot at the pin point.

(*ii*) *Stimulated emission.* The process in which the excited atoms emit photons and drop to the lower levels is known as spontaneous emission. The process is called absorption if the atoms get excited due to the absorption of photons. If the photons of suitable energy size shower down on excited atoms in such a way that for the each incident photon two photons of size identical to the incident photon are emitted. The process is called *stimulated emission* and the incident photon the *stimulating photon*.

(*iii*) *Population inversion and pumping.* It is well-known fact that the population of atoms at the ground state is always largest at all possible temperatures. If by some means the population at

higher energy levels is made larger than that in the lower energy levels, the process is called *population inversion.* At this stage the chain reaction of emission will start, the method by which such a population inversion is affected is called *pumping.*

Here we see that the population inversion is essential to produce the stimulated emission and is done by pumping the matter by photons. The excited atoms in the active medium should be in excess so that the stimulated emission might predominate over absorption.

The first maser was constructed in 1954 by C.H. Townes and others. It is called the *ammonia gas maser.* The power output of the gas beam masers has been low as the molecules are not closely packed in the gas. The *solid state masers* are used now-a-days.

The materials in which the atoms are having unpaired electron are called paramagnetic substances and the unpaired electron can exist at a number of energy levels. The *ruby crystal* is a synthetic crystal in which a small percentage of aluminium atoms is replaced by chromium or phosphorous atoms (who have unpaired electron). The chromium atoms can be tuned magnetically to certain transition frequencies. For the operation, we proceed in the following steps :

The ruby crystal is subjected to a magnetic field that will fit a suitable energy gap between the lower energy and higher energy states. The crystal is then cooled by placing it in liquid helium. The most of chromium atoms thus fall to lower energy state. The crystal is then subjected to high power microwave signal. The frequency of which is varied. As the frequency passes the frequency corresponding to the energy difference between the chromium atoms in the lower and higher energy states, the chromium atoms raised to high energy state, *i.e.*, the population reversal takes place. These excited chromium atoms fall to lower energy state giving off radiations. Thus the stimulated emission starts at the input microwave signal frequency equal to the energy difference between the chromium atoms in lower and higher energy states. The maser action will therefore start. Such a maser is called *inversion maser.*

(B) Extension of maser principle to optical range (Laser). The theory of laser was introduced by Townes and Schawlow (1958). The materials used in lasers have some peculiar energy levels having comparatively large lifetimes ($\sim 10^{-8}$ sec), known as *metastable states.* The number of electrons in such a metastable states goes on increasing with time and a stage is soon reached when the number of electrons in the higher metastable state becomes greater than that in the ground state, *i.e.*, the population inversion is achieved. A medium prepared to have population inversion between two desired energy levels is called an *active medium.* To

achieve population inversion highly reflecting surfaces on either side of the active medium. The beam bounces back and forth between the mirrors on account of reflection and gets thus amplified. To obtain output laser beam, one reflector is made partially reflecting and partially transmitting, *i.e.*, *half silvered.* Such an arrangement is called '*resonant cavity*'. Between these mirrors, the light wave propagates in both directions and gradually gets amplified in the active medium. Due to interference, stationary waves are thus formed. The sides of the resonant cavity are open (transparent).

Active material is energized by the pulses of light from flash tube. The emitted photons that are not parallel to the axis of the resonator will leave the cylinder through the transparent sides. The others which are moving parallel to the axis of the tube will suffer several reflections between the reflectors. Due to the chain reaction of stimulated emission, the more and more electrons are produced. The intensity of photon beam parallel to the axis of tube is thus increased. Some of the coherent beam (stimulated photon beam parallel to the axis of the tube) is transmitted and serves as output laser beam. The simple resonator (resonant cavity) is shown in Fig 6.14.

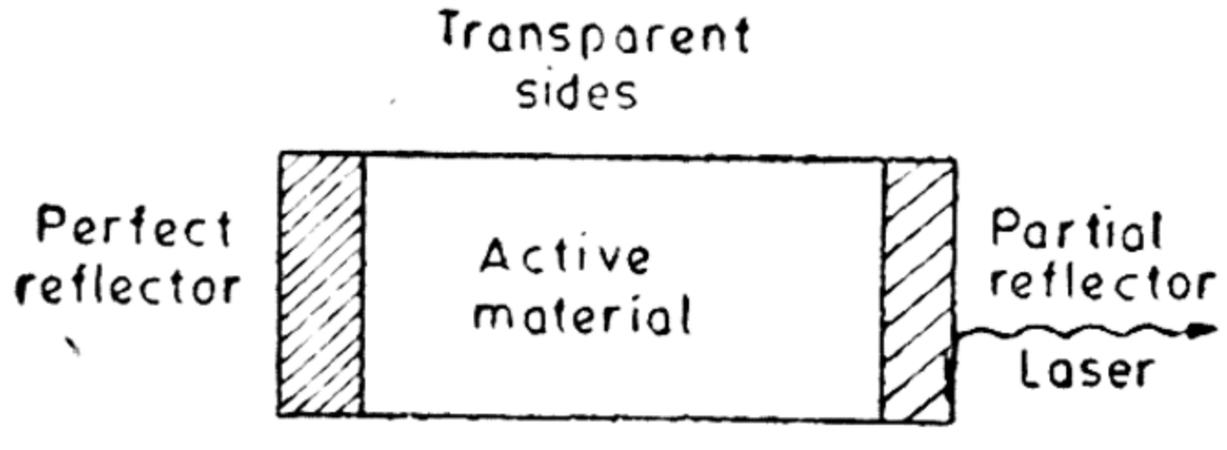


Fig. 6.14.

The basic problem in designing a laser is to prepare an active medium. This process is called *optical pumping.* In the different types of lasers, the different pumping methods are used.

Gas Laser. The *He-Ne* laser was first fabricated by Ali Javan and his colleagues in *USA.* The apparatus consists of a very long and narrow gas discharge tube filled with a mixture of helium and

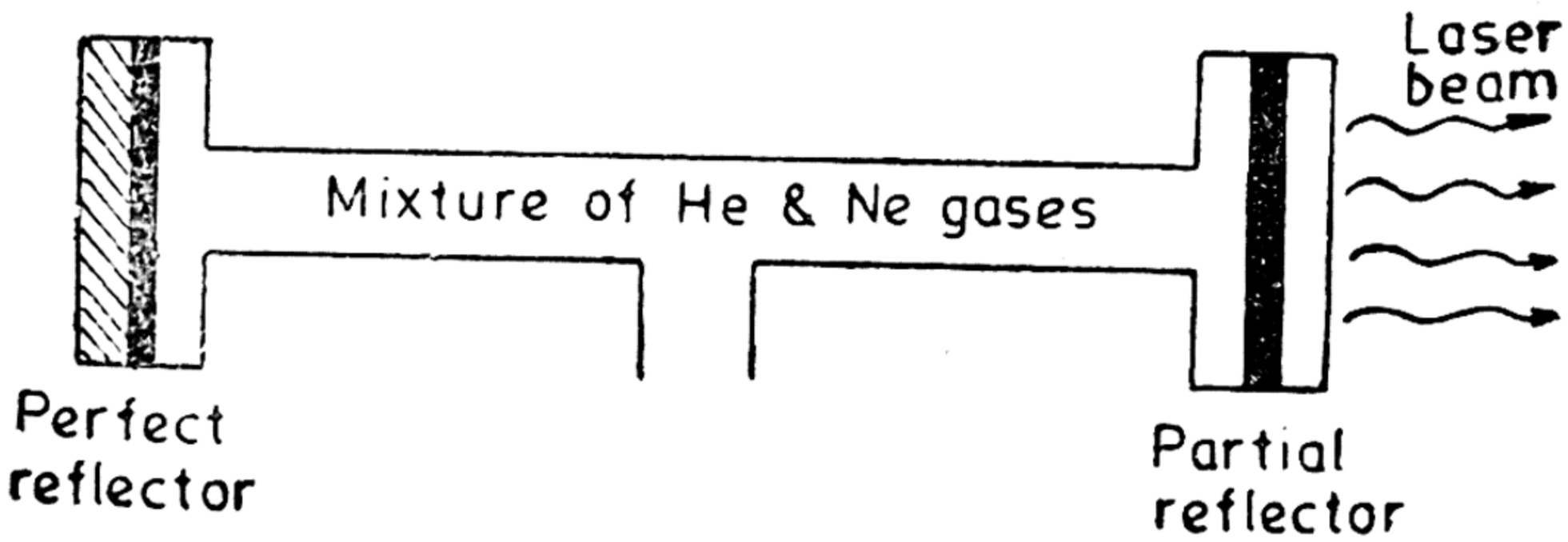


Fig. 6.15.

neon gases (*10 : 1*) at a pressure of 0.1 cm of mercury. At one end of the tube is a perfect reflector while at the other end is a partial reflector to permit the laser beam. When a discharge is passed

through the gas, the accelerated electrons moving along th discharge tube collide with the helium atoms and excite them t the higher levels. These excited helium atoms then collide with th neon atoms and excite them. The helium atoms return to the grou level. After sometimes there is a sizeable population of electrons a a population inversion state is achieved. The *Ne*-atom then dro to the lower level by spontaneous emission. The desired laser be is obtained by chosing the reflectivity of the end plates. The processes may be shown in Fig. 6.16.

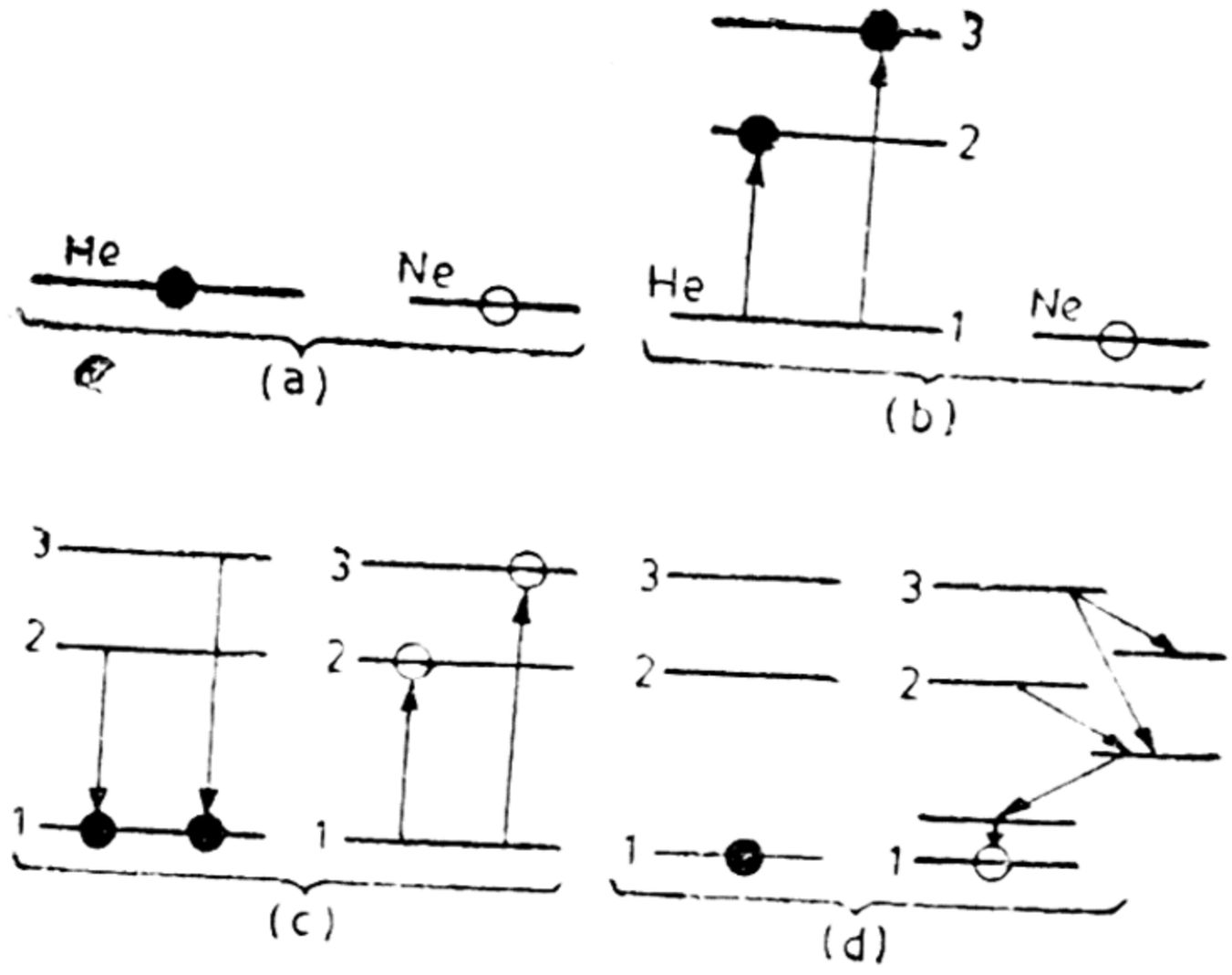


Fig. 6.16.

Ruby Laser. T.H. Maiman constructed in 1960 the first laser using a ruby crystal, which is an aluminium oxide (Al_2O_3) containing a small percentage of *Cr*-atoms (~0.05%). The Al_2O_3 is known as host crystal and the *Cr*-atoms are called activator atoms. The ruby crystal is a small cylinder of pink colour, about n long and about half a centimeter in diameter. The ends are perfectly flat and parallel to each other, one end (*A*) being pletely silvered while the other (*B*) partially silvered. Th crystal itself serves as resonant cavity. It is pumped by a p bursts of light from a flash tube (*T*), a discharge tube fi xenon gas. The tube is connected to a capacitor *C* w charged to a potential of few kilovolts and discharged through xenon tube. This results in a output power of few mega w Part of this power is utilised to excite chromium atoms. The pump thus gives a fairly broad band pumping radiation. chromium atoms absorb these radiations and get excited (E_1. states). These atoms immediately transit to the metastable (*M.S.*). The number of atoms in metastable states goes on incre ing and the population inversion is obtained. The transition from the metastable state to the ground state radiates photons, which after multiple reflections in the resonant cavity amplifies largely to an intense beam. This beam emerges from the partially silvered end of the ruby rod as a laser beam. The whole arrangement is shown in Fig. 6.17.

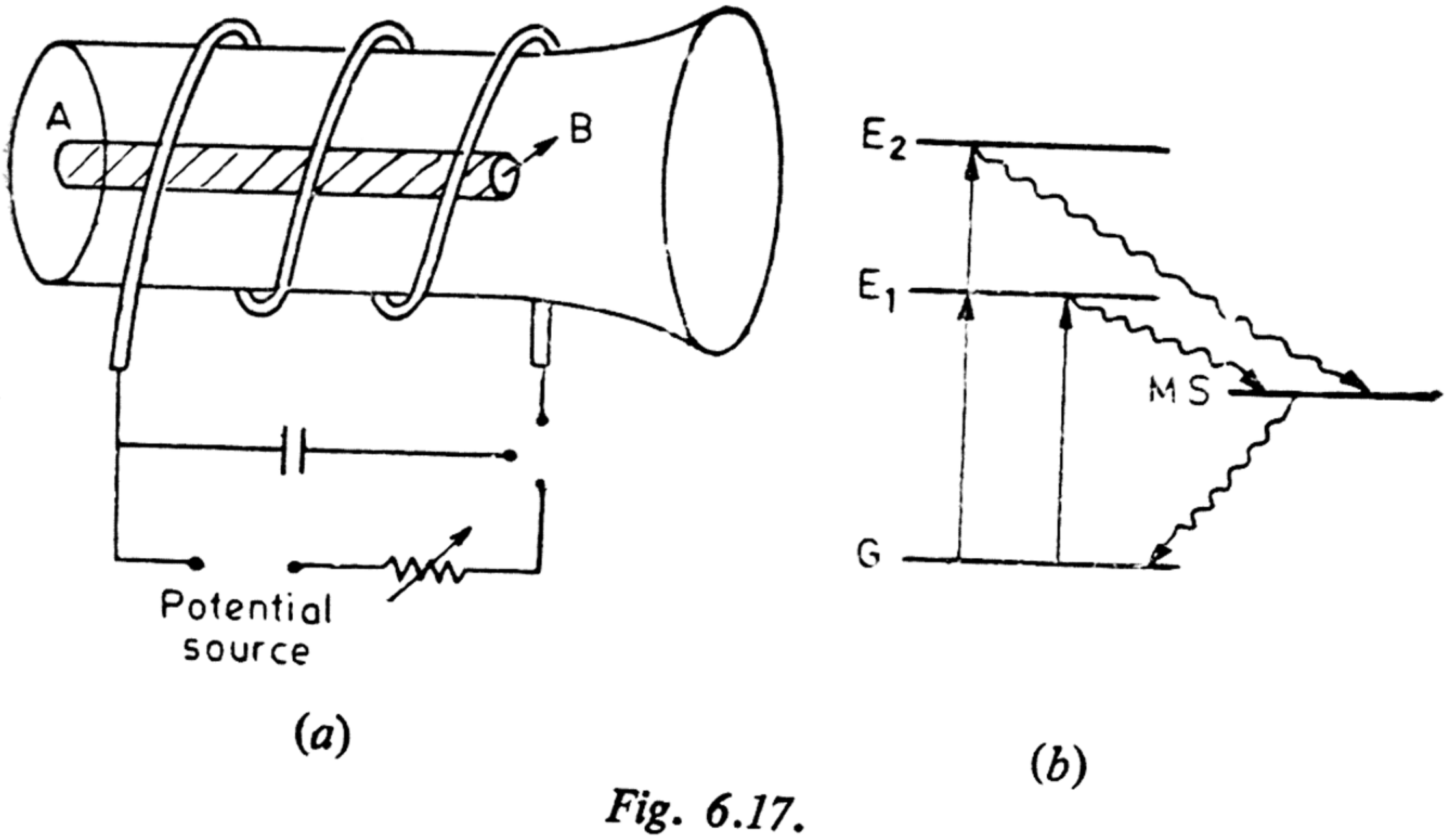


Fig. 6.17.

The main drawback of solid state or ruby laser is that the outbeam is not continuous on account of the imperfection of crystal, thermal distortion, scattering, etc. These factors are absent in the gas lasers. Thus the gas lasers emit more directional and perfectly monochromatic beam. Besides it the soild state lasers need covering while the gas lasers can operate continuously without any need of cooling. The solid state lasers produce intense beam. Thus the solid state lasers are commonly used as *amplifiers* while the gas lasers are used as *oscillators.*

Applications of Laser—(1) A controlled laser beam is used in medical surgery, *e.g.*, photo-coagulator (treatment of detached retinas), dentistry (treatment of teeth), controlled dosages (destroying cancerous cells, tumours and skin scars).

(2) The laser is used in industry, *e.g.*, in machining, welding cutting operations.

(3) The laser beam is used in chemical applications, *e.g.*, to or initiate chemical reactions.

) The laser beam has been widely used in communications ...en earth and satellites on account of narrow bandwidth, high ...ensity and high frequency.

(5) The laser beam is being developed in military to serve as ...ar weapon and as blind vision detectors for directing planes and ...ssiles.

(6) The monochromaticity of a laser is important in spectro... ...y and photography.

6.7. COMPUTERS

Computer is a technical device for accepting an output of information, processing this information according to some

prescribed programme of operations and supplying an output of processed information. The computers are millions of times faster than human beings at computing and are now very much useful in science, business, defence, government and other activities.

The computers may be classified according to the improvements in the systems as :

(*a*) **First generation computers (1940—52).** Such computers used vacuum tubes as the electronic components. The electrostatic tubes or mercury delay lines were used for storage. Paper taps and punched cards were used for the input data. The memory access time was 1 ms and the electronic time per operation ranged from 0.1 to 1 ms.

(*b*) **Second generation computers (1952—64).** Such computers used a magnetic core memory and later the transistors instead of the vacuum tubes. In this way they were more compact, reliable and fast. The memory access time from the magnetic drum or core was 1-10 μs. The electronic time per operation ranged from 1 to 10 μs.

(*c*) **Third generation computers (1964—70).** Such computers used the integrated solid state circuits, improved secondary storage devices and new input-output devices. The multi-processing and multi-programming were made possible. The memory access time was 0.1 to 10 μs. The electronic time per operation was 0.1 to 1 μs.

(*d*) **Fourth generation computers (1970—present).** Such computers used the microprocessor device, which led to the following developments—(*i*) much faster, less expensive and of much greater capacity large computers, (*ii*) less expensive minicomputers, and (*iii*) microcomputers. Such computers have advanced input-output devices : The optical readers by which whole documents can be fed into the computer ; audio response terminals by which an operator can vocally communicate with the computer and graphics display terminals by which pictures can be obtained from the computers.

(*e*) **Fifth generation computers (under development).** These are called thinking computers, expected to have capabilities of reasoning and making judgements. They may be capable of providing invaluable assistance to man in the field of medicine. They may help in new discoveries about galaxy.

The computers may be classified according to size and memory as :

(1) **Microcomputers.** A micro or personal computer is the smallest general purpose computer system. It can execute pro-

grams to perform a variety of instructions. Such computers usually have a 8, 16 or a 32 bit microprocessor. By an 8-bit microprocessor means that it can process 8 bits or 1 byte of data at a single given time.

(2) **Minicomputers.** A minicomputer system is a small general purpose computer. The minicomputers are multi-user computers in contrast to the micros. They are very useful in distributed data processing networks. Most of the minicomputers designed now-a-days are with the 32 bit microprocessor.

(3) **Mainframe computers.** A mainframe computer system is a fast computer, can process data much faster, as several microprocessors are used in place of the single one used in micro and minicomputer systems. 2 to 8 bytes can be operated automatically in the same unit of time.

(4) **Super computers.** Super computers are the most powerful and expensive computers. The time required to execute a single operation may even be of the order of nanoseconds. Such computers are very much useful in research and defence.

The computers may also be classified according to the data they process, as :

1. **Analog computers.** The analog computers handle data that is represented by physical quantities of continuously variable size such as voltage, current, temperature, length, etc.

2. **Digital computers.** The digital computers handle actual numbers expressed in digits and the quantities in the problem are represented by discrete numbers. The pulse circuitary is used in such computers.

3. **Hybrid computers.** The hybrid computers are those in which the desirable characteristics of both the analog and digital computers are integrated.

Analog computers are not so accurate as digital computers, but they are much simpler and therefore less expensive.

Analog computers make use of circuit, such as addition, subtraction, differentiating, integration, multiplication, division, etc. Analog computer can stimulate the response of any system whose response can be described mathematically. To give some understanding of working of analog computer we include very easy example of problem.

Consider the general case of simultaneous equations

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2.$$

When the coefficients a_1, b_1, etc., are greater than one, an amplifier must be used for multiplication by the constant, but when multiplier is a fraction, a simple voltage divider may be used. To make the problem easy, let us write above equations in the following way.

$$(1/b_1)\,x+(1/a_1)\,y-c_1/a_1b_1=0$$
$$(1/b_2)\,x+(1/a_2)\,y-c_2/a_2b_2=0. \qquad \ldots(18)$$

These equations are solved by using two summing amplifiers. The value of the output voltage of amplifier 1 is the numerical solution of Eq. (18) for x and amplifier 2 gives the solution for y. Amplifier 1 sums a fraction $1/b_1$ of x, a fraction $1/a_1$ of y and a constant $-c_1/a_1b_1$ to satisfy the first equation. Amplifier 2 is setup to satisfy the second equation. The outputs give the solution of the equations.

The computer system is composed of several sections, as outlined in the block diagram of Fig. 6.18. The input unit receives the instructions and parts of a problem from punched cards, taps

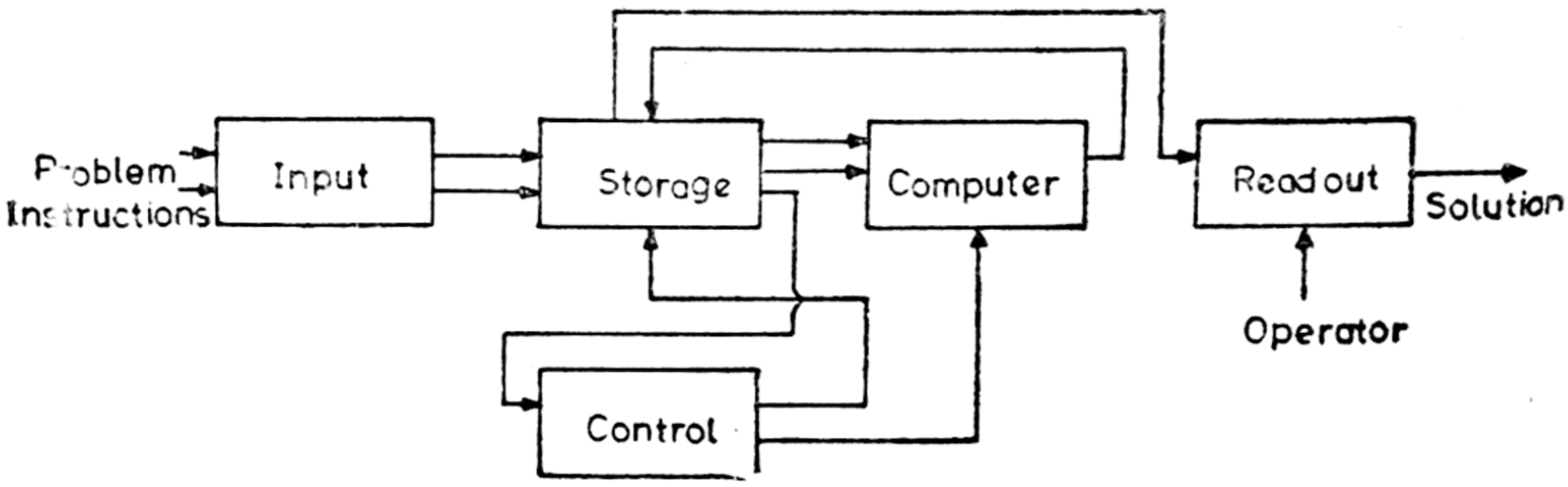


Fig. 6.18.

or switch setting. The control unit responds to the instructions from the input unit and selects the required functions. The sequence of operation is determined by the order in which the instructions are placed in the input. The storage unit provides a memory for the instructions, initial problem, and final answer. The readout unit receives the solution from the storage unit and puts it on tape, prints it. The computer unit contains the mathematical circuits required to solve a problem. It is a heart of the machine. The circuits for addition of numbers are formed of rows of two stable state multivibrators. Subtraction is accomplished by adding the complement of the subtrahend to the minuend. More complicated circuits are used for multiplication, division, etc. The computer unit complies with the instructions and is ready for another problem. Readout occurs when the operator instructs the machine to present the information contained in store.

6.8. DIGITAL CIRCUITS

Digital circuits have been very much used in the recent past in the digital systems such as digital computers, digital multimeters,

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